Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 21, 1996, 133–150

NEGATIVELY CURVED GROUPS AND THE CONVERGENCE PROPERTY II: TRANSITIVITY IN NEGATIVELY CURVED GROUPS

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Abstract. Let G be a negatively curved group. This paper continues the classification of limit points of G that began in part I. A probability measure is constructed on the space at infinity and with respect to this measure almost every point at infinity is shown to be line transitive.

0. Introduction

Let M^n be a closed riemannian n-manifold with all sectional curvatures -1. Then $\pi_1(M^n)$ is a discrete Möbius group acting properly discontinuously on the universal cover $\mathbf{H^n}$ and conformally on the boundary $\mathbf{S^{n-1}}$. Since the action of $\pi_1(M^n)$ is cocompact, every $x \in \mathbf{S^{n-1}}$ is a point of approximation (conical limit point) for $\pi_1(M^n)$ (see 2.4.9 of [Nic] for instance). Furthermore, Γ (any Cayley graph for $\pi_1(M^n)$) is quasi-isometric to $\mathbf{H^n}$ (see [Ca]) and so $\partial\Gamma$, the boundary at infinity of Γ, is homeomorphic to $\mathbf{S^{n-1}}$. Therefore every $x \in \partial\Gamma$ is a point of approximation.

Weakening the above hypotheses, assume M^n is any hyperbolic manifold. Let $x \in \mathbf{S^{n-1}}$ and suppose $L \subset \mathbf{H^n}$ is an oriented hyperbolic line with x as one endpoint. If for any $b \in \mathbf{H^n}$ there exists a sequence of distinct deck transformations $\{g_n\} \subset G = \pi_1(M^n)$ such that the images g_nL come arbitrarily close to b, then x is called *point transitive.* If for any oriented hyperbolic line L' there are distinct g_n such that $g_nL \to L'$ preserving orientation, then x is *line transitive* or a *Myrberg point.* Clearly, every Myrberg point is also point transitive, but the converse is false in general (see [Sh]). Myrberg [M] first showed that for $n = 2$ the set of line transitive points has full Lebesgue measure in the boundary at infinity. More recently, Tukia has proved that in all dimensions the collection of conical limit points that are not Myrberg points is a nullset for any conformal G measure [T].

Let G be any negatively curved (Gromov hyperbolic) group with Cayley graph Γ and space at infinity $\partial\Gamma$. In I.3.7, the author showed that every $x \in \partial\Gamma$ is a

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1991 Mathematics Subject Classification: Primary 20F32; Secondary 57S05.

This research was supported in part by the Geometry Center, University of Minnesota, an STC funded by the NSF, DOE, and Minnesota Technology, Inc.; and by an NSF research grant; also by Brigham Young University and Utah State University.

point of approximation. In this paper, almost every $x \in \partial\Gamma$ is shown to be line transitive. The strategy is based on an old idea due to Artin. In [A], Artin showed that the line transitive points for the modular group $\mathrm{SL}(2,\mathbf{Z})$ are exactly those real numbers whose continued fraction expansion contains each finite sequence of integers. The continued fraction of a real number ξ can be obtained as the cutting sequence of a geodesic ray $R \subset \mathbf{H^n}$ tending to ξ (see 5.4 of [Se]). In the language of geometric group theory, a cutting sequence for R corresponds to reading the label of an equivalent geodesic ray R' in the Cayley graph for $\mathrm{SL}(2,\mathbf{Z})$. Since $\mathrm{SL}(2,\mathbf{Z}) \simeq \mathbf{Z_2} * \mathbf{Z_3}$ is nearly a free group, cutting sequences and labels of geodesic rays are very similar [Se].

The concept of a geodesic ray containing each finite geodesic subsegment is somewhat similar to the idea of a normal number. Recall that a real number η is *normal to base* r if each block B_k of k digits occurs with frequency $1/r^k$. It is well known that almost all real numbers are normal to every base (Chapter 8 [Niv] or 9.3–9.13 in [HW]). Hedlund and Morse generalized a similar concept to strings of symbols in their paper [HM]. Given a finite set of generating symbols and certain concatenation rules they considered infinite sequences containing a copy of every possible finite string of symbols. Such sequences were labelled *transitive.* Yet a third related subject concerns Markov chains, see [Fel]. A state (symbol, outcome, event, etc.) is *persistent* if with probability one, that state will recur (infinitely often) within the chain. This paper links all of the above.

Example 0.1. Suppose $F = \langle a, b\rangle$ is the free group of rank 2. Embed the associated Cayley graph in $\mathbf{H^2}$. Each geodesic ray represents a unique point at infinity. Let w be a freely reduced word of length k in the generators. Emulating the counting estimate of [Niv], consider the ratio of the number of rays (from 0) of length nk containing w to the total number of rays of length nk. As $n \to \infty$ this ratio tends to 1. The conclusion is that with respect to a certain natural (Cantor) measure, almost every ray contains w, and in fact infinitely often. By a *transitive ray*, I mean a geodesic ray from 0 that contains every finite geodesic word as a subpath. Theorem 1.4 below shows that any point represented by such a ray is line transitive (the proof of Theorem 1.4 in the case of a free group is much simpler than what I have written). The set of all geodesic segments form a Markov chain with four states: a, b, a^{-1}, b^{-1}. A given state can be succeeded by any state other than its inverse, with probability $\frac{1}{3}$. In the language of [Fel], each state is aperiodic, persistent, and has finite mean recurrence time, i.e., all states are *ergodic.*

In the case of a generic negatively curved G, relators (of perhaps arbitrarily long length) destroy the Markov aspect and vastly complicate the counting process. Showing that most points at infinity can be represented by an actual geodesic transitive ray seems to be difficult in the general case; in fact it may not be true! The difficulty is avoided by considering quasigeodesic rays.

The reader is expected to be familiar with [Fr], the first installment of "Negatively curved groups have the convergence property". Results from that text will be prefaced by the the roman numeral I. Section 1 deals with definitions and preparatory material. Section 2 introduces the key simplification that reduces everything to symbolic dynamics. In Section 3 a measure on $\partial\Gamma$ and the main theorems are established. I thank Jim Cannon for the idea behind Section 2.

1. Preliminaries

Throughout the remainder of the text, let G be a group with a fixed finite generating set such that the inverse of each generator is also a generator. (I do not allow the neutral element of G to be in the generating set.) Let Γ be the Cayley graph of G with respect to the given generators. Assume that Γ has thin triangle constant δ, boundary at infinity $\partial\Gamma$, and that G is non-elementary. Rays, halfspaces, etc. were defined in I.1.1–I.1.11. The ball with center $a \in \Gamma$ and radius $\rho > 0$ is indicated by $B(a, \rho)$. For $\varepsilon > 0$ and ray R, the neighborhood $\{a \in \Gamma : \mathrm{d(a, R)} < \varepsilon\}$ occurs so often that I will refer to it as the ε *corridor* about R.

Exercise 1.1. Let $\varepsilon > 0$. If R: $[0, \infty) \to \Gamma$ is a ray and $r', r \in (0, \infty)$ with $r' \geq r + 2\varepsilon$, then $B(r', \varepsilon) \subset H(R, r)$.

Definition 1.2. Suppose H is a non-elementary convergence group of the first kind (every point of X a limit point) acting on a compact Hausdorff space X. A point $x \in X$ is said to be *line transitive* or a *Myrberg point* if given any two distinct points $u, v \in X$ there exists a sequence of group elements $\{h_n\}$ such that

$$h_n(x) \to u \quad \text{and} \quad h_n(y) \to v \quad \text{for all } y \neq x.$$

By passing to a subsequence we may suppose that the convergence is locally uniform away from x. If G is a non-elementary negatively curved group acting properly discontinously, cocompactly, and by isometry on a proper geodesic space Y, then G acts as a convergence group of the first kind on $X = \partial Y$ and the above definition is equivalent to the following. If L is any geodesic line with one endpoint at a Myrberg point x and $u \neq v$ are points at infinity, then there exist $\{g_n\}$ such that $g_n L$ converges to a geodesic joining u and v with $g_n(x) \to u$. This explains the term "line transitive". (If Y is classical hyperbolic m-space, such a geodesic is unique, whereas in a Cayley graph there are in general many geodesics joining a pair of points at infinity.) Definition 1.2 requires no domain of discontinuity for G, merely a non-elementary limit set. (I have been unable to find an analogous generalization for point transitivity.)

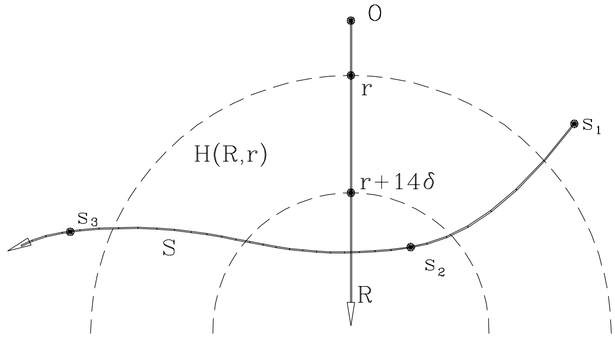


Figure 1.1

Lemma 1.3. *Let $R, S \subset \Gamma$ be rays, $r \in R$, and $s_1 \in S \cap H^-(R,r)$. If $s_2 \in S \cap H(R, r+14\delta)$ with $s_2 > s_1$, then $S^+ \subset H(R,r)$ for the sub-ray $S^+ = S\big([s_2,\infty)\big)$ of S.*

Proof. Suppose, by way of contradiction, that S exits $H(R,r)$ after s_2 (see Figure 1.1). Let s_3 be a point in $S^+ \cap H^-(R,r)$. Then $\overline{s_1 s_3}$ is a geodesic segment with both endpoints in $H^-(R,r)$ with $s_2 \in \overline{s_1 s_3}$. By Lemma I.2.4, the halfspace $H^-(R,r)$ is quasiconvex(2δ) so $\mathrm{d}\big(s_2, H^-(R,r)\big) < 2\delta$. On the other hand, Lemma I.2.5 implies that $\mathrm{d}(s_2, H^-(R,r)$ is greater than $(14-12)\delta = 2\delta$, a contradiction. ▫

Theorem 1.4. *Suppose $x \in \partial\Gamma$ is represented by a ray R from 0, and that for every finite geodesic word, R contains a directed subpath labelled with the given word. Then x is a Myrberg point.*

Proof. Let $u \neq v \in \partial\Gamma$ be represented by rays S, T respectively. Denote the vertices (group elements) of S, T tending from 0 by $s_1, s_2, \ldots$ and $t_1, t_2, \ldots$. Given $n > 0$, the literal edge path $t_n^{-1} s_n$ is almost surely not geodesic; I will abuse notation and denote by $\overline{t_n^{-1} s_n}$ any geodesic path between the vertices t_n and s_n. Using an easy thin triangles argument (see 7.5 of [G] or p. 19 of [CDP]) there is a bound $M > 0$ independent of n and an intermediate point a_n of $t_n^{-1} s_n$ such that $\mathrm{d}(a_n, 0) \le M$.

By hypothesis, the ray R contains $\overline{t_n^{-1} s_n}$ as a segment, meaning $R = r_n \overline{t_n^{-1} s_n} \cdots$ for some initial geodesic segment r_n. For each $n > 0$ define $g_n \colon \Gamma \to \Gamma$ as left multiplication by $t_n r_n^{-1}$. Then $g_n(0) = t_n r_n^{-1}$, $g_n(r_n) = t_n$, and $g_n(r_n t_n^{-1} s_n) = s_n$ (refer to Figure 1.2). Pick any $s \in S$, $t \in T$ so that both halfspaces $H(S,s)$, $H(T,t)$ are disjoint from the ball $B(0,M)$. Choose n so large that $s_n \ge s + 22\delta$ and $t_n \ge t + 14\delta$. Evidently $g_n(R)$ is a ray passing through t_n, a_n, and s_n, in that order, where a_n is an intermediate point of the subsegment $\overline{t_n^{-1} s_n}$ such that $\mathrm{d}(a_n, 0) \le M$. Note that $a_n \in H^-(T,t)$ and $t_n \in H(T, t+14\delta)$. From the proof of Lemma 1.3, $g_n(0) = t_n r_n^{-1} \in H(T,t)$, and by the same argument, the tail of $g_n(R)$ starting at s_n lies entirely within $H(S, s+8\delta)$. Therefore the endpoint $g_n(x) \in \overline{D(S, s+8\delta)} \subset D(S,s)$, the last inclusion following from

1.12 of [Sw]. This shows that $g_n(0) \to v$ and $g_n(x) \to u$ as $n \to \infty$. Use the convergence property to find a subsequence so that $g_j \to v$ locally uniformly on $\overline{\Gamma} \setminus \{x\}$. □

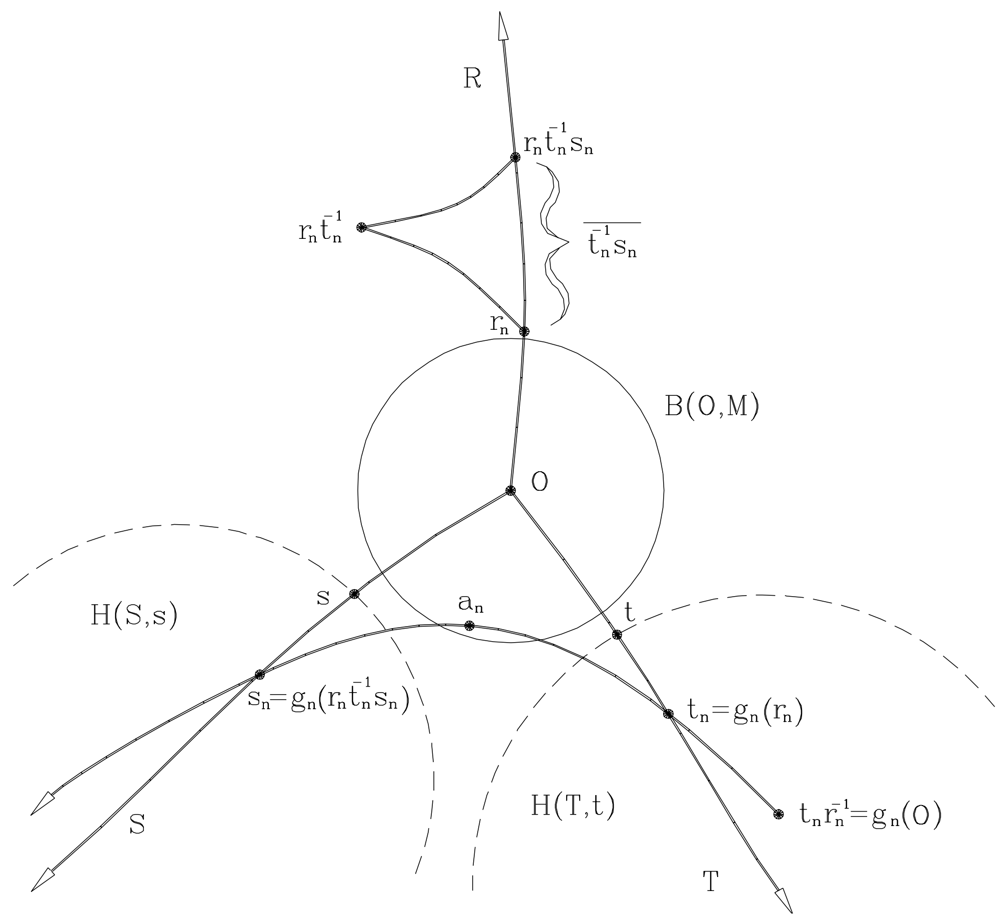


Figure 1.2

2. Symbols, words, and quasigeodesics

For each positive integer p, let $\mathscr{D}_p$ denote the cover of $\partial\Gamma$ by open combinatorial disks $\{D(R,p) : R \text{ is a ray from } 0\}$. Lemma 1.14 of [Sw] shows that if R, S are rays from 0 with $\mathrm{d}\big(R(p), S(p)\big) > 18\delta$ then the corresponding disks $D(R,p)$, $D(S,p)$ are disjoint. Furthermore, 1.15 of [Sw] says that $D(R,p)$ is uniquely determined by the segment $R\big([p-4\delta, p+4\delta]\big) \subset R$. These two results imply that there is some $P > 0$ such that for any fixed $p > 4\delta$, the closure of a given $D \in \mathscr{D}_p$ intersects at most P other closed disks in the collection $\mathscr{D}_p$. Let $\{\mathscr{D}_p\}_{p>4\delta}^{\infty}$ be such a sequence of disk covers of $\partial\Gamma$. Fix $p > 7\delta$ and for each $D(R,p) \in \mathscr{D}_p$, let $O(R,p)$ denote an open subset of $\bar{\Gamma} = \Gamma \cup \partial\Gamma$ such that

$$D(R,p) \subset O(R,p) \subset \big[H(R,p) \cup D(R,p)\big].$$

Using any global metric on $\bar{\Gamma}$ we may suppose that $O(R,p)$ is so close (pointwise) to $D(R,p)$ that

$$D(R,p) \cap D(S,p) = \emptyset \quad \text{implies} \quad O(R,p) \cap O(S,p) = \emptyset.$$

Set $\mathscr{O}_p = \{O(R,p)\}$. We may assume that p is so large that there are at least $3P$ members of $\mathscr{O}_p$.

Lemma 2.1. *The distance from any $O(R,p)$ to 0 is at least 4δ.*

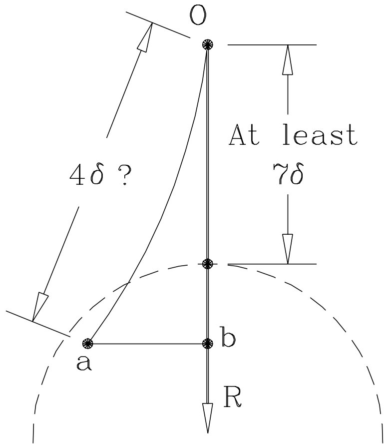


Figure 2.1

Proof. It is sufficient to show that the distance from $H(R,p)$ to the origin is at least 4δ (see Figure 2.1). Suppose that $a \in H(R,p)$ is within 4δ of 0. Then a projects to some point $b \in R[p,\infty)$. Consider the triangle with vertices $0, a, b$. Since a is in the halfspace, the length of $\overline{ab}$ is at most $4\delta = \mathrm{d}(a,0)$. A simple thin triangles exercise (made explicit in 1.3 of [Sw]) shows that

$$7\delta \le \mathrm{d}(0,b) \le \mathrm{d}(0,a) + 2\delta \le 4\delta + 2\delta = 6\delta$$

which is a contradiction. □

Pick $q > p$ so large that the complement of $B(0,q) \subset \Gamma$ is contained in $\cup\mathscr{O}_p$. Assign a distinct symbol to each element in G of length q. Let $\mathscr{A}_q$ be the *alphabet of symbols* having (reduced word) length q. Assign to each $s \in \mathscr{A}_q$ a shortest representative word in G, or equivalently, a geodesic edge path from 0 to s in the Cayley graph. Define a formal concatenation of symbols as follows. Let $s,t \in \mathscr{A}_q$. The product $s \cdot t$ is *admissible* if s^{-1} and t, considered as edge paths from 0 in the Cayley graph, terminate in $O(R,p)$, $O(S,p)$ respectively, with $O(R,p) \cap O(S,p) = \emptyset$. A word in the alphabet $\mathscr{A}_q$ is admissible if each subword of length 2 is admissible. Let $\mathscr{W}_q$ be the set of all admissible words in the alphabet $\mathscr{A}_q$.

Definition 2.2 (see 11.15 of [Ca]). Let $N > 0$, $a, b, c \in \Gamma$, and suppose $\overline{ab}, \overline{bc}$ are geodesic segments. Define the N-*deviation* of the broken geodesic segment $\overline{ab} \cup \overline{bc}$ at b by

$$\mathrm{dev}_N(\,\overline{ab} \cup \overline{bc}\,) = \sup\big\{\mathrm{d}(x,b), \mathrm{d}(y,b) : \mathrm{d}(x,y) \le N,\ x \in \overline{ab},\ y \in \overline{bd}\big\}.$$

The deviation is a measure of the angle at which $\overline{ab}$ and $\overline{bc}$ meet. If $\overline{ab} \cup \overline{bc}$ is geodesic, then the N-deviation of their union is N.

Deviation Lemma 2.3. *Given the collection $\mathscr{O}_p$, there exists $d > 0$ such that for all sufficiently large q and for all $s, t \in \mathscr{A}_q$ with $s \cdot t$ admissible, it is true that the 3δ-deviation of $s^{-1} \cup t$ is less than d.*

Proof. Suppose the conclusion is false. Then there exists an unbounded increasing sequence of positive integers q_n with corresponding alphabets and word sets $\mathscr{A}_{q_n}$, $\mathscr{W}_{q_n}$, respectively, and symbols $s_{q_n}, t_{q_n} \in \mathscr{A}_{q_n}$ such that $s_{q_n} \cdot t_{q_n}$ is admissible and

$$\mathrm{dev}_{3\delta}(s_{q_n}^{-1} \cup t_{q_n}) \ge n.$$

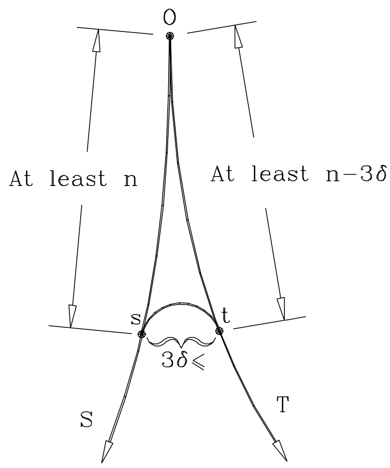


Figure 2.2

By passing to subsequences we may assume that $s_{q_n}^{-1}$, t_{q_n} are represented by edge paths that are initial substrings of the edge paths representing $s_{q_{n+1}}^{-1}$, $t_{q_{n+1}}$, respectively, for all n. Thus the increasing unions

$$\cup s_{q_n}^{-1} \quad \text{and} \quad \cup\, t_{q_n}$$

correspond to geodesic rays S, T in the Cayley graph. Given a fixed $q_n > 0$ there are points $s \in S$ and $t \in T$ with $\mathrm{d}(s,t) \le 3\delta$ and (without loss of generality) $\mathrm{d}(0,s) \ge n$ (refer to Figure 2.2). Using the fact that triangle$(s0t)$ is δ-thin it is evident that $\overline{0s}$ is in the 4δ corridor about T. Furthermore, $\mathrm{d}(0,t)$ is at least $n - 3\delta$. Since the above argument works for arbitrarily large n it follows that S is in the 4δ corridor about T, T is in the 4δ corridor about S, and thus S and T are in fact equivalent rays. Evidently S and T end in the same $D(R,p) \subset O(R,p)$. This contradicts the fact that each $s_{q_n} \cdot t_{q_n}$ is admissible. ▫

Given d from the deviation lemma fix $q > \max\{p, 2d + 12\delta\}$ so large that $\Gamma \setminus B(0,q)$ is contained in the union of all the $O(R,p)$. Let $\mathscr{A} = \mathscr{A}_q$, $\mathscr{W} = \mathscr{W}_q$, $\mathscr{O} = \mathscr{O}_p$ denote the corresponding alphabet, set of admissible words, and collection of open sets, respectively.

Recall that a path Q in Γ is *quasigeodesic(K)* if each subpath $Q' = \overline{ab}$ of Q having length at least K satisfies

$$\mathrm{d}(a,b) \geq \left(\frac{1}{K}\right) \mathrm{length}(Q').$$

Corollary 2.4. *There exists $K > 0$ such that each $w \in \mathscr{W}$ is quasigeodesic(K).*

Proof. By 11.16 of [Ca] and the deviation lemma, each $w \in \mathscr{W}$ is quasigeodesic(K) with $K = q - (2d + 12\delta)$. □

Furthermore, if a, b denote the endpoints of w as a path in Γ and $\overline{ab}$ is any geodesic segment from a to b, then there exists some $N > 0$ depending only on K and Γ such that w lies in the N corridor about $\overline{ab}$ (see 11.20 of [Ca]).

Definition 2.5. Let $\mathscr{W}_\infty$ be the set of all infinite sequences of symbols $a_0 \cdot a_1 \cdot a_2 \cdots$ such that $a_{i-1} \cdot a_i$ is admissible for all $i \geq 1$. A *cylinder* is a tree; namely a set of the form

$$\mathscr{S} = \{a_0 \cdot a_1 \cdots a_i \cdot a_{i+1} \cdots \in \mathscr{W}_\infty : a_0 \cdot a_1 \cdots a_i \text{ is fixed } \}.$$

The initial fixed edges $a_0 \cdot a_1 \cdots a_i$ form the *stem.* Unless otherwise noted, all cylinders are to be regarded as having stems based at the origin. The other end of the stem is the *branch vertex.* The *level* of a (non-stem) vertex v in a cylinder is the number of edges between the branch vertex and v.

Other authors define a more general cylinder, see 2.4.1 of [K]. It is not hard to see that the cylinders form a base for a topology that makes $\mathscr{W}_\infty$ a Cantor set. (The reader should realize at this point that $\mathscr{W}_\infty$ is an abstract topological space distinct from the Cayley graph of G. There is an obvious quotient map from $\mathscr{W}_\infty$ onto $\partial\Gamma$ that will be used in the next section.) The usual middle thirds Cantor set can be visualized as the ends of a regularly branching tree while the tree for $\mathscr{W}_\infty$ ostensibly branches in an irregular fashion (see Figure 2.3).

Corollary 2.4 implies that each $a_0 \cdot a_1 \cdot a_2 \cdots \in \mathscr{W}_\infty$ is a quasi-ray (quasi-geodesic ray) contained in the N corridor about some ray R. The converse is also true.

Approximation Lemma 2.6. *Given any geodesic ray $R \subset \Gamma$ from 0, there exists a quasi-ray $W = a_0 \cdot a_1 \cdot a_2 \cdots \in \mathscr{W}_\infty$ such that R lies inside the δ corridor about W and W is in the δ corridor about R.*

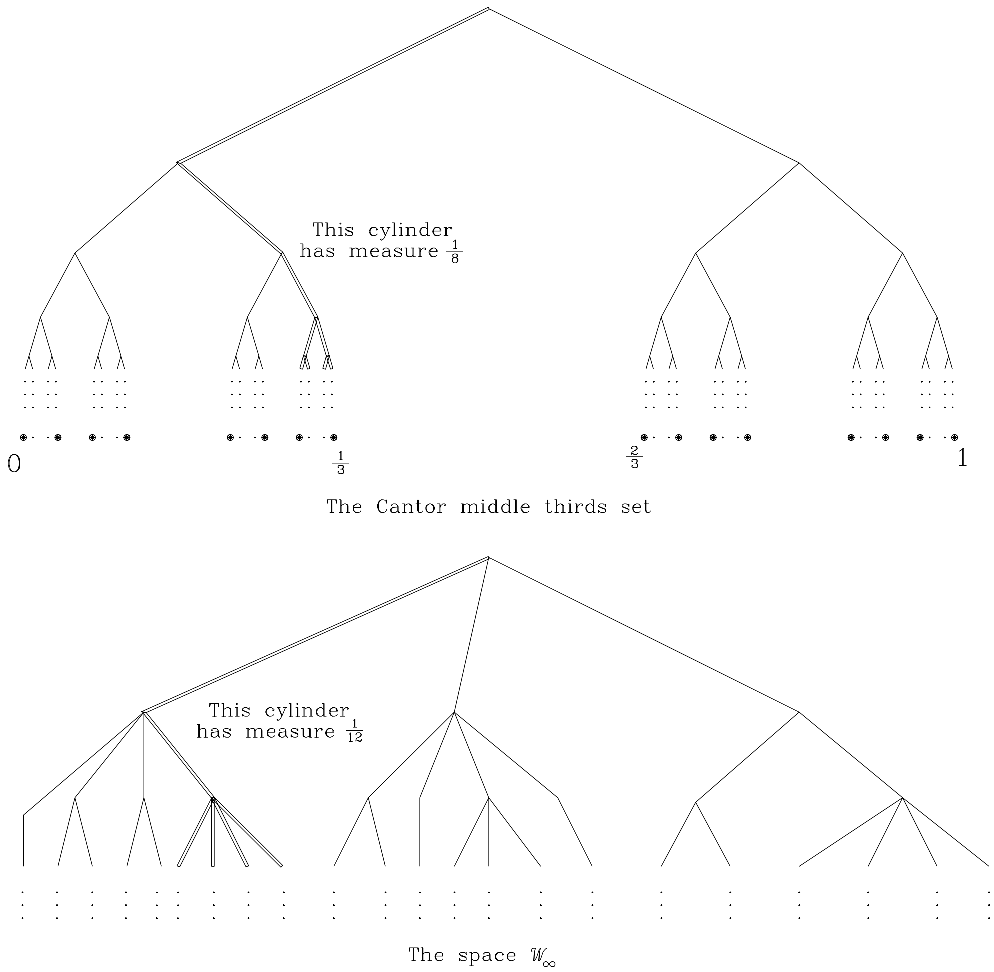


Figure 2.3

Proof. The quasi-ray W is defined inductively. For $n = 0$, pick v_0 to be the unique vertex (group element) on R satisfying $\mathrm{d}(0, v_0) = q$. There is a symbol $a_0 \in \mathscr{A}$ which, interpreted as an edge path from 0, terminates at v_0. Assume the string $a_0 \cdot a_1 \cdots a_{n-1}$ has been constructed. Let v_n be the unique vertex on R satisfying $\mathrm{d}(0, v_n) = (n+1)q$. There is a symbol $a_n \in \mathscr{A}$ such that the edge path $a_0 \cdot a_1 \cdots a_{n-1} \cdot a_n$ terminates at v_n.

Suppose, by way of contradiction, that $a_{n-1} \cdot a_n$ is not admissible. Then there exist $O, O' \in \mathscr{O}$ such that a_{n-1}^{-1}, a_n are vertices in O, O' respectively, with $O \cap O' \neq \emptyset$. Pick $b \in O \cap O'$ and consider any geodesic segments connecting a_{n-1}^{-1}, a_n with b. Note that a_{n-1}^{-1} and a_n are joined by a translated segment of R passing through 0 (refer to Figure 2.4). By thinness of the triangle with vertices a_{n-1}^{-1}, b, and a_n, it follows that $\mathrm{d}(0, \overline{a_{n-1}^{-1}b} \cup \overline{ba_n}) \leq \delta$. Without loss of generality suppose that $\mathrm{d}(0, \overline{a_{n-1}^{-1}b}) \leq \delta$.

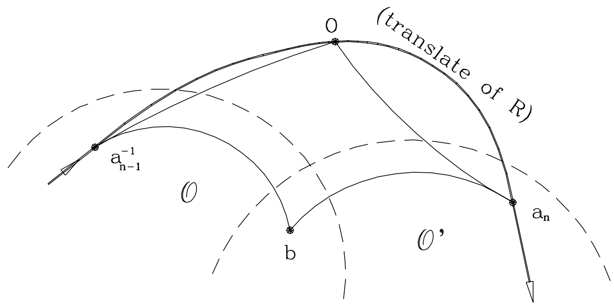


Figure 2.4

By construction, O is contained in a halfspace H of the form $H(R,p)$. Using I.2.4, the latter halfspace is quasiconvex(2δ), meaning that the entire segment $\overline{a_{n-1}^{-1}b}$ stays within 2δ of H. Therefore the distance from the origin to the open set O is at most $\delta + 2\delta = 3\delta$. This contradicts Lemma 2.1, therefore $a_{n-1}\cdot a_n$ is admissible.

The infinite word $a_0\cdot a_1\cdot a_2\cdots$ labels a geodesic ray in Γ which at levels $0, q, 2q, \cdots$ coincides with the vertices $0, v_0, v_1, \cdots$ of R. It is clear that the subpath labelled by a_n can stray no farther than $\frac{1}{2}q$ from R. In fact, this bound can be improved to δ: Let v_{n-1}, v_n be the vertices on R corresponding to the endpoints of the edge path a_n. The geodesic digon with vertices v_{n-1}, v_n and edges from R and W is δ-thin. □

Corollary 2.7. *If $w \neq w' \in \mathscr{W}$, there is some $a \in \mathscr{A}$ such that $w\cdot a\cdot w' \in \mathscr{W}$.*

Proof. Let $s, t \in \mathscr{A}$ denote the last symbol in w and the first symbol in w', respectively. By design, there are at most P inadmissible suffixes to s and P inadmissible prefixes for t, but at least $3P$ symbols in $\mathscr{A}$. Let $a \in \mathscr{A}$ be any one of the (at least $3P - P - P = P$) symbols such that $s\cdot a\cdot t$ is admissible. Then $w\cdot a\cdot w' \in \mathscr{W}$. □

The hypotheses of Theorem 1.4 may be impossible to satisfy for arbitrary negatively curved groups. The weaker condition that $x \in \partial\Gamma$ is represented by a quasi-ray $R' \in \mathscr{W}_\infty$ satisfying the property that R' contains every finite admissible word is sufficient to guarantee that x is line transitive.

Theorem 2.8. *Suppose $x \in \partial\Gamma$ is represented by a quasi-ray $R' \in \mathscr{W}_\infty$, and that every $w \in \mathscr{W}$ occurs as a substring of R'. Then x is a Myrberg point.*

Proof. The proof is modelled on Theorem 1.4. Quasigeodesics behave in the large very much like geodesics and only some of the constants need be changed.

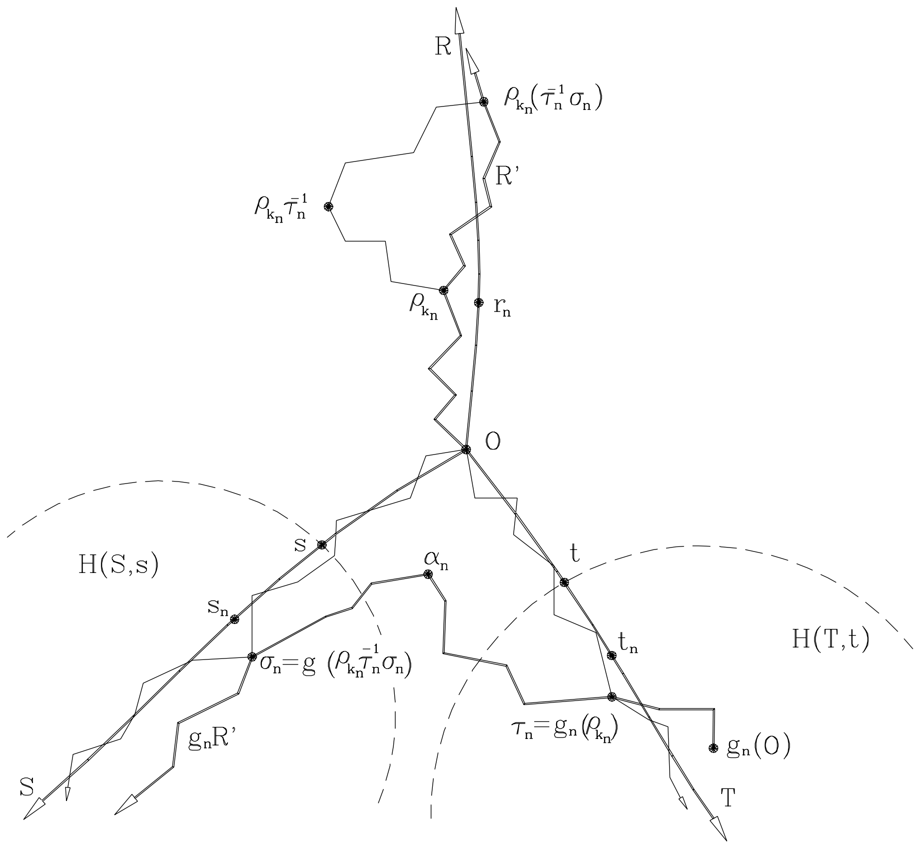


Figure 2.5

Let $R' = \rho_1 \cdot \rho_2 \cdot \rho_3 \cdots \in \mathscr{W}_\infty$ be any transitive quasi-ray from 0 representing x in the N corridor about some equivalent geodesic ray R. Suppose $u \neq v \in \partial\Gamma$, and let S, T be rays from 0 representing u, v respectively. Using the approximation Lemma 2.6 there are quasi-rays $\sigma_1 \cdot \sigma_2 \cdot \sigma_3 \cdots$ and $\tau_1 \cdot \tau_2 \cdot \tau_3 \cdots$ in the δ corridors about S, T respectively.

By Corollary 2.7 there exist admissible quasigeodesic segments connecting each τ_n to σ_n, considered as vertices in Γ. I will abuse notation and refer to any such segment as $\overline{\tau_n^{-1}\sigma_n}$. By hypothesis, the admissible quasi-ray $\rho_1 \cdot \rho_2 \cdot \rho_3 \cdots$ contains $\overline{\tau_n^{-1}\sigma_n}$ as a subsegment, i.e., the quasi-ray R' contains $\rho_{k_n} \cdot (\tau_n^{-1}\sigma_n)$ as an initial segment for some $\rho_{k_n} \in \mathscr{W}$. For each n, define g_n as left multiplication by $\tau_n \rho_{k_n}^{-1}$. Then

$$g_n(0) = \tau_n \rho_{k_n}^{-1} \quad \text{and} \quad g_n\big(\rho_{k_n}(\tau_n^{-1}\sigma_n)\big) = \sigma_n.$$

Since R' behaves very much like a geodesic, it is not hard to see that there is a constant $M > 0$ independent of n and a point $\alpha_n \in g_n(R')$ such that $\mathrm{d}(\alpha_n, 0) \leq M$ for each n. Also there exist s_n, t_n in S, T respectively, such that $\mathrm{d}(s_n, \sigma_n) < \delta$ and $\mathrm{d}(t_n, \tau_n) < \delta$. Clearly $s_n \to u$ and $t_n \to v$ as $n \to \infty$. Let $s \in S$ and $t \in T$

so that $H(S,s)\cap B(0,M+N)=\emptyset=H(T,t)\cap B(0,M+N)$. Pick n so large that $t_n>t+2N+16\delta$ and $s_n>s+2N+24\delta$ and choose $r_n\in R$ within N of $\rho_{k_n}=g_n^{-1}(\tau_n)\in R'$. Then $g_n(r_n)\in g_n(R)$ is within $N+\delta$ of $t_n\in T$. Exercise 1.1 implies that $g_n(r_n)\in B(t_n,N+\delta)\subset H\bigl(T,t_n-2(N+\delta)\bigr)\subset H(t,t+14\delta)$, while an intermediate point of $g_n(R)$ near α_n is inside $B(0,M+N)\subset H^-(T,t)$. Lemma 1.3 says the image of R between 0 and ρ_{k_n} is inside $H(T,t)$. Therefore $g_n(0)\to v$.

Similar changes in constants show that a tail of $g_n(R)\subset H(S,s)$. The details are left to the reader. ▫

Although the word space $\mathscr{W}$ is a groupoid rather than a group, it possesses an automatic structure in the sense that there exists a deterministic finite state automaton that decides if $w\cdot s\in\mathscr{W}$ for any $s\in\mathscr{A}$, $w\in\mathscr{W}$. In fact all the decision making data can be stored in a square $(0,1)$-matrix $M=(m_{ij})$ of order $|\mathscr{A}|$, the cardinality of the q-sphere in Γ. Enumerate $\mathscr{A}=\{s_1,s_2,\ldots\}$. Set

$$m_{ij}=\begin{cases}1, & \text{if the product } s_i\cdot s_j \text{ is admissible;}\\ 0, & \text{else.}\end{cases}$$

This construction makes $\mathscr{W}$ a *Markov chain*, see pp. 2, 88 of [R] or Chapter XV of [Fel]. Corollary 2.7 implies that the matrix M is primitive of exponent 3 (i.e., each entry of M^3 is strictly positive). Further links between graph and matrix theories are given in [B].

The approximation lemma implies that every point of $\partial\Gamma$ is represented by at least one quasi-ray $W\in\mathscr{W}_\infty$. The boundary at infinity is complicated by the local interaction of relators in Γ. These difficulties can be reduced by considering the simpler structure of $\mathscr{W}_\infty$.

3. Measures on $\mathscr{W}_\infty$ and $\partial\Gamma$

There is a natural Cantor measure ν on $\mathscr{W}_\infty$ such that $\nu(\mathscr{W}_\infty)=1$. To each $a\in\mathscr{A}$ let $C(a)$ denote the number of admissible suffixes for a. If $\mathscr{S}_1\subset\mathscr{W}_\infty$ is the cylinder of all quasi-rays with fixed initial symbol a_1, define $\nu(\mathscr{S}_1)=1/|\mathscr{A}|$. Inductively, if $\mathscr{S}_n\subset\mathscr{W}_\infty$ is the set of quasi-rays whose first n entries consist of the fixed symbols $a_1\cdots a_n$ and $\mathscr{S}_{n+1}\subset\mathscr{S}_n$ consists of those quasi-rays satisfying the additional condition that the $n+1^{\text{st}}$ position contains the symbol a_{n+1}, define

$$\nu(\mathscr{S}_{n+1})=\nu(\mathscr{S}_n)\frac{1}{C(a_n)}=\left(\frac{1}{|\mathscr{A}|}\right)\left(\frac{1}{C(a_1)}\right)\cdots\left(\frac{1}{C(a_n)}\right).$$

The cylinders of the type $\mathscr{S}_n$ are analogous to open intervals in the construction of Lebesgue measure on the unit interval (refer to Figure 2.3). For any Borel set $\mathscr{E}\subset\mathscr{W}_\infty$ define $\nu(\mathscr{E})$ as the infimum of $\sum\nu(\mathscr{S})$ where $\cup\mathscr{S}$ is any countable covering of $\mathscr{E}$ by basic open cylinders of the above types. In this way ν becomes a non-atomic regular Borel measure on $\mathscr{W}_\infty$ with total mass 1.

By construction of $\mathscr{O}, \mathscr{A}, \mathscr{W}$ there is a uniform upper bound of C for $C(a)$. Suppose $W = a_1 \cdot a_2 \cdot a_3 \cdots \in \mathscr{W}_\infty$. For a given $s \in \mathscr{A}$, let α_{2n} be the probability that the first occurrence of s in W is $s = a_{2n-1}$ or $s = a_{2n}$. Then the probability that s occurs as a symbol in W is $\sum_{n=1}^{\infty} \alpha_{2n}$. Define $\beta_{2n} = \sum_{i=1}^{n} \alpha_{2i}$. The claim is that $B = \lim_{n\to\infty} \beta_{2n}$ is in fact 1.

Let $\mathscr{W}_{2n}$ be the set of all words in $\mathscr{W}$ of symbolic length at most $2n$. Let $\mathscr{U}_{2n} \subset \mathscr{W}_{2n}$ be those words containing s, and let $\mathscr{V}_{2n} = \mathscr{W}_{2n} \setminus \mathscr{U}_{2n}$. By definition, $\beta_{2n} = \nu(\mathscr{U}_{2n})$.

Suppose $v \in \mathscr{V}_{2n}$. By Corollary 2.7, there is some $a \in \mathscr{A}$ such that $v \cdot a \cdot s \in \mathscr{W}$. Hence the measure of the set of quasi-rays beginning in v and having s as symbol $2n+1$ or $2n+2$ is at least $1/C^2$ times the measure of the cylinder set

$$\mathscr{S}(v) = \{W \in \mathscr{W}_\infty : W \text{ has } v \text{ as initial segment}\}.$$

Consequently

$$\alpha_{2n} \geq \frac{1}{C^2}\nu(\mathscr{V}_{2n}) = \frac{1}{C^2}(1-\beta_{2n-2}) \geq \frac{1}{C^2}(1-B).$$

But $\sum \alpha_{2n} \leq \nu(\mathscr{W}_\infty) = 1$, so that $\alpha_{2n} \downarrow 0$. Therefore $1 - B = 0$. This proves

Lemma 3.1. *Let $s \in \mathscr{A}$ and suppose $W \in \mathscr{W}_\infty$ is a quasi-ray. Then with probability 1, W contains s as a symbol.*

Scholium 3.2. *Suppose that $s \in \mathscr{W}$. Then with probability 1, W contains s as a subword. (The set of all $W \in \mathscr{W}_\infty$ containing s has full ν measure.)*

Proof. The argument given above need only be modified slightly. Let k be the length of s in the alphabet $\mathscr{A}$. Note that if $w \in \mathscr{W}$, then $w \cdot s$ is admissible as soon as $w \cdot s_1$ is admissible, where s_1 is the first symbol of s. Write $W = w_1 \cdot w_2 \cdots$ where each $w_n \in \mathscr{W}$ consists of $k+1$ symbols. Define α_n as the probability that s first appears in W as a substring of w_n. The probability that s occurs in at least one of the blocks w_i is $\sum_{n=1}^{\infty} \alpha_n$. Let β_n be the n^{th} partial sum. As in the previous argument, given a prefix $w_1 \cdot w_2 \cdots w_{n-1}$ there is at least one admissible word $w_{n-1} \cdot a \cdot s$ for some $a \in \mathscr{A}$. Thus

$$\alpha_n \geq \frac{1}{C^{k+1}}(1-\beta_{n-1}).$$

If $B = \lim_{n\to\infty} \beta_n < 1$, then just as in Lemma 3.1, $\alpha_i \geq (1-B)/C^{k+1} > 0$ and $\sum \alpha_i$ diverges, a contradiction. □

In the language of symbolic dynamics, the measure ν makes $\mathscr{W}_\infty$ an *ergodic Markov process* with respect to the shift map that sends a quasi-ray $W \in \mathscr{W}_\infty$ to the same ray less its first symbol. All the relevant information for ν can be encoded in a symmetric doubly-stochastic matrix $\Pi = (\pi_{ij})$ defined by

$$\pi_{ij} = \begin{cases} 1/C(a_i), & \text{if } m_{ij} = 1; \\ 0, & \text{else,} \end{cases}$$

where $M = (m_{ij})$ is the matrix of admissibility discussed at the conclusion of Section 2.

Let $f\colon \mathscr{W}_\infty \to \partial\Gamma$ be the quotient map that identifies each quasi-ray to its endpoint at infinity. Then f is a proper continuous surjection between compact metric spaces. The map f induces a measure μ on $\partial\Gamma$ by

$$\mu(E) = \nu\big(f^{-1}(E)\big) \quad \text{for each } E \subset \partial\Gamma.$$

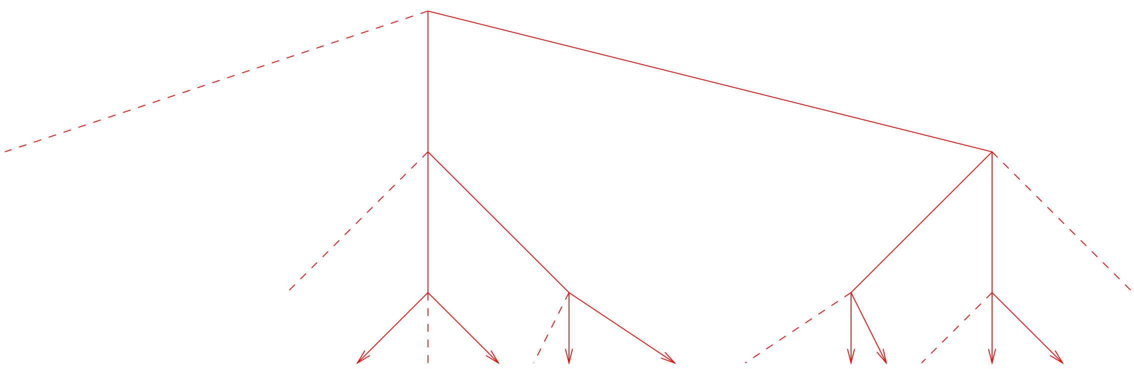
From 2.2.17 of [Fed], μ is a regular Borel (Radon) measure on $\partial\Gamma$. The next lemmas show that μ has no atoms.

Lemma 3.3. *Let $\mathscr{S} \subset \mathscr{W}_\infty$ be a cylinder and m_0 a positive integer. Divide $\mathscr{S}$ into levels of length m_0. Throw out one branch (and its descendants) from each subcylinder based at levels $0,\ m_0,\ 2m_0,\ 3m_0, \ldots$. The remaining set $\mathscr{E}$ satisfies $\nu(\mathscr{E}) = 0$.*

Proof. First consider the case when $m_0 = 1$ and $\mathscr{S}$ is a regular tree of valence C with root at the origin, i.e., $\mathscr{S}$ is the entire symbol space $\mathscr{W}_\infty$. It is easy to see (Figure 3.1) that the measure of the complement of $\mathscr{E}$ is the sum of a geometric series.

$$\nu(\mathscr{S} \setminus \mathscr{E}) = \frac{1}{C} + \frac{C-1}{C^2} + \frac{(C-1)^2}{C^3} + \cdots = 1$$

Thus $\nu(\mathscr{E}) = 0$.



The deleted branches have full measure

Figure 3.1

For $m_0 > 1$, consider $\mathscr{S}$ as a regular tree of valence $m_0 C$, where initial edges have been identified (see Figure 3.2). In terms of the measure ν, nothing has been changed:

$$\nu(\mathscr{S} \setminus \mathscr{E}) = \frac{1}{m_0 C} + \frac{m_0 C - 1}{(m_0 C)^2} + \frac{(m_0 C - 1)^2}{(m_0 C)^3} + \cdots = 1.$$

In the general case $\mathscr{S}$ has an initial segment, branches irregularly, but has valence bounded by some $C > 0$. Omitting one branch at each m_0 level results in a set $\mathscr{E}$ having ν-measure *less* than or equal to the corresponding case of a regular tree of valence C. ◻

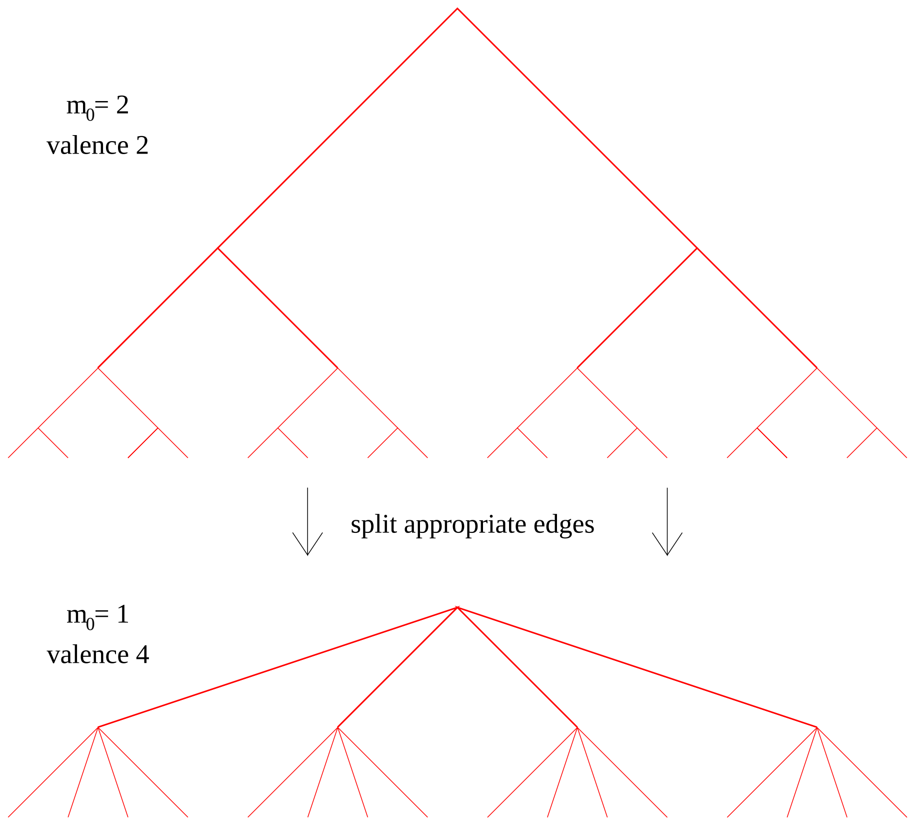


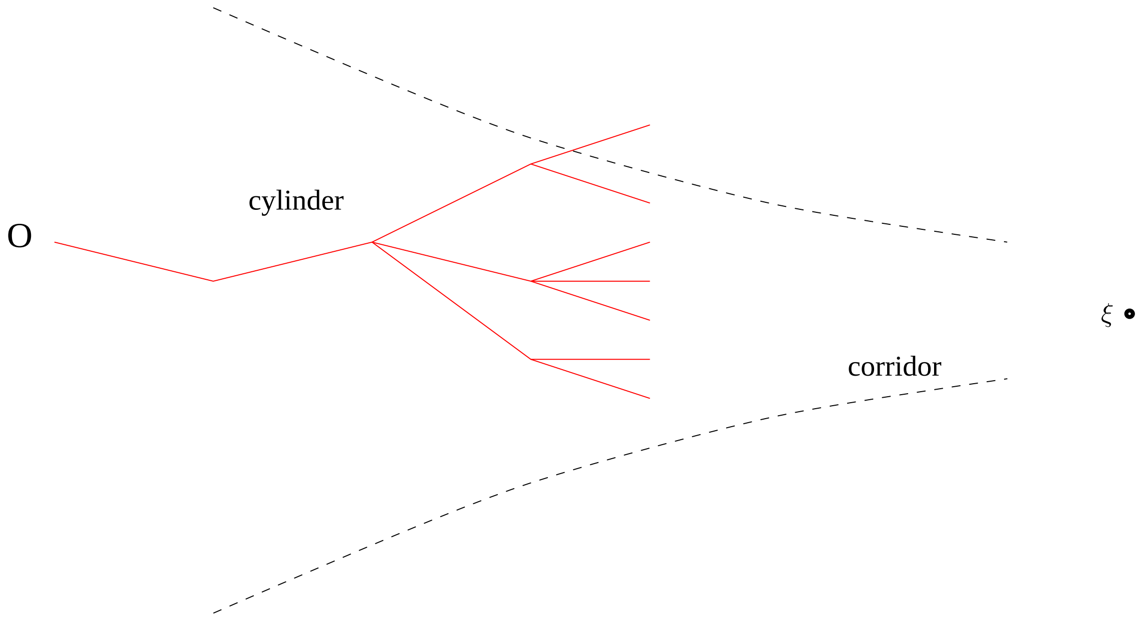
Figure 3.2

Lemma 3.4. *Let $\xi \in \partial\Gamma$, and suppose $R \subset \Gamma$ is a geodesic ray representing ξ. There exists $N > 0$ such that any quasi-ray W representing ξ lies entirely within the $N + 2\delta$ corridor about R.*

Proof. The remarks following Corollary 2.4 imply that there is some (globally defined) $N > 0$ such that any quasi-ray W representing ξ lies in the N corridor of some ray S. Clearly S represents ξ. By I.1.7 the ray S lies inside the 2δ corridor about R. ◻

Lemma 3.5. *Let $\xi \in \partial\Gamma$. There exists a positive integer m_0 such that for any geodesic ray R ending at ξ and any cylinder $\mathscr{S}$ whose branch vertex is within $N+2\delta$ of R, there exists a branch of $\mathscr{S}$ which at level m_0 is farther than $2(N+\delta)$ from R. (Translation: any cylinder whose branch point is in the $N+2\delta$ corridor of a ray has a branch that exits that corridor shortly. See Figure 3.3.)*

Sketch of proof. Following ideas in [CS], there is a global constant m_1 such that non-equivalent geodesic rays diverge more than $N+2\delta$ from each other (and continue to diverge) after traveling m_1 units from a common vertex. There exists a similar constant $m_0 \geq m_1$ for which the analogous result holds for inequivalent quasi-rays based at the same vertex (the details are left to the reader). By construction every cylinder contains non-equivalent quasi-rays (in fact the f-image of a cylinder contains some disk $D(T,t)$ and thus has non-empty interior in $\partial\Gamma$). ▫



A branch must exit

Figure 3.3

Theorem 3.6. *μ is non-atomic.*

Proof. Let $\xi \in \partial\Gamma$ be represented by a ray R. By Lemma 3.4, all quasi-rays mapping to ξ lie in the $N+2\delta$-corridor about R. Let $\{\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3, \ldots\}$ be the collection of all cylinders containing $f^{-1}(\xi)$. This set is countable since the totality of all cylinders is countable. Let $\mathscr{S}_i$ be such a cylinder. Evidently $\mathscr{S}_i$ has its branch vertex inside the $N+2\delta$ corridor about R (else no quasi-ray of $\mathscr{S}_i$ can map to ξ by Lemma 3.4). Lemma 3.5 says that at level m_0 there is a branch of $\mathscr{S}_i$ that exits the corridor, i.e., this branch is disjoint from $f^{-1}(\xi)$. As in Lemma 3.3, let $\mathscr{E}_i \subset \mathscr{S}_i$ be the set of those branches that remain to (possibly) map to ξ. Lemma 3.3 says that $\nu(\mathscr{E}) = 0$. Thus

$$0 \le \mu(\xi) = \nu\big(f^{-1}(\xi)\big) \le \sum_i \nu(\mathscr{E}_i) = 0. \square$$

Main Theorem 3.7. *The set of Myrberg points in* $\partial\Gamma$ *has full* μ *measure.*

Proof. Let $\mathscr{E}$ be the set of all quasi-rays in $\mathscr{W}_\infty$ that contain every finite subword. By Scholium 3.2, $\nu(\mathscr{E}) = 1$, and $\mathscr{E} \subset f^{-1}\big(f(\mathscr{E})\big)$ is always true. Therefore

$$\mu\big(f(\mathscr{E})\big) = \nu\big(f^{-1}\big(f(\mathscr{E})\big)\big) \ge \nu(\mathscr{E}) = 1.$$

It is sufficient to show that each $x \in f(\mathscr{E})$ is a Myrberg point. Theorem 2.8 does precisely this. □

Comments and Conjectures 3.8. Michel Coornaert has recently defined a family of *quasi-conformal measures* for the boundary of a Gromov hyperbolic group [Co]. Hausdorff measure is one such example. I conjecture that μ defined above is a quasi-conformal measure. Negatively curved groups have no parabolic elements. Does an analog of Theorem 3.7 hold for arbitrary discrete isometry groups of negatively curved (Gromov hyperbolic) geodesic spaces? Myrberg points are defined topologically (Definition 1.2) whereas my methods are geometric. Does Theorem 3.7 depend on the geometry of the Cayley graph or is there a topological proof, in particular, does Theorem 3.7 hold for convergence groups of the first kind?

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Received 22 August 1994

Received in revised form 8 January 1996