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THE SOBOLEV CAPACITY ON METRIC SPACES

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Abstract. We develop a capacity theory based on the definition of Sobolev functions on metric spaces with a Borel regular outer measure. Basic properties of capacity, including monotonicity, countable subadditivity and several convergence results, are studied. As an application we prove that each Sobolev function has a quasicontinuous representative. For doubling measures we provide sharp estimates for the capacity of balls. Capacity and Hausdorff measures are related under an additional regularity assumption on the measure.

1. Introduction

The purpose of this paper is to develop a capacity theory based on the definition of Sobolev functions on metric spaces due to Hajłasz [Ha]. His definition makes use of the fact that a smooth real-valued function u on R^n satisfies

$$|u(x)-u(y)| \le c|x-y|\big(\mathscr{M}|\nabla u|(x)+\mathscr{M}|\nabla u|(y)\big), \tag{1.1}$$

for every $x,y\in R^n$, where $\mathscr{M}f$ is the Hardy–Littlewood maximal operator of a locally integrable function f defined by

$$\mathscr{M}f(x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy.$$

Here $B(x,r)$ is the open ball with center x and radius r and $|E|$ stands for the Lebesgue measure of E. By the celebrated theorem of Hardy, Littlewood and Wiener, the maximal operator $\mathscr{M}$ is bounded in $L^p(R^n)$ for $1<p<\infty$, and hence an approximation procedure shows that for each function u belonging to the Sobolev space $W^{1,p}(R^n)$, $1<p<\infty$, the inequality (1.1) holds for every $x,y\in R^n\setminus E$, $x\ne y$, with $|E|=0$. The Sobolev space $W^{1,p}(R^n)$ consists of functions $u\in L^p(R^n)$ whose first distributional derivatives also belong to $L^p(R^n)$. Hajłasz showed in [Ha] that (1.1) also gives a sufficient condition for a function to belong to the Sobolev space $W^{1,p}(R^n)$ for $1<p<\infty$ and hence it can be taken as a definition for the Sobolev function. Let (X,d) be a metric space with a Borel regular outer measure μ. Recall, that the outer measure μ is Borel regular if it

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is a Borel measure and for every $E \subset X$ there is a Borel set $B \subset X$ such that $E \subset B$ and $\mu(E) = \mu(B)$. A function $u \in L^p(X)$ belongs to the Sobolev space $W^{1,p}(X)$, $1 < p < \infty$, if there is a non-negative function $g \in L^p(R^n)$ such that

$$|u(x) - u(y)| \le d(x,y)\big(g(x) + g(y)\big) \tag{1.2}$$

for every $x, y \in X \setminus F$, $x \ne y$, with $\mu(F) = 0$. In the Euclidean case with the Lebesgue measure this definition is equivalent to the standard definition of the Sobolev space. Observe that in this case we may choose $g = \mathscr{M}|\nabla u|$ in (1.2), but we cannot take $g = |\nabla u|$, in general. The space $W^{1,p}(X)$ is endowed with the Sobolev norm

$$\|u\|_{W^{1,p}(X)} = \big(\|u\|^p_{L^p(X)} + \|u\|^p_{L^{1,p}(X)}\big)^{1/p}, \tag{1.3}$$

where

$$\|u\|_{L^{1,p}(X)} = \inf\big\{\|g\|_{L^p(X)} : g \text{ satisfies (1.2)}\big\}.$$

This norm enables us to define the Sobolev p-capacity of an arbitrary $E \subset X$ by

$$\mathrm{C}_p(E) = \inf_{u \in \mathscr{A}(E)} \|u\|^p_{W^{1,p}(X)}, \tag{1.4}$$

where

$$\mathscr{A}(E) = \big\{u \in W^{1,p}(X) : u \ge 1 \text{ on a neighbourhood of } E\big\}.$$

Clearly, this generalizes the classical definition of the Sobolev p-capacity in the Euclidean case with the Lebesgue measure which is analogous to definition (1.4) except that the norm

$$\|u\|_{W^{1,p}(R^n)} = \big(\|u\|^p_{L^p(R^n)} + \|\nabla u\|^p_{L^p(R^n)}\big)^{1/p} \tag{1.5}$$

is used, see [FZ], [HKM, Section 2.35] and [K]. In the Euclidean case with the Lebesgue measure the norms defined by (1.3) and (1.5), and hence also the capacities, are equivalent. The capacity (1.4) shares many properties with the classical Sobolev capacity. These include monotonicity, countable subadditivity and several convergence results. All these are studied in detail in Section 3, where we also prove that each function in $W^{1,p}(X)$ has a p-quasicontinuous representative. For the classical case, see [MK] and [HKM, Chapter 4]. In Section 4 we relate the Sobolev capacity to various measures. Here an obvious choice is the measure μ. If the measure μ is doubling, which means that there is $c \ge 1$ so that

$$\mu\big(B(x,2r)\big) \le c\mu\big(B(x,r)\big)$$

holds for every $x \in X$ and $r > 0$, then we provide quite precise capacity estimates for balls in X. In this case our space is a special case of a space of homogeneous type in the sense of Coifman and Weiss, see [CW]. A nontrivial example of a metric space with a doubling measure is given by vector fields in R^n satisfying Hörmander's condition with the Lebesgue measure, see [FGW], [NSW] and [CDG]. Moreover, if μ is regular with dimension s, which means that there are $s > 0$ and $c \geq 1$ so that

$$c^{-1} r^s \leq \mu\big(B(x,r)\big) \leq c r^s$$

for all $x \in X$ and $0 < r \leq \operatorname{diam}(X)$, then we prove counterparts of the classical results relating the capacity and the Hausdorff measure. A similar problem on Carnot groups is studied in [He]. In fact, if μ is regular with dimension s, then the Hausdorff dimension of X equals s and μ and the s-dimensional Hausdorff measures are bounded by constant times each other. As examples of regular metric spaces we mention self-similar fractals with an invariant measure, the Carnot group with the Carnot–Carathéodory metric and the bi-invariant Haar measure, see [FS]. For further examples we refer to [S1] and [S2].

2. Sobolev space

Let (X,d) be a metric space and let μ be a non-negative Borel regular outer measure on X. In the following, we keep the triple (X,d,μ) fixed, and for short, we denote it by X. Let $1 < p < \infty$. Then $L^p(X)$ is the Banach space of all μ-a.e. defined μ-measurable functions $u\colon X \to [-\infty,\infty]$ for which the norm

$$\|u\|_{L^p(X)} = \left(\int_X |u|^p \, d\mu\right)^{1/p}$$

is finite. Suppose that $u\colon X \to [-\infty,\infty]$ is μ-measurable. We denote by $D(u)$ the set of all μ-measurable functions $g\colon X \to [0,\infty]$ such that

$$|u(x) - u(y)| \leq d(x,y)\big(g(x) + g(y)\big) \tag{2.1}$$

for every $x, y \in X \setminus F$, $x \neq y$, with $\mu(F) = 0$. Note that the right hand side of (2.1) is always defined for $x \neq y$. At the points $x, y \in X$, $x \neq y$, where the left hand side of (2.1) is undefined we may assume that the right hand side is $+\infty$. Following the original definition due to Hajłasz, the Dirichlet space $L^{1,p}(X)$ consists of all μ-measurable functions u with $D(u) \cap L^p(X) \neq \emptyset$; the space $L^{1,p}(X)$ is endowed with the seminorm

$$\|u\|_{L^{1,p}(X)} = \inf\big\{\|g\|_{L^p(X)} : g \in D(u) \cap L^p(X)\big\}. \tag{2.2}$$

The Sobolev space $W^{1,p}(X)$ is the space of all functions $u \in L^p(X)$ for which $D(u) \cap L^p(X) \neq \emptyset$. It is clear that (2.2) defines a seminorm in $W^{1,p}(X)$. An application of the the uniform convexity of $L^p(X)$ implies that there is a unique minimizer of (2.2); this means that the infimum is attained by a unique function in $D(u) \cap L^p(X)$. We equip the Sobolev space $W^{1,p}(X)$ with the norm

$$\|u\|_{W^{1,p}(X)} = \big(\|u\|^p_{L^p(X)} + \|u\|^p_{L^{1,p}(X)}\big)^{1/p}. \tag{2.3}$$

Then $W^{1,p}(X)$ is a linear space. In addition, it satisfies the following lattice property.

2.4. Lemma. *Suppose that* $u_1, u_2 \in W^{1,p}(X)$. *If* $g_1 \in D(u_1)$ *and* $g_2 \in D(u_2)$, *then*

(i) $u = \max(u_1, u_2) \in W^{1,p}(X)$ *and* $\max(g_1, g_2) \in D(u) \cap L^p(X)$,
(ii) $v = \min(u_1, u_2) \in W^{1,p}(X)$ *and* $\max(g_1, g_2) \in D(v) \cap L^p(X)$.

Proof. We prove the case (i) only. Let $g = \max(g_1, g_2)$ and suppose that F_1 and F_2 are the exeptional sets for u_1 and u_2 in (2.1), respectively. Clearly $u, g \in L^p(X)$. It remains to show that $g \in D(u)$. To see this, let $A = \bigl\{x \in X \setminus (F_1 \cup F_2) \colon u_1(x) \geq u_2(x)\bigr\}$. If $x, y \in A$, then

$$|u(x) - u(y)| = |u_1(x) - u_1(y)| \leq d(x,y)\bigl(g_1(x) + g_1(y)\bigr).$$

Analogously, for $x, y \in X \setminus A$ we obtain $|u(x) - u(y)| \leq d(x,y)\bigl(g_2(x) + g_2(y)\bigr)$. For the remaining cases, let $x \in A$ and $y \in X \setminus A$. If $u_1(x) \geq u_2(y)$, then

$$|u(x) - u(y)| = u_1(x) - u_2(y) \leq u_1(x) - u_1(y) \leq d(x,y)\bigl(g_1(x) + g_1(y)\bigr).$$

If $u_1(x) < u_2(y)$, then

$$|u(x) - u(y)| = u_2(y) - u_1(x) \leq u_2(y) - u_2(x) \leq d(x,y)\bigl(g_2(x) + g_2(y)\bigr).$$

The case $x \in X \setminus A$ and $y \in A$ follows by symmetry and hence

$$|u(x) - u(y)| \leq d(x,y)\bigl(g(x) + g(y)\bigr)$$

for all $x, y \in X \setminus (F_1 \cup F_2)$ with $\mu(F_1 \cup F_2) = 0$.

The following lemma shows that $W^{1,p}(X)$ is closed under μ-a.e. convergence in a certain sense.

2.5. Lemma. *The function* u *belongs to* $W^{1,p}(X)$ *if and only if* $u \in L^p(X)$ *and there are functions* $u_i \in L^p(X)$, $i = 1, 2, \dots$, *such that* $u_i \to u$ μ*-a.e. and* $g_i \in D(u_i) \cap L^p(X)$ *such that* $g_i \to g$ μ*-a.e. for some* $g \in L^p(X)$.

Proof. If $u \in W^{1,p}(X)$, then the claim of the lemma is clear. To see the converse, suppose that $u, g \in L^p(X)$, $g_i \in D(u_i) \cap L^p(X)$ and $u_i \to u$ μ-a.e. and $g_i \to g$ μ-a.e. Then

$$|u_i(x) - u_i(y)| \leq d(x,y)\bigl(g_i(x) + g_i(y)\bigr) \tag{2.6}$$

for all $x, y \in X \setminus F_i$ with $\mu(F_i) = 0$, $i = 1, 2, \dots$. Let $A \subset X$ be such that $u_i(x) \to u(x)$ and $g_i(x) \to g(x)$ for all $x \in X \setminus A$ and $\mu(A) = 0$. Write $F = A \cup \bigcup_{i=1}^{\infty} F_i$; then $\mu(F) = 0$. Let $x, y \in X \setminus F$, $x \neq y$. From (2.6) we obtain

$$|u(x) - u(y)| \leq d(x,y)\bigl(g(x) + g(y)\bigr)$$

and thus $g \in D(u) \cap L^p(X)$. This completes the proof.

Another important property of the Sobolev functions is the following Poincaré inequality. The proof can be found in [Ha], but we present it here for the sake of completeness. We use the familiar notation

$$f_E = \fint_E f\,d\mu = \frac{1}{\mu(E)}\int_E f\,d\mu$$

for μ-measurable $E \subset X$ of positive and finite measure.

2.7. Lemma. *If $u \in W^{1,p}(X)$ and $E \subset X$ is μ-measurable with $0 < \mu(E) < \infty$, then for every $g \in D(u) \cap L^p(X)$ we have*

$$\int_E |u - u_E|^p\,d\mu \le 2^p \operatorname{diam}(E)^p \int_E g^p\,d\mu.$$

Proof. By the Hölder inequality

$$|u(x) - u_E| \le \fint_E |u(x) - u(y)|\,d\mu(y) \le \left(\fint_E |u(x) - u(y)|^p\,d\mu(y)\right)^{1/p}.$$

An integration completes the proof, because

$$\begin{aligned}\int_E |u(x) - u_E|^p\,d\mu(x) &\le \int_E \fint_E |u(x) - u(y)|^p\,d\mu(y)\,d\mu(x)\\ &\le 2^{p-1}\operatorname{diam}(E)^p \int_E \left(g(x)^p + \fint_E g(y)^p\,d\mu(y)\right)d\mu(x)\\ &= 2^p \operatorname{diam}(E)^p \int_E g^p\,d\mu.\end{aligned}$$

3. Capacity

For $1 < p < \infty$, the Sobolev p-capacity of the set $E \subset X$ is the number

$$\mathrm{C}_p(E) = \inf_{u \in \mathscr{A}(E)} \|u\|^p_{W^{1,p}(X)},$$

where

$$\mathscr{A}(E) = \bigl\{u \in W^{1,p}(X) : u \ge 1 \text{ on a neighbourhood of } E\bigr\}.$$

If $\mathscr{A}(E) = \emptyset$, we set $\mathrm{C}_p(E) = \infty$. Functions belonging to $\mathscr{A}(E)$ are called admissible functions for E.

3.1. Remark. We can restrict ourselves in the definition of the capacity to those admissible functions u for which $0 \le u \le 1$. Indeed, if

$$\mathscr{A}'(E) = \bigl\{u \in \mathscr{A}(E) : 0 \le u \le 1\bigr\},$$

then $\mathscr{A}'(E) \subset \mathscr{A}(E)$ implies

$$\mathrm{C}_p(E) \le \inf_{u \in \mathscr{A}'(E)} \|u\|^p_{W^{1,p}(X)}.$$

For the reverse inequality, let $\varepsilon > 0$ and take $u \in \mathscr{A}(E)$ such that

$$\|u\|^p_{W^{1,p}(X)} \le \mathrm{C}_p(E) + \varepsilon.$$

Then $v = \max\bigl(0, \min(u,1)\bigr)$ belongs to $\mathscr{A}'(E)$ and by Lemma 2.4 we have $D(u) \subset D(v)$. Therefore

$$\inf_{u \in \mathscr{A}'(E)} \|u\|^p_{W^{1,p}(X)} \le \|v\|^p_{W^{1,p}(X)} \le \|u\|^p_{W^{1,p}(X)} \le \mathrm{C}_p(E) + \varepsilon$$

and letting $\varepsilon \to 0$ we obtain

$$\inf_{u \in \mathscr{A}'(E)} \|u\|^p_{W^{1,p}(X)} \le \mathrm{C}_p(E).$$

This completes the proof.

In the Euclidean case with the Lebesgue measure the Sobolev p-capacity enjoys many desirable properties, one of the most important of which says that it is an outer measure. This is also true in our context.

3.2. Theorem. *The Sobolev p-capacity is an outer measure.*

Proof. Clearly $\mathrm{C}_p(\emptyset) = 0$ and the definition of the capacity implies monotonicity.

To prove countable subadditivity, suppose that E_i, $i = 1, 2, \dots$, are subsets of X. Let $\varepsilon > 0$. We may assume that $\sum_{i=1}^\infty \mathrm{C}_p(E_i) < \infty$. Next we choose $u_i \in \mathscr{A}(E_i)$ and $g_{u_i} \in D(u_i) \cap L^p(X)$ so that

$$\|u_i\|^p_{L^p(X)} + \|g_{u_i}\|^p_{L^p(X)} \le \mathrm{C}_p(E_i) + \varepsilon 2^{-i},$$

for $i = 1, 2, \dots$. We show that $v = \sup_i u_i$ is admissible for $\bigcup_{i=1}^\infty E_i$ and $g = \sup_i g_{u_i} \in D(v) \cap L^p(X)$. First we observe that $v, g \in L^p(X)$. Then we define $v_k = \max_{1 \le i \le k} u_i$. By Lemma 2.4 the function $g_{v_k} = \max_{1 \le i \le k} g_{u_i}$ belongs to $D(v_k) \cap L^p(X)$. Since $v_k \to v$ μ-a.e. and $g_{v_k} \to g$ μ-a.e., Lemma 2.5 yields $v \in W^{1,p}(X)$. Clearly $v \ge 1$ in a neighbourhood of $\bigcup_{i=1}^\infty E_i$. This implies that

$$\mathrm{C}_p\Bigl(\bigcup_{i=1}^\infty E_i\Bigr) \le \|v\|^p_{W^{1,p}(X)} \le \sum_{i=1}^\infty \bigl(\|u_i\|^p_{L^p(X)} + \|g_{u_i}\|^p_{L^p(X)}\bigr) \le \sum_{i=1}^\infty \mathrm{C}_p(E_i) + \varepsilon.$$

The claim follows by letting $\varepsilon \to 0$.

3.3. Remark. The Sobolev p-capacity is an outer capacity, that is,

$$\mathrm{C}_p(E) = \inf\bigl\{\mathrm{C}_p(O) : O \supset E,\ O \text{ open}\bigr\}.$$

Indeed, by monotonicity, $\mathrm{C}_p(E) \le \inf\{\mathrm{C}_p(O) : O \text{ open},\ E \subset O\}$. To see the inequality into the other direction, let $\varepsilon > 0$ and take $u \in \mathscr{A}(E)$ such that

$$\|u\|^p_{W^{1,p}(X)} \le \mathrm{C}_p(E) + \varepsilon.$$

Since $u \in \mathscr{A}(E)$ there is an open set O containing E such that $u \ge 1$ on O, which implies

$$\mathrm{C}_p(O) \le \|u\|^p_{W^{1,p}(X)} \le \mathrm{C}_p(E) + \varepsilon.$$

The equality follows by letting $\varepsilon \to 0$.

3.4. Lemma. *If $X \supset C_1 \supset C_2 \supset \cdots$ are compact sets and $C = \bigcap_{i=1}^\infty C_i$, then*

$$\mathrm{C}_p(C) = \lim_{i\to\infty} \mathrm{C}_p(C_i).$$

Proof. First we observe that monotonicity yields $\lim_{i\to\infty} \mathrm{C}_p(C_i) \ge \mathrm{C}_p(C)$. On the other hand, let O be an open set containing C. By the compactness of C, $C_i \subset O$ for all sufficiently large i. Therefore $\lim_{i\to\infty} \mathrm{C}_p(C_i) \le \mathrm{C}_p(O)$ and since the Sobolev p-capacity is an outer capacity, see Remark 3.3, we obtain the claim by taking infimum over all open sets O containing C.

3.5. Theorem. *If $O_1 \subset O_2 \subset \cdots$ are open subsets of X and $O = \bigcup_{i=1}^\infty O_i$, then*

$$\mathrm{C}_p(O) = \lim_{i\to\infty} \mathrm{C}_p(O_i).$$

Proof. Monotonicity implies $\lim_{i\to\infty} \mathrm{C}_p(O_i) \le \mathrm{C}_p(O)$. To prove the opposite inequality, we may assume that $\lim_{i\to\infty} \mathrm{C}_p(O_i) < \infty$. Let $\varepsilon > 0$ and $u_i \in \mathscr{A}(O_i)$, $i = 1, 2, \ldots$, and $g_{u_i} \in D(u_i) \cap L^p(X)$ be such that

$$\|u_i\|^p_{L^p(X)} + \|g_{u_i}\|^p_{L^p(X)} \le \mathrm{C}_p(O_i) + \varepsilon.$$

Now (u_i) is a bounded sequence in $L^p(X)$ and hence it has a weakly convergent subsequence, which we denote again by (u_i). The sequence (g_{u_i}) is also bounded in $L^p(X)$ and hence, by passing to a subsequence, we may assume that $u_i \to u$ weakly in $L^p(X)$ and $g_{u_i} \to g$ weakly in $L^p(X)$. Using the Banach–Saks theorem we see that the sequence $v_j = j^{-1}\sum_{i=1}^j u_i$ converges to u in $L^p(X)$ and $g_{v_j} = j^{-1}\sum_{i=1}^j g_{v_i}$ converges to g in $L^p(X)$. Now there is a subsequence of (v_j) so that $v_j \to u$ μ-a.e. and $g_{v_k} \to g$ μ-a.e. The function u belongs to $W^{1,p}(X)$ by Lemma 2.5. On the other hand $v_j \to 1$ μ-a.e. in O and hence $u \ge 1$ μ-a.e. there. This means that $u \in \mathscr{A}(O)$. By the weak lower semicontinuity of norms

$$\mathrm{C}_p(O) \le \|u\|^p_{L^p(X)} + \|g\|^p_{L^p(X)} \le \liminf_{i\to\infty}\bigl(\|u_i\|^p_{L^p(X)} + \|g_{u_i}\|^p_{L^p(X)}\bigr) \le \lim_{i\to\infty} \mathrm{C}_p(O_i) + \varepsilon$$

from which the claim follows by letting $\varepsilon \to 0$.

Before proving the next lemma we recall some terminology. We say that a property holds p-quasieverywhere (p-q.e.), if it holds except of a set of p-capacity zero.

3.6. Theorem. *For each Cauchy sequence of functions in $W^{1,p}(X)\cap C(X)$ there is a subsequence which converges pointwise p-q.e. in X. Moreover, the convergence is uniform outside a set of arbitrary small p-capacity.*

Proof. There is a subsequence of (u_i), which we still denote by (u_i), such that

$$\sum_{i=1}^{\infty} 2^{ip}\|u_i-u_{i+1}\|^p_{W^{1,p}(X)}<\infty.$$

For $i=1,2,\ldots$, denote $E_i=\bigl\{x\in X\colon |u_i(x)-u_{i+1}(x)|>2^{-i}\bigr\}$ and $F_j=\bigcup_{i=j}^{\infty}E_i$. By continuity $2^i(u_i-u_{i+1})$ is admissible for E_i, which implies

$$\mathrm{C}_p(E_i)\le 2^{ip}\|u_i-u_{i+1}\|^p_{W^{1,p}(X)}$$

and by subadditivity we obtain

$$\mathrm{C}_p(F_j)\le\sum_{i=j}^{\infty}\mathrm{C}_p(E_i)\le\sum_{i=j}^{\infty}2^{ip}\|u_i-u_{i+1}\|^p_{W^{1,p}(X)}.$$

Hence

$$\mathrm{C}_p\biggl(\bigcap_{j=1}^{\infty}F_j\biggr)\le\lim_{j\to\infty}\mathrm{C}_p(F_j)=0$$

and (u_i) converges in $X\setminus\bigcap_{j=1}^{\infty}F_j$. Moreover,

$$|u_j-u_k|\le\sum_{i=j}^{k-1}|u_i-u_{i+1}|\le\sum_{i=j}^{k-1}2^{-i}\le 2^{1-j}$$

in $X\setminus F_j$ for every $k>j$, which means that u_i convergence is uniform in $X\setminus F_j$. The theorem follows.

A function $u\colon X\to[-\infty,\infty]$ is p-quasicontinuous in X if for every $\varepsilon>0$ there is a set E such that $\mathrm{C}_p(E)<\varepsilon$ and the restriction of u to $X\setminus E$ is continuous. By outer regularity (Remark 3.3), we may assume that E is open. By [Ha, Theorem 3], $W^{1,p}(X)$ is a Banach space and by [Ha, Theorem 5], $C(X)\cap W^{1,p}(X)$ is a dense subspace of $W^{1,p}(X)$ and hence completeness implies that $W^{1,p}(X)$ can be characterized as the completion of $C(X)\cap W^{1,p}(X)$ in the norm defined by (2.3). This means that $u\in W^{1,p}(X)$ if and only if there exist sequences of functions $u_i\in L^p(X)\cap C(X)$ and $g_i\in D(u_i-u)$ such that $u_i\to u$ and $g_i\to 0$ in $L^p(X)$. We deduce from the previous theorem that the limit function is p-quasicontinuous and hence each Sobolev function has a p-quasicontinuous representative.

3.7. Corollary. *For each $u\in W^{1,p}(X)$ there is a p-quasicontinuous function $v\in W^{1,p}(X)$ such that $u=v$ μ-a.e. in X.*

4. Capacity and measure

We are mainly interested in the sets of vanishing capacity, since they are in some sense exceptional sets in the theory Sobolev spaces. Our first result is rather immediate.

4.1. Lemma. *$\mu(E) \le \mathrm{C}_p(E)$ for every $E \subset X$.*

Proof. If $u \in \mathscr{A}(E)$, then there is an open $O \supset E$ such that $u \ge 1$ in O and hence

$$\mu(E) \le \mu(O) \le \|u\|^p_{L^p(X)} \le \|u\|^p_{W^{1,p}(X)}.$$

We obtain the claim by taking the infimum over all $u \in \mathscr{A}(E)$.

In particular, sets of capacity zero are also of measure zero. From now on we assume in this section that the measure is nontrivial, Borel regular and that there is $c_d \ge 1$ such that

$$\mu\big(B(x,2r)\big) \le c_d\, \mu\big(B(x,r)\big) \tag{4.2}$$

for all $x \in X$ and $r > 0$. A measure satisfying the condition (4.2) is said to be doubling and the constant c_d is called the doubling constant. If μ is doubling, then every open set has a nonzero measure. If $0 < r < R < \infty$ and $x \in X$, then iterating the doubling condition we get

$$\mu\big(B(x,R)\big) \le c\Big(\frac{R}{r}\Big)^s \mu\big(B(x,r)\big), \tag{4.3}$$

where

$$s = \frac{\log c_d}{\log 2} \tag{4.4}$$

and c depends only on the doubling constant c_d. Observe that in the Euclidean case with the Lebesgue measure s equals to the dimension of the space. Hence (4.4) defines a dimension related to the doubling measure μ.

4.5. Capacity estimates. Here we provide a sharp upper bound for the p-capacity of a ball.

4.6. Theorem. *Let $x_0 \in X$ and $0 < r \le 1$. If μ is doubling, then*

$$\mathrm{C}_p\big(B(x_0,r)\big) \le c r^{-p} \mu\big(B(x_0,r)\big) \tag{4.7}$$

where c depends only on the doubling constant and p.

Proof. Let $0 < r < R$ and define

$$u(x) = \begin{cases} \dfrac{R - d(x, x_0)}{R - r}, & x \in B(x_0, R) \setminus B(x_0, r), \\ 1, & x \in B(x_0, r), \\ 0, & x \in X \setminus B(x_0, R), \end{cases}$$

and

$$g(x) = \begin{cases} \dfrac{1}{R - r}, & x \in B(x_0, R), \\ 0, & x \in X \setminus B(x_0, R). \end{cases}$$

We show that $g \in D(u)$. Let first $x, y \in A = B(x_0, R) \setminus B(x_0, r)$. Then

$$|u(x) - u(y)| = \frac{|d(x, x_0) - d(y, x_0)|}{R - r} \le \frac{d(x, y)}{R - r}.$$

Hence (2.1) follows in this case. Next let $x \in A$ and $y \in B(x_0, r)$. Now

$$|u(x) - u(y)| = 1 - u(x) = \frac{d(x, x_0) - r}{R - r}.$$

Since $d(x, x_0) \ge r > d(y, x_0)$, we have

$$d(x, x_0) - r \le d(x, x_0) - d(y, x_0) \le d(x, y)$$

and (2.1) follows. The case $y \in A$ and $x \in B(x_0, r)$ is completely analogous. If $x, y \in B(x_0, r)$ or $x, y \in X \setminus B(x_0, R)$, then clearly (2.1) holds. For the remaining cases, let $y \in B(x_0, r)$ and $x \in X \setminus B(x_0, R)$. Now

$$|u(x) - u(y)| = 1 = \frac{R - r}{R - r} \le \frac{d(x, y)}{R - r},$$

which implies (2.1). Finally, if $x \in A$ and $y \in X \setminus B(x_0, R)$, then

$$|u(x) - u(y)| = u(x) = \frac{R - d(x, x_0)}{R - r}$$

and since $d(x, x_0) < R < d(y, x_0)$, we obtain

$$R - d(x, x_0) \le d(y, x_0) - d(x, x_0) \le d(x, y)$$

and (2.1) again holds. Thus $g \in D(u)$, u belongs to $\mathscr{A}\big(B(x_0, r)\big)$ and

$$\begin{aligned} \mathrm{C}_p\big(B(x_0, r)\big) &\le \int_{B(x_0, R)} u^p \, d\mu + \int_{B(x_0, R)} g^p \, d\mu \le \big(1 + (R - r)^{-p}\big)\mu\big(B(x_0, R)\big) \\ &\le c\big(1 + (R - r)^{-p}\big)\Big(\frac{R}{r}\Big)^s \mu\big(B(x_0, r)\big). \end{aligned}$$

This leads to (4.7) if we choose $R = 2r$.

Next we improve estimate (4.7) in the case $p = s$ where s is defined by (4.4). To this end, we need a simple equality which holds for any Borel measure μ.

4.8. Lemma. *If* $s>0$, $R>0$ *and* $x_0\in X$, *then*

$$\int_{B(x_0,R)} d(x,x_0)^{-s}\,d\mu(x) = s\int_0^R \frac{\mu\big(B(x_0,\rho)\big)}{\rho^{s+1}}\,d\rho + R^{-s}\mu\big(B(x_0,R)\big).$$

Proof. The Fubini theorem implies

$$\begin{aligned}\int_{B(x_0,R)} d(x,x_0)^{-s}\,d\mu(x) = s\int_{1/R}^{\infty} &\rho^{s-1}\mu\big(\{x\in X: d(x,x_0)^{-1}>\rho\}\big)\,d\rho\\ &+R^{-s}\mu\big(\{x\in X: d(x,x_0)<R\}\big)\end{aligned}$$

and the claim follows by changing variables in the first integral on the right hand side.

4.9. Theorem. *Let* $x_0\in X$ *and* $0<r\le\frac12$. *If* μ *is doubling and* s *is defined by* (4.4), *then*

$$\mathrm{C}_s\big(B(x_0,r)\big)\le c\Big(\log\frac1r\Big)^{1-s} r^{-s}\mu\big(B(x_0,r)\big) \tag{4.10}$$

where c *depends only on the doubling constant.*

Proof. Let $0<r\le\frac12$ and set $B=B(x_0,1)$. Define

$$u(x)=\begin{cases}\Big(\log\dfrac1r\Big)^{-1}\log\dfrac{1}{d(x,x_0)}, & x\in B\setminus B(x_0,r),\\ 1, & x\in B(x_0,r),\\ 0, & x\in X\setminus B,\end{cases}$$

and

$$g(x)=\begin{cases}\Big(\log\dfrac1r\Big)^{-1}\dfrac{1}{d(x,x_0)}, & x\in B\setminus B(x_0,r),\\ \Big(\log\dfrac1r\Big)^{-1}\dfrac1r, & x\in B(x_0,r),\\ 0, & x\in X\setminus B.\end{cases}$$

We show that $g_u\in D(u)$. To see this, let $x,y\in A=B\setminus B(x_0,r)$. If $d(y,x_0)\le d(x,x_0)$, then

$$\begin{aligned}|u(x)-u(y)| &= \Big(\log\frac1r\Big)^{-1}\log\frac{d(x,x_0)}{d(y,x_0)}\le\Big(\log\frac1r\Big)^{-1}\frac{d(x,x_0)-d(y,x_0)}{d(y,x_0)}\\ &\le\Big(\log\frac1r\Big)^{-1}\frac{d(x,y)}{d(y,x_0)}\le\Big(\log\frac1r\Big)^{-1}d(x,y)\Big(\frac{1}{d(x,x_0)}+\frac{1}{d(y,x_0)}\Big).\end{aligned}$$

Here we used the elementary inequality $\log t \le t-1$ for $t \ge 1$. By symmetry, a similar estimate holds whenever $d(x,x_0) < d(y,x_0)$. For the remaining cases, let next $x \in A$ and $y \in B(x_0,r)$. Now

$$\begin{aligned}
|u(x)-u(y)| = 1-u(x) &= \Big(\log\frac{1}{r}\Big)^{-1}\Big(\log\frac{1}{r}+\log\frac{1}{d(x,x_0)}\Big)\\
&= \Big(\log\frac{1}{r}\Big)^{-1}\log\frac{d(x,x_0)}{r} \le \Big(\log\frac{1}{r}\Big)^{-1}\frac{d(x,x_0)-r}{r}\\
&\le \Big(\log\frac{1}{r}\Big)^{-1}\frac{d(x,y)-d(y,x_0)}{r} \le \Big(\log\frac{1}{r}\Big)^{-1}\frac{d(x,y)}{r}\\
&\le \Big(\log\frac{1}{r}\Big)^{-1} d(x,y)\Big(\frac{1}{r}+\frac{1}{d(x,x_0)}\Big)
\end{aligned}$$

and hence (2.1) holds also in this case. By symmetry it holds for $x \in B(x_0,r)$ and $y \in A$. For $x,y \in B(x_0,r)$ and $x,y \in X\setminus B$ inequality (2.1) is clear. If $y \in X \setminus B$ and $x \in A$, then

$$\begin{aligned}
|u(x)-u(y)| = u(x) &= \Big(\log\frac{1}{r}\Big)^{-1}\log\frac{R}{d(x,x_0)} \le \Big(\log\frac{1}{r}\Big)^{-1}\frac{R-d(x,x_0)}{d(x,x_0)}\\
&\le \Big(\log\frac{1}{r}\Big)^{-1}\frac{d(y,x_0)-d(x,x_0)}{d(x,x_0)} \le \Big(\log\frac{1}{r}\Big)^{-1}\frac{d(x,y)}{d(x,x_0)}
\end{aligned}$$

and (2.1) holds also in this case. It remains to check (2.1) when $x \in X\setminus B$ and $y \in B(x_0,r)$. Now

$$|u(x)-u(y)| = 1 = \frac{d(x,y)}{d(x,y)} \le \frac{d(x,y)}{1-r} \le \Big(\log\frac{1}{r}\Big)^{-1}\frac{d(x,y)}{r}$$

and hence (2.1) holds. Thus $g \in D(u)$ and since $u=1$ in $B(x_0,r)$, the function u belongs to $\mathscr{A}\big(B(x_0,r)\big)$. Now

$$\int_B g^s\,d\mu = \int_{B(x_0,r)} g^s\,d\mu + \int_{B\setminus B(x_0,r)} g^s\,d\mu.$$

The first integral on the right hand side is

$$\int_{B(x_0,r)} g^s\,d\mu = \Big(\log\frac{1}{r}\Big)^{-s}\frac{\mu\big(B(x_0,r)\big)}{r^s}.$$

For the second integral we employ Lemma 4.8,

$$\begin{aligned}
\int_{B\setminus B(x_0,r)} g^s\,d\mu &= \Big(\log\frac{1}{r}\Big)^{-s}\int_{B\setminus B(x_0,r)} d(x,x_0)^{-s}\,d\mu(x)\\
&= \Big(\log\frac{1}{r}\Big)^{-s}\Big(s\int_r^1 \frac{\mu\big(B(x_0,\rho)\big)}{\rho^{s+1}}\,d\rho + \mu(B) - r^{-s}\mu\big(B(x_0,r)\big)\Big).
\end{aligned}$$

Using (4.3), we obtain

$$\begin{aligned}\int_B g^s\,d\mu &= \Big(\log\frac{1}{r}\Big)^{-s}\Big(s\int_r^1 \frac{\mu\big(B(x_0,\rho)\big)}{\rho^{s+1}}\,d\rho+\mu(B)\Big)\\ &\le cr^{-s}\Big(\log\frac{1}{r}\Big)^{-s}\Big(s\log\frac{1}{r}+1\Big)\mu\big(B(x_0,r)\big)\\ &\le cr^{-s}\Big(\log\frac{1}{r}\Big)^{1-s}\Big(s+\frac{1}{\log 2}\Big)\mu\big(B(x_0,r)\big).\end{aligned}$$

Clearly $u \le g$ and hence

$$\mathrm{C}_p\big(B(x_0,r)\big) \le 2\int_B g^s\,d\mu$$

which completes the proof.

4.11. Hausdorff measures. We recall the definition of Hausdorff measures. Let $E \subset X$ and suppose that h: $[0,\infty) \to [0,\infty)$ is a non-decreasing function so that

$$\lim_{r\downarrow 0} h(r) = h(0) = 0.$$

For $0 < \delta \le \infty$ and $E \subset X$ we define

$$\mathscr{H}^h_\delta(E) = \inf\Big\{\sum_{i=1}^\infty h(r_i) : E \subset \bigcup_{i=1}^\infty B(x_i,r_i),\ r_i \le \delta\Big\}.$$

Now

$$\mathscr{H}^h(E) = \lim_{\delta\downarrow 0}\mathscr{H}^h_\delta(E) = \sup_{\delta>0}\mathscr{H}^h_\delta(E)$$

produces the standard (spherical) h-Hausdorff measure of E. If $h(t) = t^s$ for $0 \le s < \infty$, then we obtain the s-dimensional (spherical) Hausdorff measure which we denote by $\mathscr{H}^s$. The Hausdorff dimension of a set $E \subset X$ is

$$\dim E = \inf\{s : \mathscr{H}^s(E) = 0\} = \sup\{s : \mathscr{H}^s(E) = \infty\}.$$

For the properties of Hausdorff measures we refer to [F, 2.10].

We say that a measure μ is regular with dimension $s > 0$, if there is $c \ge 1$ such that

$$c^{-1}r^s \le \mu\big(B(x,r)\big) \le cr^s \tag{4.12}$$

for each $x \in X$ and $0 < r \le \operatorname{diam}(X)$. If μ is regular with dimension s, then μ is doubling and it satisfies (4.3). Moreover, X has Hausdorff dimension s and there is a constant $c > 0$ such that $c^{-1}\mathscr{H}^s(E) \le \mu(E) \le c\mathscr{H}^s(E)$ for every $E \subset X$.

4.13. Theorem. *Let* $h\colon [0,\infty) \to [0,\infty)$,

$$h(t) = \begin{cases} t^{s-p}, & \text{for } s > p, \\ \left(\log \dfrac{1}{t}\right)^{1-s}, & \text{for } s = p. \end{cases}$$

If μ *is regular with dimension* s, *then for every* $E \subset X$,

$$\mathrm{C}_p(E) \le c\mathscr{H}^h(E).$$

The constant c *depends only on* p *and the constant in* (4.12).

Proof. Let $B(x_i, r_i)$, $i = 1, 2, \ldots$, be any covering of E such that the radii satisfy $r_i \le \frac{1}{2}$. Now (4.7) and (4.10) yield

$$\mathrm{C}_p\bigl(B(x_i, r_i)\bigr) \le \begin{cases} c\left(\log \dfrac{1}{r_i}\right)^{1-s}, & \text{for } s = p, \\ c r_i^{s-p}, & \text{for } p < s, \end{cases}$$

and subadditivity implies

$$\mathrm{C}_p(E) \le \sum_{i=1}^{\infty} \mathrm{C}_p\bigl(B(x_i, r_i)\bigr) \le c \sum_{i=1}^{\infty} h(r_i).$$

The claim follows by taking the infimum over all coverings by balls and letting the radii tend to zero.

4.14. Corollary. *Let* h *be as in Theorem* 4.13. *If* $\mathscr{H}^h(E) = 0$, *then* $\mathrm{C}_p(E) = 0$.

We next consider the converse of Theorem 4.13. We prove that if μ is regular with dimension s, then sets of p-capacity zero have Hausdorff dimension at most $s - p$.

4.15. Theorem. *If* $E \subset X$ *with* $\mathrm{C}_p(E) = 0$, *then* $\mathscr{H}^t(E) = 0$ *for all* $t > s - p$.

Proof. Let $E \subset X$ be such that $\mathrm{C}_p(E) = 0$. Then for every $i = 1, 2, \ldots$, there is $u_i \in \mathscr{A}(E)$ and $g_{u_i} \in D(u_i) \cap L^p(X)$ such that

$$\|u_i\|^p_{L^p(X)} + \|g_{u_i}\|^p_{L^p(X)} \le 2^{-i}.$$

Define $u = \sum_{i=1}^{\infty} u_i$ and $g = \sum_{i=1}^{\infty} g_i$. We show that $u \in \mathscr{A}(E)$. To this end, let $v_k = \sum_{i=1}^{k} u_i$ and $g_{v_k} = \sum_{i=1}^{k} g_{u_i}$ for $k = 1, 2, \ldots$. Then $g_{v_k} \in D(v_k) \cap L^p(X)$, $v_k \to u$ μ-a.e. and $g_{v_k} \to g$ μ-a.e. Since $u, g \in L^p(X)$ we see from Lemma 2.5 that $u \in W^{1,p}(X)$ and $g \in D(u)$. Moreover, $u \ge 1$ μ-a.e. on a neighbourhood of E which means that u is admissible for E.

For every $\lambda > 0$ and for every $x \in E$ there is r_x such that μ-a.e. point of $B(x, r_x)$ belongs to $\{x \in X : u(x) \geq \lambda\}$ and therefore

$$\limsup_{r \downarrow 0} \fint_{B(x,r)} u \, d\mu = \infty. \tag{4.16}$$

Fix $x \in E$ and assume that there is $c < \infty$ such that

$$\limsup_{r \downarrow 0} r^{-t} \int_{B(x,r)} g^p \, d\mu < c.$$

Next we choose $R > 0$ so small that

$$\int_{B(x,r)} g^p \, d\mu < c r^t$$

for every $0 < r < R$. Denote $B_i = B(x, 2^{-i} R)$, $i = 1, 2, \ldots$. Then by the Poincaré inequality (Lemma 2.7) and the n-regularity assumption (4.12) we have

$$\begin{aligned} |u_{B_{i+1}} - u_{B_i}| &\leq \fint_{B_{i+1}} |u - u_{B_i}| \, d\mu \leq \frac{\mu(B_i)}{\mu(B_{i+1})} \biggl(\fint_{B_i} |u - u_{B_i}|^p \, d\mu \biggr)^{1/p} \\ &\leq c 2^{-i} R \biggl(\fint_{B_i} g^p \, d\mu \biggr)^{1/p} \leq c (2^{-i} R)^{(p-s+t)/p}. \end{aligned}$$

Hence, for $k > j$

$$|u_{B_k} - u_{B_j}| \leq \sum_{i=j}^{k-1} |u_{B_{i+1}} - u_{B_i}| \leq c \sum_{i=j}^{k-1} (2^{-i} R)^{(p-s+t)/p}$$

and so (u_{B_i}) is a Cauchy sequence if $t > s - p$. This contradicts (4.16) and hence

$$E \subset \Bigl\{ x \in X : \limsup_{r \downarrow 0} r^{-t} \int_{B(x,r)} g^p \, d\mu = \infty \Bigr\}.$$

Let

$$E_\lambda = \Bigl\{ x \in X : \limsup_{r \downarrow 0} r^{-t} \int_{B(x,r)} g^p \, d\mu > \lambda \Bigr\}.$$

Then $E \subset E_\lambda$ for every $\lambda > 0$ and by [F, 2.10.19 (1)] we obtain

$$\mathscr{H}^t(E) \leq \frac{c}{\lambda} \int_X g^p \, d\mu$$

for every $\lambda > 0$, which implies $\mathscr{H}^t(E) = 0$.

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