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HARMONIC MEASURE OF SOME CANTOR TYPE SETS

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Abstract. We show that for some compact sets $\mathbf{K} \subset \mathbf{R}^2$ of Cantor type the harmonic measure is supported by a set whose Hausdorff dimension is strictly smaller than the dimension of $\mathbf{K}$.

1. Introduction

In [MV] Makarov and Volberg show that the Hausdorff dimension of the harmonic measure of the complement of a particular kind of Cantor set is strictly smaller than the dimension of the set, i.e. that there exists a subset of the Cantor set of full harmonic measure but of strictly smaller dimension. Furthermore, in [V1], [V2] Volberg has extended this result to cover a large class of Cantor sets on the real axis. In [C], Carleson has shown that techniques and tools of the ergodic theory could be used to study the harmonic measure of "classical" Cantor sets and [MV], [V1], [V2], [Z] contain some of the results that strongly rely on this idea. Carleson proved that the dimension of the harmonic measure of these Cantor sets in the plane is always strictly smaller than 1. A similar but more general result is proved in [JW], under the assumption of the "capacity density condition".

The purpose of this work is to prove the inequality between the dimension of the harmonic measure and the dimension of the set for a larger class of Cantor sets in the plane (or the space) without using ergodic theory. We have been motivated by an idea of Bourgain which appeared in [B], and we are also making use of some lemmas and results of the works mentioned above. Further information regarding the Hausdorff dimension and measure of Cantor type sets is provided for example in [Be], [F].

This paper is organized in five sections: first we present the main theorem and introduce some notation. In the second section we prove a number of lemmas and in the third we prove the theorem. Two examples are given in the fourth section, and in the final section we make some remarks and investigate the possibilities of the method.

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Let $\{a_j\}$ be a sequence of real numbers such that there exist two constants $\underline{A}$, $\overline{A}$, $0<\underline{A}\le\overline{A}<\frac{1}{2}$ with $\overline{A}\ge a_i\ge\underline{A}$, for all $i\in\mathbf{N}$. We construct a Cantor set $\mathbf{K}$ in the following way: we replace the square $[0,1]^2$ with four equal squares of side-length a_1 situated in the four corners, and each one of them with four new ones of side-length a_1a_2 and so on; see Figure 1. We denote $\tilde{I}^n_{i_1\dots i_n}$, where $i_j\in\{1,2,3,4\}$ for $1\le j\le n$, the 4^n squares of the nth generation constructed in this way with the enumeration shown in the figure and the usual condition that $\tilde{I}^n_{i_1\dots i_n}$ is the "father" of the sets $\tilde{I}^{n+1}_{i_1\dots i_n i_{n+1}}$, $i_{n+1}\in\{1,2,3,4\}$. It is clear that $\overline{A}\ge \operatorname{diam}\tilde{I}^{n+1}_{i_1\dots i_n i}/\operatorname{diam}\tilde{I}^n_{i_1\dots i_n}=a_{n+1}\ge\underline{A}$, $i=1,\dots,4$. We will denote by $I^n_{i_1\dots i_n}$ the intersection of $\tilde{I}^n_{i_1\dots i_n}$ with $\mathbf{K}$. We will say that a set $F\subset\mathbf{R}^2\setminus\mathbf{K}$ is of full harmonic measure for the domain $\mathbf{R}^2\setminus\mathbf{K}$ if $\omega(F)=1$, where ω is the harmonic measure of $\mathbf{R}^2\setminus\mathbf{K}$ (see Notation 1.1).

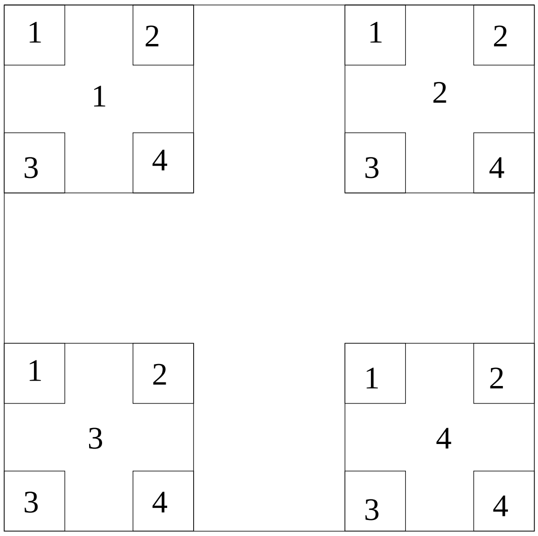


Figure 1.

Theorem 1.0. *For a Cantor set* $\mathbf{K}$ *as above there exists a subset* F *of* $\mathbf{K}$ *of full harmonic measure such that* $\dim(F)<\dim(\mathbf{K})$.

Notation 1.1. If there is a constant c independent of the parameters α,β such that $\alpha/c\le\beta\le c\alpha$, we will write $\alpha\sim\beta$ (in what follows the symbols c and C will be used to denote the constants). Suppose that Ω is a domain in $\mathbf{R}^2$ and that $F\subset\partial\Omega$. For $x\in\Omega$ we denote by $\omega(x,F,\Omega)$ the harmonic measure of F in Ω evaluated at x. We denote by $\omega(F,\Omega)$ the harmonic measure of F in Ω evaluated at infinity. If $\Omega=\mathbf{R}^2\setminus\mathbf{K}$, we will write $\omega(F)$ instead of $\omega(F,\Omega)$. For a square F we denote its side-length by $l(F)$ and, finally, h_ρ is the ρ-dimensional Hausdorff measure.

2. Preparatory lemmas

The proof of the theorem will be based on a number of lemmas some of which are already well known.

We first remark that there exists a constant $c_0 > 1$ depending only on $\underline{A}$, $\overline{A}$ such that, if $c_0 \tilde{I}^n_{i_1 \dots i_n}$ denotes the square of side-length c_0-times the side-length of $\tilde{I}^n_{i_1 \dots i_n}$ with the same center, then $c_0 \tilde{I}^n_{i_1 \dots i_n} \cap \mathbf{K} = I^n_{i_1 \dots i_n}$. For this c_0 (not depending on $i_1 \dots i_n$) we have the following classical lemma:

Lemma 2.1. *There exists a $\delta > 0$ depending neither on n nor on the choice of $i_1 \dots i_n$ such that for all $x \in \frac{1}{2}(1 + c_0)\tilde{I}^n_{i_1 \dots i_n}$*

$$\omega(x, I^n_{i_1 \dots i_n}, \mathbf{R}^2 \setminus \mathbf{K}) > \delta. \tag{1}$$

Remark 2.2. If δ_n is the side-length of the square $I^n_{i_1 \dots i_n}$, the Green's function G of the square $c_0 \tilde{I}^n_{i_1 \dots i_n}$ satisfies

$$\frac{1}{2\pi} \log \frac{C^{-1}\delta_n}{|x-y|} \le G(x,y) \le \frac{1}{2\pi} \log \frac{C\delta_n}{|x-y|}$$

for $x, y \in \frac{1}{2}(1+c_0)\tilde{I}^n_{i_1 \dots i_n}$, where the constant C depends only on c_0. Furthermore, we have

$$\omega(x, I^n_{i_1 \dots i_n}, c_0 \tilde{I}^n_{i_1 \dots i_n} \setminus \mathbf{K}) = {}^{c_0 \tilde{I}^n_{i_1 \dots i_n}}\mathbf{R}_1^{I^n_{i_1 \dots i_n}}(x) = G\mu(x)$$

where ${}^{\Omega}\mathbf{R}_1^F$ is the capacitary potential of the set F in the domain Ω and μ is the capacitary measure of $I^n_{i_1 \dots i_n}$ in $c_0 \tilde{I}^n_{i_1 \dots i_n}$, $\|\mu\| = \operatorname{cap}_{c_0 \tilde{I}^n_{i_1 \dots i_n}}(I^n_{i_1 \dots i_n})$.

Proof of Lemma 2.1. We first show that there exists a constant $c_1 > 0$ such that for $x \in \frac{1}{2}(1 + c_0)\tilde{I}^n_{i_1 \dots i_n}$

$$\omega(x, I^n_{i_1 \dots i_n}, c_0 \tilde{I}^n_{i_1 \dots i_n} \setminus \mathbf{K}) > c_1. \tag{2}$$

If μ is the probability measure on $\mathbf{K}$ charging every square of the nth generation with mass 4^{-n}, let $\mu_n = 4^n \mu_{|I^n_{i_1 \dots i_n}}$ be the restriction of the renormalized measure μ on the square $I^n_{i_1 \dots i_n}$.

Let us calculate the potential of μ_n for $y \in I^n_{i_1 \dots i_n}$:

$$\begin{aligned} G\mu_n(y) &\le 4^n \sum_{\kappa > n} 3 \cdot 4^{-\kappa} \log\left(\frac{C \prod_{i=1}^n a_i}{\prod_{i=1}^\kappa a_i} \right) = 3 \cdot 4^n \sum_{\kappa > n} 4^{-\kappa} \log\left(C \prod_{i=n+1}^{\kappa} a_i^{-1} \right) \\ &= 3 \sum_{\kappa=1}^{\infty} 4^{-\kappa} \log\left(C \prod_{i=1}^{\kappa} a_{i+n}^{-1} \right) \le \widetilde{C}(\underline{A}, \overline{A}) < \infty. \end{aligned}$$

The same reasoning provides a constant $c_2 > 0$ such that

$$\frac{1}{c_2} \le G\mu_n(y) \le c_2 \qquad \text{for all } y \in I^n_{i_1\ldots i_n}.$$

By the maximum principle we get

$$\omega(x, I^n_{i_1\ldots i_n}, c_0\tilde I^n_{i_1\ldots i_n} \setminus \mathbf{K}) \sim G\mu_n(x) \qquad \text{for} \qquad x \in c_0\tilde I^n_{i_1\ldots i_n}. \tag{3}$$

We can easily see that

$$G\mu_n(x) \ge c_3 \qquad \text{for} \qquad x \in \partial\big\{\tfrac12(1+c_0)\tilde I^n_{i_1\ldots i_n}\big\}.$$

On the other hand, the harmonic measure is non-decreasing as a function of the domain; hence

$$\omega(x, I^n_{i_1\ldots i_n}, c_0\tilde I^n_{i_1\ldots i_n} \setminus \mathbf{K}) \le \omega(x, I^n_{i_1\ldots i_n}, \mathbf{R}^2 \setminus \mathbf{K})$$

and the lemma is proved.

Lemma 2.3. *There exists a* $\delta > 0$ *not depending on* n *such that for the squares of the* n*th generation* $I^n_{1\ldots 11}$ *and* $I^n_{1\ldots 14}$ *we have*

$$\omega(I^n_{1\ldots 11}) > (1+\delta)\omega(I^n_{1\ldots 14}). \tag{4}$$

Remark 2.4. This lemma has been proved in [MV] in the case of standard planar Cantor sets. The proof given below is similar.

We will make repeated use of the following well-known formula (see for instance [Br]): If $\Omega \subset \widetilde\Omega$ are two domains and if $F \subset \partial\Omega \cap \partial\widetilde\Omega$, the harmonic measures of the domains, ω and $\tilde\omega$, are associated in the following way:

$$\omega(x, F) = \tilde\omega(x, F) - \int_{\partial\Omega\cap\widetilde\Omega} \tilde\omega(y)\, \omega(x, dy).$$

Proof of Lemma 2.3. To begin with, let us point out that the symmetry of the set implies

$$\omega(I^n_{1\ldots 11}, \mathbf{R}^2 \setminus I^{n-1}_{1\ldots 1}) = \omega(I^n_{1\ldots 14}, \mathbf{R}^2 \setminus I^{n-1}_{1\ldots 1}). \tag{5}$$

For the same reason, if x lies on the $I^{n-1}_{1\ldots 1}$ square's diagonal separating $I^n_{1\ldots 11}$ and $I^n_{1\ldots 14}$, we have

$$\omega(x, I^n_{1\ldots 11}, \mathbf{R}^2 \setminus I^{n-1}_{1\ldots 1}) = \omega(x, I^n_{1\ldots 14}, \mathbf{R}^2 \setminus I^{n-1}_{1\ldots 1}). \tag{6}$$

Let $\mathbf{H}^-$ be the half-plane limited by the line containing this diagonal, such that the square $I^n_{1\dots14}$ is contained in $\mathbf{H}^-$. Using (2), the monotony of the harmonic measure, and Harnack's inequalities, one can verify the existence of a constant $c_4 > 0$ such that

$$\omega(x, I^n_{1\dots14}, \mathbf{H}^- \setminus I^{n-1}_{1\dots1}) \ge c_4 \qquad \text{for all } x \in I^{n-1}_{1\dots4}. \tag{7}$$

By the maximum principle and (6) we obtain

$$\omega(x, I^n_{1\dots14}, \mathbf{H}^- \setminus I^{n-1}_{1\dots1}) = \omega(x, I^n_{1\dots14}, \mathbf{R}^2 \setminus I^{n-1}_{1\dots1}) - \omega(x, I^n_{1\dots11}, \mathbf{R}^2 \setminus I^{n-1}_{1\dots1})$$

for all $x \in \mathbf{H}^-$. Combining with (7),

$$\omega(x, I^n_{1\dots14}, \mathbf{R}^2 \setminus I^{n-1}_{1\dots1}) - \omega(x, I^n_{1\dots11}, \mathbf{R}^2 \setminus I^{n-1}_{1\dots1}) \ge c_4 \qquad \text{for all } x \in I^{n-1}_{1\dots4}, \tag{8}$$

and (5), (8) imply that

$$\begin{aligned} \omega(I^n_{1\dots11}, \mathbf{R}^2 \setminus \mathbf{K}) - \omega(I^n_{1\dots14}, \mathbf{R}^2 \setminus \mathbf{K}) &= \\ &= \int_{\mathbf{K}\setminus I^{n-1}_{1\dots1}} \big(\omega(y, I^n_{1\dots14}, \mathbf{R}^2 \setminus I^{n-1}_{1\dots1}) \\ &\quad - \omega(y, I^n_{1\dots11}, \mathbf{R}^2 \setminus I^{n-1}_{1\dots1})\big)\omega(dy, \mathbf{R}^2 \setminus \mathbf{K}) \\ &\ge c_4 \omega(I^{n-1}_{1\dots4}, \mathbf{R}^2 \setminus \mathbf{K}). \end{aligned} \tag{9}$$

Finally, using Harnack's principle and (7) we obtain a constant $c_5 > 0$ not depending on n, verifying

$$\omega(I^{n-1}_{1\dots4}) \ge c_5 \omega(I^n_{1\dots14}).$$

Hence, (9) turns into

$$\omega(I^n_{1\dots11}) - \omega(I^n_{1\dots14}) \ge c_4 c_5 \omega(I^n_{1\dots14})$$

and the lemma is proved.

Lemma 2.5 ([C], [MV]). *Let Ω be a domain containing ∞ and let $A_1 \subset B_1 \subset A_2 \subset B_2 \subset \cdots \subset A_n \subset B_n \subset \Omega$ be conformal discs such that the annuli $B_i \setminus A_i$ are contained in Ω, for $1 \le i \le n$. If the modules of the annuli are uniformly bounded away from zero and if $\infty \in \Omega \setminus B_n$, then, for all pairs of positive harmonic functions u, v vanishing on $\partial\Omega \setminus A_1$ and for all $x \in \Omega \setminus B_n$, we have*

$$\left| \frac{u(x)}{v(x)} : \frac{u(\infty)}{v(\infty)} - 1 \right| \le K q^n, \tag{10}$$

where $q < 1$ and K are two constants that depend only on the lower bound of the modules of the annuli.

Lemma 2.6. *There exists an* $N_0 = N_0(\delta, \underline{A}, \overline{A})$ *large enough such that for all* $n \in \mathbf{N}$ *and all squares* $I^n_{i_1\dots i_n}$

$$\omega(I^{n+N_0}_{i_1\dots i_n 11\dots 1}) > (1 + \tfrac{1}{2}\delta)\omega(I^{n+N_0}_{i_1\dots i_n 11\dots 4}), \tag{11}$$

where δ *is the positive constant defined in Lemma* 2.3*.*

Proof of Lemma 2.6. Let us first show the following estimate for $x \in \frac{1}{2}(1 + c_0)\tilde{I}^n_{i_1\dots i_n}$:

$$\omega(x, I^{n+N_0}_{i_1\dots i_n 11\dots 1}, c_0\tilde{I}^n_{i_1\dots i_n} \setminus I^n_{i_1\dots i_n}) \sim \omega(x, I^{n+N_0}_{i_1\dots i_n 11\dots 1}, \mathbf{R}^2 \setminus \mathbf{K}). \tag{12a}$$

For $N \in \mathbf{N}$ we choose x such that

$$\omega(x, I^{n+N}_{i_1\dots i_n 11\dots 1}, \mathbf{R}^2 \setminus \mathbf{K}) = \sup\bigl\{\omega(y, I^{n+N}_{i_1\dots i_n 11\dots 1}, \mathbf{R}^2 \setminus \mathbf{K}) : y \in \partial\bigl\{\tfrac{1}{2}(1+c_0)\tilde{I}^n_{i_1\dots i_n}\bigr\}\bigr\}.$$

Then,

$$\begin{aligned}
\omega(x, I^{n+N}_{i_1\dots i_n 11\dots 1}, \mathbf{R}^2 \setminus \mathbf{K}) &\geq \omega(x, I^{n+N}_{i_1\dots i_n 11\dots 1}, c_0\tilde{I}^n_{i_1\dots i_n} \setminus I^n_{i_1\dots i_n}) \\
&\geq \omega(x, I^{n+N}_{i_1\dots i_n 11\dots 1}, \mathbf{R}^2 \setminus \mathbf{K}) \\
&\quad - \int_{\partial\{c_0\tilde{I}^n_{i_1\dots i_n}\}} \omega(y, I^{n+N}_{i_1\dots i_n 11\dots 1}, \mathbf{R}^2 \setminus \mathbf{K})\omega(x, dy, c_0\tilde{I}^n_{i_1\dots i_n} \setminus I^n_{i_1\dots i_n}) \\
&\geq \omega(x, I^{n+N}_{i_1\dots i_n 11\dots 1}, \mathbf{R}^2 \setminus \mathbf{K}) \\
&\quad - \bigl(1 - \omega(x, I^n_{i_1\dots i_n}, c_0\tilde{I}^n_{i_1\dots i_n} \setminus I^n_{i_1\dots i_n})\bigr)\omega(x, I^{n+N}_{i_1\dots i_n 11\dots 1}, \mathbf{R}^2 \setminus \mathbf{K}) \\
&\geq c_1\omega(x, I^{n+N}_{i_1\dots i_n 11\dots 1}, \mathbf{R}^2 \setminus \mathbf{K})
\end{aligned}$$

because of (2).

Then (12a) follows on our using again Harnack's inequalities.

We have, of course, the same estimate for $I^{n+N_0}_{i_1\dots i_n 11\dots 4}$:

$$\omega(x, I^{n+N_0}_{i_1\dots i_n 11\dots 4}, c_0\tilde{I}^n_{i_1\dots i_n} \setminus I^n_{i_1\dots i_n}) \sim \omega(x, I^{n+N_0}_{i_1\dots i_n 11\dots 4}, \mathbf{R}^2 \setminus \mathbf{K}). \tag{12b}$$

To simplify the notation in what follows we will write $\omega_1(x)\bigl(\tilde{\omega}_1(x)\bigr)$ and $\omega_4(x)\bigl(\tilde{\omega}_4(x)\bigr)$ instead of

$$\omega(x, I^{n+N_0}_{i_1\dots i_n 11\dots 1}, \mathbf{R}^2 \setminus \mathbf{K})\bigl(\omega(x, I^{n+N_0}_{i_1\dots i_n 11\dots 1}, c_0\tilde{I}^n_{i_1\dots i_n} \setminus I^n_{i_1\dots i_n})\bigr)$$

and

$$\omega(x, I^{n+N_0}_{i_1\dots i_n 11\dots 4}, \mathbf{R}^2 \setminus \mathbf{K})\bigl(\omega(x, I^{n+N_0}_{i_1\dots i_n 11\dots 4}, c_0\tilde{I}^n_{i_1\dots i_n} \setminus I^n_{i_1\dots i_n})\bigr),$$

respectively.

By the relation (10) in Lemma 2.5, for $z \in \partial\{\frac{1}{2}(1+c_0)\tilde{I}^n_{i_1\ldots i_n}\}$

$$\frac{\omega_1(\infty)}{\omega_4(\infty)} \sim_{q^{N_0}} \frac{\omega_1(z)}{\omega_4(z)} \sim_{q^{N_0}} \frac{\omega_1(y)}{\omega_4(y)} \qquad \text{for all } y \notin \mathbf{K} \cup c_0\tilde{I}^n_{i_1\ldots i_n}. \tag{13}$$

From (13) it follows that

$$\begin{aligned}
&\Big|\frac{\tilde\omega_1(z)}{\tilde\omega_4(z)} : \frac{\omega_1(\infty)}{\omega_4(\infty)} - 1\Big| \sim \Big|\frac{\tilde\omega_1(z)}{\tilde\omega_4(z)} : \frac{\omega_1(z)}{\omega_4(z)} - 1\Big| = \frac{\omega_4(z)}{\tilde\omega_4(z)}\Big|\frac{\tilde\omega_1(z)}{\omega_1(z)} - \frac{\tilde\omega_4(z)}{\omega_4(z)}\Big| \\
&= \frac{\omega_4(z)}{\tilde\omega_4(z)} \int_{\partial\{c_0\tilde{I}^n_{i_1\ldots i_n}\}} \Big|\frac{\omega_4(y)}{\omega_4(z)} - \frac{\omega_1(y)}{\omega_1(z)}\Big|\omega(z, dy, c_0\tilde{I}^n_{i_1\ldots i_n} \setminus I^n_{i_1\ldots i_n}) \\
&\le \frac{1}{c_1} \int_{\partial\{c_0\tilde{I}^n_{i_1\ldots i_n}\}} \frac{\omega_4(y)}{\omega_4(z)}\Big|\frac{\omega_1(y)}{\omega_1(z)} : \frac{\omega_4(y)}{\omega_4(z)} - 1\Big|\omega(z, dy, c_0\tilde{I}^n_{i_1\ldots i_n} \setminus I^n_{i_1\ldots i_n}) \\
&\le Cq^{N_0}.
\end{aligned} \tag{14}$$

If we take $i_1 = \cdots = i_n = 1$, Lemma 2.3 implies $\omega_1(\infty)/\omega_4(\infty) > 1+\delta$. Then (14) shows that there exists an N_0 large enough such that $\tilde\omega_1(z)/\tilde\omega_4(z) > 1 + \frac{3}{4}\delta$. On the other hand, $\tilde\omega_1(z)/\tilde\omega_4(z)$ does not depend on the choice of $i_1, \ldots, i_n$. It follows that $\omega_1(\infty)/\omega_4(\infty) > 1 + \frac{1}{2}\delta$ for all the possible choices of $i_1, \ldots, i_n$.

Lemma 2.7. *There exists an $N_1 \in \mathbf{N}$ independent of n and of $i_1 \ldots i_n$ such that for all the squares $I^n_{i_1\ldots i_n}$ there is a square $J_m = I^{n+N_1}_{i_1\ldots i_n \ldots i_{n+N_1}} \subset I^n_{i_1\ldots i_n}$ of the $(n+N_1)$ th generation such that*

$$\omega(J_m) < \frac{1}{4}\frac{\omega(I^n_{i_1\ldots i_n})}{4^{N_1}}.$$

(In fact, $\frac{1}{4}$ could be replaced with any constant $\varepsilon > 0$).

Proof of Lemma 2.7. Choose a square $I^n_{i_1\ldots i_n}$. According to the preceding lemma, there exists an $\alpha < 1$ independent of the choice of $i_1, \ldots, i_n$ and a $J_1 = I^{n+N_0}_{i_1\ldots i_n \ldots i_{n+N_0}}$ such that

$$\omega(J_1) < \alpha\frac{\omega(I^n_{i_1\ldots i_n})}{4^{N_0}}.$$

Similarly there exists a $J_2 = I^{n+2N_0}_{i_1\ldots i_n \ldots i_{n+2N_0}} \subset J_1$ with

$$\omega(J_2) < \alpha\frac{\omega(J_1)}{4^{N_0}} < \alpha^2\frac{\omega(I^n_{i_1\ldots i_n})}{4^{2N_0}},$$

and after k steps we obtain a square J_k verifying

$$\omega(J_k) < \alpha^k\frac{\omega(I^n_{i_1\ldots i_n})}{4^{kN_0}}.$$

To finish the proof, take $k = m$ such that $\alpha^k < \frac{1}{4}$ and let $N_1 = kN_0$.

3. Proof of Theorem 1.0

The theory of martingales provides a well-known technique for proving the inequality between the dimensions of the two measures by using Lemma 2.7. However, we propose here a different path which does not involve probabilistic tools and is inspired by [B].

We introduce some more notation. For $n \in \mathbf{N}$ we will denote by $\mathscr{E}_n$ the collection of squares $\{I^n_{i_1\ldots i_n} : i_j = 1,\ldots,4,\ j = 1,\ldots,n\}$, and for $I \in \mathscr{E}_n$, $\mathscr{E}_{n+s}(I)$ will represent those squares $J \in \mathscr{E}_{n+s}$ that are contained in I.

It can be shown (see for instance Lemma 2 of [Be]) that if ρ is the Hausdorff dimension of $\mathbf{K}$, then

$$\rho = \sup\Big\{ s > 0 : \liminf_{n\to\infty} 4^n \prod_{i=1}^{n} a_i^s = \infty \Big\} = \inf\Big\{ s > 0 : \liminf_{n\to\infty} 4^n \prod_{i=1}^{n} a_i^s = 0 \Big\}$$

simply because in order to obtain the Hausdorff dimension of the Cantor set $\mathbf{K}$ it suffices to consider coverings of $\mathbf{K}$ with the squares of construction $\tilde{I}^n_{i_1\ldots i_n}$. However, the ρ-Hausdorff measure of the Cantor sets considered here could be infinite.

It easily follows that for $\varepsilon > 0$ there exists a strictly increasing sequence of integers $\{n_j\}_{j=1}^{\infty}$ such that

$$4^{n_j} \prod_{i=1}^{n_j} a_i^{\rho+\varepsilon} > 4^{n_{j+1}} \prod_{i=1}^{n_{j+1}} a_i^{\rho+\varepsilon}. \tag{15}$$

We will also assume that $n_{j+1} - n_j > 2N_1$.

Lemma 2.8. *There exists a $\beta < 1$ such that the following inequality holds for $\varepsilon > 0$ and $I \in \mathscr{E}_{n_j}$:*

$$\sum_{J\in\mathscr{E}_{n_{j+1}}(I)} \omega(J)^{1/2} l(J)^{(\rho+\varepsilon)/2} \le \beta^{n_{j+1}-n_j} \omega(I)^{1/2} l(I)^{(\rho+\varepsilon)/2}; \tag{16}$$

where n_j is the sequence corresponding to ε given by (15).

Proof of Lemma 2.8. Let us start by showing that there is a $\tilde{\beta}$ such that for $I \in \mathscr{E}_n$

$$\sum_{J\in\mathscr{E}_{n+N_1}(I)} \omega(J)^{1/2} \big(\tfrac14\big)^{(n+N_1)/2} \le \tilde{\beta}\omega(I)^{1/2} \big(\tfrac14\big)^{n/2}. \tag{17}$$

Take $J_m \in \mathscr{E}_{n+N_1}(I)$ to be the square provided by Lemma 2.7, i.e. a square such that $\omega(J_m) < \frac14 \omega(I) \cdot 4^{N_1}$. We have

$$\omega(J_m)^{1/2} \Big(\frac14\Big)^{(n+N_1)/2} \le \frac12 \frac{1}{4^{N_1}} \Big(\frac14\Big)^{n/2} \omega(I)^{1/2}$$

$$\sum_{J\in\mathscr{E}_{n+N_1}(I),\, J\neq J_m} \omega(J)^{1/2}\Big(\frac{1}{4}\Big)^{(n+N_1)/2} \leq \omega(I)^{1/2}(4^{N_1}-1)^{1/2}\Big(\frac{1}{4}\Big)^{(n+N_1)/2}$$

by the Cauchy–Schwarz inequalities. Summing up we get

$$\sum_{J\in\mathscr{E}_{n+N_1}(I)} \omega(J)^{1/2}\Big(\frac{1}{4}\Big)^{(n+N_1)/2} \leq \omega(I)^{1/2}\Big(\frac{1}{4}\Big)^{n/2}\Big(\frac{1}{2}\frac{1}{4^{N_1}}+\Big(\frac{4^{N_1}-1}{4^{N_1}}\Big)^{\frac{1}{2}}\Big) \tag{18}$$

and we may let

$$\tilde{\beta} = \frac{1}{2}\frac{1}{4^{N_1}} + \Big(\frac{4^{N_1}-1}{4^{N_1}}\Big)^{1/2} < 1.$$

Choose $\varepsilon > 0$ and let $\{n_j\}$ be a corresponding sequence given by (15). Then by (17)

$$\sum_{J\in\mathscr{E}_{n_{j+1}}(I)} \omega(J)^{1/2} l(J)^{(\rho+\varepsilon)/2} \leq 4^{n_j/2}\tilde{\beta}\left(\prod_{i=1}^{n_j} a_i^{(\rho+\varepsilon)/2}\right)\times$$

$$\times \sum_{J\in\mathscr{E}_{n_{j+1}-N_1}(I)} \omega(J)^{1/2}\Big(\frac{1}{4^{n_{j+1}-N_1}}\Big)^{1/2}.$$

We repeat the procedure and apply the Cauchy–Schwarz inequalities. We then get

$$\sum_{J\in\mathscr{E}_{n_{j+1}}(I)} \omega(J)^{1/2} l(J)^{(\rho+\varepsilon)/2} \leq 4^{n_j/2}\tilde{\beta}^{(n_{j+1}-n_j)/2N_1}\left(\prod_{i=1}^{n_j} a_i^{(\rho+\varepsilon)/2}\right)\left(\frac{\omega(I)}{4^{n_j}}\right)^{1/2}.$$

The existence of β is now obvious. For instance, one may take $\beta = \tilde{\beta}^{1/2N_1}$.

Proof of Theorem 1.0. Let $\mathscr{L}_j = \{J \in \mathscr{E}_{n_j} \mid \omega(J) > l(J)^{\rho-\varepsilon}\}$ and $\mathscr{L}'_j = \mathscr{E}_{n_j} \setminus \mathscr{L}_j$, where $\varepsilon > 0$ is to be chosen later, and let $\{n_j\}$ be a sequence corresponding to ε as above. It is clear that

$$\sum_{J\in\mathscr{L}_j} l(J)^{\rho-\varepsilon} < \sum_{J\in\mathscr{L}_j}\omega(J) \leq 1. \tag{19}$$

But, we can also estimate

$$\sum_{J\notin\mathscr{L}_j}\omega(J) = \sum_{J\notin\mathscr{L}_j}\omega(J)^{1/2}\omega(J)^{1/2} \leq \sum_{J\in\mathscr{E}_{n_j}}\omega(J)^{1/2} l(J)^{((\rho+\varepsilon)/2)-\varepsilon}$$

$$\leq \prod_{i=1}^{n_j} a_i^{-\varepsilon}\beta^{n_j-n_{j-1}} \sum_{J\in\mathscr{E}_{n_{j-1}}} \omega(J)^{1/2} l(J)^{(\rho+\varepsilon)/2}$$

because of (16). By iterating the procedure we get

$$\sum_{J\notin\mathscr{L}_j}\omega(J)\le\beta^{n_j}\prod_{i=1}^{n_j}a_i^{-\varepsilon}\le\beta^{n_j}\underline{A}^{-\varepsilon n_j}.$$

Let $\varepsilon>0$ be such that $\beta<\underline{A}^{\varepsilon}$. It is then immediate from the above that

$$\lim_{j\to\infty}\sum_{J\notin\mathscr{L}_j}\omega(J)=0. \tag{20}$$

Clearly, (19) and (20) allow us to construct a subset of $\mathbf{K}$ of Hausdorff dimension $<\rho$ but of full harmonic measure, and the proof is completed.

4. A counterexample

We state the following simple result:

Proposition 4.0. *For a Cantor set* $\mathbf{K}$ *as described in the introduction, the harmonic measure* ω *of its complement is "monodimensional", i.e. there is a dimension* σ (*the dimension of the harmonic measure*) *such that there exists a subset* $F\subset\mathbf{K}$ *of Hausdorff dimension* σ *with* $\omega(F)=1$, *and* $\omega(F')=0$ *for every set* $F'\subset\mathbf{K}$ *of dimension smaller than* σ.

The proof given below applies to all self-similar Cantor sets and therefore the proposition remains valid even for "general" Cantor sets.

Proof of Proposition 4.0. Suppose that the proposition is false. Then there is a dimension σ and a real number $0<\alpha<1$ such that

$$\sup\{\omega(F):F\subset\mathbf{K},\ \dim(F)\le\sigma\}=\alpha$$

or, equivalently, there exist a dimension σ and a $\gamma>0$ such that

$$\sup\Bigl\{\inf_{x\in\frac12(1+c_0)[0,1]^2}\omega(x,F,c_0[0,1]^2\setminus\mathbf{K}):F\text{ compact},\ F\subset\mathbf{K},\ \dim(F)\le\sigma\Bigr\}=\gamma$$

and

$$\gamma<\inf_{x\in\frac12(1+c_0)[0,1]^2}\omega(x,\mathbf{K},c_0[0,1]^2\setminus\mathbf{K}),$$

where c_0 is the constant defined in Section 2.

For every real number τ, $0<\tau<1$ there is a compact set $F\subset\mathbf{K}$ of Hausdorff dimension σ with

$$\tau\gamma<\inf_{x\in\frac12(1+c_0)[0,1]^2}\omega(x,F,c_0[0,1]^2\setminus\mathbf{K})<\frac{1}{\tau}\gamma.$$

Moreover, we can find a covering $\mathscr{F} = \{I_j\}_{j\in J}$ of F with squares I_j of the same generation of the construction of $\mathbf{K}$, satisfying

$$\tau\gamma < \inf_{x\in\frac{1}{2}(1+c_0)[0,1]^2} \omega\Big(x, \bigcup_{I\in\mathscr{F}} I, c_0[0,1]^2 \setminus \mathbf{K}\Big) < \frac{1}{\tau}\gamma.$$

There exists at least one $I_j \in \mathscr{F}$ with the following property:

"There is a compact set $F_j \subset I_j \cap \mathbf{K}$ of Hausdorff dimension σ with

$$\inf_{x\in\frac{1}{2}(1+c_0)I_j} \omega(x, F_j, c_0 I_j \setminus \mathbf{K}) > c\tau\gamma \inf_{x\in\frac{1}{2}(1+c_0)I_j} \omega(x, \mathbf{K}, c_0 I_j \setminus \mathbf{K}),$$

where c is a Harnack constant depending only on $\mathbf{K}$."

We then say that F_j is a γ-subset of I_j.

To prove this claim, we first show the existence of at least one I_j satisfying

$$\inf_{x\in\frac{1}{2}(1+c_0)[0,1]^2} \omega(x, F_j, c_0[0,1]^2 \setminus \mathbf{K}) > \tau\gamma \inf_{x\in\frac{1}{2}(1+c_0)[0,1]^2} \omega(x, \mathbf{K}\cap I_j, c_0[0,1]^2 \setminus \mathbf{K})$$

and then proceed with standard arguments, using the Brelot formula.

Recall that all squares of the same generation of the construction of $\mathbf{K}$ are identical, and therefore the preceding property is valid for any square of the generation of I_j, i.e. every such square has a γ-subset. Let $\widetilde{\mathscr{F}}$ be the collection of all squares of the same generation with I_j that do not belong to $\mathscr{F}$, and let S be the union of F with the γ-subsets of the squares in $\widetilde{\mathscr{F}}$. Thus S is a subset of $\mathbf{K}$ of Hausdorff dimension σ. By the above it is clear that

$$\begin{aligned}\inf_{x\in\frac{1}{2}(1+c_0)[0,1]^2} &\omega(x, S, c_0[0,1]^2 \setminus \mathbf{K}) \\ &> \tau\gamma + c\tau\gamma\Big(\inf_{x\in\frac{1}{2}(1+c_0)[0,1]^2} \omega(x, \mathbf{K}, c_0[0,1]^2 \setminus \mathbf{K}) - \frac{1}{\tau}\gamma\Big),\end{aligned}$$

which is greater than γ if τ is close enough to one; since γ is taken to be the maximal value of harmonic measure for subsets of $\mathbf{K}$ of Hausdorff dimension equal to σ, we have reached a contradiction. The proof is now complete.

We will now construct a Cantor set $\mathbf{K}'$ as in the introduction, except that here we replace a square J of the kth generation, $k \geq 1$, with four equal squares, $J_1, \dots, J_4$, whose size depends not only on the generation k but also on the square J; we still require $\underline{A} \leq l(J_i)/l(J) \leq \overline{A}$ with $0 < \underline{A} \leq \overline{A} < 1/2$. We will show that for an appropriate choice of the sizes of the squares *the Hausdorff dimension of* $\mathbf{K}'$ *will be equal to the dimension of its harmonic measure.* The idea of the construction was suggested to us by a remark of A. Ancona. Let us begin with the standard planar Cantor set $\mathbf{K}_{1/4}$ of dimension 1, i.e. a Cantor set as defined in the introduction with $\underline{A} = \overline{A} = 1/4$. Let D be the dimension of its harmonic measure; if F is a compact subset of $\mathbf{K}_{1/4}$ such that $\omega(F) > 1/2$, it follows from Proposition 4.0 that its dimension will be at least D. We may therefore find such a subset F of $\mathbf{K}_{1/4}$ of Hausdorff dimension D. We then construct the desired Cantor set in the following way: In each generation we replace every square J that does not intersect F with squares of size 4^{-M} times the size of J, where M is a fixed integer with $M > 1/D$, and every square J' that intersects F is replaced with four squares of size $1/4$ times the size of J'. Let $\mathbf{K}'$ be the Cantor set constructed in this way. Observe that $\mathbf{K}' \subset \mathbf{K}_{1/4}$ by construction and that $\dim \mathbf{K}' = D$ because of the choice of M. It is clear (by the monotonicity of the harmonic measure as a function of the domain) that the dimension of the harmonic measure of $\mathbf{K}'$ is also D, and the construction is complete.

We should remark here that the preceding process gives us Cantor sets whose Hausdorff dimension is equal to their harmonic measure dimension for every possible value of the dimension of the harmonic measure of a Cantor set as described in the introduction. Also, a result of [JW] implies that we cannot have $\dim \omega = 1$ for Cantor sets of this type. It is therefore natural to ask if we can have dimensions arbitrarily close to one. The following proposition answers the question.

Proposition 4.1. *For the self-similar Cantor set* $\mathbf{K}_\delta$, $0 < \delta \le \frac{1}{4}$, *as defined in the introduction with* $\frac{1}{2} - \delta = \underline{A}$ *and* $\underline{A} = \overline{A}$, *the dimension of the harmonic measure* $\dim \omega$ *is greater than* $1 - C\delta$ *for some constant* $C > 0$.

This proposition as well as the proof given below is due to Professor A. Ancona (compare with [MV, pp. 15–22, 28]).

Proof of Proposition 4.1. We will need some more notation. Let $\mathbf{K}_n$ be the nth approximation of $\mathbf{K}_\delta$ by squares of the nth generation, let g_n be the Green function of the complement of $\mathbf{K}_n$ and $\mathscr{C}_n$ its critical points. We shall rely on the formula

$$\dim \omega = 1 - \frac{\lim_{n\to\infty} n^{-1} \sum_{\mathscr{C}_n} g_n(c)}{\chi_\mu},$$

where $\chi_\mu = \log\bigl(2/(1-2\delta)\bigr)$ and the critical points in the sum are counted with their multiplicity.

This formula is a simple variant of the Carleson formula given in [MV, p. 15] (see also [C]); here we consider the sum over the critical points of g_n instead of those of the Green function of the complement of $\mathbf{K}_\delta$.

It remains to be proved that the limit in the previous formula is $O(\delta)$ as δ tends to 0.

We extend g_n on $\mathbf{K}_n$ by the value 0 and consider the critical domains of g_n, i.e. any region U which is a connected component of $\{g_n < \beta\}$ for some $\beta > 0$ and with a critical point $c \in \partial U$. Let $\mathscr{U}$ be the collection of all critical domains.

Note that if $U \in \mathscr{U}$ and if $\tau = \max\{g_n(z) : z \in \mathscr{C}_n \cap U\}$, the number of critical domains $U' \subset U$ associated with τ is exactly equal to the number of critical points $z \in U$ with $g_n(z) = \tau$ (counted with their multiplicity) plus one.

To each $U \in \mathscr{U}$ we attach a square $I = I_U$ of some stage k of the construction of the Cantor set, $k \le n$, with the following property:

(PI) We have $I \cap \mathbf{K}_n \subset U$, and if $\tilde{I}$ denotes the "father" of I, there exists a square $I' \subset \tilde{I}$ of the kth generation such that $I' \cap \mathbf{K}_n \cap U = \emptyset$ and I and I' lie on the same side of $\tilde{I}$.

The existence of I_U is easily checked. For instance, one may take for $\tilde{I}$ a minimal square such that $\tilde{I} \cap \mathbf{K}_n \cap U^c \neq \emptyset$ and $\tilde{I} \cap \mathbf{K}_n \cap U \neq \emptyset$, and then easily verify the existence of $I \subset \tilde{I}$ with the property (PI).

We now proceed with the following simple algorithm which leads to the construction of a subcollection $\mathscr{U}_0 \subset \mathscr{U}$ (the "nice" domains) and to the choice of some square $c_U \subset I_U$ of the nth generation (the last generation for $\mathbf{K}_n$), for every $U \in \mathscr{U}_0$.

Each domain $U \in \mathscr{U}$ which is maximal is "nice", and we choose the square c_U arbitrarily in I_U. For $U \in \mathscr{U}$, if the construction has been achieved for all $U' \supset U$, $U \neq U'$, we decide that $U \notin \mathscr{U}_0$ if there exists $c_{U'} \subset U$ for some $U' \supset U$, $U \neq U'$. Otherwise we say that $U \in \mathscr{U}_0$ and we associate with it some square $c_U \subset I_U$ of the nth generation.

At the end of the procedure every critical domain $U \in \mathscr{U}$ contains exactly one $c_{U'}$ for some U', $U \subset U'$, and for $U, U' \in \mathscr{U}_0$ we have $I_U = I_{U'}$ if and only if $U = U'$.

Hence we have

$$\frac{1}{n} \sum_{\mathscr{C}_n} g_n(c) = \frac{1}{n} \Bigl(\sum_{U \in \mathscr{U}_0} g_{\scriptscriptstyle U} - g_{\max} \Bigr),$$

where $g_{\scriptscriptstyle U}$ is the value of g_n on ∂U and $g_{\max}$ is the maximal critical value of g_n.

If $U \in \mathscr{U}_0$, let $I = I_U$ be the square attached to it, $\tilde{I}$ its "father", and I' as in (PI) (see Figure 2). There are at least $s = [1/4\delta]$ parallel segments, $l_1, \dots, l_s$, joining points of $\mathbf{K}_n \cap I$ with points of $\mathbf{K}_n \cap I'$, the distance between any two segments being $\ge \delta\, l(I)$. Necessarily, ∂U cuts through all these segments and therefore $\sup\{g_n(t) : t \in l_i\} \ge g_{\scriptscriptstyle U}$, $1 \le i \le s$.

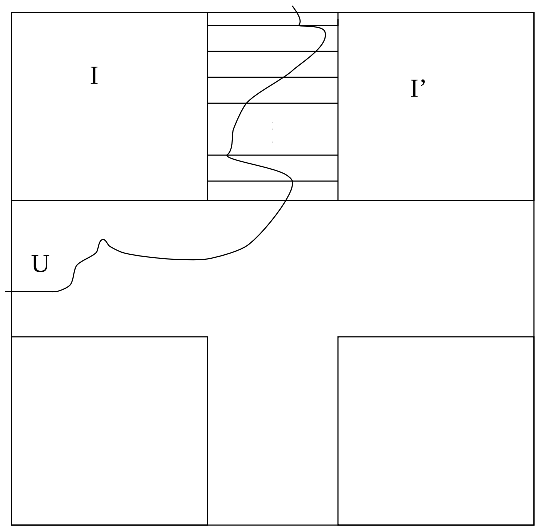


Figure 2.

For every l_i, $i = 1\dots,s$, let z_i^1, z_i^2 be the endpoints of l_i, $z_i^1 \in \mathbf{K}_n \cap I$, $z_i^2 \in \mathbf{K}_n \cap I'$. It is clear that the set $B(z_i^1, \delta l(I)/2) \cap \mathbf{K}_n \cap I$ has capacity $\geq C_0 > 0$ in the domain $B\bigl(z^1, \delta l(I)\bigr)$, with C_0 independent of $\delta \in [\frac{1}{4}, \frac{1}{2})$. By standard arguments it follows that

$$g_n(t) \leq C\omega\bigl(B\bigl(z_i^1, \delta l(I)/2\bigr), \mathbf{R}^2 \setminus K_n\bigr)$$

on the segment l_i with a constant C indepent of δ.

The above finally yields

$$\frac{1}{4\delta} g_U \leq \sum_i \sup\{g_n(t) : t \in l_i\} \leq C\,\omega(I, \mathbf{R}^2 \setminus K_n).$$

Summing up we find

$$\frac{1}{n} \sum_{\mathscr{C}_n} g_n(c) \leq \frac{1}{n} \sum_{U \in \mathscr{U}_0} g_U \leq C\delta \frac{1}{n} \sum_{I \in \mathscr{F}_n} \omega(I, \mathbf{R}^2 \setminus \mathbf{K}_n) \leq C\delta,$$

where $\mathscr{F}_n$ is the collection of all squares of some stage k, $k \leq n$, of the construction of $\mathbf{K}_\delta$. The proof of the proposition is complete.

5. Conclusion – Further remarks

It is clear that the method we developed in Sections 2 and 3 applies not only to the Cantor sets described above but also to other Cantor sets, for example those indicated by Figure 3. The proof can also be applied to some Cantor sets in higher dimensions.

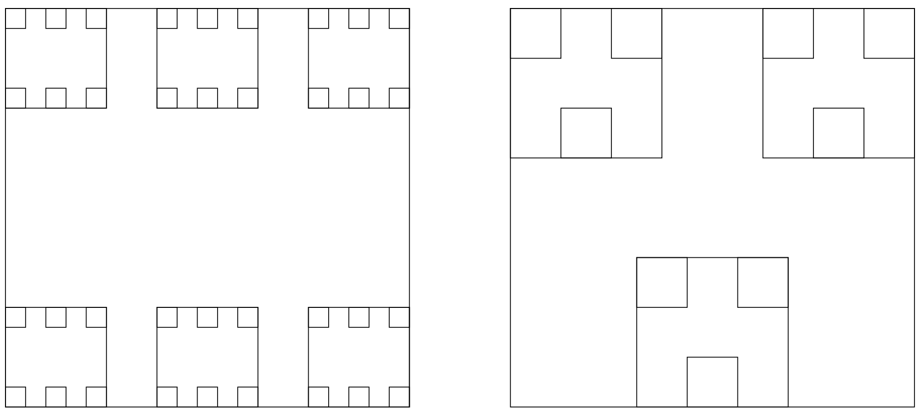
For a general Cantor set $\mathbf{K} \subset \mathbf{R}^d$, a sufficient condition to conclude that $\dim(\omega) < \rho = \dim(\mathbf{K})$ is the following: if $I^n_{i_1\dots i_n}$ is a square of the nth generation and if $I^{n+1}_{i_1\dots 1}, \dots, I^{n+1}_{i_1\dots s}$ are the squares of the next generation contained in $I^n_{i_1\dots i_n}$, there exist $0 < \alpha < 1$, $1 \le \tau \le s$, and constants $a^n_j > 0$ such that

$$(*) \qquad \omega(I^{n+1}_{i_1\dots i_n\tau}) < \alpha \frac{\operatorname{diam}(I^{n+1}_{i_1\dots i_n\tau})^\rho}{\sum_{j=1}^s \operatorname{diam}(I^{n+1}_{i_1\dots i_n j})^\rho}\omega(I^n_{i_1\dots i_n}),$$

and

$$\operatorname{diam}(I^{n+1}_{i_1\dots i_n j}) = a^n_j \operatorname{diam}(I^n_{i_1\dots i_n}), \quad 2^{-d} < \underline{A} \le a^n_j \le \overline{A} < 1 \text{ for all } j \in \{1, \dots, s\},$$

where a^n_j depends only on j, n but not on the square $I^n_{i_1\dots i_n}$ and $\underline{A}, \overline{A}$ are two constants not depending on n. Lemmas 2.7 and 2.8 can both be applied to prove a formula similar to (16), and the proof of the theorem may be completed in the same way.



Figures 3a and 3b

In general $(*)$ seems hard to check; however, under certain assumptions of symmetry on the Cantor set $\mathbf{K}$ one may verify it by proving some lemmas similar to those presented above. Even though the method presented here seems to be rather general, we have not been able to get rid of these assumptions of symmetry, and the proof of Lemma 2.3 strongly depends on them.

Added in proof. It is perhaps interesting to point out that the result of Theorem 1.0 can also be proved under some weaker assumptions on the size of the squares:

For a sequence $\{a_n\}_{n\in\mathbf{N}}$ as in the introduction we construct a Cantor set in a similar way. We allow the squares of the nth generation $I^n_{i_1\cdots i_n}$ to have sidelengths $l_{i_1\cdots i_n}$ not necessarily equal but require that $a_n(1-\varepsilon)l_{i_1\cdots i_{n-1}} \le l_{i_1\cdots i_n} \le a_n(1+\varepsilon)l_{i_1\cdots i_{n-1}}$, where $\varepsilon > 0$ and $l_{i_1\cdots i_{n-1}}$ is the sidelength of the "father" of $I^n_{i_1\cdots i_n}$. For such a Cantor set the dimension of the harmonic measure is smaller than the dimension of the set, provided ε is small enough (the proof is slightly different).

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