Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 21, 1996, 189–204

ON THE TAYLOR COEFFICIENTS OF THE COMPOSITION OF TWO ANALYTIC FUNCTIONS

Titus Hilberdink

Goldsmiths College, Department of Mathematical Sciences
New Cross, London SE14 6NW, England; map01twh@scorpio.gold.ac.uk

Abstract. We give an asymptotic formula for the Taylor coefficients f_n of $f(z) = l(h(z))$ where $l(z)$ is analytic in the unit disc whose Taylor coefficients l_n vary 'smoothly' and $h(z)$ is analytic in a larger disc. We show that under mild conditions on $h(z)$, $f_n \sim \sigma l_{[\sigma n]}$ as $n \to \infty$ where $\sigma = 1/h'(1)$. Applications to renewal theory are also discussed.

1. Introduction

Asymptotic enumeration usually involves estimating coefficients of a generating function $f(z)$ which satisfies some functional equation. In many cases such a functional equation can be reduced to the form

$$f(z) = l\bigl(h(z)\bigr),$$

where the function $l(z)$ is frequently known and its Taylor coefficients l_n are non-negative and usually satisfy a certain regularity condition (see Bender [1], Meir and Moon [11]). In this paper we make the following regularity condition on l_n which occurs in many applications:

$$(\dagger) \qquad l_{n+[\lambda\sqrt{n}\,]} \sim l_n \qquad \text{as } n \to \infty \qquad \text{for all } \lambda \in \mathbf{R}.$$

For instance, the cases $l_n = 1$ and $l_n = 1/n$ from renewal theory and harmonic renewal theory respectively both satisfy (†) for which our result applies and we will discuss the details in Theorem 2.11 at the end of the paper. It will be shown later that for a large class of functions $h(z)$ this is a natural choice to make and difficulties appear if the l_n are made any larger. We note that $l(z)$ has a radius of convergence equal to 1 and $z = 1$ is a singularity (since $e^{-\varepsilon\sqrt{n}} < l_n < e^{\varepsilon\sqrt{n}}$ follows from (†) and $l_n \geq 0$). We also assume that $h(1) = 1$, and that $h(z)$ has non-negative coefficients such that $|h(z)| = h(|z|)$ only if z is real and non-negative. This last condition is equivalent to $h(z) \neq z^p k(z^r)$ for any analytic function $k(z)$ and any $r \geq 2$, $p \geq 0$ (see Remark 2.7(3)).

We note that many authors have considered the composite function $l\bigl(h(z)\bigr)$ where the Taylor coefficients h_n are well-behaved, but we consider the case where

---

1991 Mathematics Subject Classification: Primary 30B10; Secondary 40E10, 60K05.

the coefficients l_n are well-behaved. Indeed, our setting differs significantly from the one considered by Embrechts and Omey in [4], since their '$l(z)$' is analytic in an open region *containing* the set $\{h(z) : |z| \le 1\}$. In particular, $l(z)$ would then be analytic at $z = 1$.

From an analytical viewpoint we have two quite distinct cases:

(i) $h(z)$ is analytic in a disc $D(0, 1+\delta)$ for some $\delta > 0$.

(ii) $h(z)$ is analytic in the unit disc only.

We shall first consider case (i).

Estimating the coefficients of $f(z)$ can often be done by using the analytic continuation of $f(z)$ to a region

$$\bigl\{|z| < 1 + \eta : |\arg(z-1)| > \phi \text{ for some } \eta > 0 \text{ and } 0 < \phi < \tfrac{1}{2}\pi\bigr\}.$$

For instance, such methods have been used in [7] and [12]. However, we will adopt a different approach in that we do not assume such an analytic extension, and in this respect the techniques we use seem to be more elementary.

Our main result is

Theorem 1.1. *Let $f(z) = l\bigl(h(z)\bigr)$ where the Taylor coefficients l_n of $l(z)$ satisfy (†), and $h(z)$ is analytic in $|z| < 1 + \delta$ where $\delta > 0$, has non-negative coefficients such that $|h(z)| = h(|z|)$ on $[0, 1+\delta)$ only and $h(1) = 1$. Then the coefficients f_n of $f(z)$ satisfy*

$$f_n \sim \sigma l_{[\sigma n]} \quad \text{as } n \to \infty \quad \text{where } \sigma = \frac{1}{h'(1)}.$$

We have readily the following corollary which is an extension of the theorem.

Corollary 1.2. *Let $f(z) = l\bigl(h(z)\bigr)$ with $l(z)$ and $h(z)$ as before, but with $h(z)$ analytic in $|z| < \gamma$ such that $h(\beta) = 1$ for some $0 < \beta < \gamma$. Then*

$$f_n \sim \sigma l_{[\sigma n]} \beta^{-n} \quad \text{as } n \to \infty \quad \text{where } \sigma = \frac{1}{\beta h'(\beta)}.$$

The proof of the corollary follows immediately by putting $F(z) = f(\beta z)$ and $H(z) = h(\beta z)$ so that $F(z) = l\bigl(H(z)\bigr)$ and the conditions for the theorem are satisfied. □

After the proof of Theorem 1.1, we will extend the theorem to cover the more difficult case (ii) in Theorem 2.9.

I wish to thank the referee for many helpful suggestions.

2. Proof of the theorem

We begin with the following representation of l_n.

Lemma 2.1. *Given a sequence $(l_n)_{n\geq 0}$ of positive reals such that $l_{n+[\lambda\sqrt{n}\,]} \sim l_n$ as $n \to \infty$ for all real λ, then*

$$l_n = s(e^{\sqrt{n}}\,)$$

for some slowly-varying function s.

Proof. The result will follow once we can show that $l_{n+1} \sim l_n$ as $n \to \infty$. For then $l_{n+m} \sim l_n$ as $n \to \infty$ for all integers m, so by letting $s(x) = l_{[(\log x)^2]}$ we observe that

$$s(\lambda x) \sim s(x) \qquad \text{as} \quad x \to \infty \quad \text{for all } \lambda > 0,$$

i.e., s is slowly-varying.

To show that $l_{n+1} \sim l_n$, consider $\eta_n(\lambda) = [\lambda\sqrt{n}\,] + \big[-\lambda\sqrt{n+[\lambda\sqrt{n}\,]}\,\big]$ for various $\lambda > 0$. By (†), we have $l_{n+\eta_n(\lambda)} \sim l_n$. Taking $\lambda = \sqrt{2}$ and $\lambda = 2$ in turn, one finds that $\eta_n(\sqrt{2}\,) = -2$ and $\eta_n(2) = -3$ for all n except when $|2n - m^2|$ or $|4n - m^2| <$ some constant A.

If this happens, let $k = n + [\mu\sqrt{n}\,]$ for some $0 < \mu < 1$. Then $2k$ and $4k$ are bounded away from perfect squares and hence $l_{k-2} \sim l_{k-3} \sim l_k$ from which it will be seen that $l_{n-2} \sim l_{n-3}$ as required. □

Note that the converse holds trivially, so that we have an equivalence.

Slowly-varying functions have been studied in great detail (see Bingham *et al.* [3] or Feller [5]) and it is well known that they can be represented as follows:

If $s(x)$ is slowly-varying, then

$$s(x) = c(x)\exp\bigg\{\int^x \frac{\varepsilon(t)}{t}\,dt\bigg\}$$

where $c(x) \to c > 0$ and $\varepsilon(x) \to 0$ as $x \to \infty$.

Using this, we easily obtain

Lemma 2.2. *If $l_{n+[\lambda\sqrt{n}\,]} \sim l_n$ as $n \to \infty$ for all real λ, then*

$$l_n = c_n \exp\bigg\{\sum_{m=1}^{n} \frac{\delta_m}{\sqrt{m}}\bigg\}$$

where $c_n \to c > 0$ and $\delta_n \to 0$ as $n \to \infty$.

Proof. Since $l_n = s(e^{\sqrt{n}})$, we have

$$l_n = c(e^{\sqrt{n}})\exp\left\{\int_1^{e^{\sqrt{n}}} \frac{\varepsilon(t)}{t}\,dt\right\} = c_n \exp\left\{\int_0^n \frac{\eta(s)}{\sqrt{s}}\,ds\right\}$$

by putting $t = e^{\sqrt{s}}$, $\eta(s) = \frac{1}{2}\varepsilon(e^{\sqrt{s}})$ and $c_n = c(e^{\sqrt{n}})$. This gives us

$$l_n = c_n \exp\left\{\sum_{m=1}^{n} \frac{\delta_m}{\sqrt{m}}\right\}$$

where

$$\delta_m = \int_0^1 \frac{\eta(m+t-1)}{\sqrt{1+(t-1)/m}}\,dt$$

which tends to 0, as desired. ◻

This lemma gives us a corollary which will be useful in the proof of the theorem.

Corollary 2.3. *For n and $n+k \to \infty$,*

$$l_{n+k} = l_n \exp\left\{o\left(\frac{|k|}{\sqrt{n}}\right) + o(1)\right\}.$$

Proof. Without loss of generality take $k > 0$. We have

$$\frac{l_{n+k}}{l_n} = \frac{c_{n+k}}{c_n}\exp\left\{\sum_{m=0}^{k} \frac{\delta_{m+n}}{\sqrt{m+n}}\right\}.$$

Now $c_{n+k} \sim c_n$ and $\left|\sum_0^k \delta_{m+n}/\sqrt{m+n}\right| < \delta k/\sqrt{n}$ as required. ◻

We are particularly interested in the case when $a\sqrt{n} < |k| < A\sqrt{n}\log n$ for some constants a and A in which case the $o(1)$ term becomes superfluous.

We are now in a position to prove the theorem.

Proof. We have by Cauchy's integral formula,

$$f_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{l\big(h(z)\big)}{z^{n+1}}\,dz = \sum_{m=0}^{\infty} l_m \frac{1}{2\pi i}\int_\gamma \frac{h(z)^m}{z^{n+1}}\,dz$$

where γ is inside the unit disc. Thus

$$f_n = \sum_{m=0}^{\infty} l_m J_{m,n} \tag{1}$$

where

$$J_{m,n} = \frac{1}{2\pi i} \int_\gamma \frac{h(z)^m}{z^{n+1}}\, dz$$

is the coefficient of z^n in the expansion of $h(z)^m$. Suppose h_{m_0} is the first non-zero coefficient of $h(z)$. Then $J_{m,n} = 0$ for $m > n/m_0$. (If $m_0 = 0$, $J_{m,n} > 0$ always.) The proof basically involves estimating $J_{m,n}$ for $m \geq 0$. We will show that $J_{m,n}$ is largest when m is close to σn. Recall that $\sigma = 1/h'(1)$. In fact, $(1/\sigma) J_{m,n}$ behaves like a normal distribution with mean σn and deviation $\sigma\sqrt{2\sigma\tau n}$ where τ is some positive constant dependent on $h(z)$. Note that $h'(1) = \sum_{m_0}^\infty n h_n > m_0 \sum_{m_0}^\infty h_n = m_0$, so that σn lies in the non-zero range of $J_{m,n}$. The special case where $h_n = 0$ for all $n > m_0$ so that $h'(1) = m_0$ gives $h(z) = h_{m_0} z^{m_0}$ which is not allowed.

We first find bounds for $J_{m,n}$ when $|m - \sigma n|$ is 'large'.

Lemma 2.4. (1) *For $0 \leq m \leq \frac{1}{3}\sigma n$ and $m \geq 3\sigma n$, we have*

$$J_{m,n} \leq e^{-\eta n},\ e^{-\eta' m} \quad \textit{respectively for some } \eta, \eta' > 0.$$

(2) *For $\frac{1}{3}\sigma n \leq m \leq \sigma n - 2\sigma\sqrt{n}\log n$ and $\sigma n + 2\sigma\sqrt{n}\log n \leq m \leq 3\sigma n$, we have*

$$J_{m,n} \leq A e^{-|m-\sigma n|/\sigma\sqrt{n}} \quad \textit{for some } A.$$

Proof. (1) Consider first $0 \leq m \leq \frac{1}{3}\sigma n$. We have

$$0 \leq J_{m,n} \leq \frac{h(r)^m}{r^n}$$

for any $0 < r < 1 + \delta$ since $|h(z)| \leq h(|z|)$. Take $r = e^\varepsilon$ where $\varepsilon > 0$ is chosen such that

$$h(e^\varepsilon) \leq e^{2\varepsilon/\sigma}.$$

This is possible because $h(e^t) = 1 + (t/\sigma) + O(t^2) = \exp\big((t/\sigma) + O(t^2)\big)$ as $t \to 0$. Then

$$J_{m,n} \leq e^{(2\varepsilon m/\sigma) - \varepsilon n} \leq e^{-\varepsilon n/3}$$

as required. For $m \geq 3\sigma n$, take $r = e^{-\varepsilon}$ where $\varepsilon > 0$ is chosen such that $h(e^{-\varepsilon}) \leq e^{-\varepsilon/2\sigma}$. Then again $J_{m,n} \leq e^{-(\varepsilon m/2\sigma)+\varepsilon n} \leq e^{-\varepsilon m/6\sigma}$.

(2) For the range $\frac{1}{3}\sigma n \leq m \leq \sigma n - 2\sigma\sqrt{n}\log n$ take $r = e^{1/\sqrt{n}}$ in the above inequality. Then

$$J_{m,n} \leq h\big(e^{1/\sqrt{n}}\big)^m e^{-\sqrt{n}} = e^{((1/\sigma\sqrt{n})+O(1/n))m-\sqrt{n}} = e^{-(|m-\sigma n|/\sigma\sqrt{n})+O(1)}.$$

If $\sigma n + 2\sigma\sqrt{n}\log n \leq m \leq 3\sigma n$ take $r = e^{-1/\sqrt{n}}$ and the result follows similarly. □

Let $S=\{m\mid \frac{1}{3}\sigma n<m<3\sigma n \text{ and } |m-\sigma n|\geq 2\sigma\sqrt{n}\log n\}$. Then from these bounds, we find that

$$\begin{aligned}\sum_{|m-\sigma n|\geq 2\sigma\sqrt{n}\,\log n} l_m J_{m,n} &= \sum_{0\leq m\leq \sigma n/3,\ m\geq 3\sigma n} l_m J_{m,n}+\sum_S l_m J_{m,n}\\ &\leq e^{-\eta n}\sum_{0\leq m\leq \sigma n/3} l_m+\sum_{m\geq 3\sigma n} l_m e^{-\eta' m}\\ &\quad + A\sum_S l_m e^{-|m-[\sigma n]|/\sigma\sqrt{n}}\\ &\leq e^{-\eta'' n}+2Al_{[\sigma n]}\sum_{k\geq 2\sigma\sqrt{n}\,\log n} e^{-k/\sigma\sqrt{n}+o(k/\sqrt{n}\,)}\end{aligned}$$

by putting $k=m-[\sigma n]$ and using Corollary 2.3 and the fact that $l_n=e^{o(\sqrt{n}\,)}$. The last sum is bounded for all large n by

$$\int_{2\sigma\sqrt{n}\log n}^{\infty} e^{-t/2\sigma\sqrt{n}}\,dt=\frac{2\sigma}{\sqrt{n}}.$$

Hence

$$\sum_{|m-\sigma n|\geq 2\sigma\sqrt{n}\,\log n} l_m J_{m,n}=O\Big(\frac{l_{[\sigma n]}}{\sqrt{n}}\Big).$$

Now it only remains to consider the range $|m-\sigma n|<2\sigma\sqrt{n}\log n$. Here $J_{m,n}$ behaves as follows:

Lemma 2.5. *For $|m-\sigma n|<2\sigma\sqrt{n}\log n$, we have*

$$J_{m,n}=\frac{1}{2\sqrt{\pi\sigma\tau n}}\exp\Big\{-\frac{(m-\sigma n)^2}{4\sigma^3\tau n}\Big\}+O\Big(\frac{(\log n)^2}{n}\Big)$$

where $\tau=\frac{1}{2}\big(h'(1)+h''(1)-h'(1)^2\big)>0$ by Remark 2.6.

Proof. We have

$$J_{m,n}=\frac{1}{2\pi i}\int_\gamma \frac{h(z)^m}{z^{n+1}}\,dz=\frac{1}{2\pi}\int_0^{2\pi} e^{-ni\theta}h(e^{i\theta})^m\,d\theta.$$

Let $\delta_n=\sqrt{2\log n/\sigma\tau n}$ with τ as before. Then

$$\Big|\frac{1}{2\pi}\int_{\delta_n}^{2\pi-\delta_n} e^{-ni\theta}h(e^{i\theta})^m\,d\theta\Big|\leq \sup_{\delta_n\leq\theta\leq 2\pi-\delta_n}|h(e^{i\theta})|^m=|h(e^{\pm i\delta_n})|^m$$

for all n sufficiently large, since $|h(e^{i\theta})|$ decreases near 0 and 2π where it is maximal. But

$$\begin{aligned} h(e^{i\theta}) &= h\big(1+i\theta-\tfrac{1}{2}\theta^2+O(\theta^3)\big) = 1+h'(1)i\theta-\tfrac{1}{2}\big(h'(1)+h''(1)\big)\theta^2+O(\theta^3) \\ &= \exp\Big(\frac{i}{\sigma}\theta-\tau\theta^2+O(\theta^3)\Big) \end{aligned}$$

so that $|h(e^{\pm i\delta_n})|^m = \exp\big(-m\tau\delta_n^2+O(m\delta_n^3)\big) = O(n^{-2})$. Hence

$$J_{m,n} = \frac{1}{2\pi}\int_{-\delta_n}^{\delta_n} e^{-ni\theta}h(e^{i\theta})^m\,d\theta + O\Big(\frac{1}{n^2}\Big).$$

In the range $|\theta|\le\delta_n$, we have

$$\begin{aligned} e^{-ni\theta}h(e^{i\theta})^m &= \exp\Big(-ni\theta+\frac{mi}{\sigma}\theta-m\tau\theta^2+O(m\theta^3)\Big) \\ &= \exp\Big\{\frac{m-\sigma n}{\sigma}i\theta-\sigma\tau n\theta^2+O\Big(\frac{(\log n)^2}{\sqrt{n}}\Big)\Big\} \\ &= \exp\Big\{\frac{m-\sigma n}{\sigma}i\theta-\sigma\tau n\theta^2\Big\}\Big(1+O\Big(\frac{(\log n)^2}{\sqrt{n}}\Big)\Big) \end{aligned}$$

by using $m=\sigma n+O(\sqrt{n}\log n)$. Thus

$$\begin{aligned} J_{m,n} &= \frac{1}{2\pi}\int_{-\delta_n}^{\delta_n} e^{(m-\sigma n i\theta/\sigma)-\sigma\tau n\theta^2}\,d\theta + O\Big(\frac{(\log n)^2}{\sqrt{n}}\int_{-\delta_n}^{\delta_n} e^{-\sigma\tau n\theta^2}\,d\theta\Big) + O\Big(\frac{1}{n^2}\Big) \\ &= \frac{1}{2\pi\sqrt{\sigma\tau n}}\int_{-\sqrt{2\log n}}^{\sqrt{2\log n}} e^{-\phi^2+(m-\sigma n)i\phi/\sigma\sqrt{\sigma\tau n}}\,d\phi + O\Big(\frac{(\log n)^2}{n}\int_{-\infty}^{\infty} e^{-\phi^2}\,d\phi\Big) \\ &= \frac{1}{2\pi\sqrt{\sigma\tau n}}e^{-(m-\sigma n)^2/4\sigma^3\tau n}\int_{-\infty}^{\infty} e^{-(\phi-i(m-\sigma n)/2\sigma\sqrt{\sigma\tau n})^2}\,d\phi \\ &\quad + o\Big(\frac{1}{n^2}\Big) + O\Big(\frac{(\log n)^2}{n}\Big) \\ &= \frac{1}{2\sqrt{\pi\sigma\tau n}}\exp\Big\{-\frac{(m-\sigma n)^2}{4\sigma^3\tau n}\Big\} + O\Big(\frac{(\log n)^2}{n}\Big) \end{aligned}$$

since

$$\Big|\int_{|\phi|\ge\sqrt{2\log n}}\Big| \le \frac{1}{\sqrt{2\log n}}\int_{\sqrt{2\log n}}^{\infty} 2\phi e^{-\phi^2}\,d\phi = o\Big(\frac{1}{n^2}\Big). \ \square$$

Remark 2.6. Note that $\tau>0$. This is because $\sum_0^\infty h_n\sum_0^\infty n^2h_n \ge \big(\sum_0^\infty nh_n\big)^2$ by Cauchy–Schwarz. Equality occurs only if $h_n=0$ for all n except perhaps for one value of n but this case has already been ruled out.

To finalize the proof of the theorem, we must consider the sum $\sum l_m J_{m,n}$ over the range $|m-\sigma n| < 2\sigma\sqrt{n}\,\log n$.

First we have

$$\begin{aligned}
\sum_{|m-\sigma n|<2\sigma\sqrt{n}\,\log n} l_m \frac{(\log n)^2}{n} &= l_{[\sigma n]} \frac{(\log n)^2}{n} \sum_{|k|<2\sigma\sqrt{n}\,\log n} e^{o(|k|\sqrt{n}\,)} \\
&= O\biggl(l_{[\sigma n]} \frac{(\log n)^2}{n} \int_0^{2\sigma\sqrt{n}\log n} e^{\varepsilon t/\sqrt{n}}\,dt \biggr) \\
&= O(l_{[\sigma n]} n^{-(1/2)+\varepsilon'})
\end{aligned}$$

where ε and ε' are arbitrarily small. Now choose an arbitrary $A>0$. Then we have

$$\frac{1}{2\sqrt{\pi\sigma\tau n}} \sum_{|m-\sigma n|<2\sigma\sqrt{n}\,\log n} l_m e^{-(m-\sigma n)^2/4\sigma^3\tau n} = \sum{}' + \sum{}'',$$

where $\sum'$ is over the range $|m-\sigma n| \le A\sqrt{n}$ and $\sum''$ is over $A\sqrt{n} < |m-\sigma n| \le 2\sigma\sqrt{n}\,\log n$. For $\sum''$ we have

$$\begin{aligned}
\sum{}'' &= \frac{l_{[\sigma n]}}{2\sqrt{\pi\sigma\tau n}} \sum_{A\sqrt{n}<|k|\le 2\sigma\sqrt{n}\,\log n} e^{-(k^2/4\sigma^3\tau n)+o(|k|/\sqrt{n})} \\
&= O\biggl(\frac{l_{[\sigma n]}}{\sqrt{n}} \int_{A\sqrt{n}}^{2\sigma\sqrt{n}\log n} e^{-(t^2 4\sigma^3\tau n)+o(t/\sqrt{n}\,)}\,dt \biggr) \\
&= O\biggl(l_{[\sigma n]} \int_A^\infty e^{-(y^2/4\sigma^3\tau)+o(y)}\,dy \biggr) \quad \text{where } t = y\sqrt{n} \\
&< \varepsilon(A) l_{[\sigma n]}
\end{aligned}$$

where $\varepsilon(A) \to 0$ as $A \to \infty$.

Lastly

$$\begin{aligned}
\sum{}' &\sim l_{[\sigma n]} \sum_{|m-\sigma n|\le A\sqrt{n}} J_{m,n} \qquad \text{by virtue of } (\dagger) \\
&= l_{[\sigma n]} \sum_{m=0}^\infty J_{m,n}\bigl(1+\delta_n(A)\bigr)
\end{aligned}$$

where, as before, $|\delta_n(A)| \le \delta(A) \to 0$ as $A \to \infty$ by using the previous known bounds on $J_{m,n}$. But

$$\sum_{m=0}^\infty J_{m,n} = \frac{1}{2\pi i} \int_\gamma \frac{1}{\bigl(1-h(z)\bigr) z^{n+1}}\,dz = \sigma + O(a^{-n})$$

for some $a > 1$, by choosing γ so that it contains the simple pole of $1/1 - h(z)$ at 1. So letting $A \to \infty$, (1) gives

$$f_n \sim \sigma l_{[\sigma n]}$$

as desired. □

Remark 2.7. (1) The result is 'best possible' in the sense that if we increase the growth rate of l_n it becomes false. In particular, by the same methods one can prove that if $l_n = r(e^{\sqrt{n}})$ where r is a regularly-varying function with index ρ, then

$$f_n \sim \sigma l_{[\sigma n]} e^{\sigma^3 \tau \rho^2/4} \qquad \text{as} \quad n \to \infty.$$

(Lemma 2.4 would have to be improved to $J_{m,n} = O\big(e^{-(m-\sigma n)^2/4\sigma^3 \tau n}\big)$ for $|m - \sigma n| = O(n^{2/3})$. This is done by taking $r = e^{-(m-\sigma n)/2\sigma^2 \tau n}$ in the proof.)

Also, if the l_n are made larger than $e^{A\sqrt{n}}$ (any A), then the arguments following Lemma 2.4 and Remark 2.6 fail and only with very precise knowledge of the behaviour of $l_{[\sigma n]+m}/l_{[\sigma n]}$ as m varies, can we obtain a simple asymptotic formula for f_n.

(2) The proof of the theorem would in many cases be substantially easier if we did not allow the l_n to be quite so large. For example, if l_n was regularly-varying (which satisfies (†)), we only need Lemma 2.4 after which

$$\sum_{|m-\sigma n| \le 2\sigma\sqrt{n}\,\log n} l_m J_{m,n} \sim l_{[\sigma n]} \sum_{|m-\sigma n| \le 2\sigma\sqrt{n}\,\log n} J_{m,n} \sim l_{[\sigma n]} \sum_{m=0}^{\infty} J_{m,n} \sim \sigma l_{[\sigma n]}$$

follows immediately. Also, with a little extra work, we can usually find an asymptotic expansion for f_n given that l_n has one.

(3) The original condition on $h(z)$ was that $|h(z)| = h(|z|)$ on $[0, 1+\delta)$ only. As stated earlier this is equivalent to saying that $h(z)$ is not of the form $z^p k(z^r)$ for any analytic $k(z)$ and $p \ge 0$, $r \ge 2$. On the other hand, if $h(z) = z^p k(z^r)$, then consider $\gcd(p, r) > 1$ and $= 1$. In the former case, the theorem fails since $f_n = 0$ if $d \not| n$. In fact, if $\gcd(p, r) = d \ge 2$ then $h(z) = k_1(z^d)$ some k_1 analytic which is equivalent to the case $p = 0$. If however $\gcd(p, r) = 1$, the result is, surprisingly enough, still true. Lemma 2.4 will hold as before but a change occurs in Lemma 2.5. Here we have

$$J_{m,n} = \begin{cases} \dfrac{r}{2\sqrt{\pi \sigma \tau n}} e^{-(m-\sigma n)^2/4\sigma^3 \tau n} + O\Big(\dfrac{(\log n)^2}{n}\Big) & \text{if } pm \equiv n \pmod{r}, \\ 0 & \text{otherwise.} \end{cases}$$

A little elementary number theory will be needed in the later stages of the proof. Thus the result stands true for all $h(z)$ not of the form $k(z^r)$ as long as there is a $\beta > 0$ within its radius of convergence such that $h(\beta) = 1$. If $h(z) = k(z^r)$ with $r \geq 2$ maximal, $\beta = 1$, then $f(z) = F(z^r)$ with $F(z) = \sum_0^\infty F_n z^n$ and the theorem applies to $F(z)$, i.e., $F_n \sim r\sigma l_{[r\sigma n]}$. The result now covers all possible forms of $h(z)$.

In case (ii), where $h(z)$ is analytic in the unit disc only, we cannot use the arguments of Lemma 2.4 (for the case $m < \sigma n$) so other methods are needed to provide the bounds we want, *viz.*

$$J_{m,n} = O\big(e^{-\lambda|m-\sigma n|/\sqrt{n}}\big) \qquad \text{for some } \lambda > 0.$$

For the result $f_n \sim \sigma l_{[\sigma n]}$ to hold, we clearly need some bound on h_n (given that we still allow l_n to be large). For example, if $h_n \neq O\big(e^{-\lambda\sqrt{n}}\big)$ for any $\lambda > 0$, we can always find an $l_n = s(e^{\sqrt{n}})$ such that $l_{[\sigma n_k]} = o(h_{n_k})$ along some sequence $n_k \to \infty$. Then since $f_n \geq l_1 h_n$ the theorem cannot hold.

However, if $h_n = O\big(e^{-\lambda\sqrt{n}}\big)$ for some $\lambda > 0$, the theorem still holds. The trick is to use an approximation to $h(z)$, namely $h_n(z) = \sum_{r=0}^n h_r z^r$, and to observe that $J_{m,n}$ is still the coefficient of z^n in the expansion of $h_n(z)^m$. Thus we can use Cauchy's inequality $J_{m,n} \leq h_n(r)^m r^{-n}$ with any $r > 0$. For r close to 1 this is useful, provided we know how $h_n(r)$ varies. Whence the following lemma.

Lemma 2.8. *Suppose $h_n(z) = \sum_{r=0}^n h_r z^r$ with $h_n = O(e^{-\lambda\sqrt{n}})$ for some $\lambda > 0$. Then for $0 < \varepsilon_n \leq \mu/\sqrt{n}$ with $\mu < \lambda$,*

$$h_n(e^{\varepsilon_n}) = \sum_{k=0}^{K-1} \frac{\varepsilon_n^k S_k^{(n)}}{k!} + O(\varepsilon_n^K)$$

for any $K \geq 1$ where $S_k^{(n)} = \sum_{r=0}^n h_r r^k$.

Proof. We have

$$(*) \qquad h_n(e^{\varepsilon_n}) = \sum_{r=0}^n h_r e^{r\varepsilon_n} = \sum_{r=0}^n h_r \sum_{k=0}^\infty \frac{r^k \varepsilon_n^k}{k!} = \sum_{k=0}^\infty \frac{\varepsilon_n^k}{k!} \sum_{r=0}^n h_r r^k = \sum_{k=0}^\infty \frac{\varepsilon_n^k S_k^{(n)}}{k!}$$

and for each k fixed, $S_k^{(n)} \to S_k = \sum_{r=0}^\infty h_r r^k$ as $n \to \infty$. Note that if the radius of convergence of $\sum h_n z^n$ is 1 then $h_n(e^\varepsilon) = \sum_{k=0}^\infty \varepsilon^k S_k^{(n)}/k!$ converges for all n and $\varepsilon > 0$, but this is unbounded as $n \to \infty$ so the series cannot possibly be an asymptotic series. To prove the lemma we must estimate the right hand side in $(*)$. To do this let $T_k^{(n)} = \sum_{r=0}^n r^k e^{-\lambda\sqrt{r}}$. We obtain the following bounds on $T_k^{(n)}$:

(1) For $k \geq \frac{1}{2}\lambda\sqrt{n}$, we have

$$n^k e^{-\lambda\sqrt{n}} \leq T_k^{(n)} \leq n^{k+1} e^{-\lambda\sqrt{n}}$$

(2) In any case,

$$T_k^{(n)} \le T_k = \sum_{r=0}^{\infty} r^k e^{-\lambda\sqrt{r}} = O\Big(k^{3/2}\Big(\frac{2k}{e\lambda}\Big)^{2k}\Big).$$

To prove the above, consider the function $f_k(t) = t^k e^{-\lambda\sqrt{t}}$. This increases for $0 \le t \le 4k^2/\lambda^2$ where it reaches a maximum of $(2k/e\lambda)^{2k}$ after which it decreases. Thus for $k \ge \frac{1}{2}\lambda\sqrt{n}$, the largest term occurs at the end of the series, that is,

$$n^k e^{-\lambda\sqrt{n}} \le T_k^{(n)} \le n.n^k e^{-\lambda\sqrt{n}}.$$

Next, let $N_k = [(2k/\lambda)^2]$ so that

$$\begin{aligned}
T_k &= \sum_{n=1}^{\infty} f_k(n) = \Bigg(\sum_{n=1}^{N_k-1} + \sum_{n=N_k+2}^{\infty}\Bigg) f_k(n) + f_k(N_k) + f_k(N_k+1) \\
&\le \Bigg(\int_0^{(2k/\lambda)^2} + \int_{(2k/\lambda)^2}^{\infty}\Bigg) f_k(t)\,dt + 2\Big(\frac{2k}{e\lambda}\Big)^{2k} \\
&= \int_0^{\infty} t^k e^{-\lambda\sqrt{t}}\,dt + 2\Big(\frac{2k}{e\lambda}\Big)^{2k} \\
&= \frac{(2k+1)!}{\lambda^{2k+2}} + 2\Big(\frac{2k}{e\lambda}\Big)^{2k} = O\Big(k^{3/2}\Big(\frac{2k}{e\lambda}\Big)^{2k}\Big).
\end{aligned}$$

Using these, we obtain

$$\begin{aligned}
\sum_{k=K}^{\infty} \frac{\varepsilon_n^k T_k^{(n)}}{k!} &= \sum_{K\le k\le\lambda\sqrt{n}/2} \frac{\varepsilon_n^k T_k^{(n)}}{k!} + \sum_{k>\lambda\sqrt{n}/2} \frac{\varepsilon_n^k T_k^{(n)}}{k!} \\
&\le A \sum_{K\le k\le\lambda\sqrt{n}/2} \frac{\varepsilon_n^k k^{3/2}}{k!}\Big(\frac{2k}{e\lambda}\Big)^{2k} + \sum_{k>\lambda\sqrt{n}\,/2} \frac{\varepsilon_n^k n^{k+1} e^{-\lambda\sqrt{n}}}{k!} \\
&\le A'\varepsilon_n^K \sum_{K\le k\le\lambda\sqrt{n}/2} k\varepsilon_n^{k-K}\Big(\frac{4k}{e\lambda^2}\Big)^k + \varepsilon_n^K n^{K+1} e^{-\lambda\sqrt{n}} \sum_{k>\lambda\sqrt{n}/2} \frac{(n\varepsilon_n)^{k-K}}{(k-K)!} \\
&\le A''\varepsilon_n^K \sum_{0\le k\le\lambda\sqrt{n}/2} k^{K+1}\Big(\frac{4k\varepsilon_n}{e\lambda^2}\Big)^k + \varepsilon_n^K n^{K+1} e^{-\lambda\sqrt{n}+n\varepsilon_n} \\
&\le A''\varepsilon_n^K \sum_{k=0}^{\infty} k^{K+1}\Big(\frac{2}{e}\Big)^k + \varepsilon_n^K n^{K+1} e^{-(\lambda-\mu)\sqrt{n}} = O(\varepsilon_n^K)
\end{aligned}$$

since $\varepsilon_n \le \mu/\sqrt{n}$, from which the result follows. □

Note that for $k=[\lambda\sqrt{n}\,]$,

$$\frac{\varepsilon_n^k T_k^{(n)}}{k!} \geq \frac{\varepsilon_n^k n^k e^{-\lambda\sqrt{n}}}{k!} \geq A\Big(\frac{n\varepsilon_n}{k}\Big)^k k^{-1/2} \geq A\Big(\frac{\varepsilon_n\sqrt{n}}{\lambda}\Big)^k n^{-1/4},$$

so that $\varepsilon_n<\lambda\sqrt{n}$ is necessary for an asymptotic series.

Now, using the above lemma with $\varepsilon_n=\mu/\sqrt{n}$ $(0<\mu<\lambda)$, we find that

$$h_n\big(e^{\mu/\sqrt{n}}\big) = S_0^{(n)} + S_1^{(n)}\frac{\mu}{\sqrt{n}} + O\Big(\frac{1}{n}\Big) = 1+\frac{\mu}{\sigma\sqrt{n}} + O\Big(\frac{1}{n}\Big) = e^{\mu/\sigma\sqrt{n}+O(1/n)}$$

since $S_k^{(n)}=S_k+O(n^{-A})$ for all A and for any fixed k while $S_0=1$ and $S_1=h'(1)$. Thus

$$J_{m,n} \leq h_n\big(e^{\mu/\sqrt{n}}\big)^m e^{-\mu\sqrt{n}} = e^{((\mu/\sigma\sqrt{n}\,)+O(1/n))m-\mu\sqrt{n}} = e^{-(\mu|m-\sigma n|/\sigma\sqrt{n}\,)+O(1)}$$

whenever $m<\sigma n$. We can now use this bound to obtain

$$\begin{aligned}
\sum_{m<\sigma n-A\sqrt{n}\,\log n} l_m J_{m,n} &= O\Bigg(\sum_{m<\sigma n-A\sqrt{n}\,\log n} l_m e^{-\mu|m-\sigma n|/\sigma\sqrt{n}}\Bigg)\\
&= O\Bigg(l_{[\sigma n]}\sum_{A\sqrt{n}\log n}^{\infty} e^{-\mu k/\sigma\sqrt{n}+o(k/\sqrt{n}\,)}\Bigg)\\
&= O\Bigg(l_{[\sigma n]}\int_{A\sqrt{n}\,\log n}^{\infty} e^{-\mu t/2\sigma\sqrt{n}}\,dt\Bigg)\\
&= O\Bigg(l_{[\sigma n]}\sqrt{n}\int_{A\mu\log n/2\sigma}^{\infty} e^{-y}\,dy\Bigg)\\
&= O\Big(l_{[\sigma n]}n^{(1/2)-(A\mu/2\sigma)}\Big) = o(l_{[\sigma n]})
\end{aligned}$$

if A is sufficiently large, for example $A=2\sigma/\mu$. As stated earlier, Lemma 2.4 is still valid for $m>\sigma n$, so now we only need to consider the range $|m-\sigma n| = O(\sqrt{n}\,\log n)$. Looking at Lemma 2.5, we find that we only used the fact that for θ real and small, $h(e^{i\theta})=e^{(i\theta/\sigma)-\tau\theta^2+O(\theta^3)}$. It is not difficult to show that this follows from $\sum n^3 h_n<\infty$—which is certainly true in our case. Also, the bound $|m-\sigma n|<2\sigma\sqrt{n}\,\log n$ can be replaced by $O(\sqrt{n}\,\log n)$ as only this fact was used in the proof. Hence Lemma 2.5 may be applied and the arguments following Remark 2.6 remain valid until we get to the sum $\sum_{m=0}^{\infty} J_{m,n}$, which is the coefficient of z^n in the expansion of $1/1-h(z)$. But by the Erdős–Feller–Pollard theorem [6], this tends to σ as desired. Thus we have the more general theorem as follows:

Theorem 2.9. *Let $f(z) = l\big(h(z)\big)$ where the Taylor coefficients l_n of $l(z)$ satisfy* (†), *and $h(z) = \sum_0^\infty h_n z^n$ has $0 \le h_n \le Ae^{-\lambda\sqrt{n}}$ (some $\lambda > 0$) and is not of the form $k(z^r)$ for any $r > 1$ with $h(1) = 1$. Then the coefficients f_n of $f(z)$ satisfy*

$$f_n \sim \sigma l_{[\sigma n]} \qquad \text{as } n \to \infty \text{ where } \sigma = \frac{1}{h'(1)}.$$

Remark 2.10. If we have $h_n \ne O\big(e^{-\lambda\sqrt{n}}\big)$, and we want to obtain some asymptotic result for f_n, then it seems we need to cut down the growth rate of l_n. We note that the theorem can be extended to the case where $l_n = s(e^{\psi(n)})$ with $\log n \prec \psi(n) \le \sqrt{n}$ given that $h_n = O(e^{-\lambda\psi(n)})$ for some $\lambda > 0$. We will discuss the details in a separate paper. The case where l_n is regularly-varying has been considered by Embrechts *et al.* ([4], [5]) and Grubel ([8], [9]), particularly for its connection with (generalised) renewal theory.

Our method in deriving the Tauberian theorem can also be applied to study generalized renewal theory.

Given a continuous probability distribution η: $(-\infty,\infty) \to [0,1]$ with $\eta(0) = 0$ and a sequence $(l_n)_{n\ge1}$ of positive numbers, a generalized renewal process is defined to be

$$\phi(t) = \sum_{n=0}^{\infty} l_n \eta^{*n}(t) \tag{2}$$

where $\eta^{*n}(t)$ is the n-fold convolution of $\eta(t)$. The special case where $l_n = 1$ for all n is the renewal process treated in [6]. In [5] and [10], Embrechts, Maejima and Omey obtained the asymptotic behaviour of $\phi(t+k) - \phi(t)$ as $t \to \infty$ for fixed $k > 0$, given that l_n varies regularly. We obtain a more general result under the assumption that $l_n = s(e^{\sqrt{n}})$ where s is slowly-varying, given certain convergence conditions on $\eta(t)$. We note that (2) has the form similar to (1) where we had $f_n = \sum_{m=0}^\infty l_m h_n^{*m}$. To make this connection more explicit, we let $h(z) = \int_0^\infty e^{-tz}\,d\eta(t)$ and $l(z) = \sum_{n=0}^\infty l_n z^n$, from which it follows formally that

$$f(z) \overset{\text{def}}{=} \int_0^\infty e^{-tz}\,d\phi(t) = \sum_{n=0}^{\infty} l_n \int_0^\infty e^{-tz}\,d\eta^{*n}(t) = \sum_{n=0}^{\infty} l_n h(z)^n = l\big(h(z)\big).$$

Theorem 2.11. *We have*

$$\frac{\phi(t+k)-\phi(t)}{k} \sim \sigma l_{[\sigma t]} \qquad \text{as} \quad t \to \infty$$

where $1/\sigma = -h'(0)$ under the following conditions:

(i) *$h(z)$ is analytic in $\mathrm{Re}(z) > -\delta$ for some $\delta > 0$,*
(ii) *$\int_A^\infty (|h(c \pm ix)|/x)\,dx$ converges for every $c > -\delta$ (where $A > 0$), and*
(iii) *$l_n = s\big(e^{\sqrt{n}}\big)$ for some slowly-varying function s.*

Condition (iii) allows us to make l_n significantly larger than those considered in [5] and [10], where l_n is regularly-varying. By the Riemann–Lebesque theorem, we have $|h(c+ix)| \to 0$ as $x \to \pm\infty$ through real values for any given $c > -\delta$, so that condition (ii) is not very restrictive.

Proof (main steps). First, by the Laplace inversion formula, we have

$$\eta^{*n}(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{tz}-1}{z} h(z)^n \, dz$$

for any $c > -\delta$. (Here $\int_{c-i\infty}^{c+i\infty} = \lim_{T\to\infty}\int_{c-iT}^{c+iT}$.) Again, we want to estimate $I_n(t) = \eta^{*n}(t+k) - \eta^{*n}(t)$ as $t \to \infty$ for various n. As in Lemma 2.4 we find that

$$I_n(t) = \begin{cases} O(e^{-\eta t}), O(e^{-\eta' n}) & \text{for } n \le \frac{1}{3}\sigma t \text{ and } n \ge 3\sigma t \text{ respectively,} \\ O\big(e^{-|n-\sigma t|/\sigma\sqrt{t}}\big) & \text{for } |n-\sigma t| \ge 2\sigma\sqrt{t}\,\log t \text{ and } \frac{1}{3}\sigma t < n < 3\sigma t \end{cases}$$

while for $|n-\sigma t| < 2\sigma\sqrt{t}\,\log t$, we have, as in Lemma 2.5,

$$I_n(t) = \frac{k}{2\sqrt{\pi\sigma\tau t}} \exp\Big\{-\frac{(n-\sigma t)^2}{4\sigma^3\tau t}\Big\} + O\Big(\frac{(\log t)^2}{t}\Big)$$

where $\tau = \frac{1}{2}\big(h''(0) - h'(0)^2\big) > 0$ by Cauchy–Schwarz.

For example, for $n \le \frac{1}{3}\sigma t$, choose $(-\delta <)\ c < 0$ such that $h(c) \le e^{-2c/\sigma}$, which is possible since $h(z) = e^{-(z/\sigma)+\tau z^2+O(z^3)}$ as $z \to 0$. Hence

$$\begin{aligned} |I_n(t)| &= \left|\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{t(c+ix)}\big(e^{k(c+ix)}-1\big)}{c+ix} h(c+ix)^n \, dx\right| \\ &\le \frac{1}{\pi} e^{tc} h(c)^{n-1} \int_{-\infty}^{\infty} \frac{|h(c+ix)|}{\sqrt{c^2+x^2}} \, dx \quad \text{since } |h(c+ix)| \le h(c) \\ &= O(e^{tc-2cn/\sigma}) = O(e^{ct/3}) \quad \text{by condition (ii)} \end{aligned}$$

as desired (note that $c < 0$).

For the ranges $\frac{1}{3}\sigma t < n \le \sigma t - 2\sigma\sqrt{t}\,\log t$ and $\sigma t + 2\sigma\sqrt{t}\,\log t \le n < 3\sigma t$, take $c = -1/\sqrt{t}$ and $1/\sqrt{t}$ respectively. We deform the integrals into

$$\int_{c-i\infty}^{c+i\infty} = \int_{c-i\pi}^{c+i\pi} + \int_{c+i\pi}^{i\pi} + \int_{i\pi}^{i\infty} + \int_{-i\pi}^{c-i\pi} + \int_{-i\infty}^{-i\pi}.$$

Call these last four integrals $\int_{\gamma_i}$ $(i = 1$ to $4)$ respectively. For $\int_{c-i\pi}^{c+i\pi}$ we proceed as in the proof of the main theorem, while on γ_1 we use the following estimate:

$$\left|\int_{\gamma_1}\right| \le \frac{e^{tc}(e^{kc}+1)|c|}{2\pi^2} \sup_{0\le x\le c} |h(x+i\pi)|^n$$

where $c = 1/\sqrt{t}$ (and similarly for $c = -1/\sqrt{t}$). But $|h(i\pi)| < h(0) = 1$, (since $|h(z)| \le h(\operatorname{Re} z)$ with equality occuring if and only if z is real) so that sup < 1 for all $|x|$ sufficiently small, i.e., for all t sufficiently large. Hence $\bigl|\int_{\gamma_1}\bigr| = O(A^{-n})$ for some $A > 1$. For $\int_{\gamma_2}$ we use a similar trick and the fact that $|h(ix)| \to 0$ as $x \to \infty$ to obtain $\bigl|\int_{\gamma_2}\bigr| = O(A^{-n})$ for some $A > 1$.

Applying the same arguments following Remark 2.6 to $\sum_{n=0}^{\infty} l_n I_n(t)$, we obtain

$$\sum_{n=0}^{\infty} l_n I_n(t) \sim l_{[\sigma t]} \sum_{n=0}^{\infty} I_n(t) \sim k\sigma l_{[\sigma t]},$$

which gives

$$\frac{\phi(t+k) - \phi(t)}{k} \sim \sigma l_{[\sigma t]} \qquad \text{as } t \to \infty. \ \square$$

For the special cases $l_n = 1$ and $l_n = 1/n$ of renewal theory and harmonic renewal theory respectively, we thus obtain

$$\phi(t+k) - \phi(t) \to k\sigma \qquad \text{as } t \to \infty$$

and

$$\phi(t+k) - \phi(t) \sim \frac{k}{t} \qquad \text{as } t \to \infty,$$

which agree with the known results.

Acknowledgements. I would like to thank Dr. C.-H. Chu for his invaluable comments and suggestions. I would also like to thank Goldsmiths' College for their financial support.

References

[1] Bender, E.A.: Asymptotic methods in enumeration. - SIAM Rev. 16, 1974, 485–515.

[2] Bingham, N.H., C.M. Goldie, and J.L. Teugels: Regular Variation. - Cambridge University Press, 1987.

[3] de Bruijn, N.G.: Asymptotic Methods in Analysis. - North-Holland Publishing Co., 1970 (3rd edition).

[4] Embrechts, P., and E. Omey: Functions of power series. - Yokohama Math. J. 32:2, 1984, 77–88.

[5] Embrechts, P., M. Maejima, and E. Omey: A renewal theorem of Blackwell type. - Ann. Probab. 12, 1984, 561–570.

[6] Feller, W.: An Introduction to Probability Theory and its Applications, Volumes I, II. - John Wiley, New York, 1968 (3rd edition).

[7] Flajolet, P., and A.M. Odlyzko: Singularity analysis of generating functions. - SIAM J. Discrete Math. 3, 1990, 216–240.

[8] Grübel, R.: Functions of discrete probability measures. - Z. Wahrsch. Verw. Gebiete 64, 1983, 341–357.

[9] Grübel, R.: On subordinated distributions and generalised renewal measures. - Ann. Probab. 15:1, 1987, 394–415.

[10] Maejima, M., and E. Omey: A generalised Blackwell renewal theorem. - Yokohama Math. J. 32, 1984, 123–133.

[11] Meir, A., and J.W. Moon: On an asymptotic method in enumeration. - J. Combin. Theory Ser. A 51, 1989, 77–89.

[12] Odlyzko, A.M.: Asymptotic enumeration methods. - Handbook of Combinatorics, preprint, 1992.

Receoived 22 November 1994