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# FUNDAMENTAL DOMAINS IN TEICHMÜLLER SPACE

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**Abstract.** We introduce a natural construction of fundamental domains for actions of subgroups of the mapping class group on Teichmüller space and investigate their properties. These domains are analogous to the classical Dirichlet polyhedra associated to the actions of discrete isometry groups on hyperbolic spaces.

## 1. Introduction

In the study of the action of the mapping class group on Teichmüller space (or on Thurston's compactification of Teichmüller space by the sphere of projective measured laminations), one expects that the actions of this group and of its subgroups will exhibit many of the interesting properties of the actions of discrete groups acting by isometries on a hyperbolic space (a simply connected space with a Riemannian metric of constant negative curvature), and its natural compactification by the sphere at infinity. This reason, of course, is parallel to our motivation for studying, in [MP], the dynamics of the actions of subgroups of the mapping class group on Thurston's sphere.

In this paper, as in [MP], we develop this line of thought. Specifically, we discuss a natural construction of fundamental domains in Teichmüller space, which are analogous to the Dirichlet polyhedra of the setting of hyperbolic manifolds (which we shall call henceforth the *classical* setting). It is well known that the actions of subgroups of the mapping class groups are properly discontinuous. Indeed, in [MP], we prove a stronger result. Namely, we can define a useful notion of a limit set for the natural extensions of these actions to Thurston's compactification of Teichmüller space, and the actions are properly discontinuous on the complement of the limit set, up to a set of measure zero on the boundary sphere. Again, this result has a direct analogy in the classical setting [Th] (without the clause concerning the set of measure zero). Our aim here is not to develop the proper discontinuity. Rather, we wish to study these "Dirichlet polyhedra".

The outline of the paper is as follows.

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In Section 2, we establish notation and recall some preliminary facts from the theory of measured foliations on a surface, Teichmüller spaces and mapping class groups, which will be used in the subsequent text.

In Section 3, we give the definition of the term "fundamental domain", which is in the same spirit of the one used for example by Beardon in his book [Be]. We then define a domain $D$ in Teichmüller space which is associated to the action of a subgroup $\Sigma$ of the mapping class group of the surface. As in the classical setting, this domain is defined in terms of equidistant loci and their associated halfspaces. It is well known that there are several metrics that one can define on Teichmüller space (*e.g.* Teichmüller metric, Weil–Petersson metric, Thurston's stretch metric). Each one of these metrics is natural from some given point of view. The one that we use is the Teichmüller metric, which has been extensively studied. In particular, we shall use a fact established by Earle [Ea] concerning the differentiability of the Teichmüller distance.

In Section 4, we study the equidistant loci and the associated halfspaces. We establish that an equidistant locus is a hypersurface separating Teichmüller space into two contractible halfspaces. In addition, we show that any two distinct equidistant loci intersect transversely.

In Section 5, we study the domains referred to above. We establish the fact that these domains are fundamental domains. Finally, we discuss the extent to which our domains share the properties of the classical Dirichlet polyhedra.

Let us insist on the fact that the definition of the fundamental domains is classical; the point here is that we are studying such a domain in the context of Teichmüller space, which is not a Riemannian manifold.

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## 2. Preliminaries

Let $S$ be a finite type surface of negative Euler characteristic which is not homeomorphic to a pair of pants (that is, a sphere with three punctures), and let $\Gamma$ be the mapping class group of $S$. $\Sigma$ will denote an arbitrary subgroup of $\Gamma$, and $\Sigma^*$ will denote the subset of nontrivial elements in $\Sigma$.

The Teichmüller space $\mathscr{T}$ of $S$ is the space of conformal structures on $S$ up to conformal automorphisms which are isotopic to the identity. By the uniformization theorem, we may also consider $\mathscr{T}$ as the space of hyperbolic metrics on $S$ up to isometries which are isotopic to the identity. For simplicity, we only consider conformal structures on $S$ in which the ends of $S$ are conformally isomorphic to punctured discs. From the geometric point of view, this means that we assume that the ends of $S$ are cusps in the corresponding hyperbolic metrics. The mapping class group $\Gamma$ acts in a natural way on $\mathscr{T}$ by the pullback construction.

Teichmüller space has a natural topology in which it is homeomorphic to an open ball of dimension $6g-6+2e$ where $g$ is the genus of $S$ and $e$ is the number of punctures of $S$. This topology may be defined by the Teichmüller metric $\varrho$, a complete metric with respect to which the mapping class group acts as a group of isometries. This metric is not Riemannian. Nevertheless, it provides some of the geometry one associates to Riemannian metrics and in fact, it behaves in some respects better than a generic Riemannian metric. We are particularly interested in the following features of $\varrho$.

First of all, as shown by Kravetz [Kr], $(\mathscr{T}, \varrho)$ is a *straight $G$-space* in the sense of Busemann ([Bu], [Ab]). In particular, any two distinct points in $\mathscr{T}$, $m$ and $n$, are joined by a unique geodesic segment (*i.e.* an isometric image of a Euclidean interval), $[m,n]$, and lie on a unique geodesic line (*i.e.* an isometric image of $\mathbf{R}$), $\gamma(m,n)$. This fact is based on Teichmüller's Theorem and the following result.

**Proposition 2.1.** *Let $m$, $n$ and $p$ be points in $\mathscr{T}$. If $\varrho(m,n) + \varrho(n,p) = \varrho(m,p)$, then $n$ lies on $[m,p]$.*

*Proof.* See [Ab, p. 122] for a proof based on Teichmüller's theorem. □

As a corollary, we have the following well-known fact which will be useful for us:

**Proposition 2.2.** *Let $g$ be a nontrivial isometry of $(\mathscr{T}, \varrho)$. Then the fixed point set of $g$ is nowhere dense.*

*Proof.* The fixed point set $F$ of $g$ is obviously closed. Hence, it suffices to show that $F$ has empty interior. Suppose, on the contrary, that $g$ fixes a nonempty open set $U$. Let $m$ be a point in $U$ and $n$ be in $\mathscr{T}$. By assumption, $g$ fixes a neighborhood of $m$ in $\mathscr{T}$ and, hence, in $\gamma(m,n)$. By uniqueness of geodesics, $g$ preserves the line $\gamma(m,n)$. But an isometry of a line which fixes an open interval is the identity. Hence, $g$ fixes $n$. Since $n$ was arbitrary, $g$ is the identity. This is the desired contradiction. □

We obtain as a corollary the following (also well known) fact which we shall be using:

**Corollary 2.3.** *There exists a point $m_0$ in $\mathscr{T}$ whose stabilizer in $\Gamma$ is trivial.*

*Proof.* Since $\Gamma$ is a countable group, this follows immediately from the Baire category theorem for complete metric spaces. □

Choose a point $m$ in $\mathscr{T}$. Let $\varrho_m$ denote distance from $m$ in $\mathscr{T}$.

(2.1) $\varrho_m \colon \mathscr{T} \to \mathbf{R},\ x \longmapsto \varrho(x,m)$.

We shall be interested in the variation of this function. In order to describe this variation, we must discuss the cotangent space of $\mathscr{T}$ at $x$. We use the notations of [Ea]. Let $X$ be a Riemann surface representing $x$. The cotangent space to $\mathscr{T}$ at $x$ is canonically the vector space $Q(X)$ of integrable holomorphic quadratic differentials on $X$ with the norm, $\|\varphi\| = \int |\varphi|/2$. It is an easy exercise to establish that this norm is strictly convex.

The norm actually defines a continuous function on the cotangent bundle $f\colon \mathscr{E} \to \mathscr{T}$. Let $\mathscr{E}_0$ denote the open set of cotangent vectors of length less than one. If $\varphi$ in $\mathscr{E}_0$ is a cotangent vector at $x$, (*i.e.*, a quadratic differential on $X$), the Teichmüller differential $\|\varphi\|\overline{\varphi}/|\varphi|$ on $X$ determines a point $g(\varphi)$ in $\mathscr{T}$. Indeed, this construction gives an "exponential" map $\Omega_x$ from the unit ball $Q_0(X)$ in $Q(X)$ to $\mathscr{T}$.

**Theorem 2.4** (Teichmüller). *The map $\Omega_x\colon Q_0(X) \to \mathscr{T}$ is a homeomorphism which maps rays in $Q_0(X)$ to geodesic rays in $\mathscr{T}$ emitting from $x$.*

Earle improved upon Teichmüller's result:

**Theorem 2.5** ([Ea]). *The map $\Omega\colon \mathscr{E}_0 \to \mathscr{T} \times \mathscr{T}$ is a homeomorphism.*

Let $\mathscr{F}\colon \mathscr{T} \times \mathscr{T} \to \mathscr{E}_0$ denote the inverse of $\Omega$. Earle computed the differential of $\varrho_m$ in terms of $\mathscr{F}$.

**Theorem 2.6** ([Ea]). *Let $m$ be a point in $\mathscr{T}$. Then $\varrho_m$ is a $C^1$ function on $\mathscr{T} \setminus \{m\}$ and its differential is the map $x \mapsto -\mathscr{F}(x,y)/\|\mathscr{F}(x,y)\|$.*

Let $Q_1(X)$ be the unit sphere in $Q(X)$. If $y$ is a point on a geodesic ray in $\mathscr{T}$ emitting from $x$ which is the image under $\Omega_x$ of a ray $\{t\varphi : 0 \le t < 1\}$ in $Q_0(X)$ for $\varphi$ in $Q_1(X)$, we shall say that $\varphi$ *points in the direction of* $y$. From the definition of the map $\Omega$, we can describe the differential at $x$.

**Corollary 2.7.** *The differential of $\varrho_m$ at $x$ is the unique unit norm quadratic differential on $X$ which points in the direction opposite to $m$.*

This description is what one would expect for a Riemannian metric. The gradient of the distance function measured from $m$ ought to be tangent to the field of geodesic rays emitting from $m$. Earle's result confirms this intuition.

Let $\mathscr{S}$ denote the set of isotopy classes of unoriented, connected and homotopically nontrivial simple closed curves on $S$ which are not homotopic to a puncture of $S$. The geometric intersection function $i(\ ,\ )$ on $\mathscr{S} \times \mathscr{S}$ is defined by the rule:

(2.2) $i(\alpha,\beta) = \min\{\text{cardinality}(a \cap b) \mid a \in \alpha, b \in \beta\}$.

$\Gamma$ acts on $\mathscr{S}$ in a natural way. This function is clearly symmetric and $\Gamma$-invariant.

The action of $\Gamma$ on $\mathscr{S}$ is a faithful action provided that $S$ is not a closed surface of genus two, a torus with one puncture or a sphere with four punctures. In each of these cases, the kernel of the action is a cyclic subgroup of order two. (These are the only nontrivial maps of surfaces of finite type and negative Euler characteristic which preserve every element of $\mathscr{S}$.) We say that the involution is hypergeometric.

(2.3) If $g$ is an element of $\Gamma$ which fixes every simple closed curve in $\mathscr{S}$, then $g$ is either the identity or the hypergeometric involution of a closed surface of genus two, a torus with one puncture or a sphere with four punctures.

By associating to each element $\alpha$ of $\mathscr{S}$ the length $l(m,\alpha)$ of the corresponding geodesic on $X$, we obtain a length function, which is evidently $\Gamma$-invariant:

(2.4) $l(gm, g\alpha) = l(m,\alpha)$ *for all* $g \in \Gamma$, $m \in \mathscr{T}$, $\alpha \in \mathscr{S}$.

We may choose a finite family $\mathscr{A}$ of simple closed curves in $\mathscr{S}$ such that the elements of $\mathscr{S}$ are parametrized by their geometric intersection with $\mathscr{A}$:

(2.5) *If* $i(\beta,\alpha) = i(\gamma,\alpha)$ *for all* $\alpha \in \mathscr{A}$, *then* $\beta = \gamma$.

We shall say that $\mathscr{A}$ is a *coordinate system.*

The $\Gamma$-invariance of $i$ implies that $\Gamma$ acts on coordinate systems. The stabilizers of this action are finite.

**Proposition 2.8.** *Let* $\mathscr{A}$ *be a coordinate system. The stabilizer of* $\mathscr{A}$ *in* $\Gamma$ *is a finite group.*

*Proof.* There is, of course, a subgroup $\Sigma$ of finite index in the stabilizer which fixes every element of $\mathscr{A}$. It suffices to show that $\Sigma$ is finite. Suppose that $g$ is an element of $\Sigma$. Then:

(1) $i(\alpha, g^{-1}\beta) = i(g\alpha,\beta) = i(\alpha,\beta)$ *for each* $\beta \in \mathscr{S}, \alpha \in \mathscr{A}$.

By the definition of a coordinate system, it follows that $g^{-1}$ fixes every curve $\beta$ in $\mathscr{S}$. By (2.3), $g$ is either the identity or the hypergeometric involution on $S$ (in the special cases). Hence, $\Sigma$ is of order at most two. ▫

As an immediate consequence, we have:

(2.6) *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be coordinate systems. There are at most finitely many mapping classes taking* $\mathscr{A}$ *to* $\mathscr{B}$.

Suppose that $\{g_n\}$ is an infinite sequence of distinct mapping classes. Let $\mathscr{A}$ be a coordinate system. By (2.6), the collection of coordinate systems $\{g_n(\mathscr{A})\}$ must be infinite (though not necessarily distinct). Hence, for some curve $\alpha$ in $\mathscr{A}$, $\{g_n(\alpha)\}$ must also be infinite. By the definition of a coordinate system, the geometric intersection of these curves with $\mathscr{A}$ must be unbounded. Therefore:

(2.7) *For any infinite sequence of distinct mapping classes* $\{g_n\}$, *there exists a pair of simple closed curves,* $\alpha$ *and* $\beta$, *such that the sequence* $\{i(g_n(\alpha),\beta)\}$ *is unbounded.*

There is a useful inequality relating geometric intersection and length.

**Lemma 2.9** ([FLP]). *Let* $m$ *be a point in* $\mathscr{T}$. *There is a constant* $C$, *depending only on* $m$, *such that, for every pair,* $\alpha$ *and* $\beta$, *of simple closed curves in* $\mathscr{S}$, *one has* $i(\alpha,\beta) \le Cl(m,\alpha)l(m,\beta)$.

There is another important inequality, due to Wolpert, relating the length function to the Teichmüller metric.

**Theorem 2.10** ([Wo]). *Assume* $m$ *and* $n$ *are points in* $\mathscr{T}$ *and* $\alpha$ *is in* $\mathscr{S}$. *Then* $l(n,\alpha) \le e^{\varrho(m,n)} l(m,\alpha)$.

From this inequality and the previous remarks, we may deduce the proper discontinuity of the $\Gamma$-action (and, hence, of the $\Sigma$-action).

**Theorem 2.11.** $\Gamma$ *acts properly discontinuously on* $\mathscr{T}$.

*Proof.* Suppose that $K$ is a compact subset of $\mathscr{T}$. We wish to show that there are only finitely many mapping classes $g$ in $\Sigma$ such that the intersection $g(K) \cap K$ is nonempty. Suppose, on the contrary, that there is an infinite sequence $\{f_n\}$ of distinct mapping classes such that, for every $n$, $f_n(K) \cap K$ is nonempty.

Since $K$ is compact, it has a bounded diameter $d$. Let $m$ be a point in $K$. Since $K$ meets $f_n(K)$:

(1) *for every* $n$, $\varrho(m, f_n(m)) \leq 2d$.

If we consider the sequence of mapping classes $\{g_n\}$, where $g_n = f_n^{-1}$, we may apply (2.7) to choose simple closed curves, $\alpha$ and $\beta$, such that $\{i(g_n(\alpha), \beta)\}$ is unbounded. By (2.4) and Lemma 2.9:

(2) $\{l(f_n(m), \alpha)\} = \{l(m, g_n(\alpha))\}$ *is unbounded.*

On the other hand, by Theorem 2.10:

(3) $l(f_n(m), \alpha) \leq e^{\varrho(m, f_n(m))} l(m, \alpha)$.

Clearly, (1), (2) and (3) give the desired contradiction. ▫

## 3. Fundamental domains

As in Beardon [Be], we define a fundamental domain for the action of $\Sigma$ on $\mathscr{T}$ as follows. A *fundamental set* for $\Sigma$ is a subset of $\mathscr{T}$ which contains exactly one point from each orbit in $\mathscr{T}$. A *domain* in $\mathscr{T}$ is an open subset in $\mathscr{T}$ which is homeomorphic to a ball. A *fundamental domain* for the action of $\Sigma$ on $\mathscr{T}$ is a subset $D$ of $\mathscr{T}$ which satisfies the following three properties:

(3.1) $D$ *is a domain in* $\mathscr{T}$,

(3.2) *there is a fundamental set,* $F$, *with* $D \subset F \subset \overline{D}$,

(3.3) $\partial D$ *is a connected, properly embedded locally flat submanifold of codimension one in* $\mathscr{T}$.

We say that the fundamental domain $D$ is *locally finite* if the following property is also satisfied:

(3.4) *Every compact subset of* $\mathscr{T}$ *meets only finitely many images of* $D$ *under elements of* $\Sigma$.

*Note*: This is not the same definition of a fundamental domain as used in [Ma] or [MP]. The present notion is more precise.

We shall now proceed with the construction of a fundamental domain $D$ for $\Sigma$. In the subsequent sections, we shall establish that $D$ is a locally finite fundamental domain for $\Sigma$ which is, in some sense, a polyhedron. (We shall be more precise below.)

For the remaining discussion, choose a point $m_0$, as in Corollary 2.3, which is not fixed by any element of $\Sigma^*$. Our domain shall be defined exactly as in the classical setting. Hence, we shall be comparing distances.

For each $g$ in $\Sigma^*$, we have the *equidistant locus from* $m_0$ *to* $gm_0$:

(3.5) $L_g(m_0) = \{m \in \mathscr{T} \mid \varrho(m, m_0) = \varrho(m, gm_0)\}$.

Note that the stabilizer of $gm_0$ is also trivial and:

(3.6) $L_g(m_0) = L_{g^{-1}}(gm_0)$.

We shall denote the set of points closer to $m_0$ as:

(3.7) $H_g(m_0) = \{m \in \mathscr{T} \mid \varrho(m, m_0) < \varrho(m, gm_0)\}$.

The remaining points in $\mathscr{T}$ are denoted as:

(3.8) $E_g(m_0) = H_{g^{-1}}(gm_0)$.

We shall refer to $H_g(m_0)$ as a *half space centered at* $m_0$ and to $E_g$ as the *exterior halfspace to* $m_0$. (Of course, if we replace $m_0$ by $gm_0$, then $E_g$ is a halfspace centered at $gm_0$ and $H_g$ is exterior to $gm_0$.)

The *Dirichlet polyhedron for* $\Sigma$ *centered at* $m_0$, $D(m_0)$, is defined by:

(3.9) $D(m_0) = \cap\{H_g(m_0) \mid g \in \Sigma^*\}$.

The *closed Dirichlet polyhedron for* $\Sigma$ *centered at* $m_0$, $\Delta(m_0)$, is defined in a similar way:

(3.10) $\Delta(m_0) = \cap\{\overline{H}_g(m_0) \mid g \in \Sigma^*\}$.

The standard argument involving invariance of distance demonstrates that these sets are natural in the following sense:

(3.11) $h\big(D(m_0)\big) = D\big(h(m_0)\big)$ *for all* $h \in \Sigma$,

(3.12) $h\big(\Delta(m_0)\big) = \Delta\big(h(m_0)\big)$ *for all* $h \in \Sigma$.

## 4. Equidistant loci and halfspaces

We shall need to consider the closed halfspaces. The following fact is central to our discussion.

**Lemma 4.1.** *Suppose that* $m$ *lies on* $L_g$. *Then the half-closed interval* $[m_0, m)$ *is contained in* $H_g(m_0)$.

*Proof.* Suppose that $L_g(m_0) \cap [m_0, m)$ is nonempty. Let $n$ be a point of this intersection. Then:

(1) $\varrho(m_0, n) + \varrho(n, m) = \varrho(m_0, m)$,

(2) $\varrho(m_0, n) = \varrho\big(g(m_0), n\big)$.

On the other hand, since $m$ is in $L_g$:

(3) $\varrho(m_0, m) = \varrho\big(g(m_0), m\big)$.

Thus, we conclude that:

(4) $\varrho\big(g(m_0), n\big) + \varrho(n, m) = \varrho\big(g(m_0), m\big)$.

By Proposition 2.1, $n$ lies on $[m_0, g(m_0)]$. Clearly, $n$ is not one of the endpoints of this interval. Hence, by uniqueness of geodesics, it follows that:

(5) $\gamma(m_0, m) = \gamma(m_0, n) = \gamma(m_0, gm_0)$.

Now, $\gamma(m_0, gm_0)$ is isometric to the real line with the standard metric. The points $m$ and $n$ are each equidistant from the distinct points of this line, $m_0$ and $gm_0$. Clearly, this implies that $m$ is equal to $n$. This is impossible, since $n$ lies on $[m_0, m)$.

Therefore, $[m_0, m)$ is contained in the complement of $L_g(m_0)$. The sets, $H_g(m_0)$ and $E_g(m_0)$, form a separation of this complement. Since $[m_0, m)$ is connected and $m_0$ is in $H_g(m_0)$, $[m_0, m)$ is contained in $H_g(m_0)$. □

**Corollary 4.2.** *Suppose that $m$ lies on $L_g$. Then the half-open interval from $gm_0$ to $m$ is contained in $E_g(m_0)$.*

*Proof.* This is immediate from (3.6), (3.8) and Lemma 4.1. □

These results yield the desired description of the closed halfspaces.

**Corollary 4.3.**

(a) $\overline{H}_g(m_0) = H_g(m_0) \cup L_g(m_0) = \{m \in \mathscr{T} \mid \varrho(m, m_0) \leq \varrho(m, gm_0)\}$,

(b) $\overline{E}_g(m_0) = E_g(m_0) \cup L_g(m_0) = \{m \in \mathscr{T} \mid \varrho(m, gm_0) \leq \varrho(m, m_0)\}$,

(c) $\partial H_g = \partial \overline{H}_g = L_g = \partial \overline{E}_g = \partial E_g$,

(d) $H_g = \operatorname{int}(\overline{H}_g)$ *and* $E_g = \operatorname{int}(\overline{E}_g)$.

Suppose that $m$ is an element of a subset $K$ of $\mathscr{T}$. We say that $K$ is *starlike* at $m$ if, for every point $n$ in $K$, the geodesic segment from $m$ to $n$ is contained in $K$.

It is not clear whether the halfspaces are geodesically convex. This seems unlikely, except for the case of genus one and one puncture, where it is true, due to the fact that Teichmüller space with the Teichmüller metric is the hyperbolic plane. We shall be using the following weaker result:

**Proposition 4.4.** *$H_g$ and $\overline{H}_g$ are starlike at $m_0$.*

*Proof.* In view of Lemma 4.1 and Corollary 4.3(a), it suffices to show that $H_g$ is starlike at $m_0$. To this end, let $m$ be a point in $H_g(m_0)$. Suppose $[m_0, m]$ intersects $L_g(m_0)$. Since $L_g$ is closed and the interval is compact, we can choose $n$ in $[m_0, m]$ such that:

(1) $n \in L_g(m_0)$ *and* $(n, m] \subset H_g(m_0) \cup E_g(m_0)$.

By Lemma 4.1 and the connectedness of $(n, m]$, we conclude that:

(2) $[m_0, n) \cup (n, m] \subset H_g(m_0)$.

Observe that we have shown that, for every point $p$ in $H_g$, the geodesic segment from $p$ to $m_0$ is contained in $\overline{H}_g$. Now consider the exponential map at $m_0$:

(3) $\Omega_0\colon Q(M_0) \to \mathscr{T}$ .

Let $\varphi$ be the preimage of $m$, $\Omega_0(\varphi) = m$. There exists a real number $s$ such that $0 < s < 1$ and $\Omega_0(s\varphi) = n$.

Choose an open neighborhood $\mathscr{U}$ of $\varphi$ such that $\Omega_0(\mathscr{U})$ is contained in $H_g$. The punctured cone on $\mathscr{U}$:

(4) $\mathscr{C}(\mathscr{U}) = \{t\psi \mid 0 < t < 1,\ \psi \in \mathscr{U}\}$

forms an open neighborhood of $s\varphi$ in $Q(M_0)$. By the observation above and Theorem 2.4, $\Omega_0\big(\mathscr{C}(\mathscr{U})\big)$ is an open neighborhood of $n$ which is contained in $\overline{H}_g$. Since $n$ is in $L_g$, Corollary 4.3(c) implies that $\Omega_0\big(\mathscr{C}(\mathscr{U})\big)$ meets $E_g$. This gives the desired contradiction. ▫

This allows us to sharpen the assertion of Lemma 4.1:

**Lemma 4.5.** *Suppose that $m$ lies on $L_g$. Let $r(m_0, m)$ be the geodesic ray from $m_0$ through $m$. Then*:

(a) $r(m_0, m) \cap H_g(m_0) = [m_0, m)$,

(b) $r(m_0, m) \cap L_g(m_0) = \{m\}$,

(c) $r(m_0, m) \cap E_g(m_0) = r(m_0, m) \setminus [m_0, m]$.

*Proof.* By Lemma 4.1 and the hypothesis:

(1) $[m_0, m) \subset r(m_0, m) \cap H_g(m_0)$,

(2) $\{m\} \in r(m_0, m) \cap L_g(m_0)$.

In particular, each point in the closed interval $[m_0, m]$ is contained in $\overline{H}_g(m_0)$.

Suppose $n$ is contained in a point on the ray which is not on this closed interval. Then $m$ is in the half-closed interval $[m_0, n)$. Hence, by Lemma 4.1, $n$ is not in $L_g$. Likewise, by Proposition 4.4, $n$ is not in $H_g$. Thus:

(3) $r(m_0, m) \setminus [m_0, m] \subset r(m_0, m) \cap E_g(m_0) = r(m_0, m)$.

Since $H_g$, $L_g$, and $E_g$ are disjoint, the result follows immediately. ▫

*Note*: One does not need the full thrust of Theorem 2.4 for the proof of Proposition 4.4. An argument based on Corollary 4.3(c) and convergence of sequences of geodesics ([Bu, Chapter 1, (8.14)]) would lead to the same contradiction.

Let $\delta_g$ denote the difference $\varrho_{gm_0} - \varrho_{m_0}$ of the distance functions, $\varrho_{gm_0}$ and $\varrho_{m_0}$. Let $J_g = \gamma(m_0, gm_0) \setminus [m_0, gm_0]$.

**Lemma 4.6.** *$\delta_g$ is nonsingular exactly on the complement of $J_g$ in $\mathscr{T}$.*

*Proof.* By Corollary 2.7, $\delta_g$ is singular at $m$ if and only if $m_0$ and $gm_0$ are on the same ray from $m$. ▫

**Proposition 4.7.** *$L_g$ is a connected, properly embedded $C^1$-submanifold of codimension one in $\mathscr{T}$.*

*Proof.* Since $L_g$ is closed, it is properly imbedded.

Clearly, no point of $L_g$ lies on $J_g$. Hence, by the implicit function theorem, $L_g$ is a $C^1$-submanifold of codimension one in $\mathscr{T}$. In particular, $L_g$ is bicollared in $\mathscr{T}$. (Alternatively, one can use the rays from $m_0$ through $L_g$ to define a global bicollar on $L_g$.)

Teichmüller space is contractible. By Proposition 4.4, Corollary 4.3 and (3.8), so are $\overline{H}_g$ and $\overline{E}_g$. Since $L_g$ is bicollared, a direct application of the Mayer–Vietoris sequence shows that $L_g$ has the homology of a point. In particular, $L_g$ is connected. ◻

**Proposition 4.8.** *If $g$ and $h$ are distinct elements of $\Sigma$, then $L_g \cap L_h$ is either empty or a properly embedded submanifold of codimension 2.*

*Proof.* Let $x$ be a point in $L_g \cap L_h$. By the implicit function theorem, it suffices to show that the differentials at $x$ of the functions $\delta_g$ and $\delta_h$ are linearly independent.

Let $\alpha$ be the unit norm quadratic differential at $x$ pointing in the direction of $m_0$, and $\beta_1$ (respectively $\beta_2$) the unit norm quadratic differential at $x$ pointing in the direction of $gm_0$ (respectively $hm_0$). Since the points $m_0$, $gm_0$ and $hm_0$ are distinct points in $\mathscr{T}$ at equal distances from $x$, the rays from $x$ to these points are distinct. Hence, the unit norm differentials, $\alpha$, $\beta_1$, and $\beta_2$, are distinct.

By Corollary 2.7, the differential at $x$ of $\delta_g$ (respectively $\delta_h$) is equal to $\alpha - \beta_1$ (respectively $\alpha - \beta_2$). Hence, if these differentials were linearly dependent, $\alpha$, $\beta_1$, and $\beta_2$ would be affinely dependent. That is, they would lie on a common line (not necessarily passing through the origin). Since they are distinct unit vectors in $Q(x)$, this would contradict the strict convexity of the unit sphere in $Q(x)$. ◻

We shall now prove some lemmas which will be used in establishing the polyhedral nature of $D$ as well as in the proof that $D$ is a locally finite fundamental domain.

**Lemma 4.9.** *Let $m$ be a point in $\mathscr{T}$ and $\{g_n\}$ be an infinite sequence of distinct mapping classes in $\Sigma$. Then the sequence of real numbers $\{\varrho(m, g_n m_0)\}$ is unbounded.*

*Proof.* The argument is implicit in the proof of proper discontinuity, Theorem 2.11. ◻

**Lemma 4.10.** *For any compact set $K$ in $\mathscr{T}$, the set $\{g \in \Sigma \mid K$ is not contained in $H_g(m_0)\}$ is finite.*

*Proof.* Suppose, for the sake of contradiction, that there exists an infinite sequence $\{g_n\}$ of distinct mapping classes in $\Sigma$ such that for every $n$ we have

$K \cap \overline{E}_{g_n}(m_0) \neq \phi$. Choose a sequence of points, $\{m_n\}$, such that $m_n$ is an element of $K \cap \overline{E}_{g_n}(m_0)$. Let $m$ be an accumulation point of the sequence $\{m_n\}$ in $K$. By taking a subsequence, we can assume that $m_n$ converges to $m$. As $m_n$ lies in $\overline{E}_{g_n}(m_0)$, we have, for every $n$:

(1) $\varrho(m_n, g_n m_0) \leq \varrho(m_n, m_0)$.

Let $\varepsilon$ be a positive real number. Since $m_n$ converges to $m$, there exists an integer $n_0$ such that, for every $n \geq n_0$, the following two properties hold:

(2) $\varrho(m_n, m_0) \leq \varrho(m, m_0) + \varepsilon$,

(3) $\varrho(m, m_n) \leq \varepsilon$.

By using the triangle inequality, we have:

(4) *for all* $n \geq n_0$, $\varrho(m, g_n m_0) \leq \varrho(m, m_0) + 2\varepsilon$,

which shows that the sequence $\{\varrho(m, g_n m_0)\}$ is bounded. This contradicts Lemma 4.9. □

The following corollary is an immediate consequence of the previous lemma.

**Corollary 4.11.** *For any compact set $K$ in $\mathscr{T}$, the set $\{g \in \Sigma \mid K \cap L_g(m_0) \neq \phi\}$ is finite.*

## 5. The structure of the fundamental domain

The following few results give some information on the topology of $D$ and $\Delta$.

**Proposition 5.1.** *$m_0$ is in the interior of $\Delta$.*

*Proof.* Let $K$ be a compact neighborhood of $m_0$ in $\mathscr{T}$. By Lemma 4.10, there is a finite subset $F$ of $\Sigma$ such that:

(1) *for each* $g \in \Sigma \setminus F$, $K \subset H_g(m_0)$.

For each element $g$ in $F$, let $K_g$ denote the intersection $K \cap H_g$. Since $H_g$ is open, $K_g$ is a neighborhood of $m_0$ in $\mathscr{T}$. The intersection (over all $g$ in $F$) of the sets $\{K_g\}$ is a neighborhood of $m_0$ in $\mathscr{T}$ which is contained in $\Delta$. □

**Proposition 5.2.** interior $(\Delta) = D$ *and* $\Delta = \overline{D}$.

*Proof.* Let $m$ be a point in the interior of $\Delta$. In particular:

(1) *for every* $g \in \Sigma^*$, $\varrho(m, m_0) \leq \varrho(m, g m_0)$.

If, for some $g$ in $\Sigma^*$, equality held, then $m$ would be on $L_g(m_0)$. By Corollary 4.3(c), interior $(\Delta)$, being a neighborhood of $m$, would contain a point $n$ satisfying the strict inequality, $\varrho(n, m_0) > \varrho(n, g m_0)$, which is impossible since interior $(\Delta)$ is contained in $\Delta$. Therefore, $m$ is in $D$.

Conversely, let $m$ be a point in $D$. By Lemma 4.10, we can find an open ball $U$ containing $m$ and a finite subset $F$ of $\Sigma$ such that, for each $g$ in $\Sigma \setminus F$, $U$ is contained in $H_g(m_0)$. For each $g$ in $F$, $m$ is in $H_g$, which is open. Let $U_g$ be equal to $U \cap H_g$. The intersection of the sets $\{U_g\}$ (over all $g$ in $F$) is an open neighborhood of $m$ contained in $\Delta$. Hence, $m$ is in the interior of $\Delta$. This proves the first assertion.

For the second assertion, we know that $\overline{D}$ is contained in $\Delta$, because $\Delta$ is closed. Let us prove that $\Delta$ is contained in $\overline{D}$.

Let $m$ be a point in $\Delta$. By definition, for each $g$ in $\Sigma^*$, we have:

(2) $m \in H_g(m_0)$ *or* $m \in L_g(m_0)$.

Consider the geodesic segment $[m_0, m]$ between $m_0$ and $m$. By Lemma 4.1 and Proposition 4.4, (2) shows that, except possibly for its endpoint $m$, this segment is contained in $D$. Therefore, $m$ is contained in $\overline{D}$. □

Since intersections of sets which are starlike at $m_0$ are starlike at $m_0$, we have an immediate corollary of Proposition 4.4.

**Proposition 5.3.** *$D(m_0)$ and $\Delta(m_0)$ are starlike at $m_0$.*

With this, we may deduce the following analogue for $\partial D$ of Proposition 4.7 (without the smoothness property). The proof follows the same plan as the proof of this previous proposition. (Note the remark in the proof concerning an alternative argument for the existence of a bicollar.)

**Proposition 5.5.** *$\partial D$ is a properly embedded, locally flat submanifold of codimension one in $\mathscr{T}$.*

Let us recall that the property of being locally flat means the existence (locally) of a bicollar, that is, $\partial D$ looks locally like $\mathbf{R}^n$ sitting in $\mathbf{R}^{n+1}$. Of course, this topological definition allows for corners, as in the classical case.

The next two lemmas imply the existence of a fundamental set satisfying (3.2).

**Lemma 5.6.** *For any point $m$ in $\mathscr{T}$, the $\Sigma$-orbit of $m$ intersects $D(m_0)$ in at most one point.*

*Proof.* Suppose that $g_1(m)$ and $g_2(m)$ are distinct points in $D$, where $g_1$ and $g_2$ are elements of $\Sigma$. Clearly, $g_1$ and $g_2$ are distinct. Hence, the assumption that $g_1$ is in $D(m_0)$ implies that:

(1) $\varrho(g_1 m, m_0) < \varrho(g_1 m, g_1 g_2^{-1} m_0) = \varrho(m, g_2^{-1} m_0) = \varrho(g_2 m, m_0)$.

Similarly, since $g_2 m$ is in $D$:

(2) $\varrho(g_2 m, m_0) < \varrho(g_1 m, m_0)$.

This gives the desired contradiction. □

**Lemma 5.7.** *For any element $m$ of $\mathscr{T}$, there exists a mapping class $h$ in $\Sigma$ such that $m$ is in $\Delta(h m_0)$.*

*Proof.* Suppose, on the contrary, that $m$ is not in $\Delta(hm_0)$ for any $h$ in $\Sigma$. Then we can find an infinite sequence $\{g_n\}$ of distinct mapping classes such that:

(1) $\cdots < \varrho(m, g_2m_0) < \varrho(m, g_1m_0) < \varrho(m, m_0)$.

This implies that the sequence of real numbers $\{\varrho(m, g_nm_0)\}$ is bounded, in contradiction to Lemma 4.9. ▫

The next result follows from Lemma 4.10.

**Proposition 5.8.** *For any compact set $K$ in $\mathscr{T}$, the set $\{g \in \Sigma \mid K \cap \Delta(gm_0) \neq \emptyset\}$ is finite.*

*Proof.* If $g$ is not the identity, then, by the definition of $\Delta(gm_0)$, (3.6), (3.8) and Corollary 4.3, we have:

(1) $\Delta(gm_0) \subset \overline{H}_{g^{-1}}(gm_0) = \overline{E}_g(m_0)$.

Again, by Corollary 4.3, the result follows immediately from Lemma 4.10. ▫

We have established all the necessary facts to deduce the main result.

**Theorem 5.9.** *$D$ is a locally finite fundamental domain for the action of $\Sigma$ on $\mathscr{T}$.*

*Proof.* Condition (3.1) is implicit in Propositions 5.2 and 5.3, (3.2) in Proposition 5.2 and Lemma 5.7, (3.3) in Proposition 5.5 and (3.4) in Proposition 5.8. ▫

We shall close with a brief discussion of the polyhedral nature of $D$.

By Proposition 5.2, every point of $\partial D$ lies on at least one hypersurface $L_g(m_0)$. On the other hand, by Corollary 4.11, every point in $\mathscr{T}$ lies on at most finitely many such hypersurfaces (for a fixed $m_0$). We shall say that a point of $\partial D$ *lies on a face of* $D$ (or of $\Delta$) if it lies on exactly one hypersurface. If $g$ is an element of $\Sigma$, the *$g$-face* $F_g(m)$ of $D$ is the set of points of $\partial D$ which lie only on the hypersurface $L_g(m_0)$. (Therefore, $m$ is on a face of $D$ if and only if, for some $g$ in $\Sigma$, $m$ is on $F_g$.)

Suppose that $x$ is a point in $\mathscr{T}$ and $n$ is in the $\Sigma$-orbit of $m_0$. We say that $x$ is closest to $n$ if $\varrho(x, n)$ is less than or equal to $\varrho(x, hm_0)$ for all $h$ in $\Sigma$.

**Proposition 5.10.** *Let $x$ be a point in $\mathscr{T}$. Then $x$ is on $F_g(m_0)$ if and only if $x$ is closest to exactly two points of the $\Sigma$-orbit of $m_0$, $m_0$ and $gm_0$.*

*Proof.* Given Proposition 5.2, the proof is a trivial exercise in the definitions. ▫

**Corollary 5.11.** $F_g(m_0) = F_{g^{-1}}(gm_0)$.

**Corollary 5.12.** $F_g(m_0) \subset \Delta(m_0) \cap \Delta(gm_0) \subset L_g(m_0)$.

**Corollary 5.13.** *Let $x$ be a point of $\Delta(m_0) \cap \Delta(gm_0)$. Then $x$ is on a face of $D(m_0)$ if and only if $x$ is on a face of $D(gm_0)$. If $x$ is on a face of $D(m_0)$, then $x$ is on $F_g(m_0)$.*

We observe that, in particular, the property of being "on a face" is independent of which copy of $D$ we consider. This, of course, is a property we would ask of any reasonable notion of a tesselation.

**Proposition 5.14.** *There exists an element $g$ of $\Sigma$ such that $F_g(m_0)$ is nonempty.*

*Proof.* This follows immediately from Propositions 4.8 and 5.5. We shall, however, give more explicit information.

Let $g$ be an element of $\Sigma^*$ such that $m_0$ is closest to $gm_0$ which, according to the definitions, means the following:

(1) $\varrho(m_0, gm_0) \leq \varrho(m_0, hm_0)$ *for all* $h$ *in* $\Sigma$.

Let $m$ be the midpoint of $[m_0, gm_0]$. Suppose that $m$ is not closest to $m_0$:

(2) $\varrho(m, hm_0) < \varrho(m, m_0)$ *for some* $h$ *in* $\Sigma$.

Then, by the triangle inequality:

(3) $\varrho(m_0, hm_0) \leq \varrho(m_0, m) + \varrho(m, hm_0) < 2\varrho(m_0, m) = \varrho(m_0, gm_0)$.

This contradicts the choice of $g$. Hence, $m$ is closest to $m_0$:

(4) $\varrho(m, m_0) = \varrho(m, gm_0) \leq \varrho(m, hm_0)$ *for all* $h$ *in* $\Sigma$.

Thus, by Proposition 5.2, $m$ is in $\partial D \cap L_g$.

Suppose that $m$ lies on another hypersurface $L_h$. Since $m_0$, $gm_0$ and $hm_0$ are distinct points equidistant from the midpoint of $[m_0, gm_0]$, $hm_0$ does not lie on $\gamma(m_0, gm_0)$. Hence, by Proposition 2.1:

(5) $\varrho(m_0, hm_0) < \varrho(m_0, m) + \varrho(m, hm_0) = \varrho(m_0, gm_0)$.

Again, this contradicts the choice of $g$. Therefore, $m$ lies on exactly one hypersurface $L_g(m_0)$. This shows that $m$ is in $F_g(m_0)$. ◻

**Proposition 5.15.** *$F_g$ is an open subset of $\partial D$.*

*Proof.* Suppose that $m$ is on $F_g(m_0)$. By Lemma 4.10, we may choose a neighborhood $U$ of $m$ and a finite subset $F$ of $\Sigma$ such that $U$ is contained in $H_h(m_0)$ for all $h$ in $\Sigma \setminus F$. For each element $h$ of $F \setminus \{g\}$, let $U_h$ denote the intersection $U \cap H_h$. Since $m$ is in $\Delta$ and lies on only one hypersurface $L_g$, $U_h$ is an open neighborhood of $m$ for each such $h$. The intersection $V$ of the sets $\{U_h\}$ (over $h$ in $F \setminus \{g\}$) is an open neighborhood of $m$ which is contained in $H_h(m_0)$ for all $h$ in $\Sigma \setminus \{g\}$. The intersection of $V$ with $\partial D$ gives a neighborhood of $m$ in $\partial D$ entirely contained in $F_g$. ◻

**Proposition 5.16.** *The union of the faces of $\partial D$ is a dense open subset of $\partial D$.*

*Proof.* The complement of this union consists of points which, by Proposition 4.8 and Corollary 4.11, lie on a locally finite union of codimension one submanifolds of $\partial D$. This set is closed and nowhere dense. Hence, the union is a dense open subset. ◻

We note that the structure which we have developed is sufficient to deduce that the group $\Sigma$ is generated by *face pairing transformations*. We say that an element $g$ of $\Sigma$ is a *face pairing transformation of* $D(m_0)$ if $F_g(m_0)$ is nonempty. By Proposition 4.8, Corollary 5.12 and Proposition 5.15, this amounts to the assertion that $\Delta(m_0) \cap \Delta(gm_0)$ has nonempty interior in $\partial D(m_0)$.

**Theorem 5.17.** *$\Sigma$ is generated by the face pairing transformations of $D(m_0)$.*

*Proof.* The proof is analogous to one that can be done in the case of discrete isometry groups acting in hyperbolic spaces:

Let $h$ be an element of $\Sigma$. Clearly, $m_0$ is in $D(m_0)$ and $hm_0$ is in $D(hm_0)$. Let $\alpha$ be a path in $\mathscr{T}$ from $m_0$ to $hm_0$. By Propositions 5.1, 5.2, 5.5 and 5.8, we may assume that, for all $g$ in $\Sigma$, $\alpha$ is transverse to $\partial D(gm_0)$ and intersects $\partial D(gm_0)$ only in points on faces of $D(gm_0)$.

As we traverse $\alpha$, we encounter various copies of our domain:

(1) $D(h_i m_0)$, $0 \le i \le n$,

such that:

(2) $h_0 = 1$ *and* $h_n = h$,

(3) $\emptyset \neq F_{g_i}(h_i m_0) \subset \Delta(h_i m_0) \cap \Delta(h_{i+1} m_0)$, $1 \le i \le n$,

where:

(4) $h_{i+1} = g_i h_i$.

We prove, by induction, that $h_i$ is a product of face pairing transformations of $D(m_0)$ for $1 \le i \le n$.

By definition, $h_1$ is a face pairing transformation. Suppose that $h_i$ is a product of face pairing transformations. By conjugating (3), we obtain:

(5) $\emptyset \neq F_{h_i^{-1} h_{i+1}}(m_0) \subset \Delta(m_0) \cap \Delta(h_i^{-1} h_{i+1} m_0)$.

This implies that $h_i^{-1} h_{i+1}$ is a face pairing transformation. But:

(6) $h_{i+1} = h_i (h_i^{-1} h_{i+1})$.

Hence, $h_{i+1}$ is a product of the desired type. □

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