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UNIVERSAL TEICHMÜLLER SPACE AND FOURIER SERIES

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Abstract. In the usual definition of the universal Teichmüller space the upper half-plane U is assumed to be the universal covering surface of Riemann surfaces under consideration. The author points out that on replacing U by the unit disk $\mathbf{D}$ new problems may arise. E.g. there is a one-to-one correspondence between the equivalence classes generated by the Ahlfors–Bers equivalence relation and 2π-periodic functions σ vanishing at $2k\pi/3$, $k \in \mathbf{Z}$, such that $x \mapsto x + \sigma(x)$ is M-quasisymmetric on the real line $\mathbf{R}$. An estimate $n|c_n| \le 2(M-1)/(M+1)$ for complex Fourier coefficients c_n of σ is established. Moreover, an analytic criterion of the Ahlfors–Bers equivalence relation is obtained.

0. Introduction. Statement of results

The notion of the universal Teichmüller space (abbreviated: UTS) has its source in two fundamental papers [1], [13]. Teichmüller's research on quasiconformal mappings of Riemann surfaces disclosed the necessity of distinguishing different homotopy classes of such mappings. While trying to put some Teichmüller statements on a firm basis Ahlfors used the representation of a compact Riemann surface W of genus $g > 1$ as a quotient surface $\mathbf{D}/G$, where the unit disk $\mathbf{D}$ is the universal covering surface of W and G is the Fuchsian group of covering Möbius transformations of $\mathbf{D}$. As discovered by Ahlfors, two quasiconformal mappings f_1, f_2 of W lifted to $\mathbf{D}$ and suitably normalized are identical on $\mathbf{T} = \partial\mathbf{D}$ if and only if f_1, f_2 are in the same homotopy class. In this way the equivalence relation between complex dilatations μ_k of f_k compatible with the homotopy equivalence of f_k could be established which justifies the following definition of UTS.

Let B denote the unit ball in the space of measurable, complex-valued and essentially bounded functions $\mu\colon \mathbf{D} \mapsto \widehat{\mathbf{C}}$ and t_k $(k = 0, 1, 2)$ be fixed points of $\mathbf{T} = \partial\mathbf{D}$. Given $\mu \in B$ there exists a unique quasiconformal self-mapping f^μ of $\mathbf{D}$ with complex dilatation μ whose homeomorphic extension to $\overline{\mathbf{D}}$ keeps the points t_k fixed. We say *the Ahlfors–Bers equivalence relation $\mu \sim \nu$ holds between $\mu, \nu \in B$, if and only if $f^\mu(t) = f^\nu(t)$ for any $t \in \mathbf{T}$.* Then UTS is defined as the unit ball B whose points are subdivided into equivalence classes $[\mu] = \{\nu \in B : \nu \sim \mu\}$.

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If f_μ is a quasiconformal self-mapping of $\widehat{\mathbf{C}}$ which is conformal in $\mathbf{D}^* = \widehat{\mathbf{C}} \backslash \overline{\mathbf{D}}$, quasiconformal with complex dilatation μ in $\mathbf{D}$ and has $t_k \in \mathbf{T}$ as fixed points, then the identity $f_\mu \mid \mathbf{D}^* = f_\nu \mid \mathbf{D}^*$ sets up the same equivalence relation $\mu \sim \nu$, cf. [8; p. 99]. Note that we may take as universal covering surface the upper half-plane U instead of $\mathbf{D}$ and $0, 1, \infty$ as fixed points on its boundary, as it is done in the excellent monograph [8], which is our standard book of reference.

We show in Section 1 that the equivalence relation in B can be also defined without any reference to the boundary values (Theorem 1.1). This implies an analytic criterion of equivalence (Proposition 1.2) and a potential-theoretic interpretation of $[\mu]$ (Proposition 1.4). In Section 2 the class $S(K)$ of K-quasiconformal self-mappings of $\mathbf{D}$ with fixed points $t_k = \exp(2k\pi i/3)$, $k = 0, 1, 2$, is introduced and the location of points $f(0)$, $f \in S(K)$, is investigated (Theorem 2.1). As an application the quasisymmetry order of $f \mid \mathbf{T}$ for $f \in S(K)$ can be estimated (Theorem 2.2). Proposition 1.5 establishes a mutual correspondence between $[\mu]$ and real-valued functions $\sigma \in E_0(M)$. This means that σ is 2π-periodic, vanishes at $2k\pi/3$, $k = 0, \pm 1, \pm 2, \ldots$ and $x \mapsto x + \sigma(x)$ is M-quasisymmetric on the real axis $\mathbf{R}$. Estimates of Fourier coefficients of $\sigma \in E_0(M)$ are found (Proposition 3.2, Theorem 3.4). Moreover, a slight improvement of a result due to M. Nowak (Theorem 3.5) is obtained.

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1. Some criteria of equivalence in B

Suppose $\mu \in B$ and put

$$\tilde{\mu}(z) = \begin{cases} \mu(z), & z \in \mathbf{D} \\ 0, & z \in \mathbf{D}^* := \widehat{\mathbf{C}} \setminus \overline{\mathbf{D}}. \end{cases} \tag{1.1}$$

The singular integral equation

$$\varphi = \tilde{\mu} + \tilde{\mu} S(\varphi), \tag{1.2}$$

where S denotes the Hilbert–Beurling transform, has a unique $L^2(\mathbf{C})$-solution φ_μ whose support is contained in $\overline{\mathbf{D}}$. Moreover,

$$\tilde{f}_\mu(z) := z - \frac{1}{\pi} P.V. \iint\limits_{\mathbf{D}} \frac{\varphi_\mu(\zeta)\, d\xi\, d\eta}{\zeta - z}, \qquad \zeta = \xi + i\eta, \tag{1.3}$$

is the unique quasiconformal self-mapping of $\widehat{\mathbf{C}}$ with complex dilatation $\tilde{\mu}(z)$ a.e., cf. [4], [9; p. 218].

With this notation we have the following

Theorem 1.1. *If* $\mu, \nu \in B$ *then* $\mu \sim \nu$ *holds if and only if*

$$\tilde{f}_\mu \mid \mathbf{D}^* = \tilde{f}_\nu \mid \mathbf{D}^*. \tag{1.4}$$

Proof. Suppose

$$t_k = \exp(2k\pi i/3), \qquad k = 0, 1, 2, \tag{1.5}$$

are distinguished points on $\mathbf{T} = \partial\mathbf{D}$ and f_μ is the unique generalized homeomorphic solution of the Beltrami equation $\overline{\partial} f = \tilde{\mu}\partial f$ in $\widehat{\mathbf{C}}$ such that $f_\mu(t_k) = t_k$ ($k = 0, 1, 2$). As pointed out earlier, the equivalence relation $\mu \sim \nu$ holds if and only if $f_\mu \mid \mathbf{D}^* = f_\nu \mid \mathbf{D}^*$. Given $\mu \in B$ consider the class of all $\nu \in B$ such that $\tilde{f}_\nu \mid \mathbf{D}^* = \tilde{f}_\mu \mid \mathbf{D}^*$. Then, for t_k as in (1.5), the points

$$w_k = \tilde{f}_\nu(t_k), \qquad k = 0, 1, 2, \tag{1.6}$$

are all different and do not depend on a particular choice of ν. Consider the mapping $\tilde{f}_\nu \circ f_\nu^{-1}$. It is obviously conformal in $f_\nu(\mathbf{D}^*)$. It is also conformal in $f_\nu(\mathbf{D})$ since both $f_\nu, \tilde{f}_\nu$ are generalized homeomorphic solutions of the same Beltrami equation in $\mathbf{D}$. Now, $f_\nu(\mathbf{T})$ is a quasicircle, i.e. a removable set, cf. [9], and therefore $\tilde{f}_\nu \circ f_\nu^{-1} = h$, where h is the Möbius transformation sending t_k into w_k. Hence $\tilde{f}_\nu \mid \mathbf{D}^* = h \circ f_\nu \mid \mathbf{D}^*$ for all $\nu \sim \mu$ and consequently, $f_\nu \mid \mathbf{D}^* = f_\mu \mid \mathbf{D}^*$ (i.e. $\nu \sim \mu$) implies $\tilde{f}_\nu \mid \mathbf{D}^* = \tilde{f}_\mu \mid \mathbf{D}^*$. The converse statement follows from the identity $f_\nu \mid \mathbf{D}^* = h^{-1} \circ \tilde{f}_\nu \mid \mathbf{D}^*$ and this ends the proof.

As an immediate consequence we obtain

Proposition 1.2. *If* $\mu, \nu \in B$ *and* φ_μ, φ_ν *are the* L^2*-solutions of* (1.2) *then* $\mu \sim \nu$ *if and only if*

$$\iint_{\mathbf{D}} \varphi_\mu(z) z^k \, dx\, dy = \iint_{\mathbf{D}} \varphi_\nu(z) z^k \, dx\, dy, \qquad z = x + iy,\ k = 0, 1, 2, \dots. \tag{1.7}$$

Proof. It follows from (1.3) that for $\mu \in B$ and $z \in \mathbf{D}^*$

$$\begin{aligned} \tilde{f}_\mu(z) &= z + \frac{1}{\pi z} \iint_{\mathbf{D}} [1 + \zeta/z + (\zeta/z)^2 + \cdots]\varphi_\mu(\zeta)\, d\xi\, d\eta \\ &= z + \sum_{n=1}^{\infty} b_n z^{-n}, \qquad \text{where } \zeta = \xi + i\eta \text{ and} \qquad (1.8)\\ b_{k+1} &= \frac{1}{\pi} \iint_{\mathbf{D}} \varphi_\mu(\zeta)\zeta^k \, d\xi\, d\eta, \qquad k = 0, 1, 2, \dots. \end{aligned}$$

However, by Theorem 1.1 the coefficients b_k are the same for any $\nu \in [\mu]$ which implies (1.7).

While proving Theorem 1.1 we have seen that $f_\mu \mid \mathbf{D}^* = h^{-1} \circ \tilde{f}_\mu \mid \mathbf{D}^*$. Suppose now that $\mathbf{\Gamma} = \tilde{f}_\mu(\mathbf{T})$ and g is the conformal mapping of the inside of $\mathbf{\Gamma}$ onto $\mathbf{D}$ sending $w_k \in \mathbf{\Gamma}$ into t_k as given by (1.5). With this notation we obtain

Corollary 1.3. *The functions f^μ, f_μ defined in the Introduction can be expressed by $\tilde{f}_\mu$ as follows*:

$$f^\mu = g \circ \tilde{f}_\mu \mid \mathbf{D}, \qquad f_\mu \mid \mathbf{D}^* = h^{-1} \circ \tilde{f}_\mu \mid \mathbf{D}^*. \tag{1.9}$$

If $\mathbf{\Gamma}$ is a Jordan curve in the finite plane then the conformal mapping f of $\mathbf{D}^*$ onto the unbounded component of $\widehat{\mathbf{C}} \setminus \mathbf{\Gamma}$ satisfying $f(\infty) = \infty$ has the form

$$f(z) = az + \sum_{n=0}^{\infty} b_n z^{-n}, \qquad z \in \mathbf{D}^*.$$

The transfinite diameter $d(\mathbf{\Gamma})$ of $\mathbf{\Gamma}$ is equal to $|a|$, whereas

$$b_0 = b_0(\mathbf{\Gamma}) = \frac{1}{2\pi i} \int_{|z|=R>1} f(z) z^{-1}\, dz = \int_{\mathbf{\Gamma}} w \, \frac{d\theta}{2\pi}$$

is *the conformal centre of gravity* of $\mathbf{\Gamma}$, cf. [12; Chapter IV, Problem 138]. Note that for any subarc α of $\mathbf{\Gamma}$ the angular measure of $f^{-1}(\alpha)$ generates a probability measure $\int_\alpha d\theta/2\pi$ on $\mathbf{\Gamma}$.

A quasicircle $\mathbf{\Gamma}$ in the finite plane such that $d(\mathbf{\Gamma}) = 1$, $b_0(\mathbf{\Gamma}) = 0$, is said to be *normalized.* We have following

Proposition 1.4. *There is a one-to-one correspondence between normalized quasicircles $\mathbf{\Gamma}$ and the classes $[\mu]$ of the* UTS.

Proof. If $\mu \in B$ then $\tilde{f}_\mu(\mathbf{T})$ is a normalized quasicircle according to the formula (1.3) and the class $[\mu]$ of the UTS is defined by the equivalence relation (1.4). If $\mathbf{\Gamma}$ is a normalized quasicircle then the unbounded component of $\widehat{\mathbf{C}} \setminus \mathbf{\Gamma}$, due to the Riemann mapping theorem, is the image domain of $\mathbf{D}^*$ under some f in the familiar class $\sum$ with constant term $b_0 = 0$. Since $\mathbf{\Gamma}$ is a quasicircle, it admits a quasiconformal reflection J (cf. [9; p. 99]) which may serve in the construction of a quasiconformal extension of f to $\mathbf{D}$. If $S\colon z \mapsto 1/\overline{z}$ then $\varphi = J \circ f \circ S$ maps $\mathbf{D}$ quasiconformally onto the inside of $\mathbf{\Gamma}$. Putting $\mu = \varphi_{\overline{z}}/\varphi_z$ we easily verify that

$$\tilde{f}_\mu = \begin{cases} \varphi(z), & z \in \overline{\mathbf{D}}, \\ f(z), & z \in \mathbf{D}^* \end{cases}$$

defines the class $[\mu]$ of the UTS.

In what follows we need a counterpart of the classical Beurling–Ahlfors theorem (cf. [3], or [9; pp. 81, 83]) for the unit disk which we quote as

Lemma A [6; p. 21, 22]. *An automorphism (= a sense preserving homeomorphic self-mapping) h of the unit circle $\mathbf{T}$ admits a quasiconformal extension to the unit disk $\mathbf{D}$ if and only if there exists M such that the inequality*

$$|h(\alpha_1)|/|h(\alpha_2)| \le M \tag{1.10}$$

holds for all pairs α_1, α_2 of disjoint adjacent open subarcs α_1, α_2 of $\mathbf{T}$ with equal length $|\alpha_1| = |\alpha_2|$.

An automorphism h of $\mathbf{T}$ satisfying (1.10) is said to be *an M-quasisymmetric function on* $\mathbf{T}$ and then we write $h \in Q(M)$. If $h(e^{i\theta}) = \exp\big(i\varphi(\theta)\big)$ then $\varphi(\theta) = \theta + \sigma(\theta)$ is an M-quasisymmetric function on $\mathbf{R}$ with the same M as in (1.10), cf. [6; p. 21], i.e. φ satisfies the M-condition

$$M^{-1} \le \frac{\varphi(\theta+d) - \varphi(\theta)}{\varphi(\theta) - \varphi(\theta-d)} \le M, \qquad 0 \ne d,\ \theta \in \mathbf{R}. \tag{1.11}$$

The difference $\sigma(\theta) := \varphi(\theta) - \theta$ is a continuous, 2π-periodic function of bounded variation which is represented by its Fourier series. It measures the deviation of $\varphi(\theta)$ from the identity. Given $\varphi(\theta)$ satisfying (1.11) the Beurling–Ahlfors construction leads to a quasiconformal extension of φ to the upper half-plane and subsequent exponentiation yields a quasiconformal automorphism h of $\mathbf{D}$ which satisfies $h(0) = 0$, cf. [6; p. 22].

The class of all 2π-periodic functions σ such that $\varphi(\theta) = \theta + \sigma(\theta)$ is M-quasisymmetric on $\mathbf{R}$, i.e. satisfies (1.11), is denoted by $E(M)$, whereas $\widetilde{Q}(M)$ will stand for the class of $\varphi(\theta) = \theta + \sigma(\theta)$ with $\sigma \in E(M)$.

We shall also consider the subclass $E_0(M) = \{\sigma \in E(M) : \sigma(2k\pi/3) = 0,\ k = 0,1,2\}$ and the corresponding subclasses $Q_0(M) \subset Q(M)$, $\tilde{Q}_0(M) \subset \tilde{Q}(M)$ consisting of functions with t_k and $2k\pi/3$, respectively, as fixed points.

Suppose $\mathbf{\Gamma}$ is a normalized quasicircle and $\mu \in B$ is associated with $\mathbf{\Gamma}$ as in Proposition 1.4. Then $F := \tilde{f}_\mu^{-1}$ is the conformal mapping of the outside of $\mathbf{\Gamma}$ onto $\mathbf{D}^*$ sending $w_k = \tilde{f}_\mu(t_k)$ into t_k, whereas the conformal mapping of the inside of $\mathbf{\Gamma}$ onto $\mathbf{D}$ sending w_k into t_k may be denoted by f. Since $\mathbf{\Gamma}$ is a quasicircle, there exists a quasiconformal reflection J in $\mathbf{\Gamma}$ and consequently $h := f \circ J \circ F^{-1} \circ S$, S: $z \mapsto 1/\overline{z}$, is a quasiconformal self-mapping of $\mathbf{D}$. This implies that $h \mid \mathbf{T} = f \circ F^{-1} \in Q_0(M)$.

Conversely, given $h \in Q_0(M)$, there exist a quasicircle γ and conformal mappings f, F of components of $\widehat{\mathbf{C}} \setminus \gamma$ onto $\mathbf{D}$ and $\mathbf{D}^*$, respectively, such that $f \circ F^{-1} \in Q_0(M)$. This is a consequence of the sewing theorem (cf. [11], or [9; p. 92], where complementary half-planes instead of $\mathbf{D}$, $\mathbf{D}^*$ are considered). After a suitable Möbius transformation γ becomes a normalized quasicircle $\mathbf{\Gamma}$, while $f \circ F^{-1}$ remains unchanged. Note that f and F are conformal mappings between Jordan domains and hence both mappings have homeomorphic extensions to the closures of relevant domains. This implies that $f\circ F^{-1}$ is a well-defined automorphism of $\mathbf{T}$ coinciding with the given h. In this way, taking into account Proposition 1.4, we obtain

Proposition 1.5. *There is a one-to-one correspondence between the quasisymmetric functions $h\in Q_0(M)$ on $\mathbf{T}$ and the classes $[\mu]$ of the* UTS.

2. The classes $S(K)$ and $Q_0(M)$

As stated in Proposition 1.5, there is a one-to-one correspondence between the classes $[\mu]$ of the UTS and the M-quasisymmetric functions $h\in Q_0(M)$. Since any $h\in Q_0(M)$ admits a K-quasiconformal extension on $\mathbf{D}$ with some $K\ge 1$ and fixed points t_k, the problem arises to establish a relation between M and K. To this end we introduce the family $S(K)$ of K-quasiconformal self-mappings f of $\mathbf{D}$ with fixed points t_k, where t_k are defined by (1.5). In what follows we are going to determine a majorant set $N(K)$ for $\{z=f(0): f\in S(K)\}$.

Suppose

$$\mathscr{K}(r)=\int_0^1[(1-t^2)(1-r^2t^2)]^{-1/2}\,dt,\qquad 0<r<1,$$

is the Legendre normal integral, cf. [9; p. 60]. The functions

$$\mu(r)=\frac{\pi}{2}\frac{\mathscr{K}(\sqrt{1-r^2}\,)}{\mathscr{K}(r)},\qquad \varphi_K(r)=\mu^{-1}\big(\mu(r)/K\big),\ K>0,$$

appear in many extremal problems concerning quasiconformal mappings, cf. [9; pp. 60–68], including the solution of the problem just announced. The latter problem can be stated as

Theorem 2.1. *Suppose that $K>1$ and x_1, x_2 $(-1<x_1<0<x_2<1)$ are unique solutions of the equations*

$$(2.1)\qquad u(x)=\varphi_{1/K}(\sqrt{3}/2),\qquad u(x)=\varphi_K(\sqrt{3}/2),$$

where

$$u(x)=\cos\Big(\operatorname{arccot}\frac{1+2x}{\sqrt{3}}-\frac{\pi}{6}\Big),\qquad x\in[-1,1].$$

Then the set $N_0(K):=\{z=f(0): f\in S(K)\}$ is contained in the compact subset $N(K)$ of $\mathbf{D}$ described as follows.

Denote by γ_x a circular arc with end-points $t_1=\exp(2\pi i/3)$, $t_2=\overline{t}_1$, which intersects the real axis at $x\in(-1,1)$ and let $A_0\subset\overline{\mathbf{D}}$ be the closed circular wedge whose boundary ∂A_0 is $\gamma_{x_1}\cup\gamma_{x_2}$. If $A_1=\exp(2\pi i/3)A_0$, $A_2=\exp(4\pi i/3)A_0$ then $N_0(K)\subset N(K):=A_0\cap A_1\cap A_2$.

Proof. Let α_0 be the smaller arc of $\mathbf{T}$ with end-points t_1, t_2. For any $z \in \gamma_x$ the harmonic measure $\omega(z, \alpha_0)$ satisfies

$$\omega(z,\alpha_0) = \omega(x,\alpha_0) = \frac{2}{\pi}\operatorname{arccot}\frac{1+2x}{\sqrt{3}} - \frac{1}{3}, \qquad x \in (-1,1). \tag{2.3}$$

Since $u(x) = \cos\frac{1}{2}\pi\omega(x,\alpha_0)$, $u(x)$ strictly increases from 0 to 1 for $x \in (-1,1)$. For $K > 1$ and $0 < x < 1$ we have $\varphi_{1/K}(x) < x < \varphi_K(x)$, hence $\varphi_{1/K}(\sqrt{3}/2) < \sqrt{3}/2 = u(0) < \varphi_K(\sqrt{3}/2)$. This implies the existence of unique solutions x_1, x_2 of the equations (2.1).

If $\mathbf{\Gamma}$ is the family of arcs in $\mathbf{D}\setminus\{z\}$ with end-points on α_0 separating z from $\mathbf{T}\setminus\alpha_0$ then its module satisfies (cf. [5])

$$M(\mathbf{\Gamma}) = \frac{1}{\pi}\mu\big[\cos\big(\tfrac{1}{2}\pi\omega(z,\alpha_0)\big)\big]. \tag{2.4}$$

By (2.2)–(2.4) we arrive at

$$M(\mathbf{\Gamma}) = \frac{1}{\pi}\mu\big(u(x)\big). \tag{2.5}$$

If $f \in S(K)$ then also $f^{-1} \in S(K)$; here f^{-1} sends z into 0 and $\mathbf{\Gamma}$ into $\mathbf{\Gamma}'$ separating 0 from $\mathbf{T}\setminus\alpha_0$. From the equality $M(\mathbf{\Gamma}') = \frac{1}{\pi}\mu\big(\cos(\pi/6)\big) = \frac{1}{\pi}\mu(\sqrt{3}/2)$ and the quasi-invariance of the module, we have for any $z = f(0) \in \gamma_x$

$$K^{-1}\mu(\sqrt{3}/2) \le \mu\big(u(x)\big) \le K\mu(\sqrt{3}/2).$$

Since $\mu(x)$ is strictly decreasing for $x \in (0,1)$, we obtain

$$\varphi_{1/K}(\sqrt{3}/2) \le u(x) \le \varphi_K(\sqrt{3}/2), \qquad x_1 \le x \le x_2. \tag{2.6}$$

It follows from the definition of A_0, together with (2.1), (2.2) and (2.4), that $f(z) = 0$ is impossible for $z \in \mathbf{D}\setminus A_0$. Similar reasoning can be applied to A_1 and A_2 so that ultimately $f(z) = 0$ implies $z \in N(K) = A_0 \cap A_1 \cap A_2$.

Remarks. 2.1.1. There exist $f_j \in S(K)$ such that $f_j(0) = x_j$, $j = 1,2$. If $\mathbf{D}^+$ is the upper half of $\mathbf{D}$ then f_j are extremal quasiconformal mappings of the quadrilateral $\mathbf{D}^+(-1,0,1,t_1)$ onto $\mathbf{D}^+(-1,x_j,1,t_1)$ extended by reflection to $\mathbf{D}$.

2.1.2. The region $N(K)$ is a circular hexagon whose boundary consists of three “major” arcs, one being a subarc of γ_{x_2} bisected by $x_2 \in N_0(K)$, two others arising under its rotations by the angles $2\pi/3$, $4\pi/3$. Three remaining “minor” arcs are a subarc of γ_{x_1} bisected by x_1 and its rotations. The vertices w_k of $N(K)$ where the “major” and “minor” arcs meet are equidistant from the origin, and the disk $\{z : |z| \le |w_j|\}$, $|w_j| = r(K)$, contains all the points $z = f(0)$, $f \in S(K)$.

We now prove

Theorem 2.2. *If $f \in S(K)$ then $f \mid \mathbf{T} \in Q_0(M)$ with $M \le \lambda(KK_0)$, where λ is the distortion function defined by the formula*

$$\lambda(K) = [\mu^{-1}(\pi K/2)]^{-2} - 1, \qquad \text{cf. [9; p. 81]}, \tag{2.7}$$

and $K_0 = (1+|z_0|)(1-|z_0|)^{-1}$, $z_0 = f(0) \in N_0(K)$. In particular, we may take $K_0 = \big(1+r(K)\big)\big(1-r(K)\big)^{-1}$, $r(K)$ being defined in Remark 2.1.2.

Proof. With $w = g(z) = i(1+z)(1-z)^{-1}$ and $z_0 \in \mathbf{D}$ define

$$F(z) = (1-|z_0|^2)^{-1}[(1-z_0)w + z_0(1-\overline{z}_0)\overline{w}].$$

It is easily verified that the function

$$z \mapsto L(z,z_0) = [F(z)-i][F(z)+i]^{-1} = g^{-1} \circ F(z) \tag{2.8}$$

maps $\mathbf{D}$ quasiconformally onto itself so that $L(z_0,z_0) = 0$ and $L(t,z_0) = t$ for any $t \in \mathbf{T}$. Thus complex dilatations of L and F are identical and so

$$\frac{\overline{\partial} L}{\partial L} = \frac{z_0(1-\overline{z}_0)}{1-z_0}\frac{\overline{g'(z)}}{g'(z)},$$

hence $|\overline{\partial} L/\partial L| = |z_0|$. Consequently, L is K_0-quasiconformal with

$$K_0 = (1+|z_0|)(1-|z_0|)^{-1}.$$

Given a K_1-quasiconformal self-mapping h of $\mathbf{D}$ satisfying $h(0) = 0$ the inequality (1.10) takes the form $|h(\alpha_1)|/|h(\alpha_2)| \le \lambda(K_1)$, where λ is defined by (2.7), cf. [6; p. 21]. Hence $h \mid \mathbf{T} \in Q(M)$ with $M \le \lambda(K_1)$.

Suppose now that $f \in S(K)$ and $f(0) = z_0$ so that $|z_0| \le r(K) < 1$ by Remark 2.1.2. The composite mapping $h = L \circ f$ has the same boundary values as f, is KK_0-quasiconformal and satisfies $h(0) = 0$. Hence $h \mid \mathbf{T} \in Q(M)$ with $M \le \lambda(KK_0)$. On the other hand, $h \mid \mathbf{T} = f \mid \mathbf{T} \in Q_0(M)$ and consequently $f \mid \mathbf{T} \in Q_0(M)$ with $M \le \lambda(KK_0)$, so we are done.

3. UTS and Fourier series

According to Proposition 1.5 there exists a one-to-one correspondence between the classes $[\mu]$ of UTS and quasisymmetric functions $h \in Q_0(M)$. On the other hand, any $h \in Q_0(M)$ is determined by an M-quasisymmetric function $\varphi(\theta) = \theta + \sigma(\theta) \in \widetilde{Q}_0(M)$, or, equivalently, by $\sigma \in E_0(M)$. Thus any continuous, 2π-periodic function σ vanishing at $2k\pi/3$, $k \in \mathbf{Z}$, such that $x+\sigma(x)$ is M-quasisymmetric on $\mathbf{R}$ may be considered as a class of UTS. A more general class $E(M)$, without the normalization $\sigma(2k\pi/3) = 0$, has been studied in [7]. We shall use two estimates proved there and quoted here as

Lemma B. *If h is M-quasisymmetric on $\mathbf{R}$ and $h(x)-x$ vanishes at the end-points of an interval I then*

$$|h(x)-x| \le |I|\,\frac{M-1}{M+1} \qquad \text{for any } x \in I, \tag{3.1}$$

and

$$\int_I |h(x)-x|\,dx \le \frac{1}{2}\,|I|^2\frac{M-1}{M+1}, \tag{3.2}$$

cf. [7; (2.7), (2.13)].

As an immediate consequence of (3.1) we obtain

Proposition 3.1. *If $\sigma \in E_0(M)$ then for any $x \in \mathbf{R}$*

$$|\sigma(x)| \le \frac{2\pi}{3}\frac{M-1}{M+1}. \tag{3.3}$$

Any $\sigma \in E(M)$ is the sum of its Fourier series:

$$\sigma(x) = a_0/2 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

If we introduce complex Fourier coefficients $c_n = b_n + i a_n$ then σ has the following representation

$$\sigma(x) = c_0 + \frac{1}{2i}\sum_{n=1}^{\infty}(c_n e^{inx} - \overline{c}_n e^{-inx}). \tag{3.4}$$

The inequality (3.2) implies at once

Proposition 3.2. *If $\sigma \in E_0(M)$ then*

$$|c_0| \le \frac{\pi}{3}\,\frac{M-1}{M+1}. \tag{3.5}$$

Proof. Since σ vanishes at the end-points of three adjacent intervals of length $2\pi/3$, we obtain by (3.2)

$$|c_0| = \Big|\frac{1}{2\pi}\int_0^{2\pi}\sigma(x)\,dx\Big| \le \frac{1}{2\pi}\int_0^{2\pi}|\sigma(x)|\,dx \le \frac{1}{2\pi}\cdot 3\cdot\frac{1}{2}\Big(\frac{2\pi}{3}\Big)^2\frac{M-1}{M+1} = \frac{\pi}{3}\frac{M-1}{M+1}.$$

In order to derive a bound for $|c_n|$, $n \in \mathbf{N}$, we need the following

Lemma 3.3. *If $\sigma \in E(M)$ has the representation* (3.4) *then there exists $h \in \widetilde{Q}(M) = \mathrm{id} + E(M)$ such that $h(0) = 0$, $h(2\pi) = 2\pi$ and*

$$\pi n |c_n| = \int_0^{2\pi} h(x) \cos x \, dx. \tag{3.6}$$

The proof will be done in three steps.

I. *There exists $h_1 \in \widetilde{Q}(M)$ such that*

$$\pi n c_n = \int_0^{2\pi} e^{-inx} \, dh_1(x). \tag{3.7}$$

Proof. Multiplying both sides of (3.4) by e^{-inx} and integrating over $[0, 2\pi]$ we obtain

$$\pi n c_n = -\int_0^{2\pi} \sigma(x) \, d(e^{-inx}). \tag{3.8}$$

Integrating by parts we get

$$\int_0^{2\pi} [e^{-inx} \, d\sigma(x) + \sigma(x) \, d(e^{-inx})] = [\sigma(x) e^{-inx}]_0^{2\pi} = 0.$$

Since $\int_0^{2\pi} e^{-inx} \, dx = 0$, we obtain (3.7) by taking $h_1(x) = x + \sigma(x)$ and using (3.8).

II. *There exists $h_0 \in \widetilde{Q}(M)$ such that*

$$\pi n c_n = \int_0^{2\pi} e^{-it} \, dh_0(t). \tag{3.9}$$

Proof. We have by (3.7) for $k \in \mathbf{N}$

$$\begin{aligned} \pi n c_n &= \int_0^{2\pi} e^{-inx} \, dh_1(x) = \int_{2k\pi/n}^{2\pi + 2k\pi/n} e^{-inx} \, dh_1(x) \\ &= \int_0^{2\pi} \exp\Big[-in\Big(x + \frac{2k\pi}{n}\Big)\Big] \, dh_1\Big(x + \frac{2k\pi}{n}\Big) = \int_0^{2\pi} e^{-inx} \, dh_1\Big(x + \frac{2k\pi}{n}\Big). \end{aligned}$$

Hence, by taking $k = 0, 1, \ldots, n-1$ and adding, we obtain $\pi n c_n = \int_0^{2\pi} e^{-inx} \, dh_2(x)$, where

$$h_2(x) = \frac{1}{n}\Big[h_1(x) + h_1\Big(x + \frac{2\pi}{n}\Big) + \cdots + h_1\Big(x + \frac{2(n-1)\pi}{n}\Big)\Big]. \tag{3.10}$$

Since $h_1 \in \widetilde{Q}(M)$, it follows from (3.10) that

$$h_2(x + 2\pi/n) = h_2(x) + 2\pi/n \tag{3.11}$$

and hence

$$\int_0^{2\pi} e^{-inx}\, dh_2(x) = n \int_0^{2\pi/n} e^{-inx}\, dh_2(x) = n \int_0^{2\pi} e^{-it}\, dh_2(t/n)$$

where $0 \le nx = t \le 2\pi$. Note that by (3.11) the integrand does not change if x increases by $2\pi/n$. It is easily verified that $h_0(t) := nh_2(t/n) \in \widetilde{Q}(M)$. Obviously h_0 is M-quasisymmetric on $\mathbf{R}$, cf. [7; p. 227]. Moreover

$$h_0(t + 2\pi) - h_0(t) = n\big[h_2(t/n + 2\pi/n) - h_2(t/n)\big] = 2\pi$$

by (3.11). Thus $h_0 \in \widetilde{Q}(M)$ and (3.9) follows.

III. We now prove (3.6). Putting $c_n = i|c_n|e^{i\alpha}$, $\alpha \in \mathbf{R}$, and using (3.9) we obtain

$$\begin{aligned}\pi n|c_n| = -ie^{-i\alpha}\pi n c_n &= -i \int_0^{2\pi} e^{-i(t+\alpha)}\, dh_0(t)\\ &= -i \int_\alpha^{2\pi+\alpha} e^{-ix}\, dh(x) = -i \int_0^{2\pi} e^{-ix}\, dh(x),\end{aligned}$$

where $x = t + \alpha$, $h(x) = h_0(t)$. Thus

$$\pi n|c_n| = \int_0^{2\pi} (-\sin x)\, dh(x). \tag{3.12}$$

Integration by parts yields

$$\int_0^{2\pi} \big[-\sin x\, dh(x) + h(x)\, d(-\sin x)\big] = \big[-h(x)\sin x\big]_0^{2\pi} = 0$$

and hence, because of (3.12), (3.6) follows. Since adding a constant to $h(x)$ does not change the right side in (3.6), we may assume that $h(0) = 0$, $h(2\pi) = 2\pi$.

We now prove the main result of this section, i.e.

Theorem 3.4. *If $x + \sigma(x)$ is M-quasisymmetric on $\mathbf{R}$ and σ has the expansion* (3.4) *then*

$$n|c_n| \le 2\,\frac{M-1}{M+1}. \tag{3.13}$$

Proof. By (3.6) it is sufficient to show that for any $h \in \widetilde{Q}(M)$ satisfying $h(0) = 0$ we have

$$\int_0^{2\pi} h(x)\cos x\,dx \le 2\pi\,\frac{M-1}{M+1}. \tag{3.14}$$

Suppose $x \in (0, \pi/2)$ and $\cos x = y$. Then also $\cos(2\pi - x) = y$, whereas $\cos(\pi \pm x) = -y$. Thus putting

$$H(x) = h(x) - h(\pi - x) - h(\pi + x) + h(2\pi - x)$$

we obtain

$$\int_0^{2\pi} h(x)\cos x\,dx = \int_0^{\pi/2} H(x)\cos x\,dx. \tag{3.15}$$

Since $h \in \widetilde{Q}(M)$, we have

$$H(x) = \pi - [h(x+\pi) - h(x)] + \pi - [h(-x+\pi) - h(-x)]. \tag{3.16}$$

The lower estimate of $h(t+\pi) - h(t)$ for $h \in \widetilde{Q}(M)$, $t \in \mathbf{R}$, is the same as the lower estimate of $h(\pi)$ for h normalized by the conditions $h(0) = 0$, $h(2\pi) = 2\pi$. Hence (cf. [2; p. 65]

$$\frac{2\pi}{M+1} \le h(t+\pi) - h(t), \qquad t \in \mathbf{R}. \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$H(x) \le 2\Big[\pi - \frac{2\pi}{M+1}\Big] = 2\pi\frac{M-1}{M+1}.$$

Using this and (3.15) we obtain

$$\int_0^{2\pi} h(x)\cos x\,dx \le 2\pi\,\frac{M-1}{M+1}\int_0^{\pi/2}\cos x\,dx = 2\pi\,\frac{M-1}{M+1}$$

and (3.14) follows which ends the proof.

The inequality (3.13) enables us to improve slightly an estimate of the sum $\sum |c_n|$ as obtained by M. Nowak, cf. [10, (3.2)]. We have

Theorem 3.5. *If $x+\sigma(x)$ is M-quasisymmetric on* $\mathbf{R}$ *and σ has the expansion* (3.4) *then*

$$\sum_{n=1}^{\infty}|c_n| < \pi\sqrt{2}\sum_{n=1}^{\infty}\Big[\Big(\frac{M}{M+1}\Big)^n - 2^{-n}\Big]^{1/2}. \tag{3.18}$$

Proof. As shown in [10, p. 98], we have an estimate

$$\sum_{n=2}^{\infty}|c_n| \le \pi\sqrt{2}\sum_{n=2}^{\infty}\Big[\Big(\frac{M}{M+1}\Big)^n - 2^{-n}\Big]^{1/2}.$$

From (3.13) we have

$$|c_1| \le 2\,\frac{M-1}{M+1} < \pi\Big(\frac{M-1}{M+1}\Big)^{1/2} = \pi\sqrt{2}\Big(\frac{M}{M+1}-\frac{1}{2}\Big)^{1/2},$$

and (3.18) readily follows.

Note that (3.18) holds without the normalization $\sigma(2k\pi/3)=0$, $k\in\mathbf{N}$. It is plausible that one could improve the estimates (3.13), (3.18) by taking the condition $\sigma(2k\pi/3)=0$ $(k=0,1,2)$ into account.

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