Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 21, 1996, 111–115

A NOTE ON THE CONVEXITY THEOREM FOR MEAN VALUES OF SUBTEMPERATURES

Miroslav Brzezina

Technical University of Liberec, Department of Mathematics
Hálkova 6, 461 17 Liberec 1, Czech Republic; miroslav.brzezina@vslib.cz

Abstract. In this note we prove the finiteness of mean values of subtemperatures over level surfaces of the Green function and we give a capacitary interpretation of the corresponding "mean value measure".

1. Introduction

In a series of papers, N.A. Watson has extended the classical result of F. Riesz that the integral mean of subharmonic functions in $\mathbf{R}^d$, $d \geq 3$, over concentric spheres of radius r is a convex function of r^{2-d}, for subsolutions of both second order, linear elliptic partial differential equations with variable coefficients ([8], [10]) and the heat equation ([6], [7], [9], [10]).

In the case of potential theory for the heat operator, we show that the function φ in Theorem 4 of N.A. Watson's paper [10, p. 253], is actually finite. To prove this, we use some ideas of [6, p. 249], and the fact that the corresponding mean value measure $\mu^D_{p_0,c}$ on the heat sphere $\partial\Omega_D(p_0,c)$ is the balayage of the Dirac measure ε_{p_0} concentrated at p_0 onto the complement of the heat ball $\Omega_D(p_0,c)$ as well as the capacitary measure (for the adjoint operator) for $\overline{\Omega_D(p_0,c)}$, see Theorem 2 below.

Following N.A. Watson, [10, p. 252], we give some definitions and notations. Let Θ denote the heat operator $\sum_{i=1}^d \partial^2/\partial x_i^2 - \partial/\partial t$ in $\mathbf{R}^{d+1}$, $d \in \mathbf{N}$, and let Θ^* denote its adjoint $\sum_{i=1}^d \partial^2/\partial x_i^2 + \partial/\partial t$. In what follows, all notions concerning the adjoint heat operator will be denoted by *, e.g., *regular, *heat potential etc. We put

$$\nabla_x u := \left(\frac{\partial}{\partial x_1}u, \dots, \frac{\partial}{\partial x_d}u\right), \qquad \nabla u := \left(\nabla_x u, \frac{\partial}{\partial t}u\right),$$

and use $\|\cdot\|$ to denote the Euclidean norm in both $\mathbf{R}^d$ and $\mathbf{R}^{d+1}$. A temperature is a solution of the heat equation, and subtemperatures and supertemperatures are corresponding subsolutions and supersolution, cf. [4].

1991 Mathematics Subject Classification: Primary 31B35, 31C15, 35K05.

Financial support by the Konferenz der deutschen Akademien der Wissenschaften is gratefully acknowledged.

Let D be an open subset of $\mathbf{R}^{d+1}$ which is *Dirichlet regular, i.e., if $\partial D \neq \emptyset$ then for every continuous function $f\colon \partial D \to \mathbf{R}$ there exists exactly one function ${}^*H_f^D\colon \overline{D} \to \mathbf{R}$ such that

$$\Theta^*({}^*H_f^D) = 0 \quad \text{on } D, \qquad {}^*H_f^D\,|_{\partial D} = f$$

and ${}^*H_f^D \geq 0$ whenever $f \geq 0$.

Further, let G_D denote the Green function for D in the sense of [4], cf. [3]. If $p_0 \in D$, then $G_D(p_0,\cdot)$ is a non-negative *supertemperature on D, and a *temperature on $D \setminus \{p_0\}$. It follows from Sard's theorem that the set

$$\big\{p \in D\,;\, G_D(p_0,p) = (4\pi c)^{-d/2}\big\} \tag{1}$$

is a smooth regular d-dimensional manifold for almost every $c > 0$. We call such a value of c a *regular value.* For an arbitrary positive value of c, we put

$$\Omega_D(p_0,c) := \big\{p \in D\,;\, G_D(p_0,p) > (4\pi c)^{-d/2}\big\}.$$

For any regular value of c, the union of $\{p_0\}$ and the set in (1) is equal to $\partial\Omega_D(p_0,c)$. The set $\overline{\Omega}_D(p_0,c)$ is a compact subset of D; see [10, p. 252]. The set $\Omega_D(p_0,c)$ is called the *heat ball* and $\partial\Omega_D(p_0,c)$ the *heat sphere* (with respect to D). If c is a regular value, we put for $p \in \partial\Omega_D(p_0,c) \setminus \{p_0\}$

$$K_D(p_0,p) := \|\nabla_x G_D(p_0,p)\|^2 \|\nabla G_D(p_0,p)\|^{-1}$$

and

$$\mathscr{M}_D(u,p_0,c) := \int_{\partial\Omega_D(p_0,c)} u K_D(p_0,\cdot)\, d\sigma$$

whenever the integral exists. Here σ denotes the surface area measure on $\partial\Omega_D(p_0,c)$. The measure $\mu^D_{p_0,c} := K_D(p_0,\cdot)\sigma$ will be called the *mean value measure.* If $D = \mathbf{R}^{d+1}$, the measure $\mu^D_{p_0,c}$ is called the Fulks–Pini measure; see, e.g., [1] for details.

If c_1, c_2 are regular values, $c_1 < c_2$, we put

$$A_D(p_0,c_1,c_2) := \Omega_D(p_0,c_2) \setminus \overline{\Omega_D(p_0,c_1)}.$$

The following result is due to N.A. Watson, [9, p. 176]; for a simpler proof, see [10, p. 253].

Theorem 1. *Let u be a subtemperature on an open superset E of $\overline{A_D(p_0,c_1,c_2)}$. Then there is a function φ, either finite and convex or identically $-\infty$, such that*

$$\mathscr{M}_D(u,p_0,c) = \varphi(c^{-d/2})$$

for all regular values c in $[c_1,c_2]$.

In the case $D = \mathbf{R}^{d+1}$, the finiteness of $\mathscr{M}_D(u,p_0,c)$ was proved in [6, p. 249]. We will show that this also holds when $D \neq \mathbf{R}^{d+1}$.

First we give some preliminary results. The *heat potential on D (adjoint heat potential) of a positive Radon measure μ on D is defined by

$$^*G_D^\mu(p) := \int_D G_D(q,p)\,\mu(dq).$$

From [3, p. 348], it follows that for every compact set $K \subset D$ there exists a uniquely determined Radon measure $^*\mu_{D,K}$ (called the **equilibrium measure* for K with respect to D) such that

$$^*G_D^{^*\mu_{D,K}} = {}^*\widehat{R}_1^K;$$

here, of course, $^*\widehat{R}_1^K$ denotes the balayage of 1 on K with respect to D and the adjoint heat theory. The number $^*\mu_{D,K}(K)$ is called the **capacity* of K (with respect to D).

If $U \subset \mathbf{R}^{d+1}$ is a set and $p_0 \in \mathbf{R}^{d+1}$, then $\varepsilon_{p_0}^U$ stands for the *balayage* of the Dirac measure ε_{p_0} concentrated at p_0 on U.

For $c > 0$, we consider the function

$$w(p) := \min\big\{(4\pi c)^{-d/2};\, G_D(p_0,p)\big\}, \qquad p \in D.$$

Lemma 1. *Let $p_0 \in D$ and c be a regular value. Put $\Omega := \Omega_D(p_0,c)$ and $\nu := \varepsilon_{p_0}^{\complement\Omega}$. Then*

$$^*G_D^\nu = w.$$

Lemma 2. *Let $p_0 \in D$ and c be a regular value. Put $\Omega := \Omega_D(p_0,c)$. Then*

$$^*G_D^{^*\mu_{D,\overline{\Omega}}} = (4\pi c)^{-d/2} w.$$

We omit the proofs of Lemma 1 and 2, since they differ from those of [2, pp. 472, 473] only in minor details.

Theorem 2. *Let $p_0 \in D$ and c be a regular value. Put $\Omega := \Omega_D(p_0,c)$. Then*

$$\mu_{p_0,c}^D = \varepsilon_{p_0}^{\complement\Omega} = (4\pi c)^{-d/2}\ {}^*\mu_{D,\overline{\Omega}}.$$

Proof. The first equality was proved in [9, p. 181]. The second one follows from the uniqueness of the representation of *heat potentials, see [3, p. 305], and from Lemma 1 and Lemma 2. □

In the case $D = \mathbf{R}^{d+1}$, the first equality was established by H. Bauer in [1], the second one in [2].

Corollary. *Let* $p_0, p_1 \in D$ *and* c *be a regular value. Then*

$$\mathscr{M}_D\big(G_D(\cdot, p_1), p_0, c\big) = \min\{(4\pi c)^{-d/2}, G_D(p_0, p_1)\}.$$

Proof. By definitions we have

$$\mathscr{M}_D\big(G_D(\cdot, p_1), p_0, c\big) = \int_D G_D(\cdot, p_1)\, d\mu^D_{p_0,c} = {}^*G_D^{\mu^D_{p_0,c}}(p_1).$$

According to Lemma 1 and Theorem 2

$$\mathscr{M}_D\big(G_D(\cdot, p_1), p_0, c\big) = w(p_1) = \min\big\{(4\pi c)^{-d/2}, G_D(p_0, p_1)\big\}. \ \square$$

In the case $D = \mathbf{R}^{d+1}$, the assertion of the Corollary was proved by N.A. Watson in [6, p. 248].

Now, we are in position to prove the finiteness of the mean values $\mathscr{M}_D(u, p_0, c)$ in Theorem 1.

Theorem 1′. *Let* u *be a subtemperature on an open superset* E *of* $\overline{A_D(p_0, c_1, c_2)}$. *Then there is a real-valued convex function* φ *such that*

$$\mathscr{M}_D(u, p_0, c) = \varphi(c^{-d/2})$$

for all regular values c *in* $[c_1, c_2]$.

Proof. Let $c \in [c_1, c_2]$ be a regular value. By Theorem 1, it is sufficient to prove that $\mathscr{M}_D(u, p_0, c)$ is finite. Let V be an open superset of $\overline{A_D(p_0, c_1, c_2)}$, $\overline{V} \subset E \cap D$, $\overline{V}$ compact. By the Riesz decomposition theorem, see [4, p. 279], there exists a finite measure μ and a temperature h on V such that $u = h - G_V^\mu$ on V. According to [9, p. 168], $\mathscr{M}_D(h, p_0, c) = h(p_0)$; obviously, $h(p_0)$ is finite. Since, for all $p, q \in V$,

$$0 \le G_V(p, q) \le G_D(p, q)$$

we have

$$\int_V G_V(p, q)\, \mu(dq) \le \int_V G_D(p, q)\, \mu(dq).$$

By Fubini's theorem and Corollary, we have

$$\mathscr{M}_D(G_V^\mu, p_0, c) \le \int_V \mathscr{M}_D\big(G_D(\cdot, q), p_0, c\big)\, \mu(dq) \le (4\pi c)^{-d/2} \cdot \mu(V) < \infty. \ \square$$

References

[1] Bauer, H.: Heat balls and Fulks measures. - Ann. Acad. Sci. Fenn. Ser. A I Math. 10, 1985, 62–82.

[2] Brzezina, M.: Capacitary interpretation of the Fulks measure. - Exposition. Math. 11, 1993, 469–474.

[3] Doob, J.L.: Classical potential theory and its probabilistic counterpart. - Springer-Verlag, Berlin, 1984.

[4] Watson, N.A.: Green functions, potentials, and the Dirichlet problem for the heat equation. - Proc. London Math. Soc. (3) 33, 1976, 251–298.

[5] Watson, N.A.: Thermal capacity. - Proc. London Math. Soc. (3) 37, 1978, 342–362.

[6] Watson, N.A.: A convexity theorem for local mean value of subtemperatures. - Bull. London Math. Soc. 22, 1990, 245–252.

[7] Watson, N.A.: Mean values and thermic majorization of subtemperatures. - Ann. Acad. Sci. Fenn. Ser. A I Math. 16, 1991, 113–124.

[8] Watson, N.A.: Mean values of subharmonic functions over Green spheres. - Math. Scand. 69, 1991, 307–319.

[9] Watson, N.A.: Mean values of subtemperatures over level surfaces of Green functions. - Ark. Mat. 30, 1992, 165–185.

[10] Watson, N.A.: Generalization of the spherical mean convexity theorem on subharmonic functions. - Ann. Acad. Sci. Fenn. Ser. A I Math. 17, 1992, 241–255.

Received 4 August 1994