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DIRICHLET POINTS, GARNETT POINTS, AND INFINITE ENDS OF HYPERBOLIC SURFACES I

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Abstract. The end of a hyperbolic surface is studied in terms of the behavior at infinity of geodesics on the surface. For a class of surfaces called untwisted flutes it is possible to give a fairly precise description of the ending geometry. From the point of view of a Fuchsian group representing such a surface this provides new information about the existence of Dirichlet and Garnett points.

0. Introduction

In the study of surfaces of finite topological type the analytic and geometric viewpoints have long been inextricably interwoven. This has been less true in the study of surfaces of infinite type, where even geometrically oriented thinking has tended to focus primarily on generic behavior. Our goal in this paper is to employ hyperbolic geometric techniques to investigate an infinite end of a hyperbolic manifold and especially a hyperbolic surface. We shall be interested in the behavior of specific classes of geodesics which carry information about the geometry of the end. Such classes of geodesics can have measure zero, or even be finite, and consequently this type of behavior is invisible to the usual analytic approaches.

Let M be a hyperbolic manifold and let $\sigma\colon [0,\infty) \to M$ be a geodesic ray, which we assumed to be parameterized by arc length. Define the function $\Delta_\sigma(t) = t - d_M\bigl(\sigma(0), \sigma(t)\bigr)$, where d_M denotes distance as defined by the hyperbolic metric. The ray σ is then said to be *horocyclic, critical*, or *subcritical* if $\Delta_\sigma(t)$ is respectively, unbounded, zero, or nonzero but bounded. A critical ray could be said to travel directly out a non-compact end of M, and a subcritical ray to travel almost directly out an end.

Critical and subcritical rays are closely related to the more familiar Dirichlet and Garnett points associated with Fuchsian and Kleinian groups. Viewed as asymptotic classes of geodesic rays one begins to see exactly how their existence and their particular travels depend on the geometry of the hyperbolic manifold on which they reside.

Most of our attention will be focused on the untwisted flute surfaces studied by Basmajian [1]. A flute is a complete hyperbolic surface which is homeomorphic to the infinite cylinder $\mathbf{K} = \mathrm{S}^1 \times (0,\infty)$ with the set of points $\{(1,n) \mid n \in \mathbf{N}\}$

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deleted. These surfaces are fundamental in that a general infinite type hyperbolic surface can be built up from flute pieces. These are also the simplest surfaces on which nontrivial critical and subcritical rays exist. A ray σ on $\mathbf{K}$ is *infinite* if $\sigma(t) = \big(\theta(t), r(t)\big)$ has $\lim_{t\to\infty} r(t) = \infty$. Such a ray is said to go out the infinite end of $\mathbf{K}$. On an untwisted flute there is an essentially unique class of infinite critical rays. This is proved in Section 2. In Section 4 we derive necessary and sufficient conditions for the existence of infinite subcritical rays in terms of certain length parameters which define the hyperbolic structure of the surface.

The paper is organized as follows. In Section 1 background on surfaces and their ends is presented and the flute surfaces are introduced. In Section 2 we derive some elementary properties of critical and subcritical rays on untwisted flutes. Section 3 contains the main technical results of the paper, which are applied to derive two important properties of subcritical rays. It is easy to see that a Dirichet ray is simple. Nicholls and Waterman [15] showed that subcritical rays are eventually simple. We prove that an infinite subcritical ray on an untwisted flute is eventually non-backtracking, in the sense that after some point it will cross each pair of pants exactly once. We also show that an infinite subcritical ray must eventually be uniformly close to some critical ray. These results together place severe restrictions on the path of a subcritical ray, and let us introduce a model for such a ray in Section 4. We then derive sufficient conditions and limited necessary conditions on the surface geometry for a ray to be subcritical. As a corollary we get necessary and sufficient conditions for the existence of such rays and their cardinality. In the final Section 5 a general correspondence is derived between the notions of critical and subcritical rays on a hyperbolic manifold and approximation properties of points on the boundary of hyperbolic space under the action of a discrete group of hyperbolic isometries. This allows for a translation of some of the earlier results into the language of Fuchsian groups. The motivation for these studies and the history of the subject is also outlined in the last section.

The background for much of this work arose in conversations with Ara Basmajian. Many thanks.

1. Background: ends and flutes

1.1. Ends. Define an *end* $\mathscr{E}$ of a manifold M as follows. Let $K_1 \subset K_2 \subset \cdots \subset M$ be a nested sequence of compact subsets of M so that $\bigcup_1^\infty K_i = M$. An end $\mathscr{E}$ is a sequence of connected components $\mathscr{E}_i$ in the complement of K_i so that $\mathscr{E}_{i+1} \subset \mathscr{E}_i$. This definition can be made independent of the given exhaustion $\{K_i\}$. A geodesic ray σ is said to *go out the end* $\mathscr{E}$, if for each integer $i > 0$ all but a compact segment of σ belongs to $\mathscr{E}_i$.

Lemma 1.1. *Let $\mathscr{E} = \{\mathscr{E}_i\}$ be an end of the hyperbolic manifold M. Then for any $a \in M$ there is a critical ray with initial point a that goes out the end $\mathscr{E}$. Furthermore, a critical or a subcritical ray always goes out some end of M.*

Proof. Let σ_i: $[0, b_i) \to M$ be a geodesic arc in M with $\sigma_i(0) = a$ for all $i > 0$. The function $\Delta_{\sigma_i}(t)$ is defined for all $t \in [0, b_i)$. The sequence of geodesics $\{\sigma_i\}$ has a convergent subsequence (with the same name); which, by definition, has the property that the sequence of unit vectors tangent to the geodesics at the point a is convergent. The geodesic σ that is the limit of this sequence has the limiting unit tangent at a and is defined on the interval $[0, b)$ where $\lim_{i\to\infty} b_i = b$. This all makes sense if $b > 0$. If we further suppose that $\Delta_{\sigma_i}(t) = 0$ for all $i > 0$ and that $b = \infty$, then we have $\Delta_\sigma(t) = 0$. In other words, σ is a critical ray.

Choose points $a_i \in \mathscr{E}_i$ and minimal length geodesic arcs σ_i: $[0, b_i) \to M$ with $\sigma(0) = a$ and $\sigma(b_i) = a_i$. Then a limiting geodesic ray σ, as defined above, is Dirichlet and goes out the end $\mathscr{E}$. □

1.2. Ends of surfaces. We shall now restrict our attention to two dimensional hyperbolic manifolds, or surfaces.

It is well known that the hyperbolic structure of a surface S uniquely determines S as a Riemann surface. The universal cover of such a surface is the upper-half plane $\mathbf{H}^2$ and the deck transformations form a discrete subgroup G of the real Möbius transformations, called a *Fuchsian group.* G also acts on $\partial\mathbf{H}^2 = \mathbf{R} \cup \infty$. The *limit set* of G, denoted $\Lambda(G)$, is the subset of points x in $\partial\mathbf{H}^2$ with the property that for any neighborhood U of x there are infinitely many $g \in G$ with $g(U) \cap U \neq \emptyset$ [12].

An end $\mathscr{E}$ of S is called a *puncture* if there is a subset D of S which is conformally equivalent to the punctured disc $\{z \mid 0 < |z| < 1\}$ and for i large $\mathscr{E}_i \subset D$. Similarly, we call $\mathscr{E}$ a *hole* if there is a subset D of S which is conformally equivalent to an annulus $\{z \mid 1 < |z| < r\}$ for some $r > 1$ and for i large $\mathscr{E}_i \subset D$. $\mathscr{E}$ is a *finite end* if it is either a puncture or a hole; otherwise it is an *infinite end.*

We define a critical ray on S to be *infinite* if it goes out an infinite end; otherwise it is *finite.* Sometimes we shall refer to a subsurface S', of a given surface S, as *containing* an end $\mathscr{E} = \{\mathscr{E}_i\}$. By this we mean that for i large $\mathscr{E}_i \subset S'$. A simple closed geodesic or piecewise geodesic α is said to *isolate* a finite end $\mathscr{E}$ if it is the only end contained in one of the components in the complement of α. Note that a finite end $\mathscr{E}$ is isolated by a simple close geodesic α if and only if $\mathscr{E}$ is a hole. Furthermore, such a geodesic α is uniquely determined up to orientation.

It is important to distinguish those aspects of an infinite end that parallel the distinction made between a puncture and a hole in the finite case. Toward that end we shall define infinite ends of the first and second kind, which respectively exhibit puncture-like and hole-like qualities.

A hyperbolic surface S is of the *second kind* if there exists an isometric embedding of a hyperbolic half-plane $\{z \mid \operatorname{Re} z > 0,\ \operatorname{Im}\ z > 0\}$ into S. Let P denote the image of the half-plane. Then we shall say that S *contains the half-plane* P. If the surface S is not of the second kind then it is of the *first kind.*

This agrees with the usual definition in terms of the limit set, in which a surface is of the first or second kind if a Fuchsian group G representing the surface has respectively, $\Lambda(G) = \partial\mathbf{H}^2$ or $\Lambda(G) \neq \partial\mathbf{H}^2$.

If S contains a half-plane P then P must actually belong to a unique end of S. Thus we say that an end $\mathscr{E} = \{\mathscr{E}_i\}$ is of the *second kind* if S contains a half-plane P and for all $i > 0$, $\mathscr{E}_i \cap P \neq \emptyset$. If an end is not of the second kind then it is of the *first kind*.

One easily checks that a puncture is of the first kind and a hole is of the second kind.

1.3. Flute surfaces. The simplest example of an infinite end occurs on what Basmajian [1] calls flute surfaces. In order to define this class of surfaces begin with the infinite cylinder $\mathrm{S}^1 \times (0, \infty)$,where S^1 is the unit circle in $\mathbf{C}$, and delete the set of points $\{(1, n) \mid n \in \mathbf{N}\}$ to produce the surface $\mathscr{S}$, called the model flute. The surface $\mathscr{S}$ has one infinite end and a finite end associated to each of the deleted points $(1, n)$ and to the ideal boundary $\mathrm{S}^1 \times \{0\}$. Let $\mathscr{F}$ denote the space of isometry classes of complete metrics of constant curvature -1, that is hyperbolic metrics, on the surface $\mathscr{S}$. Define an involution $r\colon \mathscr{S} \to \mathscr{S}$ by $r(e^{i\theta}, t) = (e^{-i\theta}, t)$. Let $\mathscr{F}_0 \subset \mathscr{F}$ be the set of isometry classes in $\mathscr{F}$ for which there exists a representative surface on which r is an isometry. Henceforth we shall treat elements of $\mathscr{F}$ and $\mathscr{F}_0$ as hyperbolic surfaces and suppose in the latter case that r is an isometry. A surface in $\mathscr{F}$ is called a *flute* and one in $\mathscr{F}_0$ is called an *untwisted flute.*

Given a flute surface $F \in \mathscr{F}$ and $n \in \mathbf{N}$ let α_n denote the simple closed geodesic on F in the free homotopy class of the curve

$$t \to (e^{it}, n + \tfrac{1}{2}), \qquad 0 \leq t \leq 2\pi.$$

Note that these are well defined with the possible exception of α_0, which only exists if the end corresponding to the ideal boundary $S^1 \times \{0\}$ is a hole. Let β_n be the geodesic arc of minimal length orthogonal to both α_n and α_{n+1} at its endpoints. Set $a_n = \beta_n \cap \alpha_n$ and $a'_n = \beta_{n-1} \cap \alpha_n$. See Figure 1.

Proposition 1.1. *The flute F is untwisted if and only if $a_n = a'_n$ for all $n > 0$.*

Proof. Each of the geodesics α_n is mapped onto itself by r. Since the minimal length geodesic orthogonal to α_n and α_{n+1} is unique it must also be mapped onto itself, and hence fixed, by r. It is easy to see what the fixed point set of r looks like. It is a union of the set $\beta^* = \{(-1, t) \mid t \in \mathbf{R}\}$ and the sets $\gamma_n = \{(1, t) \mid n < t < n + 1\}$. Also, since r is an isometry, its fixed point set is geodesic, thus we may take β^* and γ_n to be geodesics. Then the arc of β^* between α_n and α_{n+1} is the only fixed geodesic arc with endpoints on α_n and α_{n+1}. Consequently it must be the arc β_n. Then it is clear that $a_n = a'_n$ for all $n \in \mathbf{N}$. □

Figure 1.

This shows that the definition is in agreement with [1]. For future reference set $\bigcup_{n=1}^{\infty} \gamma_n = \gamma$; which we shall call the *spine* of F. Also set $R = \gamma \cup \beta^*$. The complement of R on F consists of two simply connected surfaces which are interchanged by r.

The end of F corresponding to the point $(1, n)$ is either a puncture or a hole, depending on the hyperbolic metric. If it is a hole then let μ_n denote the unique, up to orientation, simple closed geodesic on F that bounds a topological disc about the point $(1, n)$ on the infinite cylinder. If the end corresponding to $(1, n)$ or $\mathrm{S}^1 \times \{0\}$ is a puncture then we shall define the length of μ_n or α_0 to be zero.

Using the pants decomposition it is shown in [1] that a flute F is uniquely determined by the lengths of α_n, β_n, and μ_n, and the oriented distance between the points a_n and a'_n. Moreover, given a collection of numbers as above, with obvious restrictions, there is a flute surface having the corresponding lengths and oriented distances.

Remark 1.1. Using classical results of Koebe or the methods of [8] one can give the following simple description of an untwisted flute surface from a complex analytic viewpoint. Let $\mathbf{s} = \{s_i\}$ be an increasing sequence of real numbers with $s_0 = 0$, $\lim_{i \to \infty} s_i = s_\infty \leq \infty$, and let $\mathbf{r} = \{r_i\}$ be a sequence of non-negative real numbers with

$$r_i + r_{i+1} < s_{i+1} - s_i.$$

Define $D(\mathbf{s}, \mathbf{r})$ to be the subdomain of $\mathbf{C}$ complementary to the union of all of the closed discs $\{z \mid |z - s_i| \leq r_i\}$ for $i = 0, 1, 2, \ldots$, and with $|z| < s_\infty$. Then a hyperbolic surface is an untwisted flute if and only if it is conformally equivalent to one of the domains $D(\mathbf{s}, \mathbf{r})$.

2. Critical rays on untwisted flutes

We begin with an elementary but useful result about intersections of critical rays on an arbitrary hyperbolic surface. The proof is based on a cut and paste technique which is central to a number of the other arguments. Simply put, the idea is to show that a path does not realize the distance between its endpoints by replacing a geodesic segment of the path by another segment of at most the same length.

Proposition 2.1. *Let* σ *and* λ *be two distinct critical rays on* M *and suppose that neither one is a subset of the other. Then there exists at most one pair of numbers* $t, t' \in [0, \infty)$ *so that* $\sigma(t) = \lambda(t')$. *Furthermore, if* $\sigma(t) = \sigma(t')$ *then* $t = t'$.

Proof. Suppose that there exists t_1, t_1', t_2, and t_2' so that $\sigma(t_1) = \lambda(t_1')$ and $\sigma(t_2) = \lambda(t_2')$. We suppose that $t_1 < t_2$, $t_1' < t_2'$, and $t_2 - t_1 \le t_2' - t_1'$. The other cases follow by shifting subscripts. Define the piecewise geodesic arc

$$\bar{\lambda}(t) = \begin{cases} \lambda(t) & t \in [0, t_1], \\ \sigma(t - t_1 + t_1') & t \in [t_1, t_1 + t_2' - t_1']. \end{cases}$$

Since $\bar{\lambda}$ is definitely not a smooth geodesic it cannot minimize the distance between its endpoints. We therefore have

$$d\big(\lambda(0), \lambda(t_2)\big) < (t_1 + t_2' - t_1') \le t_2.$$

It follows that λ cannot be critical.

Here we have replaced a segment of a geodesic by a segment of at most the same length. The resulting piecewise geodesic can then be shortened. Similarly, to prove the simplicity of a critical ray, one observes that a ray which is not simple contains a loop. By removing the loop, as above, one can produce a shorter path.
□

Through the remainder of this and the following two sections F shall denote an *untwisted flute.*

As defined above β^* is a geodesic which goes out the finite end corresponding to the ideal boundary $\mathrm{S}^1 \times \{0\}$ as $t \to -\infty$, and out the infinite end of F as $t \to \infty$. Let $t_1 \in \mathbf{R}$ be the value for which $\beta^*(t_1) = a_1 = \alpha_1 \cap \beta^*$. It will be convenient to work with the geodesic ray β which we take to be β^* restricted to $[t_1, \infty)$ and reparameterized so that $\beta\colon [0, \infty) \to F$. It is not always the case that $\beta = \bigcup_{i=1}^{\infty} \beta_i$. In fact, the infinite end of an untwisted flute S is of the second kind if and only if $\sum_{i=0}^{\infty} l(\beta_i) < \infty$. If the sum converges then the geodesic subarc $\bigcup_{i=0}^{\infty} \beta_i$ terminates at the boundary of the maximal half-plane P. For more details and another twist see [1].

Lemma 2.1. *The geodesic ray* β *on an untwisted flute* F *is a critical ray.*

Proof. Let $b \ne a$ be points on β. To prove the lemma it will suffice to show that the distance between a and b is the length of the segment of β joining them. We argue by contradiction. Suppose that the minimal length geodesic arc $\sigma\colon [0,w] \to F$ from a to b is distinct from the arc of β. Since the complement of R on F is a union of two simply connected subsurfaces, σ must cross R at some point other than a and b; that is, for some $t_0 \in (0,w)$, $\sigma(t_0) \in R$. Define the piecewise geodesic

$$\bar{\sigma} = \begin{cases} \sigma(t) & t \in [0,t_0], \\ r \circ \sigma(t) & t \in [t_0,w]. \end{cases}$$

Then $\bar{\sigma}$ has the same length as σ. The geodesic that is freely homotopic to $\bar{\sigma}$ relative to its endpoints will then be strictly shorter than σ, giving the desired contradiction. □

In the next theorem we characterize the infinite critical rays on an untwisted flute. The upshot is that β is the paradigmatic infinite critical ray on such a surface. In order to make this precise we need to distinguish various notions of how two geodesic rays can have the "same" behavior as $t \to \infty$.

Let σ and λ be two distinct geodesic rays on S. Then σ and λ are *asymptotic* if there are lifts of the rays to $\mathbf{H}^2$ which have the same endpoint on $\partial\mathbf{H}^2$. σ and λ are *asymptotic relative to the half-plane* P if S contains a half-plane P and all but a finite length segment of each ray lies in P. There is a stronger notion of asymptotic which we shall need. σ and λ are said to be *strongly asymptotic* (*relative to* P) if σ and λ are asymptotic (relative to P) and there is a minimal length geodesic arc ν from $\sigma(0)$ to $\lambda(0)$ so that $\sigma\cup\lambda\cup\nu$ bounds a simply connected region on S. Saying that ν is minimal length means that $l(\nu) = d\big(\sigma(0), \lambda(0)\big)$. We do not discount the possibility that σ and λ have the same initial point, in which case ν is just a point. It is an easy exercise to show that if σ is a critical (subcritical) ray and σ and λ are asymptotic, perhaps relative to some half-plane, then λ is either critical or subcritical.

Theorem 2.1. 1. *Let λ be an infinite critical ray on an untwisted flute F. If the infinite end of F is of the first kind then λ is strongly asymptotic to β. If the infinite end of F is of the second kind then F contains a unique maximal half-plane P so that either λ and β are strongly asymptotic relative to P or λ intersects $R = \beta^* \cup \gamma$ in a single point and the subray of λ beginning at the intersection point is strongly asymptotic to β relative to P.*

2. *If λ is a geodesic ray on F which is either strongly asymptotic to β or strongly asymptotic to β relative to some half-plane P, then λ is a critical ray.*

Proof. Since λ is critical it cannot intersect R more than once. Otherwise, as in the proof of Proposition 2.1, one could reflect the segment lying between the intersection points, replace the original segment by the reflected one, and smooth out the edges to produce a shorter path. If $\lambda \cap R = c$ then let λ' denote the subray of λ beginning at c. Otherwise set $\lambda' = \lambda$. We shall first show that λ' is strongly asymptotic to β (perhaps relative to a half-plane P).

Except for its initial point λ' lies in one of the connected subsurfaces S of $F \setminus R$. By symmetry a minimal length arc ν from $\lambda'(0)$ to $\beta(0)$ may be chosen to lie on $\overline{S}$. Since $\overline{S}$ is simply connected and has geodesic boundary it is isometric to an infinite sided convex polygon in $\mathbf{H}^2$. One easily shows that an end of such a polygon is a maximal interval on $\partial\mathbf{H}^2$. Since β and λ' both go out the infinite end of F, they have their endpoints in the same interval on $\partial\mathbf{H}^2$. It follows that $\lambda' \cup \beta \cup \nu$ bounds a simply connected domain on S. If the interval is nontrivial then it is the ideal boundary of a half-plane P, and thus β and λ' are strongly asymptotic (relative P).

We need to show that if the infinite end of F is of the first kind then $\lambda' = \lambda$. We argue by contradiction. If not then there is a geodesic arc μ on $r(S)$ so the $\lambda = \lambda' \cup \mu$. The reflected geodesic ray $r(\lambda')$ is critical and, by the above, asymptotic to β and hence also to λ'. Let d be the distance between $\lambda(0)$ and $r\big(\lambda'(1)\big)$. Then $l(\mu) + 1 - d = \varepsilon > 0$. Since λ' and $r(\lambda')$ are asymptotic we can choose $t > 1$ so that $d\big(\lambda'(t), r\big(\lambda'(t)\big)\big) < \frac{1}{2}\varepsilon$. Then

$$d\big(\lambda(0), \lambda(t)\big) \le t - l(\mu) - 1 + d + \tfrac{1}{2}\varepsilon < t,$$

which says that λ is not critical.

Now we turn to the uniqueness of the maximal half-plane. Let P_1 be the maximal half-plane in F containing P. We must show that P_1 is the unique maximal half-plane in F. Suppose that F also contains the maximal half-plane P_2. If $P_2 \cap P_1 \neq \emptyset$ then by lifting them to intersecting half-planes in $\mathbf{H}^2$ one easily constructs a larger half-plane containing the two. Therefore we suppose that $P_2 \cap P_1 = \emptyset$.

Choose a lift $\widetilde{P}_2$ of P_2 to $\mathbf{H}^2$ and points $p \in \widetilde{P}_2$ and $p' \in \partial\widetilde{P}_2 \cap \partial\mathbf{H}^2$. The geodesic ray $\tilde{\varrho}$ in $\mathbf{H}^2$ with initial point p and endpoint p' projects to a critical ray ϱ in F. By the above ϱ and β are strongly asymptotic relative to some half-plane P. It follows that β intersects P_1 and P_2, contradicting the disjointness of the half-planes. Thus P_1 is unique.

The final assertion is argued by observing that there is a geodesic arc ν so that $\lambda \cup \beta \cup \nu$ bounds a simply connected region. Since β is one boundary curve of the region it must be contained in the closure of one of the simply connected components of $F \setminus R$. If λ were not a critical ray then the minimal arc between some pair of points on λ would not be along λ. This shorter path could then not lie entirely in the same component of $F \setminus R$ as λ, since $F \setminus R$ is simply connected. Thus the shorter path must cross R. Now, arguing as in Lemma 2.1, such an arc could not be of minimal length between its endpoints. ◻

2.1. Rigid subcritical rays. Define a subcritical ray σ to be a *rigid* subcritical ray if every geodesic ray λ that is asymptotic to σ is a subcritical ray. In other words, there is no critical ray asymptotic to σ. The following corollary, which is a consequence of Theorem 2.1, shows that there are only two types of subcritical rays: those that are asymptotic to β (perhaps relative to some half-plane P) and those that are rigid. Consequently the potential for interesting behavior lies with the rigid subcritical rays. This will be the subject of Section 4.

Corollary 2.1. *Let λ be a subcritical ray on an untwisted flute F. Then λ is a rigid subcritical ray if and only if λ is neither asymptotic to β nor asymptotic to β relative to some half-plane P.*

3. Properties of subcritical rays on untwisted flutes

In this section we shall derive two important properties of a subcritical ray out the infinite end on an untwisted flute F. Theorem 3.2 shows that a subcritical ray will eventually not backtrack on the surface. That is, for n large it crosses each α_n exactly once. In Theorem 3.3 we show that a subcritical ray eventually lies in an arbitrarily small neighborhood of a critical ray. The technical work is done in the other subsections. In Section 3.1 we develop techniques which make it possible to compare lengths of geodesic segments in the proof of Theorem 3.2. In Section 3.3 we prove a result on the limiting properties of certain piecewise geodesic rays, which becomes the main tool in the proof of Theorem 3.3.

We begin by recalling an important result due to Nicholls and Waterman. The translation of their theorem, which is given below, is a direct consequence of Theorem 5.1 and Corollary 5.1.

Theorem 3.1 [15]. *Let σ be a subcritical ray on a hyperbolic surface S. There is a value T so that for $t > T$, $\sigma(t) = \sigma(t')$ for some $t' \in [0, \infty)$ implies $t = t'$.*

In other words a subcritical ray on a surface has only finitely many self-intersections. It seems unlikely that this is true for hyperbolic manifolds M of dimension greater than two.

3.1. Cutting, pasting, and comparing lengths.

Let I denote the imaginary axis in the upper-half plane $\mathbf{H}^2$. I is a hyperbolic geodesic. We define $\widetilde{N}_w = \{re^{it} \mid \theta < t < \pi - \theta\}$ which is the set of points in $\mathbf{H}^2$ of distance less than $w = -\log(\csc\theta - \cot\theta)$ from I. Length in $\mathbf{H}^2$ is computed with the Poincaré metric $y^{-1}|dz|$.

If S is a hyperbolic surface and α is a closed geodesic on S then the covering group Γ can be chosen so that I covers α. Then $\widetilde{N}_w$ projects to a width w neighborhood N_w of α.

For a real number x define $w(x) = \log\coth\frac{1}{4}x$. If α is a closed geodesic set $w_\alpha = w\big(l(\alpha)\big)$.

Lemma 3.1. *If $\sigma\colon [a,b] \to S$ is a closed curve on S which is freely homotopic to α and $\sigma \cap N_{w_\alpha} = \emptyset$ then $l(\sigma) > 2$.*

Proof. Let $g(\mathrm{z}) = c\mathrm{z}$, where $\log c = l(\alpha)$, generate the stabilizer of I in Γ. Since σ is freely homotopic to α, there is a geodesic $\tilde{\sigma}$ covering σ which is also stabilized by g. Reparameterize a fundamental segment $\tilde{\sigma}([a,b])$ of $\tilde{\sigma}$ as $t \to r(t)e^{i\theta(t)}$ for $u < t < v$. Then

$$l(\sigma) = \int_u^v \frac{\sqrt{dr^2 + r^2\, d\theta^2}}{r \sin\theta} \geq \int_u^v \frac{|dr|}{r \sin\theta_0}$$

where $\theta_o = \sin^{-1} \tanh \frac{1}{2} l(\alpha)$. The right hand side of the inequality is the length of the orthogonal projection of $\tilde{\sigma}([a,b])$ onto the boundary of N_{w_α}. Then $cr(u) = r(v)$ and the orthogonal projection of $\tilde{\sigma}([a,b])$ contains an arc of ∂N_{w_α} of the form $t \to te^{i\theta_0}$, $r(u) < t < cr(u)$. Such a segment has length

$$\int_{r(u)}^{cr(u)} \frac{dt}{t \sin\theta_0} = \csc\theta_0 \log c = l(\alpha) \coth \tfrac{1}{2} l(\alpha) > 2.$$

Thus $l(\sigma) > 2$. □

If S is a hyperbolic surface with a puncture p then the covering group Γ can be chosen so that $g(\mathrm{z}) = \mathrm{z} + 1$ generates the stabilizer of ∞ in Γ and the region $\mathrm{Im}\,\mathrm{z} > \frac{1}{2}$ projects to a horocyclic neighborhood N of p.

Lemma 3.2. *If σ is a closed curve on S which is freely homotopic to a simple loop about the puncture p and $\sigma \cap N = \emptyset$, then $l(\sigma) > 2$.*

The proof of Lemma 3.2 is similar to the proof of the previous lemma, but simpler. We leave it to the reader.

Remark 3.1. We shall refer to the neighborhood N_{w_α} of α in Lemma 3.1 and the neighborhood N of a puncture in Lemma 3.2 as a *fundamental collar* about α or p. In this way every finite end of a hyperbolic surface has associated to it a fundamental collar.

Lemma 3.3. 1. *Let α be a simple closed geodesic on a hyperbolic surface S and let N_{w_α} be a neighborhood of α of width $w_\alpha = w\big(l(\alpha)\big)$. Suppose λ is a geodesic on S and that for numbers $a < b$, $\lambda\big((a,b)\big) \subset N$ and $\lambda(a), \lambda(b) \in \partial N$. If $\lambda \cap \alpha = \emptyset$ then λ has self-intersections.*

2. *Let μ: $[u,v] \to S$ be a geodesic arc distinct from α with $\mu(u)$, $\mu(v) \in \alpha$. Then $l(\mu) \geq w\big(l(\alpha)\big)$.*

More sophisticated results of this sort and references to the literature on collar lemmas can be found in [2] and [3].

Proof. We begin with the proof of (1). The strategy is to suppose that λ is simple and derive a contradiction.

Let the covering group Γ of S be as in the beginning of the section. There is a minimal length arc ν from α to λ in N_{w_α}, which is their common orthogonal. Lift ν to an arc $\tilde{\nu}$ with initial point on I and then lift λ to a geodesic $\tilde{\lambda}$ which is orthogonal to $\tilde{\nu}$ at its endpoint. Note that $\tilde{\lambda}([a,b])$ must lie within $\widetilde{N}_{w_\alpha}$.

Since I and $\tilde{\lambda}$ have a common perpendicular they have disjoint closures. Without loss of generality we may suppose that $\tilde{\lambda}$ has endpoints 1 and $b \geq 1$. Since $\tilde{\lambda} \cap g(\tilde{\lambda}) = \emptyset$, we must have $b \leq c$. The geodesic with endpoints 1 and c lies above $\tilde{\lambda}$ and it is tangent to the boundary of the neighborhood $\widetilde{N}_{w_1}$, where $w_1 = \sin^{-1}\big((c-1)/(c+1)\big)$. Since $\log c = l(\alpha)$, $w_\alpha = w_1$. Thus $\tilde{\lambda} \cap \widetilde{N}_{w_\alpha} = \emptyset$ contrary to an earlier statement. That completes the proof of (1).

A lift $\tilde{\mu}$ of μ beginning at I terminates on another lift $\widetilde{\alpha}$ of α with $\widetilde{\alpha} = h(\alpha)$ for some $h \in \Gamma$. By (1) it is clear that $\widetilde{\alpha} \cap \widetilde{N}_{w_\alpha} = \emptyset$ and therefore

$$\widetilde{N}_{w_\alpha/2} \cap h(\widetilde{N}_{w_\alpha/2}) = \emptyset.$$

We conclude that $l(\mu) \geq w_\alpha$. □

Lemma 3.4. *Let p be a puncture on a hyperbolic surface S and let N be the fundamental collar about p. If λ is a simple geodesic on S and λ does not go out the end p, then $\lambda \cap N = \emptyset$.*

Proof. As earlier we suppose that the puncture corresponds to the point at ∞ in $\partial\mathbf{H}^2$ and that the transformation $g(\mathrm{z}) = \mathrm{z}+1$ generates the stabilizer of ∞ in the covering group Γ of S. The hypothesis implies that no lift of λ terminates at $\infty \in \partial\mathbf{H}^2$. Let $\tilde{\lambda}$ be a lift of λ. Since λ is simple and neither of its endpoints are at ∞, $\tilde{\lambda} \cap g(\tilde{\lambda}) = \emptyset$, and thus $\tilde{\lambda}$ lies beneath the line $\mathrm{Im}\,\mathrm{z} = \frac{1}{2}$. The result follows. □

3.2. Subcritical rays eventually do not backtrack. Recall that α_n is the geodesic on F in the free homotopy class of the closed curve $\mathrm{S}^1 \times \{n+\frac{1}{2}\}$. The next result says that an infinite subcritical ray must eventually cross the loops α_n without backtracking.

Theorem 3.2. *Let λ: $[0,\infty] \to F$ be an infinite subcritical ray. Then there is an $N > 0$ so that for $n > N$, $\lambda \cap \alpha_n$ contains exactly one point.*

Proof. To prove this we shall show that for each instance of a double intersection it is possible to replace a segment of λ by a geodesic arc which is shorter than the original by a fixed amount, which in our case will turn out to be 0.9. If infinitely many such backtrackings were to occur then $\Delta_\lambda(t)$ would necessarily be unbounded: an untenable situation for a subcritical ray.

Suppose that λ meets infinitely many of the α_n in two or more points. By Theorem 3.1 λ is eventually simple, and so by deleting an initial segment and reparameterizing we may assume that λ is itself simple. There is an integer $N > 0$

so that $\lambda(0)$ lies in the region of F bounded by α_{N-1} and α_N. Certainly, $\lambda \cap \alpha_n$ is always a finite set. Suppose that $n > N$ and the intersection contains two or more elements. Let $0 < t_0 < t_1$ be the smallest two numbers with $\lambda(t_i) \in \alpha_n$. There must also be a point $t_2 > t_1$ so that $\lambda(t_2) \cap \alpha_n \neq \emptyset$, and $\lambda(t) \cap \alpha = \emptyset$ for all $t > t_2$.

The arc $\lambda([t_1, t_2])$ must cross the spine γ of F. This is because otherwise it would bound a simply connected region on F along with an arc of α_n, which cannot be. Thus there is a positive number $a \in (t_1, t_2)$ so that $\lambda(a) \in \gamma$ and for all $t \in (a, t_2)$, $\lambda(t) \notin \gamma$. Let k be the largest integer for which $\lambda([t_0, t_1]) \cap \alpha_k \neq \emptyset$. Choose $b' > t_2$ to be the smallest value for which $\lambda(b') \cap \alpha_{k+1} \neq \emptyset$, and let $b > b'$ be the smallest value for which $\lambda(b) \cap \gamma \neq \emptyset$.

The closed curve $\lambda([a, b]) \cup r\big(\lambda([a, b])\big)$ bounds a region (not necessarily connected) containing all of the finite ends that lie on the subsurface of F between α_{n-1} and α_{k+1}. As a consequence $r\big(\lambda([a, b])\big)$ must intersect the curve $\lambda([t_0, t_1])$. Let $c \in (a, b)$ be the smallest value with $r \circ \lambda(c) = \lambda(t^*)$ for some $t^* \in [t_0, t_1]$.

Define the new piecewise geodesic

$$\bar{\lambda} = \begin{cases} \lambda(t) & t \in [0, t^*), \\ r \circ \lambda(t - t^* + c) & \text{if } t \in [t^*, b - c + t^*], \\ \lambda(t - t^* + c) & t \in [b - c + t^*, \infty). \end{cases}$$

This is constructed from λ by removing the curve $\lambda([t^*, b])$ and inserting the reflected arc $r \circ \lambda([c, b])$. Consequently, the length of $\bar{\lambda}$ between $\lambda(0)$ and $\lambda(b)$ is $l\big(\lambda([0, b])\big) - l\big(\lambda([t^*, c])\big) = b - c + t^*$. For the remainder of the argument we shall focus on the curve $\lambda([t^*, c])$. In all but one case, which requires extra effort, $c - t^* = l\big(\lambda([t^*, c])\big) > 0.9$.

Consider the geodesic arcs $\lambda([t^*, a])$ and $r \circ \lambda([a, c])$. Since $\lambda(a) \in \gamma$, the spine of F, $r \circ \lambda(a) = \lambda(a)$. Furthermore, since $\lambda(t^*) = r \circ \lambda(c)$, the two arcs meet at their endpoints. As c is the smallest value for which such an intersection takes place, the arcs will be disjoint except for their endpoints. The arcs themselves are, by hypothesis, simple. We conclude that the set $\lambda([t^*, a]) \cup r\big(\lambda([a, c])\big)$ can be parameterized as a simple closed piecewise geodesic, which we denote by σ. Since reflection is length preserving $l(\sigma) = c - t^*$.

σ divides the planar surface F into two disjoint regions neither of which is simply connected. One possibility is that σ isolates a finite end of F. The pieces of σ lie on the simple geodesics λ and $r \circ \lambda$ which do not go out a finite end. We can infer from Lemmas 3.3 and 3.4 that $\sigma \cap N = \emptyset$, where N is the fundamental collar associated with the finite end isolated by σ. It follows from Lemmas 3.1 and 3.2 that $l(\sigma) = c - t^* > 2$.

The second possibility for how σ divides F is that σ is freely homotopic to a simple closed geodesic α not associated to an end of F. If $\sigma \cap N = \emptyset$, where N is the fundamental collar about α, then again by Lemma 3.1 $l(\sigma) = c - t^* > 2$.

We shall save the case $\sigma \cap N \neq \emptyset$, which is exceptional, for later and show how the proof is completed if either σ isolates a finite end or $\sigma \cap N = \emptyset$.

Recall that the strategy is to show that for each instance of a double intersection a path which is shorter by a definite amount can be found. Computing lengths we get

$$b = l(\lambda([0,b]) = b - c + t^* + c - t^* \geq l\big(\bar{\lambda}([0, b - c + t^*])\big) + 2.$$

Since $\bar{\lambda}(b - c + t^*) = \lambda(b)$ we have that $b \geq d\big(\lambda(0), \lambda(b)\big) + 2$, as we set out to show.

Now we turn to the case where σ is freely homotopic to a simple closed geodesic α and $\sigma \cap N \neq \emptyset$. If $l(\alpha) \geq 1$ then since α minimizes length in its free homotopy class $l(\sigma) > 1$ and the previous argument can be applied to show that $l([0,b]) \geq d\big(\lambda(0), \lambda(b)\big) + 1$.

In the remaining case, $l(\alpha) < 1$, we shall have to resort to a different technique for shortening λ. Since the arcs of σ lie on simple geodesics, Lemma 3.3 says that both λ and $r \circ \lambda$ cross α. The way things are set up, λ must then cross α two and, in fact, three times. We can choose values $0 < u < v < b$ so that $\lambda(u), \lambda(v) \in \alpha$ and $\lambda(t) \notin \alpha$ for $u < t < v$. By Lemma 3.3 $l\big(\lambda([u,v])\big) \geq w(1) > 1.4$. As in the construction of $\bar{\lambda}$ we can construct a piecewise geodesic arc from $\lambda(0)$ to $\lambda(b)$ by replacing the arc $\lambda([u,v])$ with the shorter arc of α between $\lambda(u)$ and $\lambda(v)$. The arc has length at most $.5$. This shows that $l([0,b]) \geq d\big(\lambda(0), \lambda(b)\big) + 1.4 - 0.5$. That completes the proof. ◻

3.3. Limiting geodesic rays. In this section we prove a technical proposition which gives sufficient conditions for concluding that a piecewise geodesic ray gets uniformly close to a geodesic ray. We shall make use of the proposition in the next section to show that a subcritical ray gets uniformly close to a critical ray.

Let $\lambda\colon [0,\infty) \to S$ be a piecewise geodesic ray. Then there is a sequence $0 = t_0 < t_1 < \cdots < \infty$ so that λ restricted to each interval $[t_i, t_{i+1}]$ is a maximal geodesic subarc λ_i of λ. Let ϕ_i be the (unsigned) angle between the vector tangent to λ_{i-1} at the point $\lambda(t_i)$ and the vector tangent to λ_i at $\lambda(t_i)$.

Proposition 3.1. *Let λ be a piecewise geodesic ray on the hyperbolic surface S. Suppose that there is a number $C_0 > 0$ so that for each $i \geq 0$, $l(\lambda_i) > C_0$. Further suppose that $\lim_{i\to\infty} \phi_i = 0$. Then there is a geodesic ray σ in S so that given $\varepsilon > 0$ there is a number $T_\varepsilon > 0$ so that $\lambda([T_\varepsilon, \infty))$ lies in an ε-neighborhood of σ.*

This proposition bears a likeness to results appearing in [7] and [6]. Our proof is largely inspired by the latter.

Lemma 3.5. *Consider the hyperbolic right triangle $\triangle ABC$ of Figure 2, where B and C are fixed and A may vary along a fixed geodesic. Then*

$$\frac{d\theta}{dt} = -\frac{\cosh t \tan\theta}{\sinh t \sec^2\theta} < -\frac{\sin 2\theta}{2}.$$

Figure 2.

Proof. For a right triangle we have $\tanh \overline{BC} = \sinh t \tan\theta$ [4]. Differentiating implicitly with respect to t gives

$$0 = \cosh t \tan\theta + \theta'(t)\sinh t \sec^2\theta \qquad \text{or} \qquad \theta'(t) = -\frac{\cosh t \tan\theta}{\sinh t \sec^2\theta}. \;\square$$

Set $C = \frac{1}{4}C_0$. By going to the universal cover we may take $S = \mathbf{H}^2$. Without loss of generality suppose that for all $i > 0$, $\phi_i < C\pi/4(1+C)$. Let $\overline{st}$ denote the geodesic arc from $\lambda(s)$ to $\lambda(t)$. For $t \neq t_i$ define $\theta(t)$ to be the angle between the vector tangent to $\overline{0t}$ at $\lambda(t)$ and the vector tangent to λ at $\lambda(t)$. At the points t_i we get two angles: $\theta(t_i^+)$ by considering the tangent to λ_i at $\lambda(t_i)$, and $\theta(t_i^-)$ by considering the tangent to λ_{i-1} at $\lambda(t_i)$. Then $|\theta(t_i^+) \pm \theta(t_i^-)| = \phi_i$. By the previous lemma θ is smooth on each interval (t_i, t_{i+1}) and is strictly decreasing there if $\theta(t_i^+) < \frac{1}{2}\pi$.

Lemma 3.6. *For all integers $i > 0$*

$$\theta(t_i^+) \leq \sum_{j=0}^{i-1}\left(\frac{1}{1+C}\right)^j \phi_{i-j}. \tag{1}$$

Proof. We shall argue by induction. Certainly $\theta(t_1^+) = \phi_1$, taking care of the case $i = 1$. Suppose that the formula holds for $i = n$. Then since

$$\theta(t_n^+) \leq \sum_{j=0}^{n-1}\left(\frac{1}{1+C}\right)^j \phi_{n-j} \leq \frac{\pi}{4}\left(\frac{C}{C+1}\right)\sum_{j=0}^{n-1}\left(\frac{1}{1+C}\right)^j < \frac{\pi}{4}$$

θ is strictly decreasing on (t_n, t_{n+1}) and $\theta'(t) < -\frac{1}{2}\sin 2\theta(t)$. This gives

$$\theta(t_{n+1}^-) = \int_{t_n}^{t_{n+1}} \theta'(t)\,dt + \theta(t_n^+) < -\tfrac{1}{2}C_0 \sin 2\theta(t_{n+1}^-) + \theta(t_n^+).$$

Consequently,

$$(1 + C)\theta(t_{n+1}^-) < \theta(t_{n+1}^-) + C \sin 2\theta(t_{n+1}^-) < \theta(t_n^+).$$

From this we conclude that

$$\begin{aligned}\theta(t_{n+1}^+) &\le \theta(t_{n+1}^-) + \phi_{n+1} < \frac{1}{1+C}\theta(t_n^+) + \phi_{n+1} \\ &< \frac{1}{1+C}\sum_{j=0}^{n-1}\Big(\frac{1}{1+C}\Big)^j \phi_{n-j} + \phi_{n+1} = \sum_{j=0}^{n}\Big(\frac{1}{1+C}\Big)^j \phi_{n-j+1}.\end{aligned}$$

That completes the proof of the lemma. □

Remark 3.2. 1. There are two consequences of the formula that we will make use of. Their proofs are immediate. First, given $\varepsilon > 0$ there exists $N > 0$ so that for $t > N$, $\theta(t) < \varepsilon$. The second is that given $\varepsilon > 0$ there is a $\delta > 0$ so that if $\phi_i < \delta$ for all $i \in \mathbf{N}$, then for all $t > 0$, $\theta(t) < \varepsilon$.

2. In the proof of Proposition 3.1 we will also need the following simple observation. Let two geodesic rays α, β in $\mathbf{H}^2$ have a common initial point x at which they make an angle θ. Let $\overline{\alpha\beta}$ be the geodesic from the endpoint of α to the endpoint of β in $\mathbf{H}^2$, and set $d(\theta) = d(\overline{\alpha\beta}, x) =$ distance from x to the arc $\overline{\alpha\beta}$. Given points $a \in \alpha$, $b \in \beta$, let $\overline{ab}$ be the geodesic arc joining a and b. Then by elementary hyperbolic trigonometry $d(\overline{ab}, x) < d(\overline{\alpha\beta}, x) = d(\theta)$, and $\lim_{\theta\to\pi} d(\theta) = 0$.

Proof of Proposition 3.1. Choose positive numbers ϕ^*, θ^*, and ϕ so that $d(\theta) < \varepsilon$ for $\theta > \pi - \phi^* - 2\theta^*$ and if $\phi_i < \phi$ for all $i > 0$, then $\theta(t) < \theta^*$ for all $t > 0$. Then for N large we can be assured that:

1. for $t > N$, $\theta(t) < \theta^*$ and
2. for $t_i > N$, $\phi_i < \min(\phi, \phi^*)$.

Figure 3.

Let ψ denote the angle at $\lambda(t_i)$ formed by the arcs $\overline{0t_i}$ and $\overline{t_it_j}$ where $j > i$, and let φ denote the angle formed by the arcs $\overline{t_it_j}$ and $\overline{t_it_{i+1}}$. One possible configuration for the relevant angles is illustrated in Figure 3. Then $\pi - \phi_i \leq \theta(t_i^-) + \psi + \varphi$. If $i > N$, as above, then $\pi - \phi^* - \theta^* - \varphi \leq \psi$. Furthermore, Remark 3.2.1 can be applied to the arc of λ oriented backwards from $\lambda(t_j)$ to $\lambda(0)$ to conclude that $\varphi < \theta^*$. Thus $\pi - \phi^* - 2\theta^* \leq \psi$.

It follows from the above and Remark 3.2.2 that given $\varepsilon > 0$ we can choose N so that the distance from $\lambda(t_i)$ to the geodesic arc $\overline{0t_j}$ is less than $\frac{1}{2}\varepsilon$ whenever $i > N$ and $j > i$. The geodesic arcs $\overline{0t_j}$ converge as $j \to \infty$ to a geodesic ray σ beginning at $\lambda(0)$, and $d\big(\sigma, \lambda(t_i)\big) < \varepsilon$ for all $i > N$. Since the distance between a geodesic arc and a geodesic ray will attain its maximum at an endpoint of the arc, we conclude that $d\big(\sigma, \lambda(t)\big) < \varepsilon$ for $t > t_N$. □

3.4. A subcritical ray gets close to a critical ray. The main result of this section is

Theorem 3.3. *Let λ be an infinite subcritical ray on an untwisted flute F. Then there is a critical ray σ on F so that for any $\varepsilon > 0$ there is a $T_\varepsilon > 0$ so that for all $t > T_\varepsilon$, $\lambda(t)$ lies in the ε-neighborhood of σ.*

The proof will build on the earlier work of this section and the following lemma, which is an easy consequence of the triangle inequality.

Lemma 3.7. *Let λ be a piecewise geodesic arc, made up of two geodesic pieces between points $a, b \in \mathbf{H}^2$. Suppose that there are numbers $l > 0$ and $0 < \theta < \pi$ so that the length of each maximal geodesic subarc of λ is greater than l and that the angles between the arcs is less than θ. Then there is a number $C(l, \theta) > 0$, so that $l(\lambda) - d(a, b) > C(l, \theta)$.*

Proof of Theorem 3.3. Let F_r denote the quotient of the untwisted flute F by the action of the reflection r. F_r is an orbifold; in this case an ideal hyperbolic polygon with mirror-like boundary. If we ignore the mirror aspect of the boundary, F_r is simply an infinite sided convex hyperbolic polygon in $\mathbf{H}^2$, which is isometric to the closure of a connected component of $F \setminus R$. The geodesic ray λ projects to a piecewise geodesic $\hat{\lambda}$ in F_r. Two geodesic arcs of $\hat{\lambda}$ meeting at a point on ∂F_r will make equal angles with the boundary arc. Thus $\hat{\lambda}$ will look like the path of a billiard ball on a hyperbolic billiard table. Since any geodesic ray in F_r lifts to a geodesic ray in F, $\Delta_{\hat{\lambda}}(t)$ will be a bounded function. Call a ray of this sort a *piecewise geodesic subcritical ray.* $\hat{\lambda}$ can also be viewed as a piecewise geodesic ray on F lying in the closure of one of the regions in the complement of $F \setminus R$.

By Theorem 3.1 there is no loss of generality in assuming that λ is simple. Furthermore, as a consequence of Theorem 3.2, $\hat{\lambda}$ is eventually simple. Thus there is no loss of generality in supposing that $\hat{\lambda}$ is a simple piecewise geodesic ray.

Let γ and β denote the projections from F to F_r of the sets with the same names. One should note that, except perhaps for the projection of the first geodesic segment of λ, a maximimal geodesic subarc of $\hat{\lambda}$ will have one endpoint on γ and the other endpoint on either β or on a geodesic arc in γ which is distinct from the first one.

To prove the theorem it will suffice to show that there is a critical ray σ on F_r so that given any $\varepsilon > 0$, $\hat{\lambda}$ is eventually within the ε-neighborhood of σ. We shall make use of the notation for piecewise geodesic rays developed in Section 3.3.

By Lemma 3.7 it is not possible for $\hat{\lambda}$ to be a piecewise geodesic subcritical ray if it contains infinitely many pairs of consecutive geodesic arcs $\hat{\lambda}_i, \hat{\lambda}_{i+1}$ so that $l(\hat{\lambda}_i)$, $l(\hat{\lambda}_{i+1}) > l$ and $\hat{\phi}_i > \theta$ for some numbers $l, \theta > 0$, where $\hat{\phi}_i$ denotes the angle between the tangents to consequtive segments of $\hat{\lambda}$. This is because each such pair of arcs can be replaced by a geodesic arc which is shorter by the fixed amount $C(l, \theta)$.

There are two possibilities for the geometry of $\hat{\lambda}$. The first is $\liminf_{i\to\infty} l(\hat{\lambda}_i) = l > 0$. Then by the above $\lim_{i\to\infty} \hat{\phi}_i = 0$. This is precisely the case dealt with in Proposition 3.1. The geodesic ray σ of Proposition 3.1, being a limit of geodesic arcs on F_r, will lie in F_r and will go out the infinite end of F. It follows easily that σ is strongly asymptotic, perhaps relative to some half-plane, to β. By Theorem 2.1 σ is a critical ray, proving the theorem in this case.

The remaining possibility is that $\liminf_{i\to\infty} l(\hat{\lambda}_i) = 0$. One immediate consequence of this is that $\liminf_{i\to\infty} l(\alpha_i) = 0$, and therefore the infinite end of surface F is of the first kind [1]. To complete the proof we shall perform several surgical modifications to $\hat{\lambda}$ in order to produce a piecewise geodesic ray $\bar{\lambda}$ with $\liminf_{i\to\infty} l(\bar{\lambda}_i) = l > 0$. Then the above considerations will again be applicable.

Given $0 < \varepsilon < 1$ let $I \subset \mathbf{N}$ be an infinite subset so that $l(\hat{\lambda}_i) < \frac{1}{4}\varepsilon$ if and only if $i \in I$. Note that if both endpoints of $\hat{\lambda}_i$ lie on γ then $\lambda_i \cup r(\lambda_i)$ is a simple closed curve on F. Arguing as in the proof of Theorem 3.2 we can infer that $l(\hat{\lambda}_i) \geq 1$, and thus $i \notin I$. We construct a sequence of piecewise geodesic rays $\{\Lambda_i\}_{i=2}^{\infty}$ as follows. Set $\Lambda_2 = \hat{\lambda}$. Suppose that Λ_i has been defined for $1 < i \leq n$. Rather than going through the exact definition for Λ_{n+1} in terms of parameterization etc., we shall indicate the pieces of Λ_n that are to be replaced and their replacements.

- If $n \notin I$ then $\Lambda_{n+1} = \Lambda_n$.
- If $n \in I$ and $\hat{\lambda}(t_n) \in \gamma$ let p denote the initial point of the geodesic subarc of Λ_n with terminal point $\hat{\lambda}(t_n)$. Then replace the two arcs of Λ_n from p to $\hat{\lambda}(t_{n+1})$ by the minimal length geodesic arc between those points to produce Λ_{n+1}.
- If $n \in I$ and $\hat{\lambda}(t_n) \in \beta$ then replace the arc of Λ_n from $\hat{\lambda}(t_n)$ to $\hat{\lambda}(t_{n+2})$ by the geodesic arc between those endpoints to produce Λ_{n+1} and set $\Lambda_{n+2} = \Lambda_{n+1}$.

Note that the third replacement followed by the second could modify the same arc twice, and that this is the only way an arc could experience two modifications. It follows that the piecewise geodesics Λ_i eventually stabilize in the following sense: for any $T > 0$ there is an $N > 0$ so that for $i, j < N$, $\Lambda_i(t) = \Lambda_j(t)$ for all $t < T$. Therefore the sequence of piecewise geodesic rays converges to a piecewise geodesic ray λ^*.

We note some properties of λ^*, which are easy consequences of the construction. First, any maximal geodesic subarc of λ^* not lying along β will have length greater than or equal to $\frac{1}{4}\varepsilon$. Second, at any stage in the construction a pair of geodesics is replaced by the geodesic arc that forms a triangle with them. Since one of the original pair has length less than $\frac{1}{4}\varepsilon$, each point on the pair is within $\frac{1}{4}\varepsilon$ of the side replacing them. Thus every point of $\hat{\lambda}$ is within $\frac{1}{2}\varepsilon$ of λ^*. Lastly, the path along λ^* from $\hat{\lambda}(0) = \lambda^*(0)$ to a point in the intersection of $\hat{\lambda}$ and λ^* is shorter than the path along $\hat{\lambda}$.

A further modification of the piecewise geodesic λ^* is necessary before the earlier results can be brought into play. Suppose that $\lambda^* \neq \beta$, for otherwise we are done. Let $\lambda^*_{i-1}\lambda^*_i$ be a pair of maximal geodesic subarcs of λ^* where λ^*_i is a subarc of β and $l(\lambda^*_i) < \frac{1}{4}\varepsilon$. The initial point of λ^*_i must be $\hat{\lambda}(t_j)$ for some $j > 0$. As observed earlier, $\hat{\lambda}(t_{j-1})$ and $\hat{\lambda}(t_j)$ make equal angles θ with β. After the modifications, λ^*_{i-1} will make an angle with β less than or equal to θ. Since λ^*_i lies along β we can infer that the angle between λ^*_{i-1} and λ^*_i is greater than $\frac{1}{2}\pi$. Therefore the geodesic joining the initial point of λ^*_{i-1} to the endpoint of λ^*_i will have length greater than $\frac{1}{4}\varepsilon$, and will lie within $\frac{3}{4}\varepsilon$ of $\hat{\lambda}$. Since any two pairs $\lambda^*_{i-1}\lambda^*_i$ as above will be disjoint, each such pair in λ^* can be replaced independently by the above geodesic arc joining the endpoints of the pair. The result is a piecewise geodesic $\bar{\lambda}$ lying in a $\frac{3}{4}\varepsilon$ neighborhood of $\hat{\lambda}$, realizing shorter paths to intersections with $\hat{\lambda}$, as above, and satisfying $l(\bar{\lambda}_i) \geq \frac{1}{4}\varepsilon$ on each maximal geodesic subarc $\bar{\lambda}_i$. In particular, $\bar{\lambda}$ is a piecewise subcritical ray.

We are now in a good position to complete the argument. Let $\bar{\phi}_i$ denote the angle between the tangents to consequtive segments of $\bar{\lambda}$, as in Section 3.3. If $\liminf_{i\to\infty} l(\bar{\phi}_i) > 0$ then by Lemma 3.7 $\bar{\lambda}$ cannot be subcritical. But $\hat{\lambda}$ is subcritical, and so it must be that $\lim_{i\to\infty} \bar{\phi}_i = 0$. Then by Proposition 3.1 there exists a geodesic ray σ on F_r so that all but a finite length arc of $\bar{\lambda}$ lies within the $\frac{1}{4}\varepsilon$-neighborhood of σ. Moreover, since σ goes out the infinite end it is strongly asymptotic to β. From Theorem 2.1 we infer that σ is a critical ray, and finally we may conclude that all but a finite length segment of λ lies within the ε-neighborhood of the critical ray σ. □

4. The existence of subcritical and rigid subcritical rays on untwisted flutes

The results of the previous sections can be applied to give sufficient conditions and, in restricted cases, necessary conditions for a geodesic ray on an untwisted flute to be subcritical. From this we can draw some general conclusions about when an untwisted flute will contain nontrivial infinite subcritical rays. These are expressed in terms of the sequence of lengths of the geodesics α_i.

The ray β and the geodesics α_i are naturally oriented in the direction of increasing parameter values. Each α_i has intersection number 1 with β, written $i(\alpha_i, \beta) = 1$. Similarly, orient each of the geodesics γ_i which comprise γ so that $i(\gamma_i, \alpha_i) = 1$.

Lemma 4.1. *Given a sequence of integers $\mathbf{n}$, with $n_i \neq 0$ for infinitely many $i \in \mathbf{N}$, there is a unique geodesic ray $\lambda(\mathbf{n})$ which is simple, non-backtracking, has initial point $\lambda(0)$, and has precisely the intersection numbers*

$$i(\lambda, \gamma_i) = (-1)^{s_i} n_i, \qquad \textit{where} \qquad s_i = \begin{cases} 0 & \textit{if } n_i \geq 0, \\ 1 & \textit{if } n_i < 0. \end{cases}$$

Proof. To prove that $\lambda(n)$ exists define a piecewise geodesic ray beginning at $\lambda(0)$ as follows: let Λ_1 be the geodesic α_1 traversed $|n_1|$ times with orientation s_1. Given Λ_i construct Λ_{i+1} by adding on the arc of β from a_i to a_{i+1} followed by the geodesic α_{i+1} traversed $|n_{i+1}|$ times with orientation s_{i+1}. Clearly the arcs Λ_i converge to a geodesic Λ. Let $\lambda_i(\mathbf{n})$ be the geodesic freely homotopic to Λ_i relative to its endpoints. Note that each $\lambda_i(\mathbf{n})$ is simple and has the desired intersection number with γ_j for $j \leq i$. A lift of Λ to $\mathbf{H}^2$ has a unique endpoint at infinity. One concludes that the geodesic arcs $\lambda_i(\mathbf{n})$ converge to a geodesic $\lambda(\mathbf{n})$, which is easily seen to have the asserted properties. ◻

By Theorems 3.1 and 3.2 a subcritical ray λ is eventually simple and eventually non-backtracking. Therefore, by altering λ on a finite segment, a subcritical ray λ' can be produced which is simple, never backtracks, and is asymptotic to λ. Moreover any geodesic ray that is asymptotic to λ' will be a subcritical ray. Thus one approach to understanding the subcritical rays is by characterizing the simple, non-backtracking subcritical rays.

For the remainder of this section λ will denote a simple, non-backtracking geodesic ray with $\lambda(0) = \beta(0)$. Since λ is non-backtracking $i(\lambda, \alpha_i) = 1$ for $i \geq 1$. Since λ is simple each of its intersections with γ_i for a fixed i will have the same orientation, and thus no cancellation occurs in the calculation of the intersection number. It follows that λ determines a sequence $\mathbf{n} = \{n_i\}$, $n_i \in \mathbf{Z}$ so that

$$i(\lambda, \gamma_i) = (-1)^{s_i} n_i, \qquad \text{where} \qquad s_i = \begin{cases} 0 & \text{if } n_i \geq 0, \\ 1 & \text{if } n_i < 0. \end{cases}$$

We have proved the following

Theorem 4.1. *If λ is a rigid subcritical ray then there exists a sequence of integers $\mathbf{n}$ with $n_i \neq 0$ for infinitely many $i \in \mathbf{N}$ so that λ is asymptotic to $\lambda(\mathbf{n})$.*

4.1. Necessary and sufficient conditions for existence. The theorem proved in this section gives sufficient conditions for certain $\lambda(\mathbf{n})$ to be subcritical. As a corollary we get necessary and sufficient conditions for the existence of subcritical rays in terms of the geometry of the flute.

There are some technical considerations which complicate matters when the infinite end of F is of the second kind. Given a point u on the closed geodesic α_1, let λ_u be the geodesic ray perpendicular to α_1 with $\lambda_u(0) = u$ and with a positive orientation relative to α_1 at u. For example, $\lambda_{a_1} \supseteq \beta$. If the flute surface F has an infinite end of the second kind then it contains a unique maximal half-plane P.

Lemma 4.2. *If the infinite end of F is of the second kind then there is an r-symmetric open subarc μ of α_1 centered at $a_1 = \beta(0)$ and a number $T_u > 0$ so that for $u \in \mu$, λ_u is a critical ray and for $t > T_u$, $\lambda_u(t) \in P$. Furthermore, if u is an endpoint of μ then λ_u is a critical ray but $\lambda_u \cap P = \emptyset$.*

Proof. Lift α_i to a geodesic $\widetilde{\alpha}_i$ and lift β to a geodesic $\tilde{\beta}$ with initial point on $\widetilde{\alpha}_1$. $\tilde{\beta}$ has its endpoint in a lift $\widetilde{P}$ of P. Let $\tilde{\mu}$ be the set of points on $\widetilde{\alpha}_1$ so that for $u \in \tilde{\mu}$ the geodesic $\tilde{\lambda}_u$ perpendicular to $\widetilde{\alpha}_1$ at u satisfies $\tilde{\lambda}_u \cap \widetilde{P} \neq \emptyset$. One easily checks that the projection of $\tilde{\mu}$ to F has all the properties of the lemma. □

To better deal with the infinite ends of the second kind we define geodesics $\widehat{\alpha}_i$, $i \in \mathbf{N}$, on F by removing part of the α_i. If the infinite end of F is of the first kind then set $\widehat{\alpha}_i = \alpha_i$. If the infinite end of F is of the second kind then let $\widehat{\alpha}_i$ be the arc of α_i which is disjoint from λ_u for each $u \in \mu$.

Theorem 4.2. *Suppose that $\mathbf{n}$ is a sequence of integers with $n_i \neq 0$ for infinitely many $i \in \mathbf{N}$.*

1. *If $\sum_{i=1}^{\infty} |n_i| l(\alpha_i)$ converges then $\lambda(\mathbf{n})$ is a rigid subcritical ray.*
2. *Suppose $n_i \in \{-1, 0, 1\}$ and that if $|n_i| = |n_j| = 1$ and $n_k = 0$ for $i < k < j$ then $s_i + s_j = 0$. If $\liminf_{i\to\infty}(\widehat{\alpha}_i) = 0$ then $\lambda(\mathbf{n})$ is a rigid subcritical ray.*

Proof. Suppose that $\sum_{i=1}^{\infty} |n_i| l(\alpha_i)$ converges. We first argue that the piecewise geodesic Λ introduced in the proof ot Lemma 4.1 is subcritical. Let $x \in \beta$ lie between a_j and a_{j+1}. Then

$$d\big(\beta(0), x\big) = (\text{length of } \Lambda \text{ between } \beta(0) \text{ and } x) - \sum_{i=1}^{j} |n_i| l(\alpha_i).$$

Then $\Delta_\Lambda(t) < \sum_{i=1}^{\infty} |n_i| l(\alpha_i)$ and consequently Λ is subcritical.

λ meets each α_i in a single point b_i. The geodesic arc of λ from $\lambda(0) = \beta(0)$ to b_i is freely homotopic relative endpoints to the piecewise geodesic arc of Λ_i between the same endpoints. The arc of λ will then be the shorter of the two. It follows that since Λ is a piecewise geodesic subcritical ray, λ is a subcritical ray.

As $n_i \neq 0$ for infinitely many i, λ is not asymptotic to β and therefore λ is in fact a rigid subcritical ray. That completes the proof of (1).

Now assume the hypothesis of (2). Observe that $\lambda(\mathbf{n})$ intersects β only once, in its initial point. $\lambda(\mathbf{n})$ weaves between the holes of F, never spiraling around the body of the flute. Consequently $\lambda(\mathbf{n})$ will intersect each ray λ_u for $u \in \overline{\mu}$ exactly once.

Define a new flute surface F' as follows: first remove from F the annular region in the complement of α_1. Then remove the set of points lying on the geodesic rays λ_u for $u \in \mu$. The remaining subsurface is bounded by α_1, λ_v, and λ_w where v, w are the endpoints of μ. Since λ_v and λ_w are orthogonal to α_1 we can identify the points $\lambda_v(t)$ and $\lambda_w(t)$ for $t \in [0, \infty)$ to produce a new hyperbolic surface F'. One easily sees that F' is a subsurface of an untwisted flute, that on F' the geodesics α_i are just the $\widehat{\alpha}_i$ with their endpoints identified, and that the infinite end of F' is of the first kind. Now from the first part it follows that $\lambda(\mathbf{n}) \cap F'$ is a subcritical ray on F', which is then easily transplanted back to a subcritical ray on F. □

Corollary 4.1. *An untwisted flute F contains one rigid subcritical ray, and hence uncountably many asymptotically distinct ones, if and only if*

$$\liminf_{i \to \infty} l(\widehat{\alpha}_i) = 0.$$

Proof. First we consider a surface F on which the infinite end is of the first kind. Suppose that $\liminf_{i\to\infty} l(\alpha_i) = k > 0$. We show that if λ is a geodesic ray on F which is non-backtracking and not asymptotic to β then λ is not a subcritical ray. It then follows by Corollary 2.1 that there are no rigid subcritical rays on F.

Since λ is not asymptotic to β there must exist distinct points $t_i \in [0, \infty)$ with $\lim_{i\to\infty} t_i = \infty$ so that $\lambda(t_i) \in \gamma$ for all $i \in \mathbf{N}$. Then there are corresponding integers j_i so that $\lambda(t_i) \in \gamma_{j_i}$. Since the arc of α_{j_i} is the minimal length common orthogonal of β and γ_{j_i} we have

$$d\big(\lambda(t_i), \beta\big) \geq \tfrac{1}{2} l(\alpha_{j_i}) > \tfrac{1}{2} k.$$

As any critical ray on F is asymptotic to β, it is a consequence of Theorem 3.3 that λ cannot be a subcritical ray.

The converse is an immediate consequence of the previous theorem. In fact, given any sequence $\mathbf{n}$ as above, if the sum $\sum_{i=1}^{\infty} |n_i| l(\alpha_i)$ converges and $n_i \neq 0$ for infinitely many $i \in \mathbf{N}$ then $\lambda(\mathbf{n})$ is a rigid subcritical ray. Thus if $\liminf_{i\to\infty} l(\alpha_i) = 0$ there will be uncountably many distinct non-asymptotic subcritical rays on F.

Now suppose that the infinite end of F is of the second kind. If $\liminf_{i\to\infty} (\widehat{\alpha}_i) = k > 0$ then there exist t_i as above, with $\lambda(t_i) \in \gamma$. By Theorem 2.1 a critical ray on F must be asymptotic to one of the critical rays λ_u for some $u \in \mu$. Then

$$\inf\big\{d\big(\lambda(t_i), \lambda_u\big) \mid i \in \mathbf{N}\big\} \geq \tfrac{1}{2} l(\widehat{\alpha}_i) \geq \tfrac{1}{2} k,$$

and again, by Theorem 3.3 λ cannot be subcritical.

As above, the converse and the existence of uncountably many distinct non-asymptotic subcritical rays follows from Theorem 4.2. □

Lemma 4.3. *Consider the hyperbolic right triangle with sides lengths a, b, c where $c > a, b$. With a fixed let $f(b) = c - b$. Then f is a decreasing function and $\lim_{b\to\infty} f(b) = \log(\cosh a)$.*

Proof. In general a, b, and c satisfy $\cosh c = \cosh a \cosh b$ [4]. Then

$$1 = \frac{\cosh\big(b + f(b)\big)}{\cosh a \cosh b} = \frac{e^{f(b)} + e^{-2b-f(b)}}{\cosh a(1 + e^{-2b})}.$$

Taking limits gives $\lim_{b\to\infty} e^{f(b)} = \cosh a$ from which the above limit follows. Writing $f(b) = \operatorname{arc\,cosh}(\cosh a \cosh b) - b$ and differentiating reveals that $f'(b) < 0$. □

The next result gives a necessary condition for a ray to be subcritical in a somewhat restrictive situation.

Theorem 4.3. *Suppose that $\lambda(\mathbf{n})$ is a rigid subcritical ray on a surface F, and that for some $I > 0$ and for all $i > I$, s_i is identically 1 or 0. Then*

$$\sum_{i-0}^{\infty} |n_i| l(\alpha_i)^2 \text{ converges.}$$

Proof. As in the proof of Corollary 4.1, for each $i \in \mathbf{N}$ there exist distinct points t_i, t_i' with $\lambda(t_i) \in \beta$ and, $\lambda(t_i') \in \gamma_{j_i}$ so that the open arcs $\lambda\big((t_i, t_i')\big)$ and $\lambda\big((t_i', t_{i+1})\big)$ are all disjoint from $\beta \cup \gamma$. The distance between the points $\lambda(t_i)$ and $\lambda(t_{i+1})$ is realized along the arc of β joining them and has the value d_i. As earlier, $l\big(\lambda\big((t_i, t')\big)\big) \le \frac{1}{2} l(\alpha_{j_i})$ and $l\big(\lambda\big((t_i', t_{i+1})\big)\big) \le \frac{1}{2} l(\alpha_{j_i})$. Then by the previous lemma

$$l\big(\lambda\big((t_i, t_{i+1})\big)\big) - d_i \ge 2\log\big(\cosh(\tfrac{1}{2} l(\alpha_{j_i}))\big).$$

Furthermore, since λ is subcritical, there is a value $M > 0$ so that for all $i > 0$

$$M > \Delta_\lambda(t_{i+1}) = \sum_{k=0}^{i} l\big(\lambda\big((t_k, t_{k+1})\big)\big) - d\big(\lambda(0), \lambda(t_{i+1})\big)$$

$$= \sum_{k=0}^{i} \big[l\big(\lambda\big((t_k, t_{k+1})\big)\big) - d_k\big] > 2\sum_{k=0}^{i} \log\big(\cosh \tfrac{1}{2} l(\alpha_{j_k})\big).$$

Consequently,

$$\sum_{k=0}^{\infty} \log\big(\cosh \tfrac{1}{2} l(\alpha_{j_k})\big) = \sum_{k=0}^{\infty} |n_k| \log\big(\cosh \tfrac{1}{2} l(\alpha_k)\big)$$

is convergent. The result follows. □

Remark 5.2. Keeping Lemma 4.3 in mind, one can easily construct an untwisted flute on which

1. there is a rigid subcritical ray $\lambda(\mathbf{n})$, where $\mathbf{s}$ satisfies the hypothesis of Theorem 4.3 and the sum $\sum_{i=0}^{\infty} |n_i| l(\alpha_i)$ diverges, and
2. there is a geodesic ray $\lambda(\mathbf{n})$, where $\mathbf{s}$ satisfies the hypothesis of Theorem 4.3 and the sum $\sum_{i=0}^{\infty} |n_i| l(\alpha_i)^2$ converges but $\lambda(\mathbf{n})$ is not subcritical.

Thus the truth is somewhere in between.

5. Dirichlet and Garnett points for untwisted flute groups

There is a a more familiar, but less geometric way to view questions about critical and subcritical rays in terms of the universal cover of a hyperbolic manifold M. The n-dimensional hyperbolic space is denoted by $\mathbf{H}^n$, and we write $\mathrm{Isom}(\mathbf{H}^n)$ for the group of isometries. $\mathbf{H}^n$ can be realized as the upper half-space in $\mathbf{R}^n$ endowed with the Poincaré metric. In this model $\mathbf{H}^n$ has a natural boundary $\partial\mathbf{H}^n$, and the action of $\mathrm{Isom}(\mathbf{H}^n)$ extends to the boundary. A *horocycle* is either the interior of an $n-1$-sphere in the upper half-space $\mathbf{H}^n$ which is tangent to $\partial\mathbf{H}^n$ or an upper half-space in $\mathbf{H}^n$.

Let Γ be a discrete subgroup of $\mathrm{Isom}(\mathbf{H}^n)$. The limit set $\Lambda(\Gamma)$ is defined as in Section 1.2. One then defines a point on the boundary of $\mathbf{H}^n$ to be horocyclic, Dirichlet, or Garnett in terms of how the point is approximated by the orbit of a given point under the action of the group Γ. More precisely, a limit point $x \in \Lambda(\Gamma)$ is called *horocyclic* if for some $a \in \mathbf{H}^n$ the orbit Γa enters every horocycle based at x. If some orbit enters every such horocycle then all orbits will. Among points $x \in \partial\mathbf{H}^n$ that are not horocyclic two sorts are generally distinguished. Fix $a \in \mathbf{H}^n$. If $x \in \partial\mathbf{H}^n$ is not a horocyclic limit point then there is a unique horocycle H_x based at x which is disjoint from Γa and such that any larger horocycle intersects Γa. H_x is called the *a-critical horocycle for x*. If a lies on the boundary of H_x then we say that the point x is *critical with respect to a*. Otherwise x is called *subcritical with respect to a*.

The following theorem gives the correspondence between the two viewpoints.

Theorem 5.1. *Let σ: $[0,\infty) \to M$ be a geodesic ray with initial point $\sigma(0) = a$. Lift σ to a geodesic ray $\tilde{\sigma}$ with initial point $\tilde{a} \in \mathbf{H}^n$ and endpoint $x \in \partial\mathbf{H}^n$. Then the ray σ is horocyclic, critical, or subcritical if and only if the point x is respectively horocyclic, critical with respect to $\tilde{a}$, or subcritical with respect to $\tilde{a}$.*

We will make use of the following lemma, which is a simple exercise in hyperbolic geometry.

Lemma 5.1. *Let h, h' be two points on the boundary of a horosphere H based at $x \in \partial\mathbf{H}^n$. Let $\tilde{\sigma}$ be the geodesic ray in $\mathbf{H}^n$ with initial point h and end-point x. Then*

1. *all* $t \in [0,\infty)$, $d\big(h,\tilde\sigma(t)\big) < d\big(h',\tilde\sigma(t)\big)$,
2. $\lim_{t\to\infty}\big[d\big(h,\tilde\sigma(t)\big) - d\big(h',\tilde\sigma(t)\big)\big] = 0$.

Proof of Theorem 5.1. First suppose that the point $x \in \partial\mathbf{H}^n$ is critical or subcritical with respect to $\tilde a \in \mathbf{H}^n$ and let H_x denote the $\tilde a$-critical horosphere based at x. The geodesic ray $\tilde\sigma$: $[0,\infty) \to \mathbf{H}^n$ from $\tilde a$ to x meets H_x orthogonally at a point h. Set $d = d(\tilde a, h)$, and let σ denote the projection of $\tilde\sigma$ to M. Let γ_t be a minimal length geodesic arc on M joining a to $\sigma(t)$. Take the lift $\widetilde{\gamma}_t$ of γ_t which ends at $\tilde\sigma(t)$. The initial point of this arc, denoted $\tilde a_t$, belongs to the Γ-orbit of $\tilde a$. Let $h_t = \gamma_t \cap H_x$. Then we have

$$\begin{aligned}\big[t - d_M\big(a,\sigma(t)\big)\big] &= [t - l(\gamma_t)] = \big[d\big(\tilde a,\tilde\sigma(t)\big) - d\big(\tilde a_t,\tilde\sigma(t)\big)\big]\\ &= \big[d(\tilde a,h) - d(\tilde a_t,h_t) + d\big(h,\tilde\sigma(t)\big) - d\big(h_t,\tilde\sigma(t)\big)\big] < d(\tilde a,h) \le d,\end{aligned}$$

where we have applied Lemma 5.1 to deduce the inequality. Thus the ray σ is either critical of subcritical.

If the point x is critical with respect to $\tilde a$ then $\tilde a = h$. It follows that $d = 0$ and σ is a critical ray. If x is rather subcritical with respect to $\tilde a$ then $\tilde a \notin H_x$ and $d = d(\tilde a, h) > 0$. Consequently, there is a point $\tilde a' \in \Gamma\tilde a$ and $h' \in \partial H_x$ so that $d' = d(\tilde a', h') < d$. Choose t' large enough so that in Lemma 5.1 $d\big(h',\tilde\sigma(t')\big) - d\big(h,\tilde\sigma(t')\big) < d - d'$. Then

$$d\big(\tilde a',\tilde\sigma(t')\big) = d(\tilde{a'},h') + d\big(h',\tilde\sigma(t')\big) < d' + \big[d - d' + d\big(h,\tilde\sigma(t')\big)\big] = d\big(\tilde a,\tilde\sigma(t')\big).$$

It follows that σ does not minimize the distance between two points along the ray and hence σ is subcritical.

Conversely, suppose that the geodesic ray σ is critical or subcritical. Let $\tilde\sigma$ be a lift to $\mathbf{H}^n$ beginning at $\tilde a$ and ending at x. We show that x is either critical with respect to $\tilde a$ or subcritical with respect to $\tilde a$. To this end we demonstrate the existence of a critical horosphere H_x. If there does not exist such a horosphere then for each integer $n > 0$ there is a horosphere H_n based at x so that for some $g_n \in \Gamma$, $g_n(\tilde a) = \tilde a_n \in \partial H_n$, and the points $h_n = \sigma \cup \partial H_n$ satisfy $d(\tilde a, h_n) > n$. For each n apply Lemma 1.1 to choose t_n so that

$$d\big(\tilde{a_n},\tilde\sigma(t_n)\big) - d\big(h_n,\tilde\sigma(t_n)\big) < 1.$$

Then we have

$$\begin{aligned}t_n - d\big(a,\sigma(t_n)\big) &\le d\big(\tilde a,\tilde\sigma(t_n)\big) - d\big(\tilde a_n,\tilde\sigma(t_n)\big)\\ &= d(\tilde a,h_n) + d\big(h_n,\tilde\sigma(t_n)\big) - d\big(\tilde a_n,\tilde\sigma(t_n)\big) > n - 1\end{aligned}$$

for any integer $n > 0$. This implies that σ is horocyclic, contrary to the initial assumption. The point x is therefore either critical with respect to $\tilde a$ or subcritical with respect to $\tilde a$. The theorem follows. □

A point $x \in \partial\mathbf{H}^n$ is a *Dirichlet* point of Γ if for any $a \in \mathbf{H}^n$ there is a group element $\gamma \in \Gamma$ so that $\gamma(x)$ is critical with respect to a. Those points of $\partial\mathbf{H}^n$ which are neither horocyclic nor Dirichlet are called Garnett points. Following Nicholls and Waterman, a point $x \in \partial\mathbf{H}^n$ is a *rigid* Garnett point if for all $a \in \mathbf{H}^n$, x is subcritical with respect to a. It follows from the definitions that if x is subcritical with respect to $a \in \mathbf{H}^n$ but not a rigid Garnett point then for some $a' \in \mathbf{H}^n$, x is Dirchlet with respect to a'.

The following corollary is an immediate consequence of the theorem.

Corollary 5.1. *Let x be a point on the boundary of $\mathbf{H}^n$, and let σ be a geodesic ray in M that has a lift ending at x. Then*

1. *x is a Dirichlet point of Γ if and only if for each point $a \in M$ there is a critical ray with initial point a which is asymptotic to σ.*
2. *x is a Garnett point of Γ if and only if for some $a \in M$ any geodesic ray with initial point a which is asymptotic to σ is subcritical.*
3. *x is a rigid Garnett point of Γ if and only if for all $a \in M$ a geodesic ray with initial point a which is asymptotic to σ is subcritical.*

A point is Dirichlet if and only if it lies on the boundary of some Dirichlet fundamental polyhedron for Γ. Consequently, the Dirichlet points of Γ are associated with the non-compactness of the quotient orbifold $\mathbf{H}^n/\Gamma$. In dimension $n = 2$, unless the group is infinitely generated, a Dirichlet point is either a regular point of the group or a fixed point of a parabolic [12]. In dimensions $n \geq 3$ there are finitely generated groups which have a Dirichlet point that is neither a regular point nor the fixed point of a parabolic [11].

The need for delineating the class of Garnett points seems to have arisen in the work of Sullivan [16], where it was shown that the set of Garnett points for a discrete set Γ has measure zero. The existence of a group with a Garnett point was first demonstrated by Nicholls [13] (see also [14]). In [15] it is shown that there exist groups containing rigid and non-rigid Garnett points.

5.1. Untwisted flute groups. We shall end by translating some of the earlier results into the language of Fuchsian groups. It should be clear that many of the results on subcritical rays have no simple translation into the group setting. This is the reason that the intrinsic geometric viewpoint has been adhered to through most of the paper.

Define an *untwisted flute group* to be a Fuchsian group representing an untwisted flute surface. Such groups can be constructed from the defining parameters described in Section 1 (see [1]). What follows are some consequences of the results of the preceeding sections.

Proposition 5.1. *Let G be an untwisted flute group and let $F = \mathbf{H}^2/G$. A point $x \in \partial\mathbf{H}^2$ is a Garnett point of G if and only if it is a rigid Garnett point of G if and only if it is the endpoint of a lift of a rigid subcritical ray in F.*

Proof. If x is a Garnett point of G then by Corollary 5.1 there is a point $a \in F$ and a geodesic ray σ in F with $\sigma(0) = a$ so that some lift of σ ends at x, and so that any geodesic ray λ in F asymptotic to σ with $\lambda(0) = a$ is also a subcritical ray. As a consequence of Lemma 2.1 and the definition of rigid subcritical ray, for any $a' \in F$ every ray λ asymptotic to σ with $\lambda(0) = a'$ is a subcritical ray. Thus from Corollary 5.1 we conclude that x is a rigid Garnett point. □

The situation for Dirichlet points is described in the next proposition. The proof is an easy consequence of Corollary 5.1 and Theorem 2.1. A set X in $\partial\mathbf{H}^2$ is *precisely invariant under the identity* in the Fuchsian group G if $g(X) \cap X = \emptyset$ for all $g \in G$ with $g \neq$ identity (see [12]).

Proposition 5.2. *Let G be an untwisted flute group. One of the following holds*:

1. *there exists a point $a \in \partial\mathbf{H}^2$ so that every Dirichlet point of Γ belongs to the orbit Γa of a, or*
2. *there exists a maximal closed interval $I \subset \partial\mathbf{H}^2$ which is precisely invariant under the identitity in G so that a point x is Dirichlet if and only if it is G-equivalent to a point in I.*

Note that the two cases in the proposition correspond to the cases where the infinite end of F is, respectively of the first kind and of the second kind. Translating Corollary 4.1 in terms of Fuchsian groups we have

Corollary 5.2. *Let G be an untwisted flute group. Then there is one Garnett point for G if and only if there are uncountably many Garnett points for G.*

One could as well translate a number of the other results from Section 4, but we shall leave this to the interested reader.

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