Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 21, 1996, 309–352

STOCHASTIC DYNAMICS MACROSCOPICALLY GOVERNED BY THE POROUS MEDIUM EQUATION FOR ISOTHERMAL FLOW

Michael Ekhaus and Timo Seppäläinen

Secure Computing Corporation, 2675 Long Lake Road
Roseville, MN 55113, U.S.A.; ekhaus@sctc.com

Iowa State University, Department of Mathematics
Ames, Iowa 50011-2066, U.S.A.; seppalai@pollux.math.iastate.edu

Abstract. We describe interacting lattice models on the torus whose special feature is that the macroscopic equation of the empirical density is a degenerate parabolic equation, namely the equation of an ideal gas flowing isothermally through a porous medium. The models come in two versions: one with continuous variables and one with particles on the sites. In the particle model a degenerate equation is obtained only if the size of the particle vanishes in the limit, otherwise the limiting equation is a nondegenerate equation that also governs the densities of certain exclusion processes with speed change. We establish basic properties of these models such as attractiveness and reversibility, and prove the hydrodynamic scaling limits for the empirical densities.

1. Introduction and results

The porous medium equation $\partial_t u = \Delta(u^m)$, $m > 1$, has for some time been among the most intensely studied partial differential equations. A rich theory has developed since the fundamental solutions were found in the early 1950's in Russia, but results connecting this equation with interesting stochastic dynamics are few. This equation is a degenerate parabolic equation in the sense that when written in the form $\partial_t u = \nabla \cdot \big(D(u)\,\nabla u\big)$, the diffusion matrix $D(u)$ vanishes for $u = 0$. In this paper we describe some simple interacting lattice models whose empirical densities obey the porous medium equation $\partial_t u = \Delta(u^2)$ in a hydrodynamical scaling limit.

The stochastic model we study comes in two versions, one with continuous variables (the stick model) and one with discrete variables (the particle model). The stick model is a relative of the linear models discussed in Chapter IX of Liggett's monograph [L], in the sense that when an event takes place at some time t, the new configuration η_t is a linear function of the old one η_{t-}. But the rates are not uniform as in the linear models of [L], for an event takes place at a site x at a rate proportional to the size of the variable $\eta(x)$. This model is not new. It has been studied earlier in [SU], where H. Tanaka is credited for suggesting the model. The particle model resembles the zero-range process introduced by Spitzer [S], in that particles jump from a site with an intensity determined by the

1991 Mathematics Subject Classification: Primary 60K35, 82C22.

number of particles at that site. The difference is that now more than one particle may jump simultaneously. The size of an individual particle will be an additional parameter of the model, and only if this size vanishes in the limit do we get a degenerate macroscopic equation.

Physically the porous medium equation with $m = 2$ represents the density of an ideal gas flowing isothermally through a porous medium. Whether these stochastic models can be given natural physical interpretations has yet to be determined.

For background on hydrodynamical scaling limits and their context in statistical physics we suggest the monograph of Spohn [Sp], and the lectures of De Masi and Presutti [DP] for a survey of part of the mathematical theory. The papers of Aronson [A] and Vasquez [V] present overviews of the theory of the porous medium equation.

Our paper is intended as the first of a series of studies of interacting particle systems that have zero range of interaction and lead to a degenerate macroscopic equation. These stochastic models will be generalized in a future paper so that porous medium equations with $m \neq 2$ appear. We present one proof here in fairly complete technical detail so that similar arguments in later work can do with sketchier proofs, but also with the hope of making our exposition an accessible entry point into the mathematics of hydrodynamic limits. With a wider audience in mind we have provided some explanations that a probabilist may find tedious.

The paper is organized as follows: In this section we describe the models and state the scaling limits in Theorems 1, 2 and 3. Theorem 1 is about the stick model with short-range interactions and the limit comes by diffusion scaling. In Theorem 2 the stick model interacts over an intermediate range and a different scaling is needed for the limit. It is possible to adjust the range of interaction so that the scaling limit comes by hyperbolic scaling. Theorem 3 is about the particle model. After presenting the theorems we describe earlier work relating the porous medium equation to stochastic models. Section 1 concludes with a sketch of the proofs. In Section 2 we study the stochastic models more closely. We show that they are attractive in the interacting particle systems sense and ergodic on certain hyperplanes of the state space. We describe the invariant measures, which are also reversible for the process. In Section 3 we prove the scaling limits for the stick model and in Section 4 for the particle model.

1.1. The short-range stick process. We begin with an informal description of the stick process. Fix a dimension d for the remainder of the paper. Let the scaling parameter N be a large natural number. Write $x = (x_1, \ldots, x_d)$ for the sites of the lattice $\mathbf{Z}^d$. Our process lives on the sites of the cube

$$\mathbf{Z}_N^d = \{x \in \mathbf{Z}^d : 0 \leq x_i < N \text{ for } i = 1, \ldots, N\}$$

with periodic boundary conditions, that is, coordinatewise addition in $\mathbf{Z}_N^d$ is performed modulo N. A state of the process is an assignment $\eta = \big(\eta(x) : x \in \mathbf{Z}_N^d\big)$ of nonnegative real numbers $0 \leq \eta(x) < \infty$ to each site x, which could be thought of as the lengths of vertical sticks sitting on the sites. Thus the state space of the process is $\Omega_N = [0,\infty)^{\mathbf{Z}_N^d}$. Fix a probability distribution $p(z)$ on $\mathbf{Z}^d$ such that $p(z) = p(-z)$, thinking of $p(y-x)$ as the step distribution of a random walk on $\mathbf{Z}^d$. Let $p_N(x,y)$ be the step distribution induced on $\mathbf{Z}_N^d$ given by

$$p_N(x,y) = \sum_{z:z\equiv y \bmod N} p(z-x)$$

where $z \equiv y \bmod N$ is understood coordinatewise. The symmetry is preserved: $p_N(x,y) = p_N(y,x)$.

The state evolves in time through events of the following type: When the state of the process is η, each site $x \in \mathbf{Z}_N^d$ has an independent exponential clock with rate $\eta(x)$. When the clock at site x rings, pick a site y with probability $p_N(x,y)$, break off a uniformly distributed random piece from the stick at x, and add this piece to the stick at y.

Here is a more precise description for the benefit of the reader not familiar with the jargon of interacting particle systems: Suppose the state of the process is η at some time t. To determine when the process moves from η to another state, imagine given a collection $\{T_z : z \in \mathbf{Z}_N^d\}$ of independent random variables, where T_z has exponential distribution with expectation $1/\eta(z)$. These are the random clocks. Let x be the site whose clock rings first: $T_x = \min_{z\in\mathbf{Z}_N^d} T_z$. (Since $\mathbf{Z}_N^d$ is finite and the T_z's have continuous distributions, this description is not problematic: The infimum of the T_z's is positive and it is realized at exactly one site, almost surely.) It is conventional to construct processes so that their paths are right-continuous in time. Thus we declare that the process remains at η for the time interval $[t, t+T_x)$, and at time $t' = t+T_x$ it resides at a new state η' determined as follows: Pick y as indicated above and pick a random quantity U uniformly distributed on $[0,\eta(x)]$, both independently of everything else. Then set $\eta'(x) = \eta(x) - U$, $\eta'(y) = \eta(y) + U$, and $\eta'(w) = \eta(w)$ for other sites w. Now repeat this cycle, starting with the state η' at time t' and with new random clocks independent of the past.

To achieve the correct parabolic or diffusion scaling, we speed up the dynamics by a factor N^2. All this is codified in the generator $\mathscr{L}_N$ that acts on bounded continuous functions f on Ω_N:

$$\mathscr{L}_N f(\eta) = N^2 \sum_{x\in\mathbf{Z}_N^d} \int_0^{\eta(x)} \sum_{y\in\mathbf{Z}_N^d} p_N(x,y)\,[f(\eta^{u,x,y}) - f(\eta)]\,du, \tag{1}$$

where the configuration $\eta^{u,x,y}$ is defined, for $x,y,w \in \mathbf{Z}_N^d$ and $0 \le u \le \eta(x)$, by

$$\eta^{u,x,y}(w) = \begin{cases} \eta(x) - u, & w = x, \\ \eta(y) + u, & w = y, \\ \eta(w), & w \ne x, y. \end{cases} \tag{2}$$

For a comprehensive treatment of the theory and machinery of interacting particle systems we refer the reader to [L].

Let $\eta_t = (\eta_t(x) : x \in \mathbf{Z}_N^d)$, with $0 \le t < \infty$ the time variable, denote the Markov process we have described, for a fixed N. Instead of running the process on larger and larger cubes $\mathbf{Z}_N^d$ as N increases, we shrink space by a factor of N and imagine that the stick configuration approximates a density on the d-dimensional torus $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$. This notion is captured by the empirical measure α_t^N determined by the heights of the sticks. Let $\mathscr{M}$ denote the space of finite nonnegative Borel measures on $\mathbf{T}^d$, topologized weakly by $C(\mathbf{T}^d)$. Then α_t^N is the $\mathscr{M}$-valued random variable defined by

$$\alpha_t^N = N^{-d} \sum_{x \in \mathbf{Z}_N^d} \eta_t(x)\, \delta_{x/N},$$

in other words, the integral of a bounded Borel function ϕ on $\mathbf{T}^d$ against α_t^N is given by

$$\alpha_t^N(\phi) = N^{-d} \sum_{x \in \mathbf{Z}_N^d} \eta_t(x)\, \phi\Big(\frac{x}{N}\Big).$$

In general we will replace 't' with '$\bullet$' to denote the whole path as a function of t as opposed to a value at a particular time.

As initial data we assume given a nonnegative, bounded Borel function u_0 on $\mathbf{T}^d$ that serves as the initial macroscopic density of the sticks. This means that, for large N, the empirical density of the sticks at time 0 approximates the measure $u_0(\xi)\,d\xi$ with high probability, where $d\xi$ denotes Lebesgue measure on $\mathbf{T}^d$, for integration purposes identifiable with $[0,1)^d$. For this and certain technical reasons we make the following precise assumption on the initial distributions μ_0^N of the processes:

Assumption 1. *The probability distributions μ_0^N on Ω_N satisfy these conditions: The variables $\eta(x)$, $x \in \mathbf{Z}_N^d$, are independent exponential random variables under μ_0^N. There is a constant K_0 bounding the expectations:*

$$\mu_0^N\{\eta(x)\} \le K_0 \text{ for all } N \text{ and for all } x \in \mathbf{Z}_N^d.$$

The expectations are chosen so that, as $N \to \infty$, $\alpha_0^N \to u_0(\xi)\,d\xi$ in μ_0^N-probability, in the topology of $\mathscr{M}$.

The precise choice of the expectations is immaterial. One can take

$$\mu_0^N\{\eta(x)\} = N^d \int_{(x/N)+[0,1/N)^d} u_0(\xi)\,d\xi,$$

or simply $\mu_0^N\{\eta(x)\} = u_0(x/N)$ if u_0 is continuous. We also need to make further assumptions about p. Let p^{*n} denote the nth convolution power of p. In terms of the random walk specified by p, $p^{*n}(x-y)$ is the probability that, after starting at x and taking n steps, the walker is at y.

Assumption 2. *The probability vector* p *satisfies* $\sum_z p(z)\|z\|^4 < \infty$ *and is irreducible in the sense that for each* $z \in \mathbf{Z}^d$ *there is an* n *such that* $p^{*n}(z) > 0$.

Fix a final time $T < \infty$ and set $Q_T = [0,T] \times \mathbf{T}^d$. Let P^N denote the distribution of the process on the path space $\mathscr{D}_{\Omega_N} = \mathscr{D}([0,T], \Omega_N)$, with the initial distribution μ_0^N. $\mathscr{D}_{\Omega_N}$ is the space of right-continuous functions from $[0,T]$ into Ω_N that have a left limit at each point. It is a Polish space under the so-called Skorokhod topology. This is a standard setting for stochastic processes, developed for example in [B] and [EK]. Similarly, $\mathscr{D}_{\mathscr{M}} = \mathscr{D}([0,T], \mathscr{M})$ is the space of $\mathscr{M}$-valued paths where $\alpha_\bullet^N$ takes its values. Set

$$a_{i,j} = \sum_{z \in \mathbf{Z}^d} p(z) z_i z_j \qquad \text{for } 1 \le i,j \le d,$$

where $z = (z_1, \ldots, z_d)$ denotes a vector in $\mathbf{Z}^d$. Write $\xi = (\xi_1, \ldots, \xi_d)$ for an element of $\mathbf{T}^d$. Here is the first scaling limit:

Theorem 1. *Assume Assumptions* 1 *and* 2. *Then there is a jointly measurable function* $u(t,\xi)$ *on* Q_T *such that, as* $N \to \infty$, $\alpha_\bullet^N \to u(\,\bullet\,,\xi)d\xi$ *in* P^N-*probability, in the topology of* $\mathscr{D}_{\mathscr{M}}$.

Furthermore, $0 \le u(t,\xi) \le \|u_0\|_\infty$, *the measure* $u(t,\xi)d\xi$ *is continuous in* t, *and* $u(t,\xi)$ *is the unique weak solution of*

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j} a_{i,j} \frac{\partial^2}{\partial \xi_i \partial \xi_j}(u^2)$$

on $\mathbf{T}^d$ *with initial condition* $u(0,\xi) = u_0(\xi)$.

The porous medium equation as a special case. Let p be the step probability of symmetric nearest-neighbor random walk: $p(\pm e_i) = 1/2d$ for $i = 1, \ldots, d$, where $e_1 = (1,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0)$, etc. Then the limiting equation is the porous medium equation $\partial_t u = (2d)^{-1}\Delta(u^2)$. The constant $(2d)^{-1}$ can be scaled away by multiplying the generator by $2d$, in other words, by letting the clocks ring at rate $2d\,\eta(x)$ instead of $\eta(x)$.

1.2. The long-range stick process. To specify the long range of interaction fix a parameter $0 < \alpha < 1$ and let

$$V_N = \{x \in \mathbf{Z}^d : |x_i| \le N^\alpha \text{ for all } i\}.$$

To avoid unnecessary technicalities we consider only the simplest step distribution: The receiving site y is chosen from $x + V_N$ uniformly at random, again observing periodic boundary conditions. Otherwise the dynamics of the Nth process follow the description given above for the short-range model. To get a meaningful scaling limit we have to compensate for the increased range by decreasing the time speed-up: N^β with $\beta = 2(1-\alpha)$ turns out to be the correct time scaling. Thus the generator of the Nth process of the long-range model is

$$(3) \qquad \mathscr{L}_N f(\eta) = N^\beta \sum_{x\in \mathbf{Z}_N^d} \int_0^{\eta(x)} \frac{1}{|V_N|} \sum_{y\in x+V_N} [f(\eta^{u,x,y}) - f(\eta)]\,du,$$

with the same conventions as in (1) above. Define the empirical measure α_t^N and the distributions μ_0^N and P^N of the process as before. For this model we have the following scaling limit:

Theorem 2. *Assume Assumption* 1. *Then the conclusion of Theorem* 1 *holds for the sequence of processes with generators* (3) *and initial distributions* μ_0^N, *and the limiting equation is*

$$\frac{\partial u}{\partial t} = \frac{1}{6}\,\Delta(u^2).$$

Remark on hyperbolic scaling. Setting $\alpha = \frac{1}{2}$ gives $\beta = 1$, so in this case the hydrodynamic limit comes by hyperbolic scaling: both space and time are scaled by the factor N. This is potentially of interest for the following reason: Suppose the step distribution $p(z)$ has nonzero expectation. We expect such an asymmetric version of the stick model to obey a nonlinear conservation law under hyperbolic scaling, as is the case for the simple exclusion and zero-range processes (see [R]). Assuming this happens, we can then superimpose on the asymmetric dynamics the long-range dynamics described by the generator (3) with $\alpha = \frac{1}{2}$ and $\beta = 1$, and obtain a system that obeys a viscous conservation law. Though not with the usual diffusive part of the linear heat equation, but instead with a nonlinear diffusive part from the porous medium equation. This superposition of short-range asymmetric and long-range symmetric stochastic dynamics would furnish a method for manufacturing stochastic models for viscous conservation laws. An alternative way is to combine the symmetric dynamics with a weakly asymmetric part, as has been done in [DPS], [G], and [KOV].

1.3. The particle process. For the particle models we shall consider only the short-range case. The setting is identical to that of the stick model except for the changes brought about by the discretization of the state space. Fix again N for the moment and a parameter $\varkappa_N > 0$, the particle size. Let $\eta(x)$ denote the height of the stack of particles at site x, hence an element of the set $\{k\varkappa_N : k = 0,1,2,\ldots\}$. The state space is

$$\Omega_N^{(\varkappa_N)} = \{k\varkappa_N : k = 0,1,2,\ldots\}^{\mathbf{Z}_N^d}.$$

The clock at site x rings at rate $\eta(x)$, and then the number k of particles to move is picked uniformly at random from $\{1,2,\ldots,\eta(x)/\varkappa_N\}$. The generator becomes

$$(4) \qquad \mathscr{L}_N f(\eta) = N^2 \sum_{x,y\in \mathbf{Z}_N^d} p_N(x,y)\,\varkappa_N \sum_{k=1}^{\eta(x)/\varkappa_N} \bigl[f(\eta^{k\varkappa_N,x,y}) - f(\eta)\bigr].$$

Concerning the particle sizes and initial distributions we make the following assumption:

Assumption 1′. *There is a number $\varkappa \geq 0$ such that $\varkappa_N \to \varkappa$ as $N \to \infty$. The probability distributions μ_0^N on $\Omega_N^{(\varkappa_N)}$ satisfy these conditions: The variables $\eta(x)$, $x \in \mathbf{Z}_N^d$, are independent geometrically distributed $\varkappa_N \mathbf{Z}_+$-valued random variables under μ_0^N, there is a constant K_0 such that $\mu_0^N\{\eta(x)\} \leq K_0$ for all N and for all $x \in \mathbf{Z}_N^d$, and the expectations are chosen so that, as $N \to \infty$, $\alpha_0^N \to u_0(\xi)\, d\xi$ in μ_0^N-probability, in the topology of $\mathscr{M}$.*

More precisely, the initial probabilities of individual stack heights are given by

$$\mu_0^N\{\eta : \eta(x) = k\varkappa_N\} = \frac{\varkappa_N \left(\mu_0^N\{\eta(x)\}\right)^k}{\left(\varkappa_N + \mu_0^N\{\eta(x)\}\right)^{k+1}} \qquad \text{for } x \in \mathbf{Z}_N^d \text{ and } k \geq 0.$$

Though the state space of the particle process changes with N, the empirical measure α_t^N is $\mathscr{M}$-valued for each N, so it makes again sense to ask about the convergence of the random $\mathscr{M}$-valued path $\alpha_\bullet^N$.

Theorem 3. *Assume Assumptions* 1′ *and* 2. *Then the statement of Theorem* 1 *holds for the particle processes with generators* (4) *and initial distributions μ_0^N, with the limiting equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j} a_{i,j} \frac{\partial^2}{\partial \xi_i \partial \xi_j} (\varkappa u + u^2). \tag{5}$$

In particular, in the case $\varkappa = 0$ we get the same equation as in Theorem 1.

1.4. Earlier related work. Let us first point out that the hydrodynamic limit of the zero-range process does not yield a degenerate equation, at least under the assumptions employed in [DP]. The diffusion constant of the macroscopic equation is given by $D(u) = z'(u)$ where z denotes fugacity (see Theorem 3.2.1 in [DP]). A computation shows that $z'(u) > 0$ for all $u \geq 0$, and in particular $z'(0) = c(1)$ which is positive by assumption.

As far as we know, the earliest constructions of stochastic models for the porous medium equation are by M. Inoue with an approach completely different from ours. He applied difference schemes to construct diffusion processes whose probability densities are solutions to the porous medium equation [I1] and particle systems converging to solutions of the equation [I2].

The hydrodynamics of the basic stick model with symmetric nearest-neighbor exchanges was studied by Y. Suzuki and K. Uchiyama [SU]. In their approach this model is embedded in a family of not necessarily attractive processes. Hence they do not make use of attractiveness and are led to arguments different from ours. Both approaches have their advantages: The result in [SU] allows for more general initial distributions than our Theorem 1, subject to the restriction that this initial distribution be absolutely continuous with respect to an i.i.d. exponential distribution on the sticks. In other words, the initial stick heights must all be strictly positive almost surely. Our Theorem 1 does not have this restriction. The equality $\mu_0^N\{\eta(x)\} = 0$ is permitted, so the stochastic model may start with some sites initially empty. This follows naturally from attractiveness, as the reader will see in Section 3.6.

It is also believed that the porous medium equation is connected with lattice gas dynamics at critical temperature, see Section II.3.3 in [Sp]. Rigorous results in this direction have been obtained in [LOP], where the actual model studied is a symmetric exclusion process with a weak asymmetry that involves interactions over a microscopically long range. In the critical case the density profile f obeys the macroscopic equation $\partial_t f = \partial_\xi\big[\big(\frac{1}{2} - 2f(1-f)\big)\partial_\xi f\big]$ provided that the initial profile f_0 satisfies either $0 \le f_0 < \frac{1}{2}$ or $\frac{1}{2} < f_0 \le 1$. If we now set $f = \frac{1}{2} - u$ or $f = \frac{1}{2} + u$, depending on the case, then u satisfies the porous medium equation.

For results about deterministic particles whose empirical density obeys the porous medium equation in an infinite particle limit, see [O], [U], and their references.

The equation (5) with $\varkappa > 0$ also governs the densities of certain exclusion processes with speed change. This example is one of a class treated by T. Funaki, K. Handa, and K. Uchiyama [FHU], also presented in Section II.3.2 of [Sp]. Consider the configuration space $\{0,1\}^N$ and the generator

$$L_N f(\eta) = N^2 \sum_{x=0}^{N-1} c(x, x+1, \eta)\,[f(\eta^{x,x+1}) - f(\eta)]$$

with exchange rates

$$c(x, x+1, \eta) = \big(\eta(x) - \eta(x+1)\big)^2\big[1 + \alpha\big(\eta(x-1) + \eta(x+2)\big)\big],$$

where α is a constant satisfying $1 + 2\alpha > 0$, $\eta^{x,x+1}$ is the configuration got from η by interchanging $\eta(x)$ and $\eta(x+1)$, and $x+1$ is taken modulo N. The macroscopic equation for the empirical density of this process is $\partial_t u = \partial_\xi^2(u + \alpha u^2)$ on the unit circle. Taking $\alpha > 0$ and multiplying the generator by α^{-1} then gives (5) with $\varkappa = \alpha^{-1}$.

1.5. The proofs at a glance. Here is a sketch of the proof of Theorem 1. Writing

$$A\phi = \sum_{1 \le i,j \le d} a_{i,j}\partial_{\xi_i}\partial_{\xi_j}\phi$$

for a smooth test function ϕ, the usual martingale arguments show that the equality

$$\alpha_t^N(\phi) - \alpha_0^N(\phi) = \int_0^t \frac{1}{4} N^{-d} \sum_{x \in \mathbf{Z}_N^d} A\phi\Big(\frac{x}{N}\Big)\,\eta_s^2(x)\,ds \tag{6}$$

holds approximately, with high probability. In a cube $x+\Lambda_{N\varepsilon}$ of intermediate scale ε, $N^{-1} \ll \varepsilon \ll 1$, the sticks are almost in equilibrium. They behave approximately as i.i.d. exponential random variables with expectation given by the local empirical mean

$$\frac{1}{|\Lambda_{N\varepsilon}|} \sum_{y \in x+\Lambda_{N\varepsilon}} \eta_t(y).$$

Since $A\phi$ is nearly constant across $\Lambda_{N\varepsilon}$, this turns (6) into

$$\alpha_t^N(\phi) - \alpha_0^N(\phi) = \int_0^t \frac{1}{2} N^{-d} \sum_{x \in \mathbf{Z}_N^d} A\phi\Big(\frac{x}{N}\Big) \bigg(\frac{1}{|\Lambda_{N\varepsilon}|} \sum_{y \in x+\Lambda_{N\varepsilon}} \eta_s(y)\bigg)^2 ds,$$

and upon introducing $\chi_{\varepsilon,\xi}$, the characteristic function of the cube $\xi+[0,\varepsilon)^d$ in $\mathbf{T}^d$ normalized by the volume ε^d, we have that

$$\alpha_t^N(\phi) - \alpha_0^N(\phi) = \frac{1}{2} \int_0^t \int_{\mathbf{T}^d} A\phi(\xi) \left[\alpha_s^N(\chi_{\varepsilon,\xi})\right]^2 d\xi \, ds,$$

again approximately and with high probability. In the limit $N \to \infty$, α_t^N has a density $u(t,\xi)$, and after also letting $\varepsilon \searrow 0$ we recover the weak form of the differential equation:

$$\int_{\mathbf{T}^d} u(t,\xi)\, \phi(\xi)\, d\xi - \int_{\mathbf{T}^d} u_0(\xi)\, \phi(\xi)\, d\xi = \frac{1}{2} \int_0^t \int_{\mathbf{T}^d} A\phi(\xi)\, u^2(s,\xi)\, d\xi \, ds.$$

To prove the local equilibrium we use the method of entropy estimates developed by Guo, Papanicolau and Varadhan [GPV]. However, since the equilibrium distribution is exponential, an entropy estimate alone cannot control higher moments. That is, if ν is an exponential distribution and μ some other probability measure on $[0,\infty)$, an entropy bound $H(\mu \,|\, \nu) \le C$ does not guarantee that $\int s^{1+\varepsilon}\, \mu(ds) < \infty$. To control moments we use a priori bounds that come from the attractiveness of the stochastic process. The degeneracy presents another source of trouble for the entropy estimate: Suppose the initial density u_0 equals zero on a set with nonempty interior. Then there will be sites at which the initial distribution is a unit mass at zero. But δ_0 is not absolutely continuous with respect to a nondegenerate exponential distribution, and we would have infinite entropy. Thus we prove Theorem 1 in two parts: We first assume that u_0 is bounded away from zero, and remove this assumption only at the very end.

The proof of Theorem 2 for the long-range model does not introduce anything conceptually new, but some estimates need different proofs for the two cases. The proof of Theorem 3 proceeds along similar lines, and in Section 4 we briefly touch on those aspects of its proof that differ from the earlier arguments.

Acknowledgements. We gratefully acknowledge valuable advice from D. Aronson and from S.R.S. Varadhan, especially for suggesting the argument utilized in Section 3.6. The second author also thanks the IMA at the University of Minnesota and the Institut Mittag-Leffler for their hospitality.

2. The stochastic processes

2.1. The stick process. We now take a closer look at the properties of the Markov processes and begin with a concrete construction of the stick process simultaneously for all initial states η on a single probability space. Such a construction is called a coupling in the probability literature. Fix N and the distribution p. The time scale factor plays no role here, so we leave out N^2 and N^β from (1) and (3), respectively, and then (3) becomes a special case of (1).

The probability space is constructed by first giving each site $x \in \mathbf{Z}_N^d$ an independent copy $\mathscr{A}^x$ of a Poisson point process on the positive quadrant $\{(t,b) : t \geq 0, b \geq 0\}$ of the plane, with Lebesgue measure as the intensity measure. (Think of the coordinate t as time and b as the height of the stick at x.) Next, for each point $(t,b) \in \mathscr{A}^x$, pick a site $y = y^x_{(t,b)}$ with probability $p_N(x,y)$ independently of everything else. These are the random choices needed.

Given an initial state η, the process η_t is defined as a $\mathscr{D}\big([0,\infty),\Omega_N\big)$-valued function of η and the random variables $\{\mathscr{A}^x\}$, $\{y^x_{(t,b)}\}$. The distribution P^η of the process starting at η is then the probability measure on $\mathscr{D}\big([0,\infty),\Omega_N\big)$ induced by this function. Let

$$T_x = \inf\big\{t : (t,b) \in \mathscr{A}^x, b \leq \eta(x)\big\}$$

be the first time t that a point of $\mathscr{A}^x$ is contained in the rectangle $[0,t]\times[0,\eta(x)]$. Take this to mean that the clock rang at x. T_x is exponentially distributed with rate $\eta(x)$ by the definition of a Poisson point process. Let x be the site whose clock rang first and $(t,b) \in \mathscr{A}^x$ the point that triggered the event. Let $y = y^x_{(t,b)}$ be the receiving site chosen for this point. Declare $\eta_s = \eta$ for $0 \leq s < t$, and the new state η_t at time t is obtained by setting $\eta_t(x) = b$, $\eta_t(y) = \eta(y) + \eta(x) - b$, and by leaving the other sticks intact. Now start over again with η_t the current state and consider the point processes on $[t,\infty)\times[0,\infty)$. Note that a.s. there is no other point with the same t-coordinate. Note also that, conditioned on $b \leq \eta(x)$, b is uniformly distributed on $[0,\eta(x)]$, and hence the piece we broke off $\eta(x)$ was uniformly distributed. Each point process has only finitely many points in each rectangle $[0,t] \times \big[0, \sum_x \eta(x)\big]$, so to determine the state at time t a new state needs to be computed only finitely many times. Thus this construction defines the process for all times $0 \leq t < \infty$. It is also not hard to derive the generator from this description and arrive at (1) without the N^2 time scale factor.

With this coupling, proof of attractiveness reduces to a mere observation, and the reader who already made this observation is invited to skip the next proof. Given η and ζ in Ω_N, let η_t and ζ_t denote the processes with initial conditions η and ζ, respectively, constructed as above as functions on the probability space of the point processes $\{\mathscr{A}^x\}$ and the receiving sites $\{y^x_{(t,b)}\}$. Let $P^{\eta,\zeta}$ be the probability measure on $\mathscr{D}\big([0,\infty),\Omega_N\times\Omega_N\big)$ induced by the function (η_t,ζ_t). $P^{\eta,\zeta}$ is the distribution of the joint process (η_t,ζ_t) starting at (η,ζ).

Lemma 2.1. *Assume $\eta \leq \zeta$ (pointwise for all x). Then the joint process constructed above satisfies $P^{\eta,\zeta}\{\eta_t \leq \zeta_t\} = 1$ for all $t \geq 0$.*

Proof. Pick and fix a realization of $\{\mathscr{A}^x\}$ and $\{y^x_{(t,b)}\}$. We shall argue that $\eta_t \le \zeta_t$, while true for $t = 0$ by assumption, continues to hold for all $t > 0$. It is true for $0 \le t < \varepsilon$ for some $\varepsilon > 0$ because there is a strictly positive time before the process jumps. Assume it is true for all times $0 \le t < s$, and a clock rings at x at time s. By assumption, $\eta_{s-} \le \zeta_{s-}$, so if (s, b) is the point that triggered the event, either $b \le \eta_{s-}(x) \le \zeta_{s-}(x)$ or $\eta_{s-}(x) < b \le \zeta_{s-}(x)$. In the first case $\eta_s(x) = \zeta_s(x) = b$ and $\eta_s(y) = \eta_{s-}(y) + \eta_{s-}(x) - b \le \zeta_{s-}(y) + \zeta_{s-}(x) - b = \zeta_s(y)$, where $y = y^x_{(t,b)}$ is the receiving site; and similarly the inequality is preserved in the second case. Thus $\eta_t \le \zeta_t$ holds for $0 \le t < s + \varepsilon$ where $\varepsilon > 0$ is the time to the next jump after s. Again because only finitely many jumps take place in a finite time interval, $\eta_t \le \zeta_t$ holds for all t and for each realization of $\{\mathscr{A}^x\}$ and $\{y^x_{(t,b)}\}$, hence in particular with probability 1. ▫

Next we characterize the invariant measures of the stick process and prove that these measures are reversible for the dynamics. The total length $\sum_x \eta(x)$ is obviously conserved under the dynamics, hence it is natural to study the process on the hyperplanes

$$\Omega_{N,\lambda} = \left\{ \eta \in \Omega_N : \sum_x \eta(x) = \lambda \right\}$$

for $\lambda \ge 0$. Each $\Omega_{N,\lambda}$ supports a probability measure m_λ that could be heuristically described as 'the conditional distribution of Lebesgue measure, given that $\sum_x \eta(x) = \lambda$'. Precisely speaking, we define the integral of a bounded Borel function g on Ω_N against m_λ, for $\lambda > 0$, by

$$\begin{aligned} m_\lambda(g) = &\frac{(N^d-1)!}{\lambda^{N^d-1}} \int_0^\lambda d\eta(x_1) \int_0^{\lambda-\eta(x_1)} d\eta(x_2) \cdots \int_0^{\lambda - \sum_{i=1}^{N^d-2} \eta(x_i)} d\eta(x_{N^d-1}) \\ &\times g\Big(\eta(x_1), \eta(x_2), \ldots, \eta(x_{N^d-1}), \lambda - \sum_{i=1}^{N^d-1} \eta(x_i) \Big). \end{aligned}$$

Here $\{x_1, x_2, \ldots, x_{N^d}\}$ is an arbitrary ordering of the sites in $\mathbf{Z}^d_N$. In the course of the proof of the next lemma we show that the process is ergodic on each hyperplane $\Omega_{N,\lambda}$ and m_λ is the unique invariant measure on $\Omega_{N,\lambda}$.

Lemma 2.2. *A probability measure μ on Ω_N is invariant for the process if and only if it is a mixture of the m_λ's. Furthermore, each such mixture is reversible for the process.*

Before proving Lemma 2.2, we wish to point out that i.i.d. exponential distributions are mixtures of m_λ's. Let

$$\gamma_r(dw) = I_{\{w>0\}} \frac{1}{r} e^{-w/r} dw \tag{7}$$

denote the exponential distribution with expectation r. Then

$$\gamma_r^{\otimes \mathbf{Z}^d_N}(d\eta) = \frac{r^{-N^d}}{(N^d-1)!} \int_0^\infty m_\lambda(d\eta)\, \lambda^{N^d-1} e^{-\lambda/r}\, d\lambda.$$

The mixing measure is the law of $\sum_x \eta(x)$ under $\gamma_r^{\otimes \mathbf{Z}_N^d}$, namely a gamma distribution with parameters $1/r$ and N^d. Furthermore, if $\lambda/N^d \to r$ as $N \to \infty$, then m_λ converges weakly to $\gamma_r^{\otimes \mathbf{Z}^d}$. This suggests that as we proceed towards the scaling limit, the only relevant equilibria are the i.i.d. exponentials and their mixtures. This becomes clear in the course of the proof: see Proposition 3.7 where the local equilibrium is established.

Now for the proof of Lemma 2.2. First we show that on $\Omega_{N,\lambda}$ any two initial states eventually couple.

Lemma 2.3. *Suppose $\eta, \zeta \in \Omega_{N,\lambda}$. Then the joint process (η_t, ζ_t) can be defined so that $P^{\eta,\zeta}\{\eta_t = \zeta_t$ for all large enough $t\} = 1$.*

Proof. We ask the reader to consider another equivalent way of constructing the process: Instead of giving each site an individual clock, we take only one clock whose rate is the sum of the stick lengths. When it rings, we pick the site x that gives off a piece with probability proportional to its stick length, and then proceed as before. Except that to couple two processes, the clock needs to ring at a rate $\overline{\lambda}_t = \sum_x \overline{\eta}_t(x)$, where $\overline{\eta}_t(x) = \eta_t(x) \vee \zeta_t(x)$. If the clock rings at time t, pick a site x with probability $\overline{\eta}_{t-}(x)/\overline{\lambda}_{t-}$, pick a random quantity B uniformly distributed on $[0, \overline{\eta}_{t-}(x)]$, and pick a site y with probability $p_N(x,y)$. The new state is defined by letting $\eta_t(x) = \eta_{t-}(x) \wedge B$ and $\eta_t(y) = \eta_{t-}(y) + [\eta_{t-}(x) - \eta_t(x)]$, and similarly for ζ_t. Then $\overline{\eta}_t$ and $\overline{\lambda}_t$ are updated appropriately. In other words, the taller one of $\eta_{t-}(x)$ and $\zeta_{t-}(x)$ gives off a piece to the site y, and the shorter may or may not, depending on whether it is above or below the cutoff height B. We leave it to the reader to verify that the marginals of this joint process on the η and ζ coordinates are indeed again processes with generator (1), without the N^2. It is obvious that, if $\eta_t = \zeta_t$ for some time t, then $\eta_t = \zeta_t$ for all later times t too. Note that earlier we denoted by U the piece of stick that was moved and now by B the piece that remains.

The argument that η_t and ζ_t eventually couple proceeds as follows. Fix a sequence $x_0, x_1, x_2, \ldots, x_s = x_0$ of sites such that every site appears at least once and $p_N(x_i, x_{i+1}) > 0$ for each i. Let $\gamma = \lambda/4s^2$. Rotate the labels of the $x_0, \ldots, x_s$ so that $\eta(x_0) \geq 4s\gamma$. Such a site x_0 must exist since $\sum_x \eta(x) = \lambda$. Let E_i be the following event: When the ith clock rings, $x = x_{i-1}$, $y = x_i$ and $B \in [\gamma, 2\gamma]$. (We continue to use x to denote the site that gives off a piece and y to denote the receiving site.) Consider what happens to η on the intersection $E_1 \cap \cdots \cap E_s$: For $i = 1, \ldots, s$, as the ith clock rings, the stick at x_{i-1} is cut and the piece is passed on to x_i. At the ith ring site x_i receives a piece of length at least $(4s - 2i)\gamma$, hence the stick left at x_i after the $(i+1)$st ring has length at least γ. Finally, after s rings, all sticks for η have length at least γ.

While we went through this cycle, somewhere a ζ-stick of length at least $4s\gamma$ was cut. The piece was passed along, and each time it lost at most 2γ of its length, hence after the s first rings, the ζ-stick at x_0 has length at least $2s\gamma$. Now repeat the cycle, replacing γ with $\frac{1}{2}\gamma$. In other words, let E_{s+j} be the event: When the $(s+j)$th clock rings, $x = x_{j-1}$, $y = x_j$ and $B \in [\frac{1}{2}\gamma, \gamma]$. Let

$E = E_1 \cap \cdots \cap E_s \cap E_{s+1} \cap \cdots \cap E_{2s}$. The point to notice is that after $2s$ rings $\eta = \zeta$ on the event E. This is because the second round equalizes the η- and ζ-sticks at each site in turn. Before the $(s+j+1)$st ring, both sticks at x_j have length at least γ, and after the $(s+j+1)$st ring they have a common length $B \in [\frac{1}{2}\gamma, \gamma]$, for $j = 0, \dots, s-1$; the $(2s)$th ring puts the remaining total length to the site x_0.

Let the superscript (i) denote a value after the $(i-1)$st but before the ith ring. At each ring $1 \le i \le s$, assuming that $E_1 \cap \cdots \cap E_{i-1}$ occurred, the probability of choosing $x = x_{i-1}$, $y = x_i$, and $B \in [\gamma, 2\gamma]$ equals

$$\frac{\overline{\eta}^{(i)}(x_{i-1})}{\overline{\lambda}^{(i)}} \cdot p_N(x_{i-1}, x_i) \cdot \frac{\gamma}{\overline{\eta}^{(i)}(x_{i-1})} \ge \frac{\gamma}{2\lambda} \cdot p_N(x_{i-1}, x_i). \tag{8}$$

Note that $B \in [\gamma, 2\gamma]$ has probability $\gamma / \overline{\eta}^{(i)}(x_{i-1})$ at the ith ring due to the conditioning on $E_1 \cap \cdots \cap E_{i-1}$ because then $\eta^{(i)}(x_{i-1}) \ge \big(4s - 2(i-1)\big)\gamma \ge 2\gamma$. Bound (8) works for $s+1 \le i \le 2s$ upon replacing γ by $\frac{1}{2}\gamma$. Thus there is a number $\varepsilon_0 > 0$ such that, conditioned on $E_1 \cap \cdots \cap E_{i-1}$ and the ith ring, E_i occurs with at least probability ε_0.

Now we include the clocks. Fix $R > 0$, and let G_i be the event: The clock rings exactly once in the time interval $\big[(i-1)R, iR\big)$ and when it rings E_i happens. Let T_i denote an exponential clock with rate $\overline{\lambda}^{(i)}$. Conditioned on $G_1 \cap \cdots \cap G_{i-1}$, the probability of G_i equals

$$\begin{aligned}\int_0^R & P^{\eta^{(i)}, \zeta^{(i)}}\big(E_i \cap \{T_{i+1} \ge R - t\}\big) P^{\eta^{(i)}, \zeta^{(i)}}(T_i \in dt) \\ &\ge \int_0^R \varepsilon_0 \exp\big[-\overline{\lambda}^{(i+1)}(R-t)\big] \overline{\lambda}^{(i)} \exp\big[-\overline{\lambda}^{(i)} t\big]\, dt \ge \varepsilon_1\end{aligned}$$

for a constant $\varepsilon_1 > 0$, uniformly over $1 \le i \le s$, $\eta^{(i)}$, and $\zeta^{(i)}$, because $\lambda \le \overline{\lambda}^{(i)} \le 2\lambda$ holds for all i. G_i is measurable with respect to the process on the time interval $\big[(i-1)R, iR\big)$, so the Markov property gives, with $T = 2sR$,

$$P^{\eta,\zeta}\{\eta_t = \zeta_t \text{ for } t \ge T\} \ge P^{\eta,\zeta}(G_1 \cap \cdots \cap G_{2s}) \ge \varepsilon_1^{2s} \equiv \varepsilon_2.$$

This bound holds uniformly over all starting states $(\eta, \zeta) \in \Omega_{N,\lambda} \times \Omega_{N,\lambda}$, hence again by the Markov property

$$P^{\eta,\zeta}\{\eta_{kT} \ne \zeta_{kT}\} \le (1 - \varepsilon_2) P^{\eta,\zeta}\{\eta_{(k-1)T} \ne \zeta_{(k-1)T}\} \le \cdots \le (1 - \varepsilon_2)^k.$$

An application of the Borel–Cantelli lemma completes the proof. ▫

Proof of Lemma 2.2. We start by verifying the formula

$$m_\lambda\Big\{\int_0^{\eta(x)} h(u,\eta^{u,x,y})\,du\Big\} = m_\lambda\Big\{\int_0^{\eta(y)} h(u,\eta)\,du\Big\}, \tag{9}$$

valid for bounded Borel functions h on $\mathbf{R}\times\Omega_N$ and all sites x and y, with $\eta^{u,x,y}$ defined by (2). The left-hand side equals

$$\begin{aligned}\int_0^\lambda d\eta(x)\int_0^{\lambda-\eta(x)} d\eta(y)\cdots\int_0^{\eta(x)} du\, h(u,\eta^{u,x,y}) \\ = \int_0^\lambda du\int_0^{\lambda-u} d\eta(y)\int_u^{\lambda-\eta(y)} d\eta(x)\cdots h(u,\eta^{u,x,y})\end{aligned}$$

where we changed the integration order and '$\cdots$' denotes the integrals over all the other sticks except $\eta(x)$ and $\eta(y)$. Now do the change of variable

$$\begin{cases}\omega(x)=\eta(x)-u,\\ \omega(y)=\eta(y)+u\end{cases}$$

and again change integration order. This yields the right-hand side of (9). To prove that m_λ is reversible, we need to show that

$$m_\lambda(f\,\mathscr{L}_N g) = m_\lambda(g\,\mathscr{L}_N f). \tag{10}$$

The left-hand side equals

$$\sum_{x,y} p_N(x,y) m_\lambda\Big\{\int_0^{\eta(x)} f(\eta)g(\eta^{u,x,y})\,du\Big\} - \sum_{x,y} p_N(x,y) m_\lambda\Big\{\int_0^{\eta(x)} f(\eta)g(\eta)\,du\Big\}.$$

Consider a term in the first sum for fixed x and y. Define

$$h(u,\eta) = f(\eta^{u,y,x})g(\eta)\,I_{\{\eta(y)\ge u\}}.$$

Then the integrand equals $h(u,\eta^{u,x,y})$, and applying (9) shows that the first sum equals

$$\sum_{x,y} p_N(x,y) m_\lambda\Big\{\int_0^{\eta(y)} f(\eta^{u,y,x})g(\eta)\,du\Big\}$$

which in turn equals

$$\sum_{x,y} p_N(x,y) m_\lambda\Big\{\int_0^{\eta(x)} f(\eta^{u,x,y})g(\eta)\,du\Big\}$$

by the symmetry of p_N. The equation (10) follows. We have proved that the m_λ's, and consequently their mixtures too, are reversible invariant measures for the dynamics.

Let now μ be an arbitrary invariant probability measure for the process on $\Omega_{N,\lambda}$ and f a bounded Borel function on $\Omega_{N,\lambda}$. By the invariance,

$$|m_\lambda(f) - \mu(f)| = \Big| \int E^{\eta,\zeta}\{f(\eta_t) - f(\zeta_t)\}\, m_\lambda(d\eta)\, \mu(d\zeta) \Big| \\ \le 2\|f\| \int P^{\eta,\zeta}\{\eta_t \ne \zeta_t\}\, m_\lambda(d\eta)\, \mu(d\zeta) \longrightarrow 0$$

as $t \to \infty$, by Lemma 2.3. Thus $\mu = m_\lambda$.

Finally, let μ be an invariant probability measure on Ω_N. Let ν be the distribution of $\sum_x \eta(x)$ under μ, and μ_λ its conditional distribution on $\Omega_{N,\lambda}$. To show that $\mu = \int m_\lambda\, \nu(d\lambda)$, it suffices to show, by the previous paragraph, that ν-almost every μ_λ is invariant. Let f be a function on Ω_N and g a function on $[0,\infty)$. In the next calculation use the definitions of ν and μ_λ, the invariance of μ and the fact that $\sum_x \eta(x)$ is conserved:

$$\nu\{\mu_\lambda(f)\, g(\lambda)\} = \mu\Big\{ f(\eta)\, g\Big(\sum_x \eta(x)\Big)\Big\} = \mu\Big\{ E^\eta \Big[f(\eta_t) g\Big(\sum_x \eta_t(x)\Big)\Big]\Big\} \\ = \mu\Big\{ E^\eta f(\eta_t)\, g\Big(\sum_x \eta(x)\Big)\Big\} = \nu\big\{\mu_\lambda\big(E^\eta f(\eta_t)\big) g(\lambda)\big\}.$$

Thus $\mu_\lambda(f) = \mu_\lambda\big(E^\eta f(\eta_t)\big)$ for ν-a.e. λ, and letting f vary over a countable collection of functions that separates measures shows that almost every μ_λ is invariant. This shows that μ is a mixture of m_λ's and completes the proof of Lemma 2.2. □

2.2. The particle process. This analysis is considerably easier for the particle models whose state spaces are countable, and we simply record the facts, leaving the details to the reader. The particle processes are attractive. The invariant hyperplanes

$$\Omega_{N,\ell} = \Big\{ \eta \in \Omega_N^{(\varkappa_N)} : \sum_x \eta(x) = \ell \varkappa_N \Big\}$$

for integral ℓ are finite sets, and the unique invariant measure on $\Omega_{N,\ell}$ is the uniform distribution m_ℓ that gives equal probability to each configuration. The formula corresponding to (10) now reads

$$m_\ell\Big\{ \sum_{k=1}^{\eta(x)/\varkappa_N} h(k\varkappa_N, \eta^{k\varkappa_N, x, y}) \Big\} = m_\ell\Big\{ \sum_{k=1}^{\eta(y)/\varkappa_N} h(k\varkappa_N, \eta) \Big\}. \tag{11}$$

The asymptotic analysis as $N \to \infty$ will be slightly different for the cases $\varkappa = 0$ and $\varkappa > 0$. The former case returns to the stick model with the relevant equilibria given by i.i.d. exponential distributions. For the latter case the equilibria will be i.i.d. geometric distributions.

3. Proofs of Theorems 1 and 2

Throughout this section, for each N at a time, the process η_t denotes either the short-range or the long-range stick process with generator (1) or (3), respectively, initial distribution μ_0^N on Ω_N and distribution P^N on the path space $\mathscr{D}_{\Omega_N}$. The distribution of the process on Ω_N at time t is denoted by μ_t^N. The proofs of Theorems 1 and 2 are carried out simultaneously with unified notation, and with separate arguments furnished only when necessary. We first treat the case where u_0 is bounded away from zero:

Assumption 3. *For some constant $\varepsilon_0 > 0$ and all $\xi \in \mathbf{T}^d$, $N \in \mathbf{N}$, and $x \in \mathbf{Z}_N^d$, $u_0(\xi) \geq \varepsilon_0$ and $\mu_0^N\{\eta(x)\} \geq \varepsilon_0$.*

Here is an outline of the proof:

3.1. *Preliminaries.* The topology of $\mathscr{M}$. The a priori estimate. The martingales governing the evolution of α_t^N.

3.2. *The distributions of $\alpha_\bullet^N$.* The distributions of $\alpha_\bullet^N$ are tight in $\mathscr{D}_{\mathscr{M}}$, and every limit point $\mathscr{P}$ is supported by elements $\omega_\bullet \in C([0,T],\mathscr{M})$ such that $\mathscr{P}(d\omega) \otimes dt$-almost every ω_t has a density with respect to Lebesgue measure. These are consequences of the a priori estimate alone.

3.3. *The local equilibrium.* We introduce an intermediate scale ε so that $N^{-1} \ll \varepsilon \ll 1$ and prove that in a cube of size $N\varepsilon$ the sticks behave almost like i.i.d. exponential random variables. The entropy bound needed for this step utilizes Assumption 3.

3.4. *Further technicalities.* The final steps needed for proving that weak limits of the distributions of $\alpha_\bullet^N$ are supported by weak solutions of the differential equation.

3.5. *Uniqueness.* The weak convergence is upgraded to convergence in probability by showing that a solution to the differential equation is unique. This step is independent of Assumption 3.

3.6. *Removing Assumption* 3. We let $\varepsilon_0 \searrow 0$ in Assumption 3 and argue that in the limit we recover Theorems 1 and 2 as stated.

3.1. Preliminaries.

Let $\{\phi_k\}$ be a countable set of smooth functions on $\mathbf{T}^d$ such that $\phi_1 \equiv 1$, $\|\phi_k\|_\infty \leq 1$ for all k and the span of $\{\phi_k\}$ is dense in the space $C(\mathbf{T}^d)$. Then the $C(\mathbf{T}^d)$-topology of $\mathscr{M}$ can be metrized by

$$r(\mu,\nu) = \sum_{k=1}^{\infty} 2^{-k-1}|\mu(\phi_k) - \nu(\phi_k)|. \tag{12}$$

Lemma 3.1. *$(\mathscr{M}, r)$ is a complete separable metric space.*

Proof. Let $\{\mu_n\}$ be a Cauchy sequence in the metric r. The sequence $\{\mu_n(\phi_1)\}$ is Cauchy, hence converges. If $\lim_{n\to\infty} \mu_n(\phi_1) = 0$, then $\mu_n \to 0$ in the metric r. So suppose $\lim_{n\to\infty} \mu_n(\phi_1) = a > 0$. An easy computation shows that the sequence $\{\tilde{\mu}_n\}$, $\tilde{\mu}_n = \mu_n/\mu_n(\phi_1)$, is also Cauchy in the metric r. The measures $\tilde{\mu}_n$ are elements of the space $\mathscr{M}_1$ of probability measures, which is known to be compact since $\mathbf{T}^d$ is compact (see [B] or [EK]). Thus r is complete as a metric on $\mathscr{M}_1$, and we have a measure $\tilde{\mu} \in \mathscr{M}_1$ such that $\tilde{\mu}_n \to \tilde{\mu}$ in the r-metric. It follows that $\mu_n \to a\tilde{\mu}$ in the r-metric. □

Next we turn to the a priori estimates that give bounds over the moments of the sticks, uniformly over t and N.

Lemma 3.2. *Under Assumption* 1, *there are constants $C_k < \infty$ such that $E^N\{\eta_t^k(x)\} \le C_k$ for all t, N, and k.*

First we show that products of exponential distributions dominate each other stochastically if the sitewise expectations dominate each other (see Section II.2 in [L] for the definitions). We continue the convention of letting (η, ζ) denote an element of $\Omega_N \times \Omega_N$. Recall (7) for the definition of γ_r.

Lemma 3.3. *Suppose $\mu = \bigotimes_{x \in \mathbf{Z}_N^d} \gamma_{r(x)}$ and $\nu = \bigotimes_{x \in \mathbf{Z}_N^d} \gamma_{s(x)}$ are two products of exponential distributions on Ω_N. If $r(x) \le s(x)$ for all x, then there exists a probability measure Q on $\Omega_N \times \Omega_N$ such that $Q\{(\eta,\zeta) : \eta \le \zeta\} = 1$, the η-marginal of Q is μ, and the ζ-marginal of Q is ν.*

Proof. For each x, define the distribution Q_x of the pair $\big(\eta(x), \zeta(x)\big)$ so that $\zeta(x)$ is distributed according to $\gamma_{s(x)}$ and $\eta(x) = \big(r(x)/s(x)\big)\zeta(x)$ almost surely. Take $Q = \otimes_{x \in \mathbf{Z}_N^d} Q_x$. □

Proof of Lemma 3.2. Set $\nu = \bigotimes_{x \in \mathbf{Z}_N^d} \gamma_{K_0}$ where K_0 is the constant appearing in Assumption 1. Let Q be the measure given by Lemma 3.3 with marginals μ_0^N and ν and supported by the set $\{(\eta,\zeta) : \eta \le \zeta\}$. For each such pair (η, ζ), construct the process $P^{\eta,\zeta}$ as in Lemma 2.1, and define a joint process P^Q with initial distribution Q by

$$E^Q\{f(\eta_\bullet, \zeta_\bullet)\} = \iint E^{\eta,\zeta}\{f(\eta_\bullet,\zeta_\bullet)\}\, Q(d\eta, d\zeta).$$

Let Q_t be the distribution of the joint process at time t, similarly μ_t^N and ν_t for the processes started with μ_0^N and ν, respectively. Then

$$Q_t\{\eta \le \zeta\} = \iint P^{\eta,\zeta}\{\eta_t \le \zeta_t\}\, Q(d\eta, d\zeta) = 1$$

by Lemma 2.1. The marginals of Q_t are μ_t^N and ν_t by construction, and $\nu_t = \nu$ because ν is invariant by Lemma 2.2 and the remark following it. Again by Theorem 2.4 on p. 74 in [L],

$$\mu_t^N\{\eta^k(x)\} \le \nu\{\eta^k(x)\} = K_0^k k! \equiv C_k. \;\square$$

For the remainder of this subsection fix a smooth test function ϕ on $\mathbf{T}^d$. We look for two processes $z_1(t) = z_1^{N,\phi}(t)$ and $z_2(t) = z_2^{N,\phi}(t)$ such that the processes

$$M_t = M_t^{N,\phi} = \alpha_t^N(\phi) - \alpha_0^N(\phi) - \int_0^t z_1(s)\,ds$$

and

$$V_t = V_t^{N,\phi} = M_t^2 - \int_0^t z_2(s)\,ds$$

are martingales. It is well-known, and not hard to verify, that these are given by $z_1(t) = \mathscr{L}_N f(\eta_t)$ and $z_2(t) = \mathscr{L}_N(f^2)(\eta_t) \; -2f(\eta_t)\mathscr{L}_N f(\eta_t)$, where $f(\eta) = N^{-d}\sum_{x\in\mathbf{Z}_N^d}\phi(x/N)\eta(x)$. We start with the short-range model, so let $\mathscr{L}_N$ be given by (1). Thinking of ϕ as a periodic function on $\mathbf{R}^d$, using the symmetry of p and by Taylor expanding, we may write

(13)
$$\begin{aligned}
z_1(t) &= N^{2-d}\sum_{x\in\mathbf{Z}_N^d}\sum_{z\in\mathbf{Z}^d} p(z)\frac{1}{2}\eta_t^2(x)\Big\{\phi\Big(\frac{x+z}{N}\Big) - \phi\Big(\frac{x}{N}\Big)\Big\}\\
&= \frac{1}{4}N^{-d}\sum_{x\in\mathbf{Z}_N^d}\eta_t^2(x)\sum_{z\in\mathbf{Z}^d}p(z)N^2\Big\{\phi\Big(\frac{x+z}{N}\Big)+\phi\Big(\frac{x-z}{N}\Big)-2\phi\Big(\frac{x}{N}\Big)\Big\}\\
&= \frac{1}{4}N^{-d}\sum_{x\in\mathbf{Z}_N^d}\eta_t^2(x)\sum_{z\in\mathbf{Z}^d}p(z)\Big\{\sum_{1\le i,j\le d}\partial_{\xi_i}\partial_{\xi_j}\phi\Big(\frac{x}{N}\Big)z_iz_j + O(N^{-1}\|z\|^3)\Big\}\\
&= \frac{1}{4}N^{-d}\sum_{x\in\mathbf{Z}_N^d}\eta_t^2(x)A\phi\Big(\frac{x}{N}\Big) + O(N^{-1})\cdot\sigma_3\frac{1}{4}N^{-d}\sum_{x\in\mathbf{Z}_N^d}\eta_t^2(x),
\end{aligned}$$

where

$$A\phi = \sum_{1\le i,j\le d} a_{i,j}\partial_{\xi_i}\partial_{\xi_j}\phi$$

with $a_{i,j} = \sum_{z\in\mathbf{Z}^d}p(z)z_iz_j$, and $\sigma_3 = \sum_{z\in\mathbf{Z}^d}p(z)\|z\|^3$, a finite constant by Assumption 2. The constant hidden in the error term $O(N^{-1})$ depends on the size of the third derivatives of ϕ. In particular, this estimate holds uniformly in t. Unraveling the definition of $z_2(t)$ gives, also uniformly in t,

$$\begin{aligned}
z_2(t) &= N^{-2d}\sum_{x\in\mathbf{Z}_N^d}\frac{1}{3}\eta_t^3(x)\sum_{z\in\mathbf{Z}^d}p(z)N^2\Big\{\phi\Big(\frac{x+z}{N}\Big)-\phi\Big(\frac{x}{N}\Big)\Big\}^2\\
&= N^{-2d}\sum_{x\in\mathbf{Z}_N^d}\frac{1}{3}\eta_t^3(x)\sum_{z\in\mathbf{Z}^d}p(z)\Big\{\Big\langle\nabla\phi\Big(\frac{x}{N}\Big),z\Big\rangle + O(N^{-1}\|z\|^2)\Big\}^2 \qquad (14)\\
&= O(N^{-d})\cdot\sigma_4N^{-d}\sum_{x\in\mathbf{Z}_N^d}\eta_t^3(x),
\end{aligned}$$

where $\sigma_4 = \sum_{z\in\mathbf{Z}^d} p(z)\|z\|^4$, another finite constant by Assumption 2. These estimates and the a priori bound are basic for everything that follows.

For the long-range model, the same computations give

$$z_1(t) = \frac{1}{12} N^{-d} \sum_{x\in\mathbf{Z}_N^d} \eta_t^2(x)\Delta\phi\Big(\frac{x}{N}\Big) + O(N^{-\alpha})\,\cdot\, N^{-d} \sum_{x\in\mathbf{Z}_N^d} \eta_t^2(x) \tag{15}$$

and

$$z_2(t) = O(N^{-d})\,\cdot\, N^{-d} \sum_{x\in\mathbf{Z}_N^d} \eta_t^3(x). \tag{16}$$

Let us declare $c = \frac{1}{4}$ for the short-range model, $c = \frac{1}{12}$ for the long-range model, and $A = \Delta$ for the long-range model. Then the leading part of $z_1(t)$ is given by

$$c\, N^{-d} \sum_{x\in\mathbf{Z}_N^d} \eta_t^2(x)\, A\phi\Big(\frac{x}{N}\Big)$$

for both models.

3.2. The distributions of $\alpha_\bullet^N$. The distributions of $\alpha_\bullet^N$ are probability measures $\mathscr{P}^N$ on $\mathscr{D}_{\mathscr{M}}$, defined by $\mathscr{P}^N(B) = P^N\{\alpha_\bullet^N \in B\}$ for Borel subsets B of $\mathscr{D}_{\mathscr{M}}$. We write $\omega_\bullet$ for a generic element of $\mathscr{D}_{\mathscr{M}}$.

Lemma 3.4. *The sequence $\{\mathscr{P}^N\}$ is tight.*

Proof. By Theorem 15.3 in [B] it is enough to show (i) compact containment, that for any $\varepsilon > 0$ there exists a compact $K \subset \mathscr{M}$ such that

$$P^N\{\alpha_t^N \in K \text{ for all } 0 \le t \le T\} \ge 1 - \varepsilon, \tag{17}$$

and (ii) that, given positive ε_1 and ε_2, there exists a $0 < \delta < 1$ and N_0 such that

$$P^N\Big\{\sup_{|s-t|\le\delta} r(\alpha_t^N, \alpha_s^N) \ge \varepsilon_1\Big\} \le \varepsilon_2 \tag{18}$$

for all $N \ge N_0$. (Theorem 15.3 in [B] is stated only for real valued processes, but it is not hard to see that its proof and the proof of Theorem 14.4 on which it is based both apply to complete metrics on arbitrary Polish spaces.)

Compact containment is immediate from conservation of total stick length: $\alpha_t^N(\mathbf{T}^d) = N^{-d}\sum_x \eta_t(x) = N^{-d}\sum_x \eta_0(x)$ P^N-a.s., and so

$$P^N\Big\{\sup_{0\le t\le T} \alpha_t^N(\mathbf{T}^d) \ge B\Big\} \le B^{-1} N^{-d} \sum_x \mu_0^N\{\eta(x)\} \le C/B,$$

where the last inequality comes from Assumption 1. The sets $\mathscr{M}^B = \{\nu \in \mathscr{M} : \nu(\mathbf{T}^d) \le B\}$ are compact for $B < \infty$ and (17) follows by taking $K = \mathscr{M}^B$ for $B > C/\varepsilon$.

Next we show that

$$\limsup_{\delta\searrow 0}\limsup_{N\to\infty} E^N\Big\{\sup_{|s-t|\le\delta} |\alpha_t^N(\phi) - \alpha_s^N(\phi)|^2\Big\} = 0 \tag{19}$$

for an arbitrary smooth function ϕ on $\mathbf{T}^d$. Then (18) follows by Markov's inequality and the definition (12) of the metric r. From

$$\alpha_t^N(\phi) - \alpha_s^N(\phi) = M_t - M_s + \int_s^t z_1(u)\,du$$

we see that the integrand in (19) is bounded above by

$$\sup_{|s-t|\le\delta} 2(M_t - M_s)^2 + \sup_{|s-t|\le\delta} 2\bigg(\int_s^t z_1(u)\,du\bigg)^2. \tag{20}$$

Bound the first term by $\sup_{0\le t\le T} 8M_t^2$, then apply Doob's inequality, the definition of the martingale V_t, the bound (14) on $z_2(u)$, and finally the a priori estimate Lemma 3.2:

$$E^N\Big\{\sup_{0\le t\le T} M_t^2\Big\} \le CE^N\{M_T^2\} = C\,E^N\bigg\{\int_0^T z_2(u)\,du\bigg\} \le CTN^{-d}. \tag{21}$$

(C denotes a constant whose value changes from line to line.) For the second term in (20) apply Schwarz's inequality and (13) to get

$$\begin{aligned}\sup_{|s-t|\le\delta}\bigg(\int_s^t z_1(u)\,du\bigg)^2 &\le \sup_{0\le s\le t\le(s+\delta)\wedge T}(t-s)\int_s^t z_1^2(u)\,du \le \delta\int_0^T z_1^2(u)\,du\\ &\le C\delta N^{-d}\sum_x\int_0^T \eta_u^4(x)\,du.\end{aligned}$$

Now integrate and use the a priori bound to bound the expectation by $CT\delta$. This together with (21) gives

$$E^N\Big\{\sup_{|s-t|\le\delta}|\alpha_t^N(\phi) - \alpha_s^N(\phi)|^2\Big\} \le CT(N^{-d}+\delta),$$

which implies (19). For the long-range model, just use (15)–(16) instead of (13)–(14). $\square$

Secondly we establish some properties of the limit points of $\{\mathscr{P}^N\}$.

Lemma 3.5. *Let $\mathscr{P}$ be a limit point of $\{\mathscr{P}^N\}$. Then $\mathscr{P}$ is supported by continuous paths, $\omega_t \ll d\xi$ for $\mathscr{P}(d\omega_\bullet) \otimes dt$-a.e. measure ω_t, and for all $1 \le p < \infty$,*

$$\mathscr{E}\left\{\int_0^T \int_{\mathbf{T}^d} u^p(t,\xi)\,d\xi\,dt\right\} < \infty, \tag{22}$$

where the $\mathscr{P}(d\omega_\bullet) \otimes dt \otimes d\xi$-a.e. defined derivative $u(\omega_\bullet, t, \xi) = (d\omega_t/d\xi)(\xi)$ is jointly measurable in the variable $(\omega_\bullet, t, \xi)$, and $\mathscr{E}$ denotes expectation under $\mathscr{P}$.

Proof. Let $\Delta(\omega_\bullet) = \sup_t r(\omega_t, \omega_{t-})$ be the maximal jump of a path $\omega_\bullet \in \mathscr{D}_{\mathscr{M}}$. It is a continuous function in the Skorokhod topology, hence

$$\mathscr{E}\{\Delta(\omega_\bullet)\} \le \limsup_{N\to\infty} E^N\{\Delta(\alpha_\bullet^N)\}.$$

It is obvious that $\Delta(\omega_\bullet) \le \sup_{|s-t|\le\delta} r(\omega_s, \omega_t)$ for all $\delta > 0$, hence (19) implies that $\mathscr{E}\{\Delta(\omega_\bullet)\} = 0$. In other words, $\mathscr{P}$-a.e. $\omega_\bullet$ is continuous.

For the second part, let $\phi \ge 0$ be a bounded continuous function on $\mathbf{T}^d$, and $p > 0$ an integer.

$$\begin{aligned}\int_0^T E^N\{\alpha_t^N(\phi)^p\}\,dt &= N^{-pd}\sum_{x_1,\dots,x_p}\phi\Big(\frac{x_1}{N}\Big)\cdots\phi\Big(\frac{x_p}{N}\Big)\int_0^T E^N\{\eta_t(x_1)\cdots\eta_t(x_p)\}\,dt\\ &\le CTN^{-pd}\sum_{x_1,\dots,x_p}\phi\Big(\frac{x_1}{N}\Big)\cdots\phi\Big(\frac{x_p}{N}\Big) = CT\Big(N^{-d}\sum_x\phi\Big(\frac{x}{N}\Big)\Big)^p,\end{aligned}$$

where the inequality comes from the a priori bound. Letting $N \to \infty$ along a suitable subsequence gives

$$\frac{1}{T}\int_0^T\int \omega_t(\phi)^p\,\mathscr{P}(d\omega)\,dt \le C\Big(\int_{\mathbf{T}^d}\phi(\xi)\,d\xi\Big)^p.$$

The proof is then completed by an application of the following lemma. ▫

Lemma 3.6. *Let X and Y be two Polish spaces, μ and κ probability measures on X and Y, respectively, and let $y \mapsto \nu^y$ be a measurable map from Y into the space of finite real-valued Borel measures on X, topologized weakly by bounded continuous functions on X. Suppose that for some constants $1 < p < \infty$ and $C < \infty$ and all bounded continuous $f \ge 0$ on X,*

$$\int_Y \nu^y(f)^p\,\kappa(dy) \le C\mu(f)^p. \tag{23}$$

Then $\nu^y \ll \mu$ for κ-a.e. y, the derivative $\varphi^y(x) = (d\nu^y/d\mu)(x)$ is jointly measurable in (x,y) and satisfies

$$\int_Y \mu\{(\varphi^y)^p\}\,\kappa(dy) \le C. \tag{24}$$

Remark. To complete the proof of Lemma 3.5, take $X = \mathbf{T}^d$, $Y = \mathscr{D}_{\mathscr{M}} \times [0,T]$, $\mu = d\xi$, $\kappa = \mathscr{P} \otimes T^{-1}\,dt$, and $\nu^y = \omega_t$ for $y = (\omega_\bullet, t)$.

Proof. We give a probabilistic proof, based on constructing a Radon–Nikodym derivative from the martingale convergence theorem. Since (23) is preserved under bounded pointwise convergence, it holds for all bounded Borel functions $f \geq 0$ on X. Let $\{\mathscr{B}_k\}$ be an increasing sequence of finite partitions that generate $\mathscr{B}_X$, the Borel field of X. By taking $f = I_A$ for $A \in \bigcup_k \mathscr{B}_k$ in (23) we see that $\nu^y \ll \mu$ on each $\mathscr{B}_k$, outside a single κ-null set of y's. Set

$$\varphi_k(x,y) = \sum_{\substack{A:A\in\mathscr{B}_k \\ \mu(A)>0}} I_A(x)\frac{\nu^y(A)}{\mu(A)}.$$

Then $\varphi_k^y = (d\nu^y/d\mu)|_{\mathscr{B}_k}$ and also $\varphi_k = (d\nu/d\mu\otimes\kappa)|_{\mathscr{B}_k\otimes\mathscr{B}_Y}$, where the measure ν on $X\times Y$ is defined by $\nu(dx,dy) = \nu^y(dx)\,\kappa(dy)$ and $\mathscr{B}_Y$ is the Borel field of Y. Under the measure $\mu\otimes\kappa$, φ_k is a nonnegative martingale with respect to the filtration $\{\mathscr{B}_k\otimes\mathscr{B}_Y\}$, hence there is a $\mu\otimes\kappa$-a.s. limit $\varphi(x,y) = \lim_{k\to\infty}\varphi_k(x,y)$.

If $\{\varphi_k\}$ were uniformly $\mu\otimes\kappa$-integrable, then $\varphi_k \to \varphi$ would also hold in $L^1(\mu\otimes\kappa)$, and consequently, for any m and $A \in \mathscr{B}_m\otimes\mathscr{B}_Y$,

$$\nu(A) = \lim_{k\to\infty}\int_A \varphi_k\,d\mu\otimes\kappa = \int_A \varphi\,d\mu\otimes\kappa.$$

The filtration $\{\mathscr{B}_k\otimes\mathscr{B}_Y\}$ generates $\mathscr{B}_X\otimes\mathscr{B}_Y$, which in turn equals $\mathscr{B}_{X\times Y}$ by the second countability of the topologies, hence the above implies that $\varphi = d\nu/d\mu\otimes\kappa$. In particular, since ν and $\mu\otimes\kappa$ have a common Y-marginal, $\varphi^y = d\nu^y/d\mu$ and $\nu^y \ll \mu$ for κ-a.e. y.

It remains to prove the uniform integrability of $\{\varphi_k\}$ and (24). For both it suffices to show that $\mu\otimes\kappa(\varphi_k^p) \leq C$ for all k.

$$\begin{aligned}\mu\otimes\kappa(\varphi_k^p) = \nu(\varphi_k^{p-1}) &= \sum_{\substack{A:A\in\mathscr{B}_k \\ \mu(A)>0}} \iint I_A(x)\,\frac{\nu^y(A)^{p-1}}{\mu(A)^{p-1}}\,\nu(dx,dy) \\ &= \sum_{\substack{A:A\in\mathscr{B}_k \\ \mu(A)>0}} \frac{1}{\mu(A)^{p-1}}\int \nu^y(A)^p\,\kappa(dy) \leq C\sum_{A\in\mathscr{B}_k}\mu(A) = C. \qquad \square\end{aligned}$$

3.3. The local equilibrium. Now fix a smooth function ϕ on $\mathbf{T}^d$. We start by showing that, for any $\delta > 0$,

(25)
$$\lim_{N\to\infty} P^N\left\{\sup_{0\leq t\leq T}\left|\,\alpha_t^N(\phi) - \alpha_0^N(\phi) - \int_0^t cN^{-d}\sum_{x\in\mathbf{Z}_N^d}\eta_s^2(x)\,A\phi\Big(\frac{x}{N}\Big)\,ds\,\right| \geq \delta\right\} = 0.$$

Using estimates (13) and (15) (for the short-range and long-range models, respectively) the expression in $|\ \ |$'s can be bounded by

$$\sup_{0\leq t\leq T}\left(|M_t| + CN^{-1}\int_0^t N^{-d}\sum_x \eta_s^2(x)\,ds\right) \leq \sup_{0\leq t\leq T}|M_t| + CN^{-1}\int_0^T N^{-d}\sum_x\eta_s^2(x)\,ds$$

for a constant C that depends on ϕ alone. The expectation of this quantity is bounded by

$$E^N\Big\{\sup_{0\le t\le T} M_t^2\Big\}^{1/2} + O(N^{-1}),$$

which vanishes as $N\to\infty$ as was shown in (21) above. This establishes (25).

Let $\Lambda_{N\varepsilon} = \{z\in\mathbf{Z}^d : 0\le z_i < N\varepsilon \text{ for } i=1,\dots,d\}$ for $\varepsilon>0$. Next we turn (25) into

$$\begin{aligned}\lim_{\varepsilon\to 0}\limsup_{N\to\infty} P^N\Big\{\sup_{0\le t\le T}\Big|\, \alpha_t^N(\phi)-\alpha_0^N(\phi) - \int_0^t cN^{-d}\sum_{x\in\mathbf{Z}_N^d} A\phi\Big(\frac{x}{N}\Big)\\ \times\Big(\frac{1}{|\Lambda_{N\varepsilon}|}\sum_{y\in x+\Lambda_{N\varepsilon}}\eta_s^2(y)\Big)\,ds\,\Big|\ge\delta\Big\} = 0.\end{aligned}\tag{26}$$

Of course, $y\in x+\Lambda_{N\varepsilon}$ is again interpreted with periodic boundary conditions of $\mathbf{Z}_N^d$ in mind. To prove (26), notice first that by changing the order of summation

$$\begin{aligned}N^{-d}\sum_x A\phi\Big(\frac{x}{N}\Big)\Big(\frac{1}{|\Lambda_{N\varepsilon}|}\sum_{y\in x+\Lambda_{N\varepsilon}}\eta_s^2(y)\Big) &= N^{-d}\sum_x \eta_s^2(x)A\phi\Big(\frac{x}{N}\Big)\\ + O\Big(\sup_{\|\xi-\xi'\|\le\sqrt{d}\varepsilon}|A\phi(\xi)-A\phi(\xi')|\Big)&\cdot N^{-d}\sum_x\eta_s^2(x).\end{aligned}$$

By the a priori bound the expectation of the error term vanishes as $\varepsilon\to 0$, uniformly in N and t, so (26) follows from (25).

The next proposition establishes a weak form of local equilibrium, sufficient for our needs: The empirical second moment of the sticks in the cube $\Lambda_{N\varepsilon}$ is asymptotically the same as for i.i.d. exponential random variables with expectation given by the empirical mean:

Proposition 3.7.

$$\begin{aligned}\lim_{\varepsilon\to 0}\limsup_{N\to\infty} E^N\Big\{\int_0^T N^{-d}\sum_{x\in\mathbf{Z}_N^d}\Big|\,\frac{1}{|\Lambda_{N\varepsilon}|}\sum_{y\in x+\Lambda_{N\varepsilon}}\eta_t^2(y)\\ -2\Big(\frac{1}{|\Lambda_{N\varepsilon}|}\sum_{y\in x+\Lambda_{N\varepsilon}}\eta_t(y)\Big)^2\,\Big|\,dt\Big\} = 0.\end{aligned}\tag{27}$$

Proposition 3.7 will be proved via a number of intermediate results. Write $S_\Lambda(\eta) = |\Lambda|^{-1}\sum_{x\in\Lambda}\eta(x)$ for any set $\Lambda\subset\mathbf{Z}_N^d$, and similarly $S_\Lambda(\eta^2)$ for the average of squares. Define the probability measure $\overline{\mu}^N$ on Ω_N by

$$\overline{\mu}^N = \frac{1}{T}\int_0^T N^{-d}\sum_{x\in\mathbf{Z}_N^d}\mu_t^N\circ\theta_x\,dt,\tag{28}$$

where μ_t^N is the distribution of the stick process at time t and θ_x are the translations defined on Ω_N by $(\theta_x\eta)_y = \eta_{x+y}$, again modulo the cube $\mathbf{Z}_N^d$. Then $\overline{\mu}^N$ is a translation invariant measure that continues, by Lemma 3.2, to satisfy the a priori bounds:

$$\overline{\mu}^N\{\eta^k(x)\} \le TC_k \quad \text{for all } k,\ x, \text{ and } N. \tag{29}$$

The claim (27) of the proposition now reads

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\overline{\mu}^N\big\{|S_{\Lambda_{N\varepsilon}}(\eta^2) - 2\,S^2_{\Lambda_{N\varepsilon}}(\eta)|\big\} = 0. \tag{30}$$

As a standing notational convention, ν always denotes a product probability measure under which the sticks are i.i.d. exponential random variables with expectation K_0. For example, on each Ω_N, $\nu = \gamma_{K_0}^{\otimes \mathbf{Z}_N^d}$, but we shall have occasion to consider other sets of sites too besides the $\mathbf{Z}_N^d$'s. The choice of K_0 for the expectation is technically convenient for then we have derivatives $f_t^N(\eta) = (d\mu_t^N/d\nu)(\eta)$ on Ω_N that are bounded uniformly over both η and t. This holds for $t=0$ by Assumption 1, and for $t>0$ by general principles: Suppose ν is invariant for a Markov process and $f_0 = d\mu/d\nu$. If $1\le p\le\infty$ and q is the dual exponent, then for $0\le g\in L^q(\nu)$

$$\mu_t(g) = \nu\{f_0 E^\bullet g(\eta_t)\} \le \|f_0\|_{L^p(\nu)}\|E^\bullet g(\eta_t)\|_{L^q(\nu)}.$$

First with $p=1$ this shows that $\mu_t \ll \nu$. Letting $f_t = d\mu_t/d\nu$ we see for all p that $\|f_t\|_{L^p(\nu)}$ is nonincreasing in t.

The relative entropy or Kullback–Leibler information $H(Q\mid P)$ of two probability measures Q and P is defined by

$$H(Q\mid P) = \begin{cases} P\Big\{\dfrac{dQ}{dP}\log\dfrac{dQ}{dP}\Big\} & \text{if } Q\ll P,\\ \infty & \text{otherwise.}\end{cases}$$

$H(Q\mid P)$ measures a certain statistical distance between Q and P. $H(Q\mid P)\ge 0$ holds always and $H(Q\mid P)=0$ if and only if $Q=P$. A straightforward computation shows that Assumptions 1 and 3 imply that the entropy bound

$$H(\mu_0^N\mid\nu) \le CN^d \tag{31}$$

holds for all N with a constant $C = C(K_0,\varepsilon_0)$ that depends only on the constants of Assumptions 1 and 3. It is for the sake of (31) that we need to make Assumption 3.

To handle simultaneously both the short-range and the long-range model, let $p_N(x,y)$ be the uniform distribution on $x+V_N$ for the long-range model, and let $\beta = 2$ for the short-range model. For Borel functions $f\ge 0$ on Ω_N, set

(32)
$$\sigma_N(f) = \sum_{x,y} p_N(x,y)\,\nu\Big\{\int_0^{\eta(x)}\big[f(\eta^{u,x,y}) - f(\eta)\big]\,\big[\log f(\eta^{u,x,y}) - \log f(\eta)\big]\,du\Big\}.$$

The functional σ_N is nonnegative, convex, and translation invariant: $\sigma_N(f) = \sigma_N(f\circ\theta_x)$. Set $H_t^N = H(\mu_t^N\mid\nu)$.

Lemma 3.8. $(d/dt)H_t^N = -\frac{1}{2}N^\beta\sigma_N(f_t^N)$.

Proof. First let $\varepsilon > 0$ and deduce the conclusion for $g_t^\varepsilon = (1-\varepsilon)f_t^N + \varepsilon$ by direct calculation:

$$\begin{aligned}\frac{d}{dt}\nu\{g_t^\varepsilon \log g_t^\varepsilon\} &= \nu\{(1+\log g_t^\varepsilon)\,\mathscr{L}_N g_t^\varepsilon\}\\ &= \nu\{\log g_t^\varepsilon\,\mathscr{L}_N g_t^\varepsilon + g_t^\varepsilon\,\mathscr{L}_N(\log g_t^\varepsilon)\} = -\frac{1}{2}N^\beta\sigma_N(g_t^\varepsilon)\end{aligned}$$

by using reversibility, by substituting in (1) or (3), and by applying (9) to one of the resulting terms. Nothing is problematic here since everything in sight is bounded. Then write, for $s < t$,

$$\nu\{g_t^\varepsilon \log g_t^\varepsilon\} - \nu\{g_s^\varepsilon \log g_s^\varepsilon\} = -\frac{1}{2}N^\beta\int_s^t \sigma_N(g_u^\varepsilon)\,du$$

and let $\varepsilon \searrow 0$. Calculus shows that the nonnegative integrand of $\sigma_N(g_t^\varepsilon)$ increases as ε decreases, hence $\sigma_N(g_t^\varepsilon) \nearrow \sigma_N(f_t^N)$ by monotone convergence as $\varepsilon \searrow 0$. On the left-hand side we may apply dominated convergence as g_t^ε is bounded above uniformly in ε and $\xi\log\xi \geq -1/e$ holds for all $\xi \geq 0$. □

Let $\overline{f}^N = d\overline{\mu}^N/d\nu$. By the convexity of $F(a,b) = (a-b)(\log a - \log b)$ and the translation invariance of σ_N,

$$\sigma_N(\overline{f}^N) \leq \frac{1}{T}\int_0^T \sigma_N(f_t^N)\,dt.$$

Hence by Lemma 3.8

$$\sigma_N(\overline{f}^N) \leq \frac{2N^{-\beta}}{T}(H_0^N - H_T^N) \leq CN^{d-\beta}, \tag{33}$$

where we used (31) and $H_T^N \geq 0$. For $x \in \mathbf{Z}_N^d$ and functions $g \geq 0$ on Ω_N define

$$D_x^N(g) = \sum_{y\in\mathbf{Z}_N^d} p_N(x,y)\,\nu\Big\{\int_0^{\eta(x)} \big[g(\eta^{u,x,y}) - g(\eta)\big]^2\,du\Big\}. \tag{34}$$

Let $g^N = \sqrt{\overline{f}^N}$. By the translation invariance of ν and $\overline{f}^N$, $D_x^N(g^N)$ does not depend on the site x. Hence, utilizing the inequality

$$(\sqrt{u} - \sqrt{v})^2 \leq (u-v)(\log u - \log v),$$

we get

$$D_x^N(g^N) = N^{-d}\sum_z D_z^N(g^N) \leq N^{-d}\sigma_N(\overline{f}^N), \tag{35}$$

and then from (33) our next fundamental bound

$$D_x^N(g^N) \le CN^{-\beta} \tag{36}$$

that controls the exchange of stick pieces between sites.

Now fix a positive integer L, large but much smaller than $N\varepsilon$, and choose $K = K(N, L, \varepsilon)$ so that $N\varepsilon - L < KL \le N\varepsilon$. Then our goal (30) is not affected by replacing $\Lambda_{N\varepsilon}$ by a union $\Lambda_{KL} = \Delta_1 \cup \cdots \cup \Delta_{K^d}$ of disjoint translates $\Delta_i = x_i + \Lambda_L$ of $\Lambda_L = \{z \in \mathbf{Z}^d : 0 \le z_i < L \text{ for } i = 1, \ldots, d\}$.

Lemma 3.9. *For a constant $C = C(K_0)$ coming from the a priori bound and for all N, K, and L,*

$$\begin{aligned} \overline{\mu}^N\{|S_{\Lambda_{KL}}(\eta^2) - 2S^2_{\Lambda_{KL}}(\eta)|\} &\le \overline{\mu}^N\{|S_{\Lambda_L}(\eta^2) - 2\,S^2_{\Lambda_L}(\eta)|\} \\ &\quad + C\biggl(K^{-2d} \sum_{1 \le i,j \le K^d} \overline{\mu}^N\{|S_{\Delta_i}(\eta) - S_{\Delta_j}(\eta)|^2\}\biggr)^{1/2}. \end{aligned} \tag{37}$$

Proof. Write

$$\begin{aligned} \bigl|S_{\Lambda_{KL}}(\eta^2) - 2S^2_{\Lambda_{KL}}(\eta)\bigr| &\le \biggl|K^{-d}\sum_{i=1}^{K^d} S_{\Delta_i}(\eta^2) - 2K^{-d}\sum_{i=1}^{K^d} S^2_{\Delta_i}(\eta)\biggr| \\ &\quad + 2\biggl|K^{-d}\sum_{i=1}^{K^d} S^2_{\Delta_i}(\eta) - \biggl(K^{-d}\sum_{i=1}^{K^d} S_{\Delta_i}(\eta)\biggr)^2\biggr|. \end{aligned}$$

By translation invariance, the $\overline{\mu}^N$-expectation of the first right-hand-side term above is bounded by the first term on the right-hand side of (37). For the second term, apply the inequality

$$\biggl|\frac{1}{n}\sum_i \xi_i^2 - \biggl(\frac{1}{n}\sum_i \xi_i\biggr)^2\biggr| \le \biggl(\frac{1}{n}\sum_i \xi_i^2\biggr)^{1/2}\biggl(\frac{1}{n^2}\sum_{i,j}(\xi_i - \xi_j)^2\biggr)^{1/2}, \tag{38}$$

take expectations and apply Schwarz's inequality. (38) is true because

$$0 \le \frac{1}{n}\sum_i \xi_i^2 - \biggl(\frac{1}{n}\sum_i \xi_i\biggr)^2 = \frac{1}{n^2}\sum_{i,j} \xi_i(\xi_i - \xi_j)$$

and Schwarz's inequality again. □

Showing that the two terms on the right-hand side of (37) vanish as first $N \to \infty$, then $\varepsilon \to 0$, and then $L \to \infty$ are called the one-block and the two-block estimate, respectively. These are our next tasks.

3.3.1. One-block estimate. Here the goal is to show that

$$\lim_{L\to\infty} \limsup_{N\to\infty} \overline{\mu}^N\big\{|S_{\Lambda_L}(\eta^2) - 2S^2_{\Lambda_L}(\eta)|\big\} = 0. \tag{39}$$

The first observation is that it suffices to show that

$$\lim_{L\to\infty} \mu_L\big\{|S_{\Lambda_L}(\eta^2) - 2S^2_{\Lambda_L}(\eta)|\big\} = 0 \tag{40}$$

for an arbitrary collection $\{\mu_L\}$ of limit points of $\{\overline{\mu}^N\}$ on $\Omega = [0,\infty)^{\mathbf{Z}^d_+}$. The justification is twofold: Firstly, the a priori bound (29) implies that $\{\overline{\mu}^N\}$ is tight, so any subsequence $\overline{\mu}^{N_j}$ has a further convergent subsequence. Secondly, since the a priori bound is uniform in N for each fixed L, the following general fact applies: If $\nu_n \to \nu$ weakly, $f \ge 0$ is a continuous function and $\sup_n \nu_n(f^2) < \infty$, then $\nu_n(f) \to \nu(f)$ (proof elementary). This implies also that a limit point μ_L continues to satisfy the a priori bound (29).

Fix a limit point μ of $\{\overline{\mu}^N\}$. For $\eta \in \Omega$, let η_Γ denote its restriction to a subset $\Gamma \subset \mathbf{Z}^d_+$. Recall also (2).

Lemma 3.10. *Suppose Γ is a finite subset of $\mathbf{Z}^d_+$ and $q = \big(q(x) : x \in \Gamma\big)$ is a fixed stick configuration on Γ. Then whenever x, $y \in \Gamma$ and $0 \le u \le q(x)$,*

$$\mu\{\eta : \eta_\Gamma \ge q\} = \mu\{\eta : \eta_\Gamma \ge q^{u,x,y}\}. \tag{41}$$

Proof. We start by proving that, for $0 \le a < b \le q(x)$,

$$\mu\{\eta : \eta_\Gamma \ge q\} = \frac{1}{b-a}\int_a^b \mu\{\eta : \eta_\Gamma \ge q^{u,x,y}\}\, du. \tag{42}$$

Pick N large enough so that $\Gamma \subset \mathbf{Z}^d_N$. As before $g^N = \sqrt{\overline{f}^N}$. By a change of variable,

$$\begin{aligned}
\Big|\int_a^b \overline{\mu}^N\{\eta : \eta_\Gamma \ge q\} - \overline{\mu}^N\{\eta : \eta_\Gamma \ge q^{u,x,y}\}\, du\Big| \qquad\qquad& \\
= \Big|\int_a^b \int_{\Omega_N} I_{\{\eta_\Gamma \ge q\}}\{\overline{f}^N(\eta) - \overline{f}^N(\eta^{u,x,y})\}\, \nu(d\eta)\, du\Big|& \\
\le \int_{\Omega_N} I_{\{\eta_\Gamma \ge q\}}\Big\{\int_a^b \big|\overline{f}^N(\eta) - \overline{f}^N(\eta^{u,x,y})\big|\, du\Big\}\, \nu(d\eta)& \\
\le \nu\Big\{\int_0^{\eta(x)} \big|\overline{f}^N(\eta^{u,x,y}) - \overline{f}^N(\eta)\big|\, du\Big\}& \\
\le \bigg(\nu\Big\{\int_0^{\eta(x)} \big[g^N(\eta^{u,x,y}) - g^N(\eta)\big]^2\, du\Big\}\bigg)^{1/2}& \\
\times \bigg(\nu\Big\{\int_0^{\eta(x)} \big[g^N(\eta^{u,x,y}) + g^N(\eta)\big]^2\, du\Big\}\bigg)^{1/2}&.
\end{aligned} \tag{43}$$

The last factor on the last line contributes a constant which we ignore in the sequel, as can be seen by bringing the square inside the [ ]'s and using the a priori bound. The first factor is controlled by the single-site Dirichlet form D_x^N. For the short-range model, pick a path $x = x_0,\ x_1,\ \ldots,\ x_R = y$ in $\mathbf{Z}^d$ such that

$$A = \inf_{1\le i\le R} p(x_i - x_{i-1}) > 0. \tag{44}$$

This induces a path on $\mathbf{Z}_N^d$, again denoted by $x = x_0, x_1, \ldots, x_R = y$, with $p_N(x_{i-1}, x_i) \ge A$ for each i. Take

$$h(u,\eta) = \big[g^N(\eta^{u,x_1,x_R}) - g^N(\eta^{u,x_1,x_0})\big]^2 I_{\{\eta(x_1)\ge u\}}$$

in (9). Then

$$\begin{aligned}
&\nu\Big\{\int_0^{\eta(x_0)} \big[g^N(\eta^{u,x_0,x_R}) - g^N(\eta)\big]^2\,du\Big\}\\
&\qquad = \nu\Big\{\int_0^{\eta(x_1)} \big[g^N(\eta^{u,x_1,x_R}) - g^N(\eta^{u,x_1,x_0})\big]^2\,du\Big\}\\
&\qquad \le 2\nu\Big\{\int_0^{\eta(x_1)} \big[g^N(\eta^{u,x_1,x_R}) - g^N(\eta)\big]^2\,du\Big\}\\
&\qquad\quad + 2\nu\Big\{\int_0^{\eta(x_1)} \big[g^N(\eta^{u,x_1,x_0}) - g^N(\eta)\big]^2\,du\Big\}.
\end{aligned}$$

Iterating this R times gives the bound

$$\begin{aligned}
&\nu\Big\{\int_0^{\eta(x)} \big[g^N(\eta^{u,x,y}) - g^N(\eta)\big]^2\,du\Big\}\\
&\qquad \le C\sum_{i=1}^R \nu\Big\{\int_0^{\eta(x_i)} \big[g^N(\eta^{u,x_i,x_{i-1}}) - g^N(\eta)\big]^2\,du\Big\}\\
&\qquad \le CA^{-1}\sum_{i=1}^R \sum_w p_N(x_i,w)\nu\Big\{\int_0^{\eta(x_i)} \big[g^N(\eta^{u,x_i,w}) - g^N(\eta)\big]^2\,du\Big\}\\
&\qquad \le CD_x^N(g^N) \le CN^{-\beta}.
\end{aligned}$$

Substituting this back into (43) proves for the short-range model that

$$\lim_{N\to\infty}\Big|\overline{\mu}^N\{\eta : \eta_\Gamma \ge q\} - \frac{1}{b-a}\int_a^b \overline{\mu}^N\{\eta : \eta_\Gamma \ge q^{u,x,y}\}\,du\Big| = 0. \tag{45}$$

For the long-range model we have to proceed differently because the constant A defined in (44) vanishes as $N \to \infty$. Let $W_N = (x + V_N) \cap (y + V_N)$. Then

$|W_N|/|V_N| \to 1$ as $N \to \infty$.

$$
\begin{aligned}
\nu\Big\{\int_0^{\eta(x)} & \big[g^N(\eta^{u,x,y}) - g^N(\eta)\big]^2\,du\Big\} \\
&= \nu\Big\{\int_0^{\eta(x)} \big(|W_N|^{-1}\sum_{w\in W_N}\big[g^N(\eta^{u,x,y}) - g^N(\eta^{u,x,w}) \\
&\qquad + g^N(\eta^{u,x,w}) - g^N(\eta)\big]\big)^2\,du\Big\} \\
&\le \frac{2}{|W_N|}\sum_{w\in W_N} \nu\Big\{\int_0^{\eta(x)} \big[g^N(\eta^{u,x,y}) - g^N(\eta^{u,x,w})\big]^2\,du\Big\} \\
&\qquad + \nu\Big\{\int_0^{\eta(x)} \big[g^N(\eta^{u,x,w}) - g^N(\eta)\big]^2\,du\Big\} \\
&\le C\frac{1}{|V_N|}\sum_{z\in V_N} \nu\Big\{\int_0^{\eta(y)} \big[g^N(\eta^{u,y,y+z}) - g^N(\eta)\big]^2\,du\Big\} \\
&\qquad + \nu\Big\{\int_0^{\eta(x)} \big[g^N(\eta^{u,x,x+z}) - g^N(\eta)\big]^2\,du\Big\} \\
&\le C D_x^N(g^N) \le C N^{-\beta},
\end{aligned}
$$

where the passage to the second last line involved two applications of (9). Thus (45) holds also for the long-range model.

From (45) we argue to (42) as follows: For all but countably many q's, $\mu\{\eta : \eta_\Gamma = q\} = 0$. For such q's (45) implies (42). Since (42) is preserved by increasing limits $q^{(n)} \nearrow q$, it holds for all q.

Lastly we go from (42) to (41). If $0 \le u < q(x)$, then let $0 \le a < b < q(x) - u$. By (42),

$$
\begin{aligned}
\mu\{\eta \ge q^{u,x,y}\} &= \frac{1}{b-a}\int_a^b \mu\{\eta \ge q^{u+w,x,y}\}\,dw \\
&= \frac{1}{b-a}\int_{a+u}^{b+u} \mu\{\eta \ge q^{w,x,y}\}\,dw = \mu\{\eta \ge q\}.
\end{aligned}
$$

If $u = q(x) > 0$, pick $0 \le a < b \le q(x)$ so that

$$
\begin{aligned}
\mu\{\eta \ge q^{q(x),x,y}\} &= \frac{1}{b-a}\int_a^b \mu\{\eta \ge (q^{q(x),x,y})^{w,y,x}\}\,dw \\
&= \frac{1}{b-a}\int_{q(x)-b}^{q(x)-a} \mu\{\eta \ge q^{w,x,y}\}\,dw = \mu\{\eta \ge q\}. \quad \square
\end{aligned}
$$

Corollary 3.11. *μ is a mixture of i.i.d. exponential distributions.*

Proof. Since interchanging the sticks $q(x)$ and $q(y)$ in Γ turns q into $q^{q(x)-q(y),x,y}$ (if $q(x) \geq q(y)$) it is clear from Lemma 3.10 that μ is exchangeable. Thus there is a probability measure Q on the probability measures on $[0,\infty)$ such that $\mu = \int \rho^{\otimes \mathbf{Z}_+^d}\, Q(d\rho)$. By Lemma 3.10,

$$\begin{aligned}\int \rho[a,\infty)^k\, Q(d\rho) &= \mu\{\eta_{\{x_1,\dots,x_k\}} \geq (a,\dots,a)\} \\ &= \mu\{\eta_{x_1} \geq ka\} = \int \rho[ka,\infty)\, Q(d\rho).\end{aligned} \tag{46}$$

It follows that $\rho[1/n,\infty) < 1$ for all n Q-a.s., for if $Q\{\rho : \rho[a,\infty) = 1\} = \delta > 0$ for some $a > 0$, then

$$\delta \leq \int \rho[a,\infty)^k\, Q(d\rho) = \mu\{\eta_{x_1} \geq ka\} \searrow 0$$

as $k \nearrow \infty$, a contradiction. Hence there is a finite function $r_n(\rho)$ such that for Q-a.e. ρ,

$$\rho\Big[\frac{1}{n},\infty\Big) = \exp\Big\{-\frac{1}{n\, r_n(\rho)}\Big\}.$$

Then by (46)

$$\begin{aligned}\int \rho\Big[\frac{k}{n},\frac{k+1}{n}\Big)\, Q(d\rho) &= \int \rho\Big[\frac{1}{n},\infty\Big)^k - \rho\Big[\frac{1}{n},\infty\Big)^{k+1} Q(d\rho) \\ &= \int e^{-k(n\,r_n(\rho))^{-1}} - e^{-(k+1)(n\,r_n(\rho))^{-1}}\, Q(d\rho) \\ &= \int \Big\{ \int_{k/n}^{(k+1)/n} \frac{e^{-w/r_n(\rho)}}{r_n(\rho)}\, dw \Big\}\, Q(d\rho).\end{aligned} \tag{47}$$

Setting $Q_n(B) = Q\{\rho : \gamma_{r_n(\rho)} \in B\}$ for Borel sets B of probability measures defines a sequence of measures Q_n supported by exponential distributions. (Recall that γ_r was defined as the exponential distribution with expectation r, see (7).) Let f be a bounded uniformly continuous function on $[0,\infty)$ and

$$\delta_n(f) = \sup\{|f(w) - f(w')| : |w - w'| \leq 1/n\}.$$

Utilizing (47) we get

$$\begin{aligned}\int \rho(f)\, Q(d\rho) &= \sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big) \int \rho\Big[\frac{k}{n},\frac{k+1}{n}\Big)\, Q(d\rho) + O\big(\delta_n(f)\big) \\ &= \int \Big\{ \sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big) \int_{k/n}^{(k+1)/n} \frac{e^{-w/r_n(\rho)}}{r_n(\rho)}\, dw \Big\}\, Q(d\rho) + O\big(\delta_n(f)\big) \\ &= \int \gamma_{r_n(\rho)}(f)\, Q(d\rho) + O\big(\delta_n(f)\big) \\ &= \int \gamma(f)\, Q_n(d\gamma) + O\big(\delta_n(f)\big).\end{aligned} \tag{48}$$

Taking $f(w) = w$ (truncate and pass to a limit in (48)) gives

$$\int \gamma(f)\, Q_n(d\gamma) = \mu\{\eta(x)\} + O\Big(\frac{1}{n}\Big) \le C$$

by the a priori bound, consequently

$$Q_n\{\gamma : \gamma(f) \ge A\} \le C/A$$

for all $A > 0$ and so $\{Q_n\}$ is tight, because $\gamma \mapsto \gamma(f)$ is a homeomorphism from the set of exponential distributions onto $[0, \infty)$. Let $\widetilde{Q}$ be a limit point of $\{Q_n\}$, still a measure supported by exponential distributions. Letting $n \to \infty$ in (48) along a suitable subsequence shows that, at least on a single site, μ behaves as a $\widetilde{Q}$-mixture of exponentials. But then for any finite set $\Gamma \subset \mathbf{Z}_+^d$ and $x \in \Gamma$, Lemma 3.10 implies that

$$\begin{aligned}\mu\{\eta : \eta_\Gamma \ge q\} = \mu\Big\{\eta : \eta_x \ge \sum_{y\in\Gamma} q(y)\Big\} &= \int \gamma\Big[\sum_{y\in\Gamma} q(y), \infty\Big)\, \widetilde{Q}(d\gamma) \\ &= \int \gamma^{\otimes \mathbf{Z}_+^d}\{\eta : \eta_\Gamma \ge q\}\, \widetilde{Q}(d\gamma)\end{aligned}$$

where the last equality used an elementary property of exponential distributions. Since the class of sets $\{\eta : \eta_\Gamma \ge q\}$ is rich enough to determine a probability measure, $\mu = \int \gamma^{\otimes \mathbf{Z}_+^d}\, \widetilde{Q}(d\gamma)$ and the corollary is proved. ▫

We are in a position to finish off the proof of the one-block estimate. Let $\nu_r = \gamma_r^{\otimes \mathbf{Z}_+^d}$ be the i.i.d. exponential distribution on Ω with expectation r. Since

$$E\bigg\{\bigg|\frac{1}{n}\sum_{i=1}^n X_i - EX_1\bigg|^2\bigg\} = \frac{1}{n} E\big\{|X_1 - EX_1|^2\big\}$$

holds for any square-integrable i.i.d. random variables, and exponential variables satisfy the formulas $E\{(X - EX)^2\} = (EX)^2$ and $E\big\{\big(X^2 - E(X^2)\big)^2\big\} = 20(EX)^4$, we can estimate as follows:

$$\begin{aligned}\nu_r\big\{|S_{\Lambda_L}(\eta^2) - 2\, S^2_{\Lambda_L}(\eta)|\big\} &\le \nu_r\big\{|S_{\Lambda_L}(\eta^2) - 2r^2|\big\} \\ &\quad + 2\nu_r\big\{|r + S_{\Lambda_L}(\eta)| \cdot |r - S_{\Lambda_L}(\eta)|\big\} \\ &\le \big\|S_{\Lambda_L}(\eta^2) - 2r^2\big\|_{L^2(\nu_r)} + Cr\big\|S_{\Lambda_L}(\eta) - r\big\|_{L^2(\nu_r)} \\ &\le CL^{-d/2}\, r^2 \le CL^{-d/2}\nu_r\{\eta^2(x)\}.\end{aligned}$$

Let $\mu = \int \nu_r\, Q(dr)$ be the decomposition of the limit point μ we have been considering. Then the above gives, together with the a priori bound,

$$\mu\big\{|S_{\Lambda_L}(\eta^2) - 2\, S^2_{\Lambda_L}(\eta)|\big\} \le CL^{-d/2}\mu\{\eta^2(x)\} \le CL^{-d/2}.$$

This estimate holds for all limit points μ with the same constant C; hence it holds uniformly over L in (40). We have established (39) and completed the one-block estimate.

3.3.2. Two-block estimate. In this subsection we prove

$$\lim_{L\to\infty}\limsup_{\varepsilon\to 0}\limsup_{N\to\infty} K^{-2d}\sum_{1\le i,j\le d}\overline{\mu}^N\big\{|S_{\Delta_i}(\eta)-S_{\Delta_j}(\eta)|^2\big\}=0. \tag{49}$$

We remind the reader that K^d is the maximal number of disjoint translates of Λ_L that fit inside $\Lambda_{N\varepsilon}$, and $\Delta_1,\dots,\Delta_{K^d}$ is a tiling of Λ_{KL} with a maximal collection of such translates. The first task is to get rid of the i,j-dependence in the integral appearing in (49). Define a two-site Dirichlet form by

$$D_{x,y}(g)=\nu\bigg\{\int_0^{\eta(x)}\big[g(\eta^{u,x,y})-g(\eta)\big]^2\,du\bigg\}.$$

Set $D_{x,y}=D_{x,y}(g^N)$ where as before $g^N=\sqrt{\overline{f}^N}$.

Lemma 3.12. *There is a constant C such that $D_{x,y}\le C\varepsilon^2$ for all N, ε, and x, $y\in\Lambda_{N\varepsilon}$.*

Proof. We begin with the short-range model. Fix $R>0$ and for all pairs z, $w\in\Lambda_R$ pick and fix a path $z=z_0,z_1,\dots,z_{b(z,w)}=w$ such that $p(z_i-z_{i-1})>0$ for $i=1,\dots,b(z,w)$. Set

$$A=\inf_{z,w\in\Lambda_R}\ \inf_{1\le i\le b(z,w)}p(z_i-z_{i-1})\qquad\text{and}\qquad B=\max_{z,w\in\Lambda_R}b(z,w).$$

Then $A>0$ and $B<\infty$. For each pair z, $w\in\Lambda_R$ and for each $N>R$ there is a path $z=\overline{z}_0,\overline{z}_1,\dots,\overline{z}_{b(z,w)}=w$ inside $\mathbf{Z}_N^d$ such that $p_N(\overline{z}_{i-1},\overline{z}_i)\ge p(z_i-z_{i-1})\ge A$.

Given x, $y\in\Lambda_{N\varepsilon}$, construct first a path $x=w_0,w_1,\dots,w_\ell=y$ so that each consecutive pair w_{i-1}, w_i is contained in a translate of Λ_R. By proceeding along each coordinate axis in turn, this can be achieved with

$$\ell\le 3dN\varepsilon/R. \tag{50}$$

Now fill in between each pair w_{i-1},w_i with translates of the paths constructed earlier. This results in the path $x=x_0,x_1,\dots,x_m=y$ with $m\le B\ell$ and $p(x_i-x_{i-1})\ge A>0$ for each i.

$$\begin{aligned}\big[g^N(\eta^{u,x,y})-g^N(\eta)\big]^2&=\bigg\{\sum_{i=1}^m\big[g^N(\eta^{u,x_0,x_i})-g^N(\eta^{u,x_0,x_{i-1}})\big]\bigg\}^2\\&\le m\sum_{i=1}^m\big[g^N(\eta^{u,x_0,x_i})-g^N(\eta^{u,x_0,x_{i-1}})\big]^2,\end{aligned}$$

hence an application of (9) and the above development gives

$$\begin{aligned}
&\nu\Big\{\int_0^{\eta(x)}\big[g^N(\eta^{u,x,y})-g^N(\eta)\big]^2\,du\Big\}\\
&\qquad\le m\sum_{i=1}^m \nu\Big\{\int_0^{\eta(x_{i-1})}\big[g^N(\eta^{u,x_{i-1},x_i})-g^N(\eta)\big]^2\,du\Big\}\\
&\qquad\le m\sum_{i=1}^m \frac{1}{p(x_i-x_{i-1})}\sum_z p_N(x_{i-1},z)\\
&\qquad\quad\times\nu\Big\{\int_0^{\eta(x_{i-1})}\big[g^N(\eta^{u,x_{i-1},z})-g^N(\eta)\big]^2\,du\Big\}\\
&\qquad\le A^{-1}m^2\,D_x^N(g^N)\le C\varepsilon^2
\end{aligned}$$

by (36) and (50), remembering that $\beta=2$ for this model. This proves the lemma for the short-range model.

For the long-range model, arrange a sequence $E_0,E_1,\dots,E_m$ of cubes with side length $\frac{1}{4}N^\alpha$ so that $x\in E_0$, $y\in E_m$, and $E_i\subset z+V_N$ for each $z\in E_{i-1}$ for all i. This can be achieved with $m\le CN^{1-\alpha}\varepsilon$. Consider a path $x=x_0,x_1,\dots,x_m=y$ with $x_i\in E_i$ for all i. Reasoning as above, we get

$$D_{x,y}\le m\sum_{i=1}^m D_{x_{i-1},x_i}.$$

Sum over $x_m\in E_m$ to get

$$\begin{aligned}
D_{x,y}&\le m\sum_{i=1}^{m-1}D_{x_{i-1},x_i}+4^dN^{-\alpha d}m\sum_{z\in E_m}D_{x_{m-1},z}\\
&\le m\sum_{i=1}^{m-1}D_{x_{i-1},x_i}+CmD^N_{x_{m-1}}(g^N).
\end{aligned}$$

Now iterate: sum over $x_{m-1}\in E_{m-1},\dots,x_1\in E_1$ in turn, and use (36) together with $2\alpha+\beta=2$. ▫

As far as this estimate and the a priori bound goes, the relative positions of x and y, and consequenly of Δ_i and Δ_j, are immaterial, so we can simply think of two disjoint cubes Λ_L and Λ_L' and a probability measure on the sticks in the union $\Lambda_L\cup\Lambda_L'$. N disappears, and instead of (49) we prove

$$\lim_{L\to\infty}\lim_{\varepsilon\to0}\sup_{\mu\in\mathscr{N}_{L,\varepsilon}}\mu\big\{|S_{\Lambda_L}(\eta)-S_{\Lambda_L'}(\eta)|^2\big\}=0, \tag{51}$$

where $\mathscr{N}_{L,\varepsilon}$ is the class of probability measures μ on $\Omega_{\Lambda_L\cup\Lambda_L'}$ that satisfy $\mu\ll\nu$, $D_{x,y}(\sqrt{f}\,)\le C\varepsilon^2$ for $f=(d\mu/d\nu)$, and the a priori bound $\mu\{\eta^k(x)\}\le C_k$, for all $x,y\in\Lambda_L\cup\Lambda_L'$.

For each L, let μ_L be a measure that satisfies

$$\mu_L\big\{|S_{\Lambda_L}(\eta) - S_{\Lambda'_L}(\eta)|^2\big\} = \lim_{\varepsilon\to 0} \sup_{\mu\in\mathscr{N}_{L,\varepsilon}} \mu\big\{|S_{\Lambda_L}(\eta) - S_{\Lambda'_L}(\eta)|^2\big\}.$$

Existence of the μ_L's is justified as in the discussion following (40). Take $\Gamma = \Lambda_L \cup \Lambda'_L$ in Lemma 3.10, and observe that the last bound in (43) is precisely what we have control over with Lemma 3.12. Thus the proof of Lemma 3.10 goes through again, and (41) holds for μ_L. In particular, the variables $(\eta(x) : x \in \Lambda_L \cup \Lambda'_L)$ are exchangeable under μ. (But finite exchangeability does not imply a decomposition into product measures, so Corollary 3.11 does not follow yet.)

Fix R, and for $L \gg R$ replace Λ_L with a disjoint union $\Gamma_1 \cup \cdots \cup \Gamma_{K^d}$ of translates of Λ_R, maximal with respect to the property of fitting inside Λ_L. Let Γ'_i be the translate inside Λ'_L that sits relative to Λ'_L as Γ_i sits relative to Λ_L. Then (51) follows from

$$\lim_{R\to\infty} \limsup_{L\to\infty} K^{-d} \sum_i \mu_L\big\{|S_{\Gamma_i}(\eta) - S_{\Gamma'_i}(\eta)|^2\big\} = 0, \tag{52}$$

which by exchangeability is equivalent to

$$\lim_{R\to\infty} \limsup_{L\to\infty} \mu_L\big\{|S_{\Lambda_R}(\eta) - S_{\Lambda'_R}(\eta)|^2\big\} = 0. \tag{53}$$

But any limit point of the μ_L's is infinitely exchangeable, hence we may prove

$$\lim_{R\to\infty} \sup_{\mu} \mu\big\{|S_{\Lambda_R}(\eta) - S_{\Lambda'_R}(\eta)|^2\big\} = 0, \tag{54}$$

where the supremum is over infinitely exchangeable μ that satisfy the moment conditions $\mu\{\eta^k(x)\} \le C_k$ for all k. The proof can be completed as was done for the one-block estimate.

We are ready to prove the local equilibrium:

Proof of Proposition 3.7. Combine (30), (37), (39), and (49). □

Before utilizing (27) we wish to change it slightly. For $\xi, \theta \in \mathbf{T}^d$, let

$$\chi_{\varepsilon,\xi}(\theta) = \varepsilon^{-d} I_{\xi+[0,\varepsilon)^d}(\theta), \tag{55}$$

i.e. the indicator function of the ε-cube on $\mathbf{T}^d$ with lower left corner at ξ, normalized by the volume. Since $y \in x + \Lambda_{N\varepsilon}$ if and only if $y/N \in x/N + [0,\varepsilon)^d$, (27) is equivalent to

(56)

$$\lim_{\varepsilon\to 0} \limsup_{N\to\infty} E^N\bigg\{\int_0^T N^{-d} \sum_{x\in\mathbf{Z}_N^d} \bigg|\frac{1}{|\Lambda_{N\varepsilon}|} \sum_{y\in x+\Lambda_{N\varepsilon}} \eta_t^2(y) - 2\big[\alpha_t^N(\chi_{\varepsilon,x/N})\big]^2\bigg|\, dt\bigg\} = 0.$$

By inserting (56) into (26) we get as conclusion of this subsection:

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty} P^N\Big\{\sup_{0\le t\le T}\Big|\alpha_t^N(\phi)-\alpha_0^N(\phi) - 2c\int_0^t N^{-d}\sum_{x\in\mathbf{Z}_N^d} A\phi\Big(\frac{x}{N}\Big)\big[\alpha_s^N(\chi_{\varepsilon,x/N})\big]^2\,ds\Big|\ge\delta\Big\}=0. \tag{57}$$

3.4. Further technicalities. First we turn (57) into

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty} P^N\Big\{\sup_{0\le t\le T}\Big|\alpha_t^N(\phi)-\alpha_0^N(\phi) - 2c\int_0^t \int_{\mathbf{T}^d} A\phi(\xi)\,\big[\alpha_s^N(\chi_{\varepsilon,\xi})\big]^2\,d\xi\,ds\Big|\ge\delta\Big\}=0. \tag{58}$$

This step requires the a priori estimate and the continuity of $A\phi$: For $\xi\in x/N+[0,N^{-1})^d$,

$$\Big|A\phi(\xi)\big[\alpha_s^N(\chi_{\varepsilon,\xi})\big]^2 - A\phi\Big(\frac{x}{N}\Big)\big[\alpha_s^N(\chi_{\varepsilon,x/N})\big]^2\Big| \le 2\|A\phi\|_\infty\Big(N^{-d}\sum_y\eta(y)\Big)$$
$$\times\Big(N^{-d}\sum_{y\in\Lambda(x,N\xi,N\varepsilon)}\eta(y)\Big)+\Big(N^{-d}\sum_y\eta(y)\Big)\Big|A\phi(\xi)-A\phi\Big(\frac{x}{N}\Big)\Big|,$$

where $\Lambda(x,y,K)=(x+\Lambda_K)\Delta(y+\Lambda_K)$. Integrate this bound over $\mathbf{T}^d$ and note that $|\Lambda(x,N\xi,N\varepsilon)|=O\big((N\varepsilon)^{d-1}\big)$. Then (58) follows from (57).

Lemma 3.13. *For $\psi\in C(\mathbf{T}^d)$ and fixed $\varepsilon>0$, $\nu\mapsto\int_{\mathbf{T}^d}\psi(\xi)\nu(\chi_{\varepsilon,\xi})^2\,d\xi$ is a continuous function of $\nu\in\mathscr{M}$.*

Proof. Suppose $\nu_n\to\nu$ in the topology of $\mathscr{M}$. Let $0\le f_k\le g_k\le\varepsilon^{-d}$ be bounded continuous functions such that $f_k\nearrow\varepsilon^{-d}I_{(0,\varepsilon)^d}$ and $g_k\searrow\varepsilon^{-d}I_{[0,\varepsilon]^d}$ on $\mathbf{T}^d$, and let $f_k^\xi(\theta)=f_k(\theta-\xi)$, similarly for g_k^ξ. Then for all k,

$$\int\psi(\xi)\nu(f_k^\xi)^2\,d\xi\le\liminf_{n\to\infty}\int\psi(\xi)\nu_n(\chi_{\varepsilon,\xi})^2\,d\xi$$
$$\le\limsup_{n\to\infty}\int\psi(\xi)\nu_n(\chi_{\varepsilon,\xi})^2\,d\xi\le\int\psi(\xi)\nu(g_k^\xi)^2\,d\xi.$$

Thus it suffices to show that $\lim_{k\to\infty}\big[\nu(g_k^\xi)-\nu(f_k^\xi)\big]=0$ for Lebesgue a.e. ξ. This comes by a simple Fubini argument:

$$\int\lim_{k\to\infty}\big[\nu(g_k^\xi)-\nu(f_k^\xi)\big]\,d\xi=\int\Big\{\int I_{[0,\varepsilon]^d}(\theta-\xi)-I_{(0,\varepsilon)^d}(\theta-\xi)\,\nu(d\theta)\Big\}\,d\xi$$
$$=\int\Big\{\int I_{[0,\varepsilon]^d}(\theta-\xi)-I_{(0,\varepsilon)^d}(\theta-\xi)\,d\xi\Big\}\nu(d\theta)=0. \quad\square$$

Let $\mathscr{P}$ be any limit point of $\{\mathscr{P}^N\}$. The probability in (58) equals

$$\mathscr{P}^N\{\omega_\bullet \in \mathscr{D}_{\mathscr{M}} : \Phi_\varepsilon(\omega_\bullet) \geq \delta\}$$

for a certain function Φ_ε that is continuous on the support of $\mathscr{P}$.

Since (58) holds for all $\delta > 0$, it implies that

$$\lim_{\varepsilon\to 0} \mathscr{P}\{\omega_\bullet : \Phi_\varepsilon(\omega_\bullet) \geq \delta\} = 0, \tag{59}$$

again for all $\delta > 0$. Now recall that according to Lemma 3.5 there exists a $\mathscr{P}(d\omega_\bullet) \otimes dt \otimes d\xi$-a.e. defined jointly measurable function $u(\omega_\bullet, t, \xi)$ such that $\omega_t(d\xi) = u(\omega_\bullet, t, \xi)\,d\xi$. We suppress the $\omega_\bullet$-argument from $u(\omega_\bullet, t, \xi)$, and then (59) may be written as

$$\begin{aligned}\lim_{\varepsilon\to 0} \mathscr{P}\Bigg\{ \sup_{0\leq t\leq T} \Bigg| &\omega_t(\phi) - \omega_0(\phi) \\ &- 2c\int_0^t \int_{\mathbf{T}^d} A\phi(\xi)\Big(\varepsilon^{-d}\int_{\xi+[0,\varepsilon)^d} u(s,\theta)\,d\theta\Big)^2 d\xi\,ds\Bigg| \geq \delta\Bigg\} = 0.\end{aligned} \tag{60}$$

For a.e. $\omega_\bullet$ and s,

$$\lim_{\varepsilon\to 0}\Big(\varepsilon^{-d}\int_{\xi+[0,\varepsilon)^d} u(s,\theta)\,d\theta\Big)^2 = u^2(s,\xi)$$

for a.e. ξ by Lebesgue's differentiation theorem and (22). To extend this convergence to the integral inside (60), introduce the maximal function

$$M_p(s,\xi) = \sup_{\varepsilon>0} \varepsilon^{-d}\int_{\xi+[0,\varepsilon)^d} u^p(s,\theta)\,d\theta.$$

Write $\mathscr{E}$ for expectation under the measure $\mathscr{P}$.

Lemma 3.14.

$$\mathscr{E}\Big\{\int_0^T \int_{\mathbf{T}^d} M_2(s,\xi)\,d\xi\,ds\Big\} < \infty.$$

Proof. Since $M_2^2(s,\xi) \leq M_4(s,\xi)$, the maximal theorem gives (see p. 91 in [Fo]):

$$\big|\{\xi : M_2(s,\xi) > r\}\big| \leq \big|\{\xi : M_4(s,\xi) > r^2\}\big| \leq Cr^{-2}\int_{\mathbf{T}^d} u^4(s,\xi)\,d\xi,$$

where $|\cdot|$ denotes Lebesgue measure on $\mathbf{T}^d$. Hence by (22)

$$\begin{aligned}\mathscr{E}\Big\{\int_0^T \int_{\mathbf{T}^d} M_2(s,\xi)\,d\xi\,ds\Big\} &\leq \mathscr{E}\Big\{\int_0^T \int_1^\infty \big|\{\xi : M_2(s,\xi) > r\}\big|\,dr\,ds\Big\} + T \\ &\leq C\int_1^\infty r^{-2}\,dr \cdot \mathscr{E}\Big\{\int_0^T \int_{\mathbf{T}^d} u^4(s,\xi)\,d\xi\,ds\Big\} + T < \infty. \quad\square\end{aligned}$$

This lemma and the dominated convergence theorem imply that

$$\lim_{\varepsilon\to 0}\mathscr{E}\Big\{\sup_{0\le t\le T}\Big|\int_0^t\int_{\mathbf{T}^d}A\phi(\xi)\Big(\varepsilon^{-d}\int_{\xi+[0,\varepsilon)^d}u(s,\theta)\,d\theta\Big)^2\,d\xi\,ds - \int_0^t\int_{\mathbf{T}^d}A\phi(\xi)u^2(s,\xi)\,d\xi\,ds\Big|\Big\}=0,$$

and consequently (60) turns into the statement

$$\mathscr{P}\Big\{\sup_{0\le t\le T}\Big|\omega_t(\phi)-\omega_0(\phi)-2c\int_0^t\int_{\mathbf{T}^d}A\phi(\xi)u^2(s,\xi)\,d\xi\,ds\Big|\ge\delta\Big\}=0. \tag{61}$$

Letting $\delta\searrow 0$ shows that $\mathscr{P}$-a.e. $\omega_\bullet$ is a continuous $\mathscr{M}$-valued path with a derivative for a.e. t, and a weak solution of the equation $\partial_t u = 2cA(u^2)$. By the uniqueness lemma of the next subsection $\mathscr{P}$ is supported by a single path. This has two consequences: (i) It implies that ω_t has a density $u(t,\xi)$ for all t. For we can reprove Lemma 3.5 for a fixed time t and conclude that $\omega_t \ll d\xi$ for $\mathscr{P}$-a.e. $\omega_\bullet$, in particular, for the unique $\omega_\bullet$ supporting $\mathscr{P}$. (ii) It promotes the weak convergence $\mathscr{P}^N\to\mathscr{P}$ to convergence in probability of the $\mathscr{D}_{\mathscr{M}}$-valued random variables $\alpha^N_\bullet$ to the path $u(\bullet,\xi)\,d\xi$ as stated in Theorem 1. This comes from a general fact: Weak convergence to a degenerate distribution implies convergence in probability.

For Theorems 1 and 2, it remains to prove that $u(t,\xi)\le\|u_0\|_\infty$ for all $(t,\xi)\in Q_T$. For any $0\le\phi\in C(\mathbf{T}^d)$,

$$\int_{\mathbf{T}^d}\phi\,u(t)=\lim_{N\to\infty}E^N\{\alpha^N_t(\phi)\}=\lim_{N\to\infty}N^{-d}\sum_x\phi\Big(\frac{x}{N}\Big)E^N\{\eta_t(x)\}\le K_0\int_{\mathbf{T}^d}\phi,$$

where the precise constant K_0 of Assumption 1 comes from the last line of the proof of the a priori estimate Lemma 3.2. But now note that it is perfectly possible to choose the initial distribution so that $K_0\le\|u_0\|_\infty$.

Thus with the uniqueness lemma we have proved Theorems 1 and 2 under Assumption 3, that the initial density is bounded away from 0. This assumption will be lifted after the uniqueness proof.

3.5. Uniqueness lemma.

Lemma 3.15. *Let $(a_{i,j})$ be a symmetric positive semidefinite matrix and set*

$$A\phi=\sum_{i,j}a_{i,j}\partial_{\xi_i}\partial_{\xi_j}\phi.$$

Suppose $t\mapsto\omega(t,d\xi)$ is a continuous $\mathscr{M}$-valued path on $0\le t\le T$ such that

(i) *there exists a jointly measurable function* $u(t,\xi)$ *on* Q_T *such that* $\omega(t,d\xi) = u(t,\xi)\,d\xi$ *for a.e.* t,

(ii) $\int_0^T \int_{\mathbf{T}^d} u^3(t,\xi)\,d\xi\,dt < \infty$, *and*

(iii) $\int_{\mathbf{T}^d} \phi(\xi)\,\omega(t,d\xi) - \int_{\mathbf{T}^d} \phi(\xi)\,\omega(0,d\xi) = \int_0^t \int_{\mathbf{T}^d} A\phi(\xi)\,u^2(s,\xi)\,d\xi\,ds$ *for all smooth* ϕ *on* $\mathbf{T}^d$ *and for all* $0 \le t \le T$.

Then if $\sigma(t,d\xi)$ *is another continuous* $\mathscr{M}$*-valued path that satisfies the analogues of* (i)–(iii) *and* $\sigma(0) = \omega(0)$, *then* $\sigma(t) = \omega(t)$ *for all* $0 \le t \le T$.

Proof. Let $\sigma(t,d\xi) = v(t,\xi)\,d\xi$ when the density exists. Let $f_\varepsilon(\xi) = \varepsilon^{-d} f(\varepsilon^{-1}\xi)$ be a compactly supported, symmetric, smooth approximation to the identity. Set

$$u_\varepsilon(t,\xi) = [\omega(t) * f_\varepsilon](\xi) = \int_{\mathbf{T}^d} f_\varepsilon(\xi-\theta)\,\omega(t,d\theta)$$

and similarly define $v_\varepsilon(t,\xi)$. Then u_ε (and also v_ε) satisfies

$$\int_{\mathbf{T}^d} \psi(\xi) u_\varepsilon(t,\xi)\,d\xi - \int_{\mathbf{T}^d} \psi(\xi) u_\varepsilon(0,\xi)\,d\xi = \int_0^t \int_{\mathbf{T}^d} A\psi(\xi)[u^2(s) * f_\varepsilon](\xi)\,d\xi\,ds$$

for all smooth ψ and *all* t. We wrote $u^2(s)$ for the function $u^2(s)(\theta) = u^2(s,\theta)$. Subtracting the equation for v_ε from the equation for u_ε and writing $\phi_\varepsilon^s = u^2(s) * f_\varepsilon - v^2(s) * f_\varepsilon$ (a well-defined smooth function on $\mathbf{T}^d$ for a.e. s) gives

$$\int_{\mathbf{T}^d} \psi\big[u_\varepsilon(t) - v_\varepsilon(t)\big] = \int_{\mathbf{T}^d} \sum_{i,j} a_{i,j}(\partial_{\xi_i}\partial_{\xi_j}\psi)\Big(\int_0^t \phi_\varepsilon^s\,ds\Big). \tag{62}$$

Now take $\psi = \phi_\varepsilon^t$ for those a.e. t for which this makes sense and integrate over $0 \le t \le T$ to render the exceptional set of t's harmless. After an integration by parts on the right-hand side we have

$$\begin{aligned}\int_0^T dt \int_{\mathbf{T}^d} \phi_\varepsilon^t\big[u_\varepsilon(t) - v_\varepsilon(t)\big] &= -\int_{\mathbf{T}^d} \sum_{i,j} a_{i,j} \int_0^T (\partial_{\xi_i}\phi_\varepsilon^t)\Big(\int_0^t \partial_{\xi_j}\phi_\varepsilon^s\,ds\Big)\,dt \\ &= -\int_{\mathbf{T}^d} \sum_{i,j} a_{i,j} \int_0^T \int_0^T (\partial_{\xi_i}\phi_\varepsilon^t)(\partial_{\xi_j}\phi_\varepsilon^s) I_{\{t \ge s\}}\,dt\,ds.\end{aligned}$$

By the symmetry of $(a_{i,j})$

$$2\int_0^T dt \int_{\mathbf{T}^d} \phi_\varepsilon^t[u(t)*f_\varepsilon - v(t)*f_\varepsilon] = -\int_{\mathbf{T}^d} \sum_{i,j} a_{i,j}\Big(\int_0^T \partial_{\xi_i}\phi_\varepsilon^t\,dt\Big)\Big(\int_0^T \partial_{\xi_j}\phi_\varepsilon^t\,dt\Big).$$

The conclusion we derive from this, by $(a_{i,j})$'s positive semidefiniteness, is that for all $\varepsilon > 0$,

$$\int_0^T dt \int_{\mathbf{T}^d} [u^2(t) * f_\varepsilon - v^2(t) * f_\varepsilon]\,[u(t) * f_\varepsilon - v(t) * f_\varepsilon] \le 0.$$

Hypothesis (ii) gives sufficient integrability to let $\varepsilon \searrow 0$ and recover

$$\int_{Q_T} [u^2 - v^2]\,[u - v] \le 0$$

which implies that $u(t,\xi) = v(t,\xi)$ almost everywhere. Hence $\omega(t) = \sigma(t)$ for a.e. t, and by continuity for all t. □

3.6. Removing Assumption 3. Assume given an initial density u_0, not necessarily bounded away fom zero, and initial distributions μ_0^N satisfying Assumption 1. For $\varepsilon > 0$, the versions of the theorems thus far proved apply to the initial densities $u_0^\varepsilon = u_0 + \varepsilon$ with initial distributions $\mu_0^{\varepsilon,N}$ arranged to satisfy $\mu_0^{\varepsilon,N}\{\eta(x)\} = \mu_0^N\{\eta(x)\} + \varepsilon$. Fix $\varepsilon > 0$ for the moment. Let Q^N be the coupling of μ_0^N and $\mu_0^{\varepsilon,N}$ given in Lemma 3.3. Let $\mathbf{P}^N$ be the distribution of the joint process with initial distribution Q^N, constructed as in the proof of Lemma 3.2 so that $\mathbf{P}^N\{(\eta_\bullet,\zeta_\bullet) : \eta_\bullet \le \zeta_\bullet\} = 1$, and the $\eta_\bullet$ and $\zeta_\bullet$ marginals of $\mathbf{P}^N$ are the processes P^N and $P^{\varepsilon,N}$ with initial distributions μ_0^N and $\mu_0^{\varepsilon,N}$, respectively. Let $\mathfrak{P}^N$ be the distribution of

$$(\alpha_\bullet^N, \alpha_\bullet^{\varepsilon,N}) = \Big(N^{-d}\sum_x \eta_\bullet(x)\delta_{x/N},\, N^{-d}\sum_x \zeta_\bullet(x)\delta_{x/N}\Big)$$

on the space $\mathscr{D}_{\mathscr{M}} \times \mathscr{D}_{\mathscr{M}}$. The tightness proof did not depend on Assumption 3, hence $\{\mathfrak{P}^N\}$ has tight marginals and consequently is itself tight. Let $\mathfrak{P}$ be a limit point with marginals $\mathscr{P}$ and $\mathscr{P}^\varepsilon$. We know that $\mathscr{P}^\varepsilon$ is supported by the unique path $u^\varepsilon(t,\xi)\,d\xi$ described in Theorem 1 or 2, whichever model we are talking about. Nor did Lemma 3.5 depend on Assumption 3, and so $\mathscr{P}(d\omega_\bullet) \otimes dt$-a.e. ω_t has a density $v(t,\xi)$. For all $0 \le a < b \le T$ and $0 \le \phi \in C(\mathbf{T}^d)$, the coupling implies

$$\mathbf{P}^N\Big\{\int_a^b \alpha_t^N(\phi)\,dt \le \int_a^b \alpha_t^{\varepsilon,N}(\phi)\,dt\Big\} = 1,$$

hence in the limit

$$\mathscr{P}\Big\{\int_a^b\int_{\mathbf{T}^d} v(t,\xi)\phi(\xi)\,d\xi\,dt \le \int_a^b\int_{\mathbf{T}^d} u^\varepsilon(t,\xi)\phi(\xi)\,d\xi\,dt\Big\} = 1. \tag{63}$$

(The random variable inside the probability is $v(t,\xi)$.) By considering all rational a, b and a suitable countable set of functions ϕ we see that, for $\mathscr{P}$-almost every $\omega_\bullet$, $v(t,\xi) \le u^\varepsilon(t,\xi)$ almost everywhere on Q_T.

Coupling processes for different values of ε shows that u^ε is increasing in ε, hence the limit $u^\varepsilon(t,\xi) \searrow u(t,\xi)$ exists as $\varepsilon \searrow 0$, and by applying dominated convergence to the weak form of the differential equation we see that $u(t,\xi)$ satisfies the equation with initial data $u_0(\xi)$. It remains to show that

$$\mathscr{P}\{\omega_t = u(t,\xi)\,d\xi\} = 1$$

for all t. Since ω_t and $u(t,\xi)\,d\xi$ are continuous in t (by Lemma 3.5 which is valid for $\mathscr{P}$), it suffices to get this for a.e. t. Letting $\varepsilon \searrow 0$ through a countable set of values gives a shrinking sequence of events in (63), hence $\mathscr{P}$-a.s.,

$$v(t,\xi) \le u(t,\xi) \qquad \text{a.e. on } Q_T. \tag{64}$$

Conversely, since the process preserves total stick length, $\mathscr{P}$-a.s.

$$\int_{\mathbf{T}^d} v(t) = \omega_t(\mathbf{T}^d) = \omega_0(\mathbf{T}^d) = \int_{\mathbf{T}^d} u_0 = \int_{\mathbf{T}^d} u(t)$$

for those t for which the density $v(t)$ exists. This and (64) force

$$v(t,\xi) = u(t,\xi) \qquad \text{a.e. on } Q_T,$$

and we are done. In conclusion, $\mathscr{P}$ is supported by the unique path $u(\,\bullet\,,\xi)\,d\xi$, and Theorems 1 and 2 follow for the density u_0 and initial distributions μ_0^N.

4. Proof of Theorem 3

The proof for the particle models is basically the same as for the stick models. The a priori estimate is again a consequence of attractiveness. Proceeding as in Section 3.1 gives

$$\begin{aligned} z_1(t) &= \frac{1}{4} N^{-d} \sum_{x\in \mathbf{Z}_N^d} \eta_t(x)\big(\varkappa_N + \eta_t(x)\big) A\phi\Big(\frac{x}{N}\Big) \\ &\quad + O(N^{-1}) \cdot N^{-d} \sum_{x\in \mathbf{Z}_N^d} \eta_t(x)\big(\varkappa_N + \eta_t(x)\big) \end{aligned}$$

and $E^N\{z_2(t)\} = O(N^{-d})$. The results of Section 3.2 on tightness and the properties of the limit points of the distributions of $\alpha_\bullet^N$ follow as before.

For each N, ν^N denotes a product measure under which the particle stacks at different sites are $\{k\varkappa_N : k = 0, 1, 2, \dots\}$-valued, i.i.d. geometric random variables with expectation K_0. Again we have derivatives $f_t^N(\eta) = d\mu_t^N/d\nu^N(\eta)$, bounded by some constant K_1 uniformly over N, η and t.

The entropy bound (31) is valid as long as $\varkappa_N + \mu_0^N\{\eta(x)\} \ge \varepsilon_0 > 0$ holds for some constant ε_0, uniformly over N and $x \in \mathbf{Z}_N^d$. Thus for the case $\varkappa = 0$ we need to proceed as for the stick model, by first assuming u_0 bounded away from zero and then removing this assumption in the end, but for the case $\varkappa > 0$ no such assumption is needed. The quantities $\sigma_N(g)$ and $D_x^N(g)$ are defined as in (32) and (34) except that the integral $\int_0^{\eta(x)} du$ and u are replaced by the sum $\varkappa_N \sum_{k=1}^{\eta(x)/\varkappa_N}$ and $k\varkappa_N$, respectively. Then (36) follows from the entropy bound as before, with $\beta = 2$.

For the local equilibrium we need to prove

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty} E^N\Big\{\int_0^T N^{-d}\sum_{x\in\mathbf{Z}_N^d}\Big|\frac{1}{|\Lambda_{N\varepsilon}|}\sum_{y\in x+\Lambda_{N\varepsilon}}\eta_t(y)\big(\varkappa_N+\eta_t(y)\big) - 2\big(\varkappa S_{x+\Lambda_{N\varepsilon}}(\eta_t)+S^2_{x+\Lambda_{N\varepsilon}}(\eta_t)\big)\Big|\,dt\Big\}=0. \tag{65}$$

As before $S_\Gamma(\eta)=|\Gamma|^{-1}\sum_{y\in\Gamma}\eta(y)$. Equation (65) follows from showing

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty} E^N\Big\{\int_0^T N^{-d}\sum_{x\in\mathbf{Z}_N^d}\Big|S_{x+\Lambda_{N\varepsilon}}(\eta_t^2) - \big(\varkappa S_{x+\Lambda_{N\varepsilon}}(\eta_t)+2S^2_{x+\Lambda_{N\varepsilon}}(\eta_t)\big)\Big|\,dt\Big\}=0. \tag{66}$$

Repeating Lemma 3.9, our tasks are again to prove the one-block estimate

$$\lim_{L\to\infty}\limsup_{N\to\infty}\overline{\mu}^N\big\{|S_{\Lambda_L}(\eta^2)-\varkappa S_{\Lambda_L}(\eta)-2S^2_{\Lambda_L}(\eta)|\big\}=0 \tag{67}$$

and the two-block estimate

$$\lim_{L\to\infty}\limsup_{\varepsilon\to 0}\limsup_{N\to\infty} K^{-2d}\sum_{1\le i,j\le d}\overline{\mu}^N\big\{|S_{\Delta_i}(\eta)-S_{\Delta_j}(\eta)|^2\big\}=0. \tag{68}$$

The measure $\overline{\mu}^N$ was defined by (28).

4.1. One-block estimate, case $\varkappa=0$. Letting μ_L denote a limit point of a subsequence of $\{\overline{\mu}^N\}$ that realizes the $\limsup_{N\to\infty}$ in (67), we need to prove

$$\lim_{L\to\infty}\mu_L\{|S_{\Lambda_L}(\eta^2)-2S^2_{\Lambda_L}(\eta)|\}=0 \tag{69}$$

exactly as in Section 3.3.1. Thus it suffices to show that Lemma 3.10 holds for an arbitrary limit point μ of $\{\overline{\mu}^N\}$. Given $q=(q(x):x\in\Gamma)\in[0,\infty)^\Gamma$, define the configuration q_N by $q_N(x)=[q(x)/\varkappa_N]\varkappa_N$, where $[\,\cdot\,]$ denotes integer part. Then we get

$$\int_a^b\overline{\mu}^N\{\eta_\Gamma\ge q^{u,x,y}\}\,du=\varkappa_N\sum_{k=[a/\varkappa_N]}^{[b/\varkappa_N]-1}\overline{\mu}^N\{\eta_\Gamma\ge q_N^{k\varkappa_N,x,y}\}+O(\varkappa_N).$$

The error is $O(\varkappa_N)$, because for any x and any fixed constants a_0 and a_1,

$$\overline{\mu}^N\{a_0\le\eta(x)<a_0+a_1\varkappa_N\}\le K_1a_1\sup_k\nu^N\{\eta(x)=k\varkappa_N\}=O(\varkappa_N).$$

Reasoning as in (43) yields

$$\Big|\int_a^b\overline{\mu}^N\{\eta:\eta_\Gamma\ge q\}-\overline{\mu}^N\{\eta:\eta_\Gamma\ge q^{u,x,y}\}\,du\Big| \le C\Big(\nu^N\Big\{\varkappa_N\sum_{k=1}^{\eta(x)/\varkappa_N}\big[g^N(\eta^{k\varkappa_N,x,y})-g^N(\eta)\big]^2\Big\}\Big)^{1/2}+O(\varkappa_N). \tag{70}$$

Now proceed as in the proof of Lemma 3.10.

4.2. One-block estimate, case $\varkappa > 0$. Let μ be any limit point of $\{\overline{\mu}^N\}$. It is a probability measure on $\{k\varkappa : k = 0, 1, 2, \ldots\}^{\mathbf{Z}_+^d}$. Let Γ be a finite subset of $\mathbf{Z}_+^d$ and $q = (k(x)\varkappa : x \in \Gamma)$ a particle configuration on Γ, where $k(x)$ are nonnegative integers.

Lemma 4.1. *For any $x, y \in \Gamma$ and $k \le k(x)$,*

$$\mu\{\eta : \eta_\Gamma = q\} = \mu\{\eta : \eta_\Gamma = q^{k\varkappa, x, y}\}. \tag{71}$$

Proof. Let $q_N = (k(x)\varkappa_N : x \in \Gamma)$ be the corresponding configuration for the Nth process. Proceeding as in (43), then constructing a suitable path from x to y and using the bound (36) give, as in the proof of Lemma 3.10,

$$\begin{aligned}
&\left|\overline{\mu}^N\{\eta_\Gamma = q_N\} - \overline{\mu}^N\{\eta_\Gamma = q_N^{k\varkappa_N, x, y}\}\right| \\
&\qquad \le C\varkappa_N^{-1}\Biggl(\nu^N\Biggl\{\varkappa_N \sum_{k=1}^{\eta(x)/\varkappa_N} \bigl[g^N(\eta^{k\varkappa_N, x, y}) - g^N(\eta)\bigr]^2\Biggr\}\Biggr)^{1/2} \le CN^{-1}.
\end{aligned}$$

This suffices for (71). ▫

Corresponding to Corollary 3.11 we now have:

Corollary 4.2. *μ is a mixture of i.i.d. geometric distributions.*

Proof. Exchangeability is immediate from (71), so there is a representation

$$\mu = \int \rho^{\otimes \mathbf{Z}_+^d}\, Q(d\rho)$$

of μ in terms of i.i.d. distributions. The integration variable ρ is a probability measure on $\{k\varkappa : k = 0, 1, 2, \ldots\}$. Consider first a single site x, and a set $\{x_1, \ldots, x_m\}$ of mutually distinct sites distinct from x. By (71),

$$\begin{aligned}
\mu\{\eta(x) = m\varkappa\} &= \sum_{k_1, \ldots, k_m \ge 0} \mu\{\eta(x) = m\varkappa, \eta(x_1) = k_1\varkappa, \ldots, \eta(x_m) = k_m\varkappa\} \\
&= \sum_{k_1, \ldots, k_m \ge 0} \mu\{\eta(x) = 0, \eta(x_1) = (k_1 + 1)\varkappa, \ldots, \eta(x_m) = (k_m + 1)\varkappa\} \\
&= \mu\{\eta(x) = 0, \eta(x_1) \ge \varkappa, \ldots, \eta(x_m) \ge \varkappa\} \\
&= \int \rho(0)\bigl(1 - \rho(0)\bigr)^m\, Q(d\rho),
\end{aligned}$$

so the distribution $\mu\{\eta(x) \in \cdot\,\}$ is a mixture of geometric distributions. But then

$$\begin{aligned}
\mu\{\eta_\Gamma \ge q\} = \mu\Biggl\{\eta(x) \ge \varkappa \sum_{y \in \Gamma} k(y)\Biggr\} &= \int \bigl(1 - \rho(0)\bigr)^{\sum_{y \in \Gamma} k(y)}\, Q(d\rho) \\
&= \int \prod_{y \in \Gamma} \bigl(1 - \rho(0)\bigr)^{k(y)}\, Q(d\rho),
\end{aligned}$$

and we see that μ is indeed a mixture of i.i.d. geometrics. ▫

The one-block estimate is now completed as was done in Section 3.1.1, utilizing the fact that if X is a $\varkappa\mathbf{Z}_+$-valued geometric random variable, then $E(X^2) = \varkappa EX + 2(EX)^2$.

4.3. Two-block estimates. This time we have to define the two-site Dirichlet form separately for each N:

$$D^N_{x,y} = \nu^N\Big\{\varkappa_N \sum_{k=1}^{\eta(x)/\varkappa_N} \big[g^N(\eta^{k\varkappa_N,x,y}) - g^N(\eta)\big]^2\Big\}.$$

The argument of Lemma 3.12 works again to give

$$D^N_{x,y} \le C\varepsilon^2$$

for all N, ε, and $x, y \in \Lambda_{N\varepsilon}$. As in Section 3.3.2, this bound allows us to deduce (68) by proving

$$\lim_{L\to\infty} \limsup_{\varepsilon\to 0} \limsup_{N\to\infty} \sup_{\mu} \mu\big\{|S_{\Lambda_L}(\eta) - S_{\Lambda'_L}(\eta)|^2\big\} = 0, \tag{72}$$

where the supremum is over the class of probability measures μ on $\Omega^{(\varkappa_N)}_{\Lambda_L\cup\Lambda'_L}$ that satisfy $\mu \ll \nu^N$, $D^N_{x,y} \le C\varepsilon^2$, and $\mu\{\eta^k(x)\} \le C_k$ for all $x, y \in \Lambda_L \cup \Lambda'_L$.

For each L, let μ_L be a measure that satisfies

$$\mu_L\big\{|S_{\Lambda_L}(\eta) - S_{\Lambda'_L}(\eta)|^2\big\} = \limsup_{\varepsilon\to 0} \limsup_{N\to\infty} \sup_{\mu} \mu\big\{|S_{\Lambda_L}(\eta) - S_{\Lambda'_L}(\eta)|^2\big\}.$$

For the case $\varkappa = 0$ the computation done in (70) shows that

$$\Big|\int_a^b \overline{\mu}^N\{\eta : \eta_\Gamma \ge q\} - \overline{\mu}^N\{\eta : \eta_\Gamma \ge q^{u,x,y}\}\, du\Big| \le C\varepsilon + O(\varkappa_N).$$

Thus, as we let first $N \to \infty$ and then $\varepsilon \to 0$, Lemma 3.10 holds for μ_L and we can complete the proof of the two-block estimate for the case $\varkappa = 0$ as was done in Section 3.3.2.

The pattern is clear by now so we leave the details of the two-block estimate for the case $\varkappa > 0$ to the reader.

After establishing the local equilibrium we have the analogue of (57) for the particle model:

$$\lim_{\varepsilon\to 0} \limsup_{N\to\infty} P^N\Big\{\sup_{0\le t\le T}\Big|\alpha^N_t(\phi) - \alpha^N_0(\phi) - 2c\int_0^t N^{-d} \sum_{x\in\mathbf{Z}^d_N} A\phi\Big(\frac{x}{N}\Big)\big(\varkappa\alpha^N_s(\chi_{\varepsilon,x/N}) + [\alpha^N_s(\chi_{\varepsilon,x/N})]^2\big)\, ds\Big| \ge \delta\Big\} = 0.$$

The remaining technical steps follow as before, and with this we consider Theorem 3 proved.

References

[A] Aronson, D.G.: The porous medium equation. - Lecture Notes in Math. 1224, Springer-Verlag, Berlin, 1986.

[B] Billingsley, P.: Convergence of probability measures. - Wiley & Sons, New York, 1968.

[DP] De Masi, A., and E. Presutti: Mathematical methods for hydrodynamic limits. - Lecture Notes in Math. 1501, Springer-Verlag, Berlin, 1991.

[DPS] De Masi, A., E. Presutti, and E. Scacciatelli: The weakly asymmetric simple exclusion process. - Ann. Inst. H. Poincaré A 25, 1989, 1–38.

[EK] Ethier, S.N., and T.G. Kurtz: Markov processes: Characterization and convergence. - Wiley & Sons, New York, 1985.

[F] Folland, G.B.: Real analysis. - John Wiley & Sons, New York, 1984.

[FHU] Funaki, T., K. Handa, and K. Uchiyama: Hydrodynamic limit of one-dimensional exclusion processes with speed change. - Ann. Probab. 19, 1991, 245–265.

[G] Gärtner, J.: Convergence towards Burger's equation and propagation of chaos for weakly asymmetric exclusion processes. - Stochastic Process. Appl. 27, 1988, 233–260.

[GPV] Guo, M.Z., G.C. Papanicolau, and S.R.S. Varadhan: Nonlinear diffusion limit for a system with nearest neighbor interactions. - Comm. Math. Phys. 118, 1988, 31–59.

[I1] Inoue, M.: Construction of diffusion processes associated with a porous medium equation. - Hiroshima Math. J. 19, 1989, 281–297.

[I2] Inoue, M.: Derivation of a porous medium equation from many Markovian particles and the propagation of chaos. - Hiroshima Math. J. 21, 1991, 85–110.

[KOV] Kipnis, C., S. Olla, and S.R.S. Varadhan: Hydrodynamics and large deviation for simple exclusion processes. - Comm. Pure Appl. Math. XLII, 1989, 115–137.

[LOP] Lebowitz, J.L., E. Orlandi, and E. Presutti: A particle model for spinodal decomposition. - J. Statist. Phys. 63, 1991, 933–974.

[L] Liggett, T.M.: Interacting particle systems. - Springer-Verlag, New York, 1985.

[O] Oelschläger, K.: Large systems of interacting particles and the porous medium equation. - J. Differential Equations 88, 1990, 294–346.

[R] Rezakhanlou, F.: Hydrodynamic limit for attractive particle systems on $\mathbf{Z}^d$. - Comm. Math. Phys. 140, 1991, 417–448.

[S] Spitzer, F.: Interaction of Markov processes. - Adv. Math. 5, 1970, 246–290.

[Sp] Spohn, H.: Large scale dynamics of interacting particles. - Springer-Verlag, Berlin, 1991.

[SU] Suzuki, Y., and K. Uchiyama: Hydrodynamic limit for a spin system on a multidimensional lattice. - Probab. Theory Related Fields 95, 1993, 47–74.

[U] Uchiyama, K.: Scaling limit for a mechanical system of interacting particles. - Preprint.

[V] Vasquez, J.L.: An introduction to the mathematical theory of the porous medium equation. In: Shape optimization and free boundaries, edited by M.C. Delfour, Kluwer Academic Publ. (to appear)

Received 29 March 1995