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REGULARITY AND EXTREMALITY OF QUASICONFORMAL HOMEOMORPHISMS ON CR 3-MANIFOLDS

Puqi Tang

Purdue University, Department of Mathematics
West Lafayette, IN 47907, USA; tang@math.purdue.edu

Abstract. This paper first studies the regularity of conformal homeomorphisms on smooth locally embeddable strongly pseudoconvex Cauchy–Riemann manifolds. Then moduli of curve families are used to estimate the maximal dilatations of quasiconformal homeomorphisms. On certain CR 3-manifolds, namely, CR circle bundles over flat tori, extremal quasiconformal homeomorphisms in some homotopy classes are constructed. These extremal mappings have similar behavior to Teichmüller mappings on Riemann surfaces.

1. Introduction

A contact manifold M is a manifold of odd dimension with a nowhere integrable distribution HM of tangent hyperplanes. A Cauchy–Riemann (CR) manifold is a contact manifold M endowed a complex structure on the contact bundle HM. Two CR manifolds are equivalent if there is a diffeomorphism between them which respects both contact and CR structures. Generally, between any two CR structures assigned on the same contact manifold, there may be no such so-called CR diffeomorphism between them. Therefore we consider those diffeomorphisms between CR manifolds which preserve the underlying contact structures and distort the CR structures in a bounded fashion. They are called quasiconformal diffeomorphisms. Quasiconformality can be generalized to certain homeomorphisms with much weaker regularity. In a class of homeomorphisms between two CR manifolds, an extremal mapping is a quasiconformal homeomorphism which distorts the CR structures in a minimal way. This paper studies regularity of quasiconformal homeomorphisms and extremal quasiconformal homeomorphisms on smooth strongly pseudoconvex CR manifolds.

The notion of quasiconformal homeomorphisms is a new tool to study CR structures as initiated by Korányi and Reimann. In this paper, we use an analytic definition of quasiconformal homeomorphisms given in [13] which is a generalization of the one given by Korányi and Reimann in [5]. We restrict ourselves to the 3-dimensional case here not only because the notion of quasiconformality is not invariant under CR transformations in higher dimensional cases, but also because

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3-dimensional CR structures are among the most interesting objects in the theory of CR manifolds. We refer to [7] for details about the second point.

The purpose of defining quasiconformal homeomorphisms is to introduce variants of CR diffeomorphisms. But quasiconformal homeomorphisms are variants of conformal homeomorphisms by default. Hence it is important to know if conformal homeomorphisms are actually CR diffeomorphisms. Korányi and Reimann proved that C^4 conformal homeomorphisms on Heisenberg groups must be smooth and CR ([6, Theorem 8]). By applying a regularity theorem of weak CR mappings of Pinchuk and Tsyganov, we generalize Korányi and Reimann's result to that a conformal homeomorphism f between two smooth, strongly pseudoconvex, locally embeddable CR manifolds must be smooth and CR, if f has L^1_{loc} horizontal derivatives (Theorem 2.3).

To study the extremality of quasiconformal homeomorphisms is a global problem. But our analytic definition of quasiconformality is proposed infinitesimally. Therefore we need some global notion to describe the quasiconformality. The one best fitting our later developments is the notion of moduli of curve families. We prove that a C^2 diffeomorphism is quasiconformal if and only if it preserves moduli of certain curve families up to a fixed bounded multiple (Theorem 3.3). On the other hand, a homeomorphism satisfying this property is absolutely continuous on lines (ACL) (Theorem 3.4) and has L^4_{loc} horizontal derivatives (Corollary 3.5).

Between CR circle bundles over flat tori, we construct extremal quasiconformal homeomorphisms in certain homotopy classes (Theorem 4.2). There are two transversal Legendrian foliations such that the extremal homeomorphism constructed preserves these two foliations. More precisely, it is a stretching by a constant factor along leaves of one foliation and a compressing by the same factor along leaves of another foliation. This behavior is analogous to that of Teichmüller mappings on Riemann surfaces (see [14] or [1]). The generator T of the circle action is transversal to the contact bundle. In this transversal direction, the extremal mappings are equivariant under the circle action.

But on an arbitrary CR 3-manifold with a transversal free circle action, an extremal quasiconformal homeomorphism is not necessarily equivariant under the circle action. Such CR manifolds are constructed in [13] so that no extremal quasiconformal mapping between them is equivariant.

This work is heavily influenced by the theory of quasiconformal homeomorphisms on Riemann surfaces and Teichmüller theory. Numerous proofs in this paper are motivated by the proofs of the analogous facts on Riemann surfaces. For example, the construction of the extremal homeomorphisms made in Theorem 4.2, one of the main results of this paper, can find its root in Grötzsch's theorem which is proved by a length-area argument [9]. Teichmüller generalized this result to closed Riemann surfaces, in particular tori, by an ergodic version of the length-area argument [14]. The notion of modulus of a curve family is a formalism of the length-area (volume) argument. In Section 4, we reformulate

Teichmüller's method to the CR setting by computing moduli of some special families of curves and successfully find the extremal quasiconformal homeomorphisms in certain homotopy classes of mappings between CR circle bundles over flat tori.

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2. Regularity of conformal homeomorphisms

For $j=1,2$, let M_j be a smooth strongly pseudoconvex CR 3-manifolds. The contact bundle HM_j is assumed to be smooth and orientable, that is, there exists smooth global 1-form η_j on M_j, which is called a contact form, so that $HM_j = \operatorname{Ker}\eta_j$. Let $J_j\colon HM_j \to HM_j$ denote the CR structure on M_j. $H^{0,1}M_j \triangleq \{X+iJ_jX \mid X \in HM_j\} \subset \mathbf{C}\otimes HM_j$ is the $(0,1)$ tangent bundle on M_j. $\wedge^{1,0}M_j \triangleq \{\text{linear functionals } \psi\colon \mathbf{C}\otimes HM_j \to \mathbf{C} \mid \psi(J_jX) = i\psi(X), \text{ for } X \in HM_j\}$. For a non-zero $X \in HM_j$, $\{X, J_jX\}$ determines an orientation of HM_j.

A continuous mapping $f\colon M_1 \to M_2$ is said to be absolutely continuous on lines (ACL) if for any open set with a smooth contact fibration, f is absolutely continuous along all fibers in this fibration except a subfamily of measure zero. Here a subfamily of fibers of the fibration is said to have measure zero if intersections of these fibers with any transversal smooth surface has measure zero on the surface (see [12] for more details).

Definition 2.1. A homeomorphism $f\colon M_1 \to M_2$ is said to be K-quasiconformal for a finite constant $K \geq 1$ if

(i) f is ACL;

(ii) f is differentiable almost everywhere, and its differential f_* preserves the contact structures, i.e., $f_*(H_qM_1) \subset H_{f(q)}M_2$, for almost every $q \in M_1$.

(iii) for norms $|\cdot|_1$ and $|\cdot|_2$ defined by any Hermitian metrics on HM_1 and HM_2 respectively,

$$K(f)(q) = \frac{\max\limits_{X\in H_qM_1,|X|_1=1} |f_*(X)|_2}{\min\limits_{X\in H_qM_1,|X|_1=1} |f_*(X)|_2} \leq K < \infty, \tag{2.1}$$

for almost all $q \in M_1$. The number $K(f) = \operatorname{ess\,sup}_{q\in M_1} K(f)(q)$ is called the maximal dilatation of f. f is conformal if $K(f) = 1$. $K(f) = \infty$ if f is not K-quasiconformal for any finite $K \geq 1$.

It is easy to see that $K(f)(q)$ does not depend on the choice of the Hermitian norms $|\cdot|_1$ and $|\cdot|_2$. This is not true if the underlying CR manifolds have dimensions > 3.

A mapping $f\colon M_1 \to M_2$ is said to have L^p_{loc} horizontal derivatives for $p \geq 1$ if for any smooth function $h\colon M_2 \to \mathbf{R}$ and any smooth local section X of HM_1 on an open set $U \subset\subset M_1$, the function $h\circ f$ has derivative at almost all points along the trajectory of X passing through q for almost all $q \in U$, and the derivative $X(h\circ f)$ is in $L^p(U)$.

A mapping $f\colon M_1 \to M_2$ is said to have $L^p_{\rm loc}$ weak horizontal derivatives for some $p \geq 1$ if for any smooth function $h\colon M_2 \to \mathbf{R}$ and any open set $U \subset\subset M_1$ with a smooth local section X of HM_1 on U, there exists a function $g \in L^p_{\rm loc}(M_1)$ so that

$$\int_U X\phi\cdot(h\circ f)\,dv_1 = -\int_U g\,\phi\,dv_1 \tag{2.2}$$

for all $\phi \in C_0^\infty(U)$, where dv_1 is a smooth volume form on M_1. Certainly, the function g depends on the choice of dv_1.

For the proof of the next theorem, we fix a norm $|\cdot|$ on HM_1. A C^1 curve on a contact manifold is called Legendrian if it is tangent to the contact structure. For any Legendrian curve $\gamma\colon I \to M_1$ with an interval $I \subset \mathbf{R}$ and a function g defined on an open neighborhood of γ, define the line integral

$$\int_\gamma g = \int_I g\big(\gamma(t)\big)|\gamma'(t)|\,dt.$$

Theorem 2.2. *A mapping $f\colon M_1 \to M_2$ is ACL and has $L^p_{\rm loc}$ horizontal derivatives for some $p \geq 1$ if and only if f has $L^p_{\rm loc}$ weak horizontal derivatives.*

Proof. First assume that the homeomorphism $f\colon M_1 \to M_2$ is ACL and has $L^p_{\rm loc}$ horizontal derivatives. Let $h\colon M_2 \to \mathbf{R}$ be a smooth function and $U \subset\subset M_1$ be any open set with a smooth section $X \neq 0$ of HM_1 on it.

We can assume that the trajectories $\Gamma = \{\gamma\}$ of X form a contact fibration of U by shrinking U appropriately. Let Γ_1 be the subfamily of those $\gamma \in \Gamma$ along which f is absolutely continuous. Then $\Gamma \setminus \Gamma_1$ has measure zero. Along $\gamma \in \Gamma_1$, $h\circ f$ is absolutely continuous, so $X(h\circ f)$ exists almost everywhere on γ and

$$\int_\gamma X\phi\cdot(h\circ f) + \int_\gamma \phi\cdot X(h\circ f) = \int_\gamma X(\phi\cdot h\circ f) = 0, \qquad \forall\ \phi \in C_0^\infty(U). \tag{2.3}$$

We have topological and differential structures on Γ such that the natural projection $p\colon U \to \Gamma$ is open and smooth. Then Γ becomes a smooth surface. Let t be the parameter (with respect to a local coordinate system on U) of the flow generated by X and ω be any area form on Γ. Then $dv_1 \triangleq p^*\omega \wedge dt$ is a volume form of U. Integrating the expressions in (2.3) against ω with respect to $\gamma \in \Gamma_1$, by the ACL property of f, local L^p integrability of $X(h\circ f)$, and Fubini's theorem, we obtain

$$\int_U X\phi\cdot(h\circ f)\,dv_1 = -\int_U \phi\cdot X(h\circ f)\,dv_1. \tag{2.4}$$

So the weak derivative of $h\circ f$ in the X direction is given by $X(h\circ f)\in L^p_{\mathrm{loc}}$.

Conversely, assume f has L^p_{loc} weak horizontal derivatives. For any open set $U\subset\subset M_1$ and a smooth contact fibration Γ of U, let X be the nonzero horizontal vector field on U so that X_q, for any $q\in U$, is the tangent vector at q of the fiber $\gamma\in\Gamma$ passing through q. Let $B\subset\subset U$ be an open set with coordinate system $\{(x,y,t)\,|\,a_1<x<a_2,\ b_1<y<b_2,\ c_1<t<c_2\}$, such that $X=\partial/\partial t$.

For any smooth function h on M_1, $h\circ f$ has $L^p(U)$ weak derivative in the direction X. Denote it by $\psi\in L^p(U)$. Hence there exists a sequence of C^1 functions g_n on B such that g_n converges to $h\circ f$ uniformly in B and Xg_n converges to ψ in $L^p(B)$. Let $B_{x,y,t}=(a_1,x)\times(b_1,y)\times(c_1,t)$, $R_{x,y}=(a_1,x)\times(b_1,y)$.

$$\int_{B_{x,y,t}} Xg_n(u,v,w)\,du\,dv\,dw=\int_{R_{x,y}}\big(g_n(u,v,t)-g_n(u,v,c_1)\big)\,du\,dv. \tag{2.5}$$

Hence by taking limits, we have

$$\int_{B_{x,y,t}} \psi(u,v,w)\,du\,dv\,dw=\int_{R_{x,y}}\big((h\circ f)(u,v,t)-(h\circ f)(u,v,c_1)\big)\,du\,dv. \tag{2.6}$$

Let $\{t_n\}$ be a countable dense set of (c_1,c_2). (2.6) implies for each t_n, there exists a set $E_n\subset R_{a_2,b_2}$ such that $R_{a_2,b_2}\setminus E_n$ is of measure zero and

$$\int_{c_1}^{t_n}\psi(x,y,w)\,dw=(h\circ f)(x,y,t_n)-(h\circ f)(x,y,c_1),\qquad \forall\ (x,y)\in E_n. \tag{2.7}$$

Then (2.7) is true for all $(x,y)\in E\triangleq\cap E_n$ and all t_n. Hence by continuity of both sides in t,

$$\int_{c_1}^{t}\psi(x,y,w)\,dw=(h\circ f)(x,y,t)-(h\circ f)(x,y,c_1),\ \forall\,(x,y)\in E,\ t\in(c_1,c_2). \tag{2.8}$$

So $(h\circ f)(x,y,t)$ is absolutely continuous in t for $(x,y)\in E$. Note $R_{a_2,b_2}\setminus E$ has measure zero. So f is ACL since $B\subset\subset U$ is an arbitrary rectangular coordinate chart and $U\subset\subset M_1$ is arbitrary in M_1. Moreover, $X(h\circ f)$ exists almost everywhere and $X(h\circ f)=\psi\in L^p(U)$ on U. So f has L^p_{loc} horizontal derivatives. □

Remark An equivalent result on Heisenberg groups was proved by Korányi and Reimann ([4, Proposition 9]). Our proof of the second part is different from theirs.

When M_2 is locally embeddable into some Euclidean space $\mathbf{C}^k$, a homeomorphism $f\colon M_1\to M_2$ is said to be weakly CR if $\int_U(h\circ f)\cdot\overline{Z}\phi=0$ for any smooth CR function $h\colon M_2\to\mathbf{C}$, open set $U\subset\subset M_1$, $\phi\in C_0^\infty(U)$ and $\overline{Z}\in H^{0,1}M_1$. Here h is said to be CR if $\overline{Z}f=0$ for all $\overline{Z}\in H^{0,1}M_1$.

Theorem 2.3. *Let M_1 and M_2 be two smooth, strongly pseudoconvex, locally embeddable CR 3-manifolds. Assume $f\colon M_1 \to M_2$ is a conformal homeomorphism with $L^1_{\rm loc}$ horizontal derivatives and at almost all points of differentiability f_* preserves the orientation of the contact bundles determined by CR structures. Then f is smooth and CR.*

Proof. A simple linear algebra argument shows that at almost all points $q \in M_1$ where f is differentiable

$$K(f)(q) = \frac{1+|\mu(q)|}{1-|\mu(q)|} \qquad \text{with} \qquad |\mu(q)| = \Big|\frac{\langle f^*\psi_2, \overline{Z}\,\rangle}{\langle f^*\psi_2, Z\rangle}\Big|(q), \tag{2.9}$$

for any nonzero $\overline{Z} \in H^{0,1}M_1$ and nonzero $\psi_2 \in \wedge^{1,0}M_2$.

Thus $K(f) = 1$ implies that for almost every $q \in M_1$, $\overline{Z} \in H^{0,1}_q M_1$ and $\psi_2 \in \wedge^{1,0}_{f(q)}M_2$, we have $\langle f^*\psi_2, \overline{Z}\,\rangle = 0$. Hence for any CR function h on M_2,

$$\overline{Z}(h\circ f) = \langle d(h\circ f), \overline{Z}\,\rangle = \langle f^*\,dh, \overline{Z}\,\rangle = 0, \tag{2.10}$$

since $dh|_{\mathbf{C}\otimes HM} \in \wedge^{1,0}M_2$. Theorem 2.2 says that f is weakly CR. M_1 and M_2 are locally embeddable implies they are locally embeddable into $\mathbf{C}^2$ as hypersurfaces. A theorem of Pinchuk and Tsyganov ([10, Theorem 2]) asserts that such f must be smooth, hence CR. ▫

3. Moduli of curve families

Let M be a smooth, compact, contact 3-manifold. We always assume HM is smooth and oriented. A sub-Riemannian metric on M with respect to HM is a smooth positive definite quadratic form on HM. Fix a sub-Riemannian metric on M with respect to HM momentarily, and denote by $|\cdot|$ the corresponding norm on HM. For the general theory of sub-Riemannian geometry, we refer to [12] and [2].

The sub-Riemannian metric on M can be extended to a Riemannian metric on M canonically as follows. Let ω be the oriented area form on HM corresponding to the sub-Riemannian metric. There exists a unique contact form η so that $d\eta|_{HM} = \omega$. Let T be the characteristic vector field of η, namely, T is the unique vector field satisfying that $T \lrcorner \eta = 1$ and $T \lrcorner d\eta = 0$. Declaring T a unit vector orthogonal to HM, we obtain a Riemannian metric which is called the canonical extension of the sub-Riemannian metric. The positive volume form of this Riemannian metric is $dv \triangleq d\eta \wedge \eta$.

A curve $\gamma\colon I_\gamma \to M$ with an interval $I_\gamma \subset \mathbf{R}$ is called locally rectifiable if γ is absolutely continuous and $\gamma'(t)$ is tangent to HM for almost all $t \in I_\gamma$. γ is called rectifiable if γ is locally rectifiable and the length

$$l(\gamma) \triangleq \int_{I_\gamma} |\gamma'(t)|\,dt < \infty. \tag{3.1}$$

We set $l(\gamma) = \infty$ if γ is not rectifiable. For a locally rectifiable curve γ and a non-negative Borel-measurable function σ on M, define the line integral

$$\int_\gamma \sigma = \int_{I_\gamma} \sigma\big(\gamma(t)\big)|\gamma'(t)|\,dt. \tag{3.2}$$

Definition 3.1. Let Γ be a family of curves $\gamma\colon I_\gamma \to M$. An admissible measure for Γ is a Borel-measurable function $\sigma\colon M \to \mathbf{R}$ such that $\sigma \geq 0$ and $\int_\gamma \sigma \geq 1$, for all locally rectifiable $\gamma \in \Gamma$. Denote the set of admissible measures for Γ by $A(\Gamma)$. The modulus of Γ is defined by

$$\operatorname{Mod}_M(\Gamma) = \inf_{\sigma\in A(\Gamma)} \int_M \sigma^4\,dv. \tag{3.3}$$

Remark. (1) It is easy to see that if two sub-Riemannian metrics on M with respect to HM define the same conformal structure on HM, then they give the same value for $\operatorname{Mod}_M(\Gamma)$.

(2) If $\Gamma_r \subset \Gamma$ consists of all locally rectifiable curves of Γ, then $\operatorname{Mod}_M(\Gamma_r) = \operatorname{Mod}_M(\Gamma)$.

The following proposition shows that modulus, regarded as a measure of (locally rectifiable) curve families, generalizes the concept of measure zero used in the definition of ACL property. Henceforth if a property holds for all curves in a family Γ except a subfamily with zero modulus, we say this property is true for almost all curves in Γ.

Proposition 3.2. *Let U be an open set of M, Γ a contact fibration of U, $\Gamma_1 \subset \Gamma$. Then Γ_1 has measure zero if and only if* $\operatorname{Mod}_M(\Gamma_1) = 0$.

Proof. Without loss of generality, we assume U is a domain of the coordinate system $\{(x,y,t) \mid a_1 < x < a_2,\ b_1 < y < b_2,\ c_1 < t < c_2\}$ and $X = \partial/\partial t$ is tangent to Γ. Let $E \subset (a_1,a_2)\times(b_1,b_2)$ so that $\Gamma_1 = \{\text{curves } t \mapsto (x,y,t) \mid (x,y) \in E\}$.

If Γ_1 has measure zero, then $\int_E dx\,dy = 0$. Notice

$$\sigma_0 = \begin{cases} 1/(c_2-c_1), & \text{when } (x,y) \in E,\ t \in (c_1,c_2), \\ 0, & \text{otherwise,} \end{cases} \tag{3.4}$$

is an admissible measure for Γ_1. Therefore

$$\operatorname{Mod}_M(\Gamma_1) \leq c \int_U \sigma_0^4\,dx\,dy\,dt = 0, \tag{3.5}$$

where c is a constant upper bound of the Jacobian J on U with $J\,dx\,dy\,dt = dv$.

If $\operatorname{Mod}_M(\Gamma_1) = 0$, then for any $\sigma \in A(\Gamma_1)$ with $\sigma = 0$ outside U, $\gamma \in \Gamma_1$,

$$1 \leq \left(\int_\gamma \sigma\right)^4 \leq \left(\int_\gamma \sigma^4\right)\left(\int_\gamma 1\right)^3 = (c_2-c_1)^3 \int_\gamma \sigma^4. \tag{3.6}$$

Taking the integral over E,

$$\int_E dx\,dy \le c'(c_2 - c_1)^3 \int_M \sigma^4. \tag{3.7}$$

Hence

$$\int_E dx\,dy \le c'(c_2 - c_1)^3 \mathrm{Mod}_M(\Gamma_1) = 0, \tag{3.8}$$

that is, Γ_1 has measure zero. □

Let M_1 and M_2 be two compact, smooth, strongly pseudoconvex CR 3-manifolds with smooth contact forms η_1 and η_2 respectively. Here the roles of sub-Riemannian metrics on HM_1 and HM_2 are played by Hermitian metrics with respect to the CR structures J_1 and J_2 respectively.

Theorem 3.3. *A C^2 homeomorphism $f\colon M_1 \to M_2$ is K-quasiconformal for a constant $K \ge 1$ if and only if for any family Γ of C^1 Legendrian curves on M_1*

$$\frac{1}{K^2}\mathrm{Mod}_{M_1}(\Gamma) \le \mathrm{Mod}_{M_2}\big(f(\Gamma)\big) \le K^2\mathrm{Mod}_{M_1}(\Gamma), \tag{3.9}$$

where $f(\Gamma) = \{f(\gamma) \mid \gamma \in \Gamma\}$.

Proof. Assume $f\colon M_1 \to M_2$ is K-quasiconformal. Then f is contact, i.e., for contact forms η_1 and η_2 on M_1 and M_2 respectively, $f^*\eta_2 = \lambda\eta_1$ with a C^1 function λ on M_1. Then

$$f^*(d\eta_2) = d\lambda \wedge \eta_1 + \lambda\, d\eta_1. \tag{3.10}$$

Therefore

$$f^*(d\eta_2|_{HM_2}) = \lambda\, d\eta_1|_{HM_1}. \tag{3.11}$$

For $j = 1, 2$, the Levi form L_j on M_j is a symmetric bilinear form on HM_j defined by

$$L_j(X, Y) = d\eta_j(X, J_jY), \qquad \text{for } X, Y \in HM_j. \tag{3.12}$$

By replacing η_j by $-\eta_j$, if necessary, we can always assume that L_j is positive definite. Hence L_j is a Hermitian form on HM_j. With respect to the Levi forms on M_1 and M_2, we define

$$\begin{aligned} \lambda_1(q) &= \max_{Y \in H_qM_1,\, |Y|_1 = 1} |f_*(Y)|_2, \\ \lambda_2(q) &= \min_{Y \in H_qM_1,\, |Y|_1 = 1} |f_*(Y)|_2. \end{aligned} \tag{3.13}$$

Then (3.11) implies that $|\lambda| = \lambda_1\lambda_2$. Moreover, (3.10) implies that

$$f^*(d\eta_2 \wedge \eta_2) = \lambda^2\, d\eta_1 \wedge \eta_1. \tag{3.14}$$

Thus the Jacobian of f with respect to the volume forms $dv_1 = d\eta_1 \wedge \eta_1$ on M_1 and $dv_2 = d\eta_2 \wedge \eta_2$ on M_2 is $J(f) = (\lambda_1\lambda_2)^2$.

For any non-negative Borel-measurable function σ_2,

$$\begin{aligned}\int_{f(\gamma)} \sigma_2 &= \int_{I_\gamma} \big|f_*\big(\gamma'(t)\big)\big|_2 \sigma_2\big(f\big(\gamma(t)\big)\big)\, dt \\ &\le \int_{I_\gamma} \lambda_1 |\gamma'(t)|_1 \sigma_2\big(f\big(\gamma(t)\big)\big)\, dt = \int_\gamma \lambda_1 \cdot \sigma_2 \circ f.\end{aligned} \tag{3.15}$$

Hence $\sigma_2 \in A\big(f(\Gamma)\big)$ implies that $\lambda_1 \cdot \sigma_2 \circ f \in A(\Gamma)$.

On the other hand,

$$\int_{f(\gamma)} \sigma_2 \ge \int_{I_\gamma} \lambda_2 |\gamma'(t)|_1 \sigma_2\big(f\big(\gamma(t)\big)\big)\, dt = \int_\gamma \lambda_2 \cdot \sigma_2 \circ f. \tag{3.16}$$

Hence $\sigma_1 \in A(\Gamma)$ implies $(\sigma_1/\lambda_2) \circ f^{-1} \in A\big(f(\Gamma)\big)$. Therefore,

$$\begin{aligned}\operatorname{Mod}_{M_2}\big(f(\Gamma)\big) &= \inf_{\sigma_2 \in A(f(\Gamma))} \int_{M_2} \sigma_2^4\, dv_2 \le \inf_{\sigma_1 \in A(\Gamma)} \int_{M_2} \Big(\Big(\frac{\sigma_1}{\lambda_2}\Big) \circ f^{-1}\Big)^4 dv_2 \\ &= \inf_{\sigma_1 \in A(\Gamma)} \int_{M_1} \frac{\sigma_1^4}{\lambda_2^4} \cdot (\lambda_1\lambda_2)^2\, dv_1 \le K^2 \operatorname{Mod}_{M_1}(\Gamma).\end{aligned} \tag{3.17}$$

$$\begin{aligned}\operatorname{Mod}_{M_1}(\Gamma) &= \inf_{\sigma_1 \in A(\Gamma)} \int_{M_1} \sigma_1^4\, dv_1 \le \inf_{\sigma_2 \in A(f(\Gamma))} \int_{M_1} (\lambda_1 \cdot \sigma_2 \circ f)^4\, dv_1 \\ &= \inf_{\sigma_2 \in A(f(\Gamma))} \int_{M_2} \frac{(\lambda_1 \circ f^{-1} \cdot \sigma_2)^4}{(\lambda_1\lambda_2)^2 \circ f^{-1}}\, dv_2 \le K^2 \operatorname{Mod}_{M_2}\big(f(\Gamma)\big).\end{aligned} \tag{3.18}$$

Assume that f satisfies the inequalities (3.9). Then f must respect contact structure in the sense that $f_*(HM_1) \subset HM_2$. Otherwise, there will be a point $q \in M_1$ with a tangent vector $X_q \in H_qM_1$ so that $f_*(X_q) \notin H_{f(q)}M_2$. f is C^1, so there is a neighborhood U of q with a nonzero smooth section X of HM_1 on U so that $f_*(X_{q'})$ is not contact for each $q' \in U$. Let Γ be the family of trajectories of X, and shrink U appropriately so that $\operatorname{Mod}_{M_1}(\Gamma) \neq 0$. On the other hand, curves in $f(\Gamma)$ are not Legendrian, hence not locally rectifiable. So $\operatorname{Mod}_{M_2}\big(f(\Gamma)\big) = 0$ by Definition 3.1. Thus f cannot satisfy (3.9) for this family Γ. This contradiction shows that f must be contact.

If f is not K-quasiconformal, there exists an open set $U \subset M_1$ such that $\lambda_1/\lambda_2 \geq K+\varepsilon$, for some $\varepsilon > 0$. Let X be the vector field on U so that $|X|_1 = 1$, $|f_*(X)|_2 = \lambda_2$. Let Γ be the family of trajectories of X in U. Then for any function σ_2 on $f(U)$, non-negative Borel-measurable,

$$\int_{f(\gamma)} \sigma_2 = \int_\gamma \sigma_2 \circ f \cdot |f_*(X)|_2 = \int_\gamma \lambda_2 \cdot \sigma_2 \circ f. \tag{3.19}$$

So $\sigma_2 \in A\bigl(f(\Gamma)\bigr)$ if and only if $\sigma_1 \triangleq \lambda_2 \cdot \sigma_2 \circ f \in A(\Gamma)$. Hence

$$\begin{aligned}\mathrm{Mod}_{M_2}\bigl(f(\Gamma)\bigr) &= \inf_{\sigma_2 \in A(f(\Gamma))} \int_{M_2} \sigma_2^4 \, dv_2 \\ &= \inf_{\sigma_1 \in A(\Gamma)} \int_{f(U)} \Bigl(\Bigl(\frac{\sigma_1}{\lambda_2}\Bigr) \circ f^{-1}\Bigr)^4 dv_2 \\ &= \inf_{\sigma_1 \in A(\Gamma)} \int_U \Bigl(\frac{\sigma_1}{\lambda_2}\Bigr)^4 \cdot (\lambda_1\lambda_2)^2 \, dv_1 \\ &= \inf_{\sigma_1 \in A(\Gamma)} \int_U \Bigl(\frac{\lambda_1}{\lambda_2}\Bigr)^2 \sigma_1^4 \, dv_1 \geq (K+\varepsilon)^2 \, \mathrm{Mod}_{M_1}(\Gamma).\end{aligned} \tag{3.20}$$

Therefore, (3.9) implies that

$$(K+\varepsilon)^2 \mathrm{Mod}_{M_1}(\Gamma) \leq K^2 \mathrm{Mod}_{M_1}(\Gamma). \tag{3.21}$$

But we can always choose U such that $\mathrm{Mod}_{M_1}(\Gamma) \neq 0$. So (3.9) cannot be true for such Γ. Hence $\lambda_1/\lambda_2 \leq K$ on M_1, i.e., f is K-quasiconformal. □

It would be important to know if we can use (3.9) to define quasiconformality for a homeomorphism $f\colon M_1 \to M_2$. The rest of this section is devoted to this problem.

Theorem 3.4. *If $f\colon M_1 \to M_2$ is a homeomorphism so that for a constant $K \geq 1$ and any curve family Γ which forms a smooth contact fibration of an open set in M_1*

$$\mathit{Mod}_{M_1}(\Gamma) \leq K^2 \, \mathit{Mod}_{M_2}\bigl(f(\Gamma)\bigr), \tag{3.22}$$

then f is ACL.

Proof. Let $U \subset M_1$ be an open set with a smooth contact fibration Γ, $X \neq 0$ be a horizontal vector field on U tangent to Γ. By replacing X by $X/|X|_1$, we can assume $|X|_1 = 1$. Shrink U, if necessary, so that there is a smooth surface $S \subset U$ which intersects each fiber of Γ transversally once and only once. Choose a local coordinate system such that fibers $\gamma \in \Gamma$ is parametrized by t, $X = \partial/\partial t$ and $\gamma(0) \in S$. Let $p\colon U \to S$ denote the natural projection given by $\gamma(t) \mapsto \gamma(0)$.

Recall that Hermitian metrics on HM_1 and HM_2 can be extended canonically to Riemannian metrics on M_1 and M_2 respectively. Restricting the Riemannian metric on M_1 to S makes S a Riemannian 2-manifold. Let ω be the area form on S, and $A_\omega(E) \triangleq \int_E \omega$ the ω-area of a measurable set $E \subset S$. Define a set function F for measurable set $E \subset S$ by letting $F(E) = \mathrm{vol}\big(f\big(p^{-1}(E)\big)\big)$, where vol refers to the Riemannian volume on M_2. Then Lebesgue's theorem ([15, Theorem 23.5]) asserts that F has finite derivatives at all points in S_1 with respect to ω-area, for a subset $S_1 \subset S$ with $A_\omega(S \setminus S_1) = 0$. We want to prove f is absolutely continuous along the fibers of Γ passing through points in S_1.

For $q \in S_1$, let $D_r \subset S$ be a disc centered at q with radius r, $\gamma_q\colon I \to U$ be the fiber of γ passing through q. Take any sequence $t_1', t_1'', t_2', t_2'', \dots, t_k', t_k'' \in I$ such that

$$t_1' < t_1'' < t_2' < t_2'' \cdots < t_k' < t_k''. \tag{3.23}$$

Let $\Delta t_j \triangleq t_j'' - t_j'$. When $\max_{1\le j\le k} \Delta t_j$ is small enough,

$$d_{r,j} \le \Delta t_j \le 2\, d_{r,j}, \tag{3.24}$$

where $d_{r,j}$ is the sub-Riemannian distance between the set $B_{r,j}' \triangleq \{\gamma(t_j') \mid \gamma \in \Gamma,\ \gamma(0) \in D_r\}$ and the set $B_{r,j}'' \triangleq \{\gamma(t_j'') \mid \gamma \in \Gamma,\ \gamma(0) \in D_r\}$. For $j = 1, 2, \dots, k$, denote

$$R_{r,j} \triangleq \{\gamma(t) \mid \gamma \in \Gamma,\ \gamma(0) \in D_r,\ t_j' \le t \le t_j''\}$$

and let

$$\Gamma_{r,j} \triangleq \{\gamma_j = \gamma|_{[t_j', t_j'']} \mid \gamma \in \Gamma,\ \gamma(0) \in D_r\},$$

a contact fibration of $R_{r,j}$. Then by (3.22),

$$\mathrm{Mod}_{M_1}(\Gamma_{r,j}) \le K^2\, \mathrm{Mod}_{M_2}\big(f(\Gamma_{r,j})\big). \tag{3.25}$$

Next we use length-volume argument to give an estimate for $\mathrm{Mod}_{M_1}(\Gamma_{r,j})$. For $\sigma \in A(\Gamma_{r,j})$ and $\gamma_j \in \Gamma_{r,j}$,

$$1 \le \Big(\int_{\gamma_j} \sigma\Big)^4 \le \Big(\int_{\gamma_j} \sigma^4\Big)\Big(\int_{\gamma_j} 1\Big)^3 = (\Delta t_j)^3 \int_{\gamma_j} \sigma^4. \tag{3.26}$$

Integrating each term against ω over D_r, then

$$A_\omega(D_r) \le c(\Delta t_j)^3 \int_{R_{r,j}} \sigma^4\, dv_1. \tag{3.27}$$

where c is constant, dv_1 is the volume form of the Riemannian metric on M_1. By (3.3) and (3.24),

$$A_\omega(D_r) \le 8c{d_{r,j}}^3\, \mathrm{Mod}_{M_1}(\Gamma_{r,j}). \tag{3.28}$$

On the other hand, let $\delta_{r,j}$ be the sub-Riemannian distance between $f(B'_{r,j})$ and $f(B''_{r,j})$. Then $1/\delta_{r,j} \in A\big(f(\Gamma_{r,j})\big)$. Thus

$$\mathrm{Mod}_{M_2}\big(f(\Gamma_{r,j})\big) \le \frac{1}{\delta_{r,j}^4}\mathrm{vol}\big(f(R_{r,j})\big). \tag{3.29}$$

Combining (3.25), (3.28) and (3.29), we have

$$\big(A_\omega(D_r)\big)^{1/4}\,\delta_{r,j} \le (8cK^2)^{1/4} {d_{r,j}}^{3/4}\big(\mathrm{vol}\big(f(R_{r,j})\big)\big)^{1/4}. \tag{3.30}$$

Summing (3.30) over j,

$$\begin{aligned}\big(A_\omega(D_r)\big)^{1/4}\sum_{j=1}^{k}\delta_{r,j} &\le (8cK^2)^{1/4}\sum_{j=1}^{k}{d_{r,j}}^{3/4}\big(\mathrm{vol}\big(f(R_{r,j})\big)\big)^{1/4}\\ &\le (8cK^2)^{1/4}\Big(\sum_{j=1}^{k}d_{r,j}\Big)^{3/4}\Big(\sum_{j=1}^{k}\mathrm{vol}\big(f(R_{r,j})\big)\Big)^{1/4}\\ &\le (8cK^2)^{1/4}\Big(\sum_{j=1}^{k}d_{r,j}\Big)^{3/4}\big(\mathrm{vol}\big(f\big(p^{-1}(D_r)\big)\big)\big)^{1/4}.\end{aligned} \tag{3.31}$$

So by (3.24),

$$\sum_{j=1}^{k}\delta_{r,j} \le (8cK^2)^{1/4}\Big(\sum_{j=1}^{k}\Delta t_j\Big)^{3/4}\Big(\frac{F(D_r)}{A_\omega(D_r)}\Big)^{1/4}. \tag{3.32}$$

Letting $r\to 0$,

$$\sum_{j=1}^{k}\delta_j \le \big(8cK^2F'(q)\big)^{1/4}\Big(\sum_{j=1}^{k}\Delta t_j\Big)^{3/4}, \tag{3.33}$$

where $\delta_j \triangleq \lim_{r\to 0}\delta_{r,j}$, i.e., the sub-Riemannian distance between $f\big(\gamma_q(t_j)\big)$ and $f\big(\gamma_q(t_{j+1})\big)$. Hence f is absolutely continuous along γ_q, for $q\in S_1$. So f is ACL. □

Corollary 3.5. *If a homeomorphism* $f\colon M_1\to M_2$ *satisfies* (3.22), *then* f *has horizontal derivatives almost everywhere on* M_1 *in the sense that for any open set* $U\subset M_1$ *with a smooth section* X *of* HM_1 *on* U *and any smooth function* h *on* $f(U)$, $X(h\circ f)$ *exists almost everywhere on* U. *Moreover* f *has* L^4_{loc} *horizontal derivatives.*

Proof. We need only prove the last statement for non-zero X. Using notations as in the proof of Theorem 3.4, we will show $X(h\circ f)\in L^4_{\mathrm{loc}}$ along γ_q for $q\in S$ where $F'(q)$ exists. Without loss of generality, we assume $\gamma_q\colon I=[a,b]\to M_1$ is a closed curve. Let $t_0=a,\ t_1,\dots,t_N=b$ be a partition of I. For $j=0,1,\dots,N-1$, define $\delta_{r,j},\delta_j,d_{r,j}$, and d_j similarly. Then the same proof of (3.30) shows that

$$\left(\frac{\delta_{r,j}}{d_{r,j}}\right)^4 d_{r,j}\le 8cK^2\frac{\mathrm{vol}\left(f(R_{r,j})\right)}{A_w(D_r)}, \tag{3.34}$$

when the partition of I is fine enough. Then

$$\sum_{j=1}^{N-1}\left(\frac{\delta_{r,j}}{d_{r,j}}\right)^4 d_{r,j}\le 8cK^2\frac{\mathrm{vol}\left(f\left(p^{-1}(D_r)\right)\right)}{A_w(D_r)}. \tag{3.35}$$

Letting $r\to 0$, we obtain

$$\sum_{j=1}^{N-1}\left(\frac{\delta_j}{d_j}\right)^4 d_j\le 8cK^2F'(q). \tag{3.36}$$

Letting $\max_{0\le j\le N-1}|t_{j+1}-t_j|\to 0$, by an elementary argument based on Fatou's lemma [11],

$$\int_{\gamma_q}|f_*(X)|^4\le 8cK^2F'(q). \tag{3.37}$$

Note F' exists for almost all $q\in D_r$ for a fixed small $r>0$ and F' is integrable over D_r by Lebesgue's theorem. Taking integral of both sides of (3.37) over D_r against ω,

$$\int_{p^{-1}(D_r)}|f_*(X)|^4\,dv_1\le 8c_1K^2\int_{D_r}F'\omega<\infty. \tag{3.38}$$

Hence f has L^4_{loc} horizontal derivatives. □

4. Extremal quasiconformal homeomorphisms

In this section we will construct CR 3-manifolds M_1, M_2 and a quasiconformal diffeomorphism $f_0\colon M_1\to M_2$ such that $K(f_0)\le K(f)$, for any C^2 quasiconformal homeomorphism $f\colon M_1\to M_2$ homotopic to f_0. M_1 to be constructed is the quotient of the 3-dimensional Heisenberg group by a lattice. So we start with the Heisenberg group.

The 3-dimensional Heisenberg group $\mathbf{H}^3$ is the space $\mathbf{R}^3$ endowed with the group structure defined by

$$(x,y,t)\,(u,v,s) = (x+u, y+v, t+s+2yu-2xv). \tag{4.1}$$

The standard contact structure on $\mathbf{H}^3$ is given by the contact form

$$\tilde\eta = -\tfrac{1}{2}y\,dx + \tfrac{1}{2}x\,dy + \tfrac{1}{4}\,dt. \tag{4.2}$$

The contact bundle $H\mathbf{H}^3 = \operatorname{Ker}\tilde\eta$ has two global sections

$$\widetilde X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \qquad \widetilde Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t} \tag{4.3}$$

which span $H\mathbf{H}^3$ everywhere and are invariant under the left group translation. The standard CR structure on $\mathbf{H}^3$ is

$$\tilde J\colon H\mathbf{H}^3 \to H\mathbf{H}^3, \qquad \widetilde X \mapsto \widetilde Y, \qquad \widetilde Y \mapsto -\widetilde X. \tag{4.4}$$

Note that $d\tilde\eta = dx\wedge dy$, thus the sub-Riemannian metric on $H\mathbf{H}^3$ determined by the area form $d\tilde\eta$ and the complex structure $\tilde J$, i.e., the Levi form, is the one making $\{\widetilde X, \widetilde Y\}$ orthonormal.

Next we study the geodesics on $\mathbf{H}^3$ with respect to this sub-Riemannian metric. Since the metric is invariant under left group translations, it suffices to study geodesics joining the origin $O=(0,0,0)$ and a generic point q. Denote by p the projection from $\mathbf{H}^3$ to the horizontal plane $P \triangleq \{t=0\}$. It is easy to see that if γ is a rectifiable curve in $\mathbf{H}^3$ with respect to the sub-Riemannian metric, $p(\gamma)$ is rectifiable with respect to the Euclidean metric on $\{t=0\}$, and the respective lengths of γ and $p(\gamma)$ coincide. The characterization of the geodesics given in the next theorem was first given by Korányi by studying the Euler–Lagrange equations [3]. The following proof, which is due to Lempert [8], is a geometric one.

Proposition 4.1. *On $\mathbf{H}^3$, any minimal geodesic connecting the origin O and $q\in\mathbf{H}^3$ is either a straight line segment if $q\in P$, or an arc of a helix whose projection to P is a circle if $q\notin P$.*

Proof. The conclusion is obvious if $q\in P$. Next we assume $q\notin P$. We first consider the case $p(q)=O$, i.e., $q=(0,0,t)$ for some $t\neq 0$. Let

$$\gamma\colon [0,l]\to\mathbf{H}^3, \qquad s\mapsto \big(x(s),y(s),t(s)\big) \tag{4.5}$$

be any oriented rectifiable curve joining O and q. Then

$$-2yx' + 2xy' + t' = 0, \tag{4.6}$$

almost everywhere on γ by (4.2). Thus

$$t = \int_0^l t'\,ds = \int_{p(\gamma)} 2y\,dx - 2x\,dy = -4\int_\Omega dx \wedge dy, \tag{4.7}$$

where Ω is the 2-chain on P so that $\partial\Omega = p(\gamma)$. Now the sub-Riemannian length of γ is equal to the Euclidean length of $p(\gamma)$. Obviously, the length of $p(\gamma)$ is the minimal only when Ω is a simply connected domain with fixed area $\frac{1}{4}|t|$. Furthermore, $p(\gamma)$ must be a circle if γ is a length minimizing geodesic joining O and q, by the isoperimetric property on the Euclidean plane.

If $p(q) \neq O$, the above reasoning with some slight modifications can be applied. For instance, Ω is a 2-chain bounded by the union of $p(\gamma)$ and the straight line segment from $p(q)$ to O. □

Take $\sigma, \tau \in \mathbf{R}$, $\tau \neq 0$. Let $A = (1,0,0)$, $B = (\sigma,\tau,0)$. Let Γ be the discrete subgroup of $\mathbf{H}$ generated by $\{A, B\}$. Left group translations by elements of Γ define a Γ-action. The CR structure on $\mathbf{H}$ is invariant under this Γ-action. We consider the quotient space $M_1 \triangleq \mathbf{H}/\Gamma$. Let $\pi\colon \mathbf{H} \to M_1$ denote the natural projection. M_1 has a contact structure with the contact form η satisfying $\pi^*\eta = \tilde\eta$ and a CR structure J_1 inherited from $\tilde J$. Hence M_1 becomes a CR manifold. Let $X = \pi_*\widetilde X$, $Y = \pi_*\widetilde Y$. Then with respect to the sub-Riemannian metric uniquely determined by η and J_1, $\{X, Y\}$ is orthonormal. Note also $\overline{Z}_1 = \frac{1}{2}(X + iY) \in T^{0,1}M_1$. There is an $\mathbf{R}$-action on $\mathbf{H}$ defined by

$$(x, y, t) \mapsto (x, y, t + t'), \qquad \text{for } t' \in \mathbf{R}. \tag{4.8}$$

This $\mathbf{R}$-action is free and preserves the CR structures on $\mathbf{H}^3$. The plane P can be regarded as the quotient space of this $\mathbf{R}$-action. The natural projection $p\colon \mathbf{H}^3 \to P$ is isometric between the sub-Riemannian metric on $\mathbf{H}^3$ and the Euclidean metric on P. The free $\mathbf{R}$-action on $\mathbf{H}^3$ commutes with the Γ-action. Thus it induces a free S^1-action on M_1 so that M_1 becomes a circle bundle over the torus $\Sigma_1 \triangleq \pi(P)$. The torus Σ_1 has a flat Riemannian metric inherited from the Euclidean metric on P. Denote the natural projection by $\pi_1\colon M_1 \to \Sigma_1$. Note the S^1-action preserves the CR structure on M_1. We call such a bundle $M_1 \xrightarrow{\pi_1} \Sigma_1$ a CR circle bundle over a flat torus.

Given a constant $K \geq 1$, define a new CR structure M_2 on the contact structure of M_1 by declaring that

$$\overline{Z}_2 \triangleq \frac{1}{2}\Big(\sqrt{K}\,X + \frac{i}{\sqrt{K}}Y\Big) = \frac{K+1}{4\sqrt{K}}\Big(\overline{Z}_1 + \frac{K-1}{K+1}Z_1\Big) \tag{4.9}$$

is an $(0,1)$ tangent vector. Obviously, the identity mapping $f_0\colon M_1 \to M_2$ is quasiconformal and $K(f_0)(q) = K$ for all $q \in M_1$.

Theorem 4.2. *Let $f\colon M_1 \to M_2$ be a C^2 homeomorphism homotopic to $f_0\colon M_1 \to M_2$. Then $K(f) \geq K$.*

Note the CR structure on M_2 is also invariant under the circle action. So if S_2 denotes the same smooth torus as S_1, but endowed with the complex structure induced from CR structure on M_2, then $M_2 \stackrel{\pi_1}{\to} S_2$ is also a CR circle bundle.

Lemma 4.3 (Strichartz, see [12, Lemma 3.2]). *For any q in a sub-Riemannian manifold M, there is an $\varepsilon > 0$ so that if $q_1, q_2 \in M$ with $d(q_1, q) \leq \varepsilon$ and $d(q_2, q) \leq \varepsilon$, there exists a length minimizing curve joining q_1 and q_2.*

If α is a curve, denote by $[\alpha]$ the homotopy class with fixed end points of α. Recall that $l(\alpha)$ is the length of α with respect to the sub-Riemannian metric on M.

Lemma 4.4. *Let $\alpha\colon [0,1] \to M$ be a curve on a compact sub-Riemannian manifold connecting two points q_1 and q_2, $\inf l(\beta)$ be taken over all $\beta \in [\alpha]$. Then this infimum is attained at a rectifiable curve $\tilde{\alpha} \in [\alpha]$.*

Proof. Let $\varepsilon_q > 0$ be a number for $q \in M$ determined by Lemma 4.3. Since M is compact, we can cover M with finitely many balls $B(q_j, \frac{1}{3}\varepsilon_{q_j})$, $j = 1, 2, \ldots, k$ with respect to the sub-Riemannian metric. Note any two points in the same $B(q_j, \frac{1}{3}\varepsilon_{q_j})$ can be joined by a length minimizing curve within $B(q_j, \varepsilon_{q_j})$.

Let $L = \inf l(\beta)$ over all $\beta \in [\alpha]$. Let $\beta_n \in [\alpha]$ be curves such that $\lim_{n\to\infty} l(\beta_n) = L$. Divide each β_n by points $p_{n0} = q_1, p_{n1}, p_{n2}, \ldots, p_{nN} = q_2 \in \beta_n$ into subcurves $\tau_{n1}, \tau_{n2}, \ldots, \tau_{nN}$ so that each $\tau_{nj} \subset B(q_{m_{nj}}, r)$ for some integer m_{nj} with $1 \leq m_{nj} \leq k$. Here r denotes the radius $\frac{1}{3}\varepsilon_{q_{m_{nj}}}$ for simplicity. Note this N is uniform for all n by appropriately choosing β_n.

Let σ_{nj} be a length minimizing curve in $B(q_{m_{nj}}, 3r)$ to join the end points of τ_{nj} and $\sigma_n = \sigma_{n1}\sigma_{n2}\cdots\sigma_{nN}$. Then since $B(q_{m_{nj}}, 3r)$ is simply connected, $\sigma_{nj} \in [\tau_{nj}]$, and hence $\sigma_n \in [\beta_n] = [\alpha]$. Furthermore

$$L \leq l(\sigma_n) \leq \sum_{j=1}^{N} l(\sigma_{nj}) \leq \sum_{j=1}^{N} l(\tau_{nj}) = l(\beta_n). \tag{4.10}$$

Therefore $\lim_{n\to\infty} l(\sigma_n) = L$.

Since M is compact, there is a sequence $\{n_j\} \subset \mathbf{N}$ so that for each $j = 1, 2, \ldots, N-1$, p_{nj} is convergent to some point p_j, when $n_j \to \infty$. Note $p_{j-1}, p_j \in B(q_{m_j}, r)$ for some integer m_j with $1 \leq m_j \leq k$. Connect p_{j-1}, p_j by a length minimizing curve $\alpha_j \subset B(q_{m_j}, 3r)$ and let $\tilde{\alpha} = \alpha_1\alpha_2\cdots\alpha_N$. Then $\tilde{\alpha} \in [\alpha]$ and

$$l(\tilde{\alpha}) = \lim_{n_i\to\infty} l(\sigma_{n_i}) = L. \ \square \tag{4.11}$$

The next two lemmas will be crucial to establish the extremality of f_0 described by Theorem 4.2.

Lemma 4.5. *The flow of diffeomorphisms h_t generated by X on M_1 preserves the volume form $dv_1 = d\eta \wedge \eta$.*

Proof. This is obvious since the the flow of diffeomorphisms $\tilde{h}_s$ on $\mathbf{H}^3$ generated by $\widetilde{X}$ is given by

$$\tilde{h}_s\colon (x, y, t) \mapsto (x + s, y, t + 2sy), \qquad s \in \mathbf{R} \tag{4.12}$$

which preserves the volume form $d\tilde{\eta} \wedge \tilde{\eta} = dx \wedge dy \wedge dt$. ▫

We define a sub-Riemannian metric on M_2 such that $\{\sqrt{K}\,X, Y/\sqrt{K}\}$ is orthonormal. Note this sub-Riemannian metric induces the same CR structure on M_2. Denote the corresponding curve length by l_2. On a sub-Riemannian manifold M, a geodesic $\gamma\colon \mathbf{R} \to M$ is said to be (homotopically) conjugate point free if for any $[a, b] \subset \mathbf{R}$ the curve $\gamma|_{[a,b]}$ is the unique length minimizing curve in its homotopy class of curves with end points $\gamma(a)$ and $\gamma(b)$.

Lemma 4.6. *Integral curves of the vector field $\sqrt{K}\,X$ are conjugate point free geodesics on M_2.*

Proof. Lifting the sub-Riemannian metric on M_2 to its universal covering $\mathbf{H}$, we obtain a new sub-Riemannian metric on $\mathbf{H}$. Let us call this new sub-Riemannian manifold $\mathbf{H}_2$. On $\mathbf{H}_2$ there is a characterization of geodesics similar to Proposition 4.1.1. The trajectories of $\sqrt{K}\,\widetilde{X}$ are exactly the straight geodesics on $\mathbf{H}_2$. Indeed, the sub-Riemannian metric is invariant under the $\mathbf{R}$-action (4.8), hence a rectifiable curve γ in $\mathbf{H}_2$ has the same length as $p(\gamma)$ on P_2. Here P_2 is the plane P endowed with the flat Riemannian metric chosen such that $\sqrt{K}\,\partial/\partial x, (1/\sqrt{K}\,)(\partial/\partial y)$ are orthonormal. Therefore the x-axis, which is the trajectory of $\sqrt{K}\,\widetilde{X}$ passing through O, is a conjugate point free geodesic. Note $\sqrt{K}\,\widetilde{X}$ and the sub-Riemannian metric are invariant under the group left translation. Hence any other trajectory of $\sqrt{K}\,\widetilde{X}$ is also a conjugate point free geodesic. The lemma follows from this fact. ▫

Proof of Theorem 4.2. Let $f\colon M_1 \to M_2$ be a C^2 quasiconformal homeomorphism. For $q \in M_1$, let $\gamma_{q,a}\colon [-a, a] \to M_1$ be the integral curve of X so that $\gamma_{q,a}(0) = q$. Let $\Gamma_a = \{\gamma_{q,a} \,|\, q \in M_1\}$. Then by Theorem 3.4,

$$\operatorname{Mod}_{M_1}(\Gamma_a) \le K(f)^2 \operatorname{Mod}_{M_2}\big(f(\Gamma_a)\big). \tag{4.13}$$

First note the length of $\gamma_{q,a}$ is $2a$, so $1/2a \in A(\Gamma_a)$ and

$$\operatorname{Mod}_{M_1}(\Gamma_a) \le \int_{M_1} \Big(\frac{1}{2a}\Big)^4 dv_1 = \Big(\frac{1}{2a}\Big)^4 \operatorname{vol}(M_1). \tag{4.14}$$

For $\sigma \in A(\Gamma_a)$,

$$1 \le \Big(\int_{\gamma_{q,a}} \sigma\Big)^4 \le \Big(\int_{\gamma_{q,a}} 1\Big)^3 \Big(\int_{\gamma_{q,a}} \sigma^4\Big) = (2a)^3 \int_{\gamma_{q,a}} \sigma^4. \tag{4.15}$$

That implies

$$\frac{1}{(2a)^3} \le \int_{\gamma_{q,a}} \sigma^4 = \int_{-a}^{a} \big(\sigma\big(\gamma_{q,a}(s)\big)\big)^4 \, ds. \tag{4.16}$$

Integrating both sides of (4.16) against dv_1 with respect to q over M_1,

$$\begin{aligned}\frac{1}{(2a)^3}\operatorname{vol}(M_1) &\le \int_{M_1}\bigg(\int_{-a}^{a}\big(\sigma\big(\gamma_{q,a}(s)\big)\big)^4\,ds\bigg)\,dv_1\\ &= \int_{-a}^{a}\bigg(\int_{M_1}\big(\sigma\big(\gamma_{q,a}(s)\big)\big)^4\,dv_1\bigg)\,ds\\ &= 2a\int_{M_1}\sigma^4\,dv_1.\end{aligned} \tag{4.17}$$

The last equality in (4.17) is due to Lemma 4.5, which implies that

$$\int_{M_1}\big(\sigma\big(\gamma_{q,a}(s)\big)\big)^4\,dv_1 = \int_{M_1}\sigma^4\,h_s^*(dv_1) = \int_{M_1}\sigma^4\,dv_1 \tag{4.18}$$

is independent of s. Therefore

$$\Big(\frac{1}{2a}\Big)^4\operatorname{vol}(M_1) \le \int_{M_1}\sigma^4\,dv_1, \qquad \forall\,\sigma\in A(\Gamma_a). \tag{4.19}$$

Combining (4.14) and (4.19), we get

$$\operatorname{Mod}_{M_1}(\Gamma_a) = \Big(\frac{1}{2a}\Big)^4\operatorname{vol}(M_1). \tag{4.20}$$

Next we estimate $\operatorname{Mod}_{M_2}\big(f(\Gamma_a)\big)$ in (4.13) for a quasiconformal homeomorphism f homotopic to $f_0\colon M_1\to M_2$. Since f is homotopic to f_0, there is a continuous map $H\colon[0,1]\times M_1\to M_2$ so that

$$H(0,q)=f_0(q), \qquad H(1,q)=f(q), \qquad \forall\,q\in M_1. \tag{4.21}$$

Let $\alpha_q\colon[0,1]\to M_2$ be a curve given by $t\mapsto H(t,q)$. Define $G(q)=\inf l_2(\beta)$ over all $\beta\in[\alpha_q]$. Then by Lemma 4.4, $G(q)$ is attained at a rectifiable curve $\tilde\alpha_q\in[\alpha_q]$.

Now we prove that $G(q)$ is continuous on M_1. When q_1 is close enough to q on M_1, $f_0(q_1)$ is close enough to $f_0(q)$ so that there exist a length minimizing curve δ_1 joining $f_0(q)$, $f_0(q_1)$ according to Lemma 4.3. Let $\delta_2=f(\delta_1)$. Then

$$\delta_1\tilde\alpha_{q_1}\delta_2^{-1}\in[\alpha_q], \qquad \delta_1^{-1}\tilde\alpha_q\delta_2\in[\alpha_{q_1}]. \tag{4.22}$$

This implies

$$l_2(\tilde{\alpha}_q) \le l_2(\delta_1) + l_2(\tilde{\alpha}_{q_1}) + l_2(\delta_2^{-1}), \\ l_2(\tilde{\alpha}_{q_1}) \le l_2(\delta_1^{-1}) + l_2(\tilde{\alpha}_q) + l_2(\delta_2). \tag{4.23}$$

In other words,

$$|G(q_1) - G(q)| \le l_2(\delta_1) + l_2(\delta_2). \tag{4.24}$$

Thus there exists $A > 0$ so that

$$G(q) \le A < \infty, \qquad \forall\, q \in M_1, \tag{4.25}$$

since M_1 is compact.

For a curve γ: $[a, b] \to M_1$, $\tilde{\alpha}_{\gamma(a)} f(\gamma) \tilde{\alpha}^{-1}_{\gamma(b)} \in [f_0(\gamma)]$. By Lemma 4.6 and (4.25),

$$l_2\big(f_0(\gamma)\big) \le l_2\big(\tilde{\alpha}_{\gamma(a)}\big) + l_2\big(f(\gamma)\big) + l_2\big(\tilde{\alpha}^{-1}_{\gamma(b)}\big) \le 2A + l_2\big(f(\gamma)\big). \tag{4.26}$$

Applying (4.26) to $\gamma = \gamma_{q,a} \in \Gamma_a$, we obtain

$$2\sqrt{K}\, a \le 2A + l_2\big(f(\gamma_{q,a})\big). \tag{4.27}$$

Hence $1/(2\sqrt{K}\, a - 2A) \in A\big(f(\Gamma_a)\big)$. So we have the following estimate:

$$\mathrm{Mod}_{M_2}\big(f(\Gamma_a)\big) \le \int_{M_2} \Big(\frac{1}{2\sqrt{K}\, a - 2A}\Big)^4 dv_2 = \Big(\frac{1}{2\sqrt{K}\, a - 2A}\Big)^4 \mathrm{vol}\,(M_2). \tag{4.28}$$

Note $\mathrm{vol}\,(M_1) = \mathrm{vol}\,(M_2) \ne 0$, so by (4.13), (4.20) and (4.28), we have

$$\Big(\sqrt{K} - \frac{A}{a}\Big)^2 \le K(f). \tag{4.29}$$

Letting $a \to \infty$, we get that $K \le K(f)$. □

Remark. In horizontal directions, the extremal quasiconformal homeomorphism f_0 behaves as a stretching by the constant factor $\sqrt{K}$ along trajectories of X and a compressing by the same factor along trajectories of JX. The generator T of the circle action is transversal to the contact bundle. In this transversal direction T, f_0 is equivariant under the circle action since it is simply the identity mapping while M_1 and M_2 have the same circle action.

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