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INVARIANT SETS FOR A-HARMONIC MEASURE

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Abstract. We prove that the zero capacity is a sufficient condition for invariant sets for the $\mathscr{A}$-harmonic measure, i.e., if $\operatorname{cap}_p F = 0$ then $\omega(E \cup F, \Omega; \mathscr{A}) = \omega(E, \Omega; \mathscr{A})$ for any closed $E \subset \partial\Omega$.

1. Introduction

The $\mathscr{A}$-harmonic measure ω is a function similar to the classical harmonic measure. However, it is associated with a more general, possibly non-linear, elliptic partial differential equation $\nabla \cdot \mathscr{A}(x, \nabla u) = 0$ than the Laplace equation. An invariant set is a set $F \subset \partial\Omega$ such that F does not change the $\mathscr{A}$-harmonic measure of the original set E, i.e., $\omega(E \cup F, \Omega; \mathscr{A}) = \omega(E, \Omega; \mathscr{A})$. If $\mathscr{A}(x, \nabla u) = \nabla u$, then invariant sets are, of course, nothing else but sets of harmonic measure zero. The p-harmonic case, i.e., $\mathscr{A}(x, \nabla u) = |\nabla u|^{p-2}\nabla u$, is studied by P. Aviles and J. Manfredi [AM]. They proved that if F is a closed set such that the Hausdorff dimension of F is small enough, then $\omega(E \cup F, \Omega; p) = \omega(E, \Omega; p)$. The linearization method employed by Aviles and Manfredi does not work for arbitrary $\mathscr{A}$. In this paper we derive the following sufficient condition for invariant sets:

Theorem 1.1. *Let $E, F \subset \partial\Omega$ and let E be closed. If $\operatorname{cap}_p F = 0$, then*

$$\omega(E \cup F, \Omega; \mathscr{A}) = \omega(E, \Omega; \mathscr{A}).$$

For $p < n$ it is known that $\dim_H F < n - p$ implies $\operatorname{cap}_p F = 0$ where $\dim_H F$ refers to the Hausdorff dimension of F. Hence $\dim_H F < n - p$ for a set F yields that F is invariant in the sense of Theorem 1.1. Bounds of this type have been obtained in [AM]. These, however, depend on the set Ω. By the paper of Tukia [T] it is easy to see that the result is the best possible involving a general class of equations and Hausdorff dimensions. In particular, for each $\gamma < p = 2$, there are compact sets K, on the boundary of unit disks $B \subset \mathbf{R}^2$, such that $\dim_H K < 2 - \gamma$ and $\omega(K, B, \mathscr{A}) > 0$ for some operator $\mathscr{A}$.

For $p > n$ no non-empty set is of p-capacity zero and Theorem 1.1 gives nothing in this case.

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2. Definitions for $\mathscr{A}$-harmonic measure

Throughout this paper we assume that Ω is an open, bounded and connected set in $\mathbf{R}^n$. We also assume that the operator $\mathscr{A}$ satisfies assumptions 2.1–2.5 below for some $1 < p < \infty$ and $0 < \alpha \le \beta < \infty$:

$$\begin{aligned} &x \to \mathscr{A}(x,h) \text{ is measurable for all } h \in \mathbf{R}^n \quad \text{and} \\ &h \to \mathscr{A}(x,h) \text{ is continuous for a.e. } x \in \Omega, \end{aligned} \tag{2.1}$$

and for all $h \in \mathbf{R}^n$ and a.e. $x \in \Omega$

$$\mathscr{A}(x,h) \cdot h \ge \alpha |h|^p \tag{2.2}$$

$$|\mathscr{A}(x,h)| \le \beta |h|^{p-1} \tag{2.3}$$

$$\big(\mathscr{A}(x,h_1) - \mathscr{A}(x,h_2)\big) \cdot (h_1 - h_2) > 0, \text{ whenever } h_1 \ne h_2, \text{ and} \tag{2.4}$$

$$\mathscr{A}(x,\lambda h) = |\lambda|^{p-2} \lambda \mathscr{A}(x,h) \text{ for all } \lambda \in \mathbf{R} \setminus \{0\}. \tag{2.5}$$

A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution of the equation

$$\nabla \cdot \mathscr{A}(x, \nabla u) = 0 \tag{2.6}$$

if

$$\int_\Omega \mathscr{A}(x,\nabla u) \cdot \nabla \varphi \, dx = 0$$

for all $\varphi \in C_o^\infty(\Omega)$. Any solution of (2.6) can be redefined in a set of measure zero so that it becomes continuous in Ω. This redefined continuous solution of (2.6) is said to be $\mathscr{A}$-harmonic in Ω. We denote by $\mathscr{H}(\Omega)$ the set of all $\mathscr{A}$-harmonic functions in Ω. If $v\colon \Omega \to \mathbf{R} \cup \{\infty\}$ is lower semicontinuous and if v is not identically infinite in Ω, then v is $\mathscr{A}$-superharmonic if for each domain $D \subset\subset \Omega$ and for each $u \in \mathscr{H}(D) \cap C(\bar{D})$ the condition $u \le v$ in ∂D implies that $u \le v$ in D. We let $\mathscr{S}(\Omega)$ denote the family of all $\mathscr{A}$-superharmonic functions.

Definition 2.1. Let $f\colon \partial\Omega \to \mathbf{R} \cup \{\pm\infty\}$ be any function and

$$\begin{aligned} \bar{H}_f(x) = \inf\{v(x) \,|v \in \mathscr{S}(\Omega), &\text{ bounded below and} \\ &\liminf_{z \to y} v(z) \ge f(y) \text{ for all } y \in \partial\Omega\}. \end{aligned}$$

The function $\bar{H}_f$ is called the upper Perron solution of f.

Now $\bar{H}_f \in \mathscr{H}(\Omega)$ if it is bounded in Ω. Let $E \subset \partial\Omega$ and let χ_E be the characteristic function of E. The function $\omega(E,\Omega;\mathscr{A}) = \bar{H}_{\chi_E}$ is called the $\mathscr{A}$-harmonic measure of set E with respect to Ω. For these constructions see [HKM].

The next lemma is employed in the proof of Theorem 1.1. The lemma is proved in [HKM, Theorem 9.3].

Lemma 2.2 *Let $f_j \colon \partial\Omega \to \mathbf{R}$ be a decreasing sequence of continuous functions and let $f = \lim f_j$. Then*

$$\bar{H}_f = \lim_{j\to\infty} \bar{H}_{f_j}.$$

Let $\theta \in W^{1,p}(\Omega)$. We write

$$\mathscr{K}_\theta = \{v \in W^{1,p}(\Omega) : v \geq \theta \text{ a.e.}, v - \theta \in W_o^{1,p}(\Omega)\}.$$

We call a function v a solution to the obstacle problem with obstacle and boundary values θ if $v \in \mathscr{K}_\theta$ and if

$$\int_\Omega \mathscr{A}(x, \nabla v) \cdot \nabla(\varphi - v)\, dx \geq 0$$

whenever $\varphi \in \mathscr{K}_\theta$.

Lemma 2.3. *Let $\phi_j \in W^{1,p}(\Omega)$ be a decreasing sequence such that $\phi_j \to \phi$ in $W^{1,p}(\Omega)$. Let $u_j \in W^{1,p}(\Omega)$ be a solution to the obstacle problem with obstacle and boundary values ϕ_j. Then the sequence u_j is decreasing and $u = \lim u_j$ is a solution to the obstacle problem with ϕ as an obstacle and boundary value.*

Lemma 2.3 is proved in [HKM, Theorem 3.79]. For further details see [HKM, Chapter 3].

3. Proof of Theorem 1.1

Let I be the set of all irregular points, for the p-Dirichlet problem, in the boundary of Ω. We may assume that $I \subset F$ because I is also a set of p-capacity zero [HKM, Theorem 9.11].

Let $\varphi_i \in C_o^\infty(\mathbf{R}^n)$ be a decreasing sequence of non-negative functions such that $\varphi_i \searrow \chi_E$. Let the function $\bar{H}_{\varphi_i}$ be as in Definition 2.1.

Let $B \subset \mathbf{R}^n$ be a ball such that $\Omega \subset\subset \frac{1}{2}B$. Because $\operatorname{cap}_p F = 0$, there exists a sequence of open sets U_j such that $F \subset U_j$ and $\operatorname{cap}_p(U_j, B) < 1/j$. Let $\psi_j = \hat{R}^1_{U_j}(B)$, where $\hat{R}^1_{U_j}(B)$ is the $\mathscr{A}$-potential of U_j in B (see [HKM, Chapter 8]). By using the estimates in [HKM] we get that $\psi_j = 1$ in U_j, $\psi_j \in W_o^{1,p}(B)$ and $\int_B |\nabla\psi_j|^p\, dx < c/j$ where the constant c depends only on α, β and p. Let $v_{ij} \in \mathscr{S}(\Omega)$ be the solution to the obstacle problem with the function $\bar{H}_{\varphi_i} + \psi_j$ as an obstacle and boundary value. Now the continuity of $\bar{H}_{\varphi_i}$ yields $v_{ij} \geq \bar{H}_{\varphi_i}$ in Ω and $\psi_j \equiv 1$ in U_j gives $v_{ij} \geq 1$ in $U_j \cap \Omega$. It follows that

$$\liminf_{x\to y} v_{ij}(x) \geq \chi_{E\cup F}(y)$$

for all $y \in \partial\Omega$ and for all i and j. Thus $v_{ij} \geq \omega(E \cup F, \Omega; \mathscr{A})$ for all i and j. The solution to the obstacle problem with obstacle and boundary values φ_i is clearly $\bar{H}_{\varphi_i}$. By Lemma 2.3 the limit function of the sequence v_{ij} is $\bar{H}_{\varphi_i}$ as $j \to \infty$. Hence $\bar{H}_{\varphi_i} \geq \omega(E \cup F, \Omega; \mathscr{A})$ for all i. Lemma 2.2 says that $\bar{H}_{\varphi_i} \searrow \omega(E, \Omega; \mathscr{A})$ as $i \to \infty$. So $\omega(E, \Omega; \mathscr{A}) \geq \omega(E \cup F, \Omega; \mathscr{A})$ and the theorem follows since the opposite inequality is obvious. □

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