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ON A THEOREM OF BEARDON AND MASKIT

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Abstract. We refine a theorem of Beardon and Maskit by showing that a Kleinian group is geometrically finite if and only if its limit set consists entirely of conical limit points and parabolic fixed points.

1. Introduction

Consider a group G of Möbius transformations acting on the two sphere S^2. Such transformations are identified with elements of $\mathrm{PSL}(2,\mathbf{C})$ in a natural way and G is called *Kleinian* if it is discrete in this topology (i.e., the identity is isolated in G). Such a group G also acts as isometries on the hyperbolic 3-ball $\mathbf{B}^3$. The group G is called geometrically finite if there is a finite sided fundamental polyhedron for the group action in $\mathbf{B}^3$. The limit set $\Lambda \subset S^2$ is the accumulation set of the orbit $G(0)$ of $0 \in \mathbf{B}^3$. For $z \in S^2$ let $\Gamma(x,r)$ be the convex hull (in Euclidean geometry) of $\{z\}$ and $\{w \in \mathbf{B}^3 : |w| < r\}$. The point z is called a conical limit point if there is a $r < 1$ such that z is an accumulation point of $\Gamma(r,x) \cap G(0)$. The set of all conical limit points is denoted Λ_c.

The purpose of this note is to prove

Theorem 1.1. *G is geometrically finite if and only if every point of Λ is either a conical limit point or a fixed point of a parabolic element of G.*

The proof actually shows something slightly stronger.

Corollary 1.2. *G is geometrically finite if and only if $\Lambda \setminus \Lambda_c$ is countable* (*possibly empty*).

Our result is very similar to a result of Beardon and Maskit. To explain the difference, we need to recall a few facts about parabolic elements of G. A non-identity Möbius transformation on S^2 must have either one or two fixed points on S^2. If it has two fixed points it must be conjugate to a transformation of the form $z \to \lambda z$ and is called elliptic or loxodromic depending on whether $|\lambda| = 1$ or $|\lambda| \neq 1$. If the Möbius transformation has only one fixed point it is called

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parabolic and must be conjugate to the transformation $z \to z+1$. If p is fixed by some parabolic element of G then the set of parabolic elements in G fixing p is a subgroup which is isomorphic to either $\mathbf{Z}$ or $\mathbf{Z}^2$; the point p is called rank one or rank two respectively. Let $\Omega = S^2 \setminus \Lambda$ denote the ordinary set of G. If $p \in \Lambda$ is a parabolic fixed point of an element $g \in G$, we say p is doubly cusped if there are disjoint open disks D_1, D_2 in Ω which are tangent at p and which are both invariant under g. If p is not doubly cusped, but there is a g-invariant disk $D_1 \subset \Omega$, we say p is singly cusped. If neither condition holds we say p is not cusped.

The fixed point of a parabolic element $g \in G$ is clearly a limit point of G, but it is never a conical limit point. Beardon and Maskit's theorem [1] says that a Kleinian group is geometrically finite if and only if every limit point is either a conical limit point, a rank two parabolic fixed point or a rank 1, doubly cusped parabolic fixed point. Thus for groups with no parabolics our result says nothing new, but it gives a cleaner statement when parabolics are present.

The idea for the proof of Theorem 1.1 comes from the work of J.L. Fernández and M.V. Melián (personal communication) where they show that $\dim(\Lambda \setminus \Lambda_c) = 1$ when G is a Fuchsian group covering a Riemann surface of infinite area with no Green's function. I am grateful to Maria Melián for explaining their results to me and to Dick Canary, Bernie Maskit and Yair Minsky for conversations about the problem. I also thank the referee for carefully reading the manuscript and making several helpful suggestions.

2. Proof of Theorem 1.1

The proof follows from two facts; the first is an easy lemma about hyperbolic geometry in $\mathbf{B}^3$ and the second is a more difficult result of F. Bonahon [3] on closed geodesics in hyperbolic 3-manifolds.

Lemma 2.1. *Given $\theta > 0$ there are $C, M < \infty$ so that the following is true. Suppose γ is a piecewise geodesic path from a to b, that is, $\gamma = \bigcup_{j=1}^n \gamma_j \subset \mathbf{B}^3$ is a union of disjoint (except for endpoints) geodesic arcs, each of hyperbolic length at least M and such that γ_j and γ_{j+1} meet at an angle $\geq \theta$. Then γ is within hyperbolic distance C of the geodesic arc connecting a and b. In particular, a (one sided) infinite path with this property is within hyperbolic distance C of a infinite geodesic, hence terminates at a single point of S^2 and approaches that point inside some nontangential cone.*

The proof is left to the reader. (We will only use the case $\theta = \pi/2$.) The axis A_g of a loxodromic element $g \in G$ is the (infinite) hyperbolic geodesic in $\mathbf{B}^3$ which connects its two fixed points. Let ρ denote the hyperbolic metric on $\mathbf{B}^3$ and let $\operatorname{dist}_\rho(A, B)$ denote the hyperbolic distance between two sets. The following is Bonahon's result that in a geometrically infinite hyperbolic three manifold, there is a sequence of closed geodesics which leaves every compact set (we also wish to allow Kleinian groups which may have elliptic elements, but Bonahon's proof is still valid in this case).

Theorem 2.2 [3]. *If G is a geometrically infinite Kleinian group then there is a sequence of loxodromic elements g_j so that* $\operatorname{dist}_\rho\big(A_{g_j}, G(0)\big) \to \infty$.

Proof of Theorem 1.1. By Beardon and Maskit's theorem, if G is geometrically finite then every limit point is either conical or a parabolic fixed point. Therefore we need only show that if G is geometrically infinite, then there is a limit point which is neither conical nor a parabolic fixed point.

Let $F = \big\{w \in \mathbf{B}^3 : \rho(w,0) < \operatorname{dist}_\rho\big(w, G(0)\big)\big\}$. This is a convex fundamental domain for G. Take the sequence of loxodromics $\{g_j\} \subset G$ given by Bonahon's theorem and let $\{A_j\}$ denote the corresponding axes. By replacing each element by a conjugate, we may assume that $\operatorname{dist}_\rho(A_j, 0)$ is minimal over all choices of the conjugate (the minimum occurs because G is discrete).

Since $\operatorname{dist}_\rho(A_j, 0) \to \infty$ as $j \to \infty$, the Euclidean diameter of the sets A_j tends to zero. By passing to a subsequence (which we still denote by $\{A_j\}$) we may assume that the sets A_j converge to a point $x \in S^2$. Passing to yet another subsequence, if necessary, we may also assume that

$$\operatorname*{dist}_\rho\big(A_{j+1}, G(A_j)\big) \geq 10 \operatorname*{dist}_\rho(A_j, 0) \geq 10\max(M, 10), \tag{2.1}$$

where M is the constant in 2.1 with $\theta = \pi/2$.

Let $x_1 \in A_1$ be the point closest to 0 and let L_1 be the geodesic arc connecting 0 to x_1. In general, let $x_j \in A_j$ be the point closest to x_{j-1} and let L_j be the geodesic arc which connects x_{j-1} to x_j. Note that by the minimality of its length, L_j meets A_j at angle $\pi/2$ (but we can make no estimate of its angle with A_{j-1}).

Next we want to see that condition (2.1) implies that x_j is "near the top" of A_j, more precisely, there is a $C < \infty$ such that

$$x_j \in \{w \in A_j : \rho(w,0) \leq \operatorname*{dist}_\rho(A_j, 0) + C\}. \tag{2.2}$$

Using simply hyperbolic geometry it is enough to show that the angle between A_j and the geodesic arc $[0, x_j]$ is bounded away from zero. To do this we use hyperbolic trigonometry. Given a hyperbolic geodesic triangle with vertices a, b, c, opposite edge lengths A, B, C and vertex angles α, β, γ, we have (cosine rule I, p. 148 of [2]),

$$\cos(\gamma) = \frac{\cosh(A)\cosh(B) - \cosh(C)}{\sinh(A)\sinh(B)}.$$

Now let $a = 0$, $b = x_{j-1}$ and $c = x_j$. By (2.1), $A, B, C \geq 100$ and $A \geq 10C$, so $\cos(\gamma)$ is bounded uniformly away from 0 (in fact, is $\geq 9/10$). Hence γ (the angle between $[x_{j-1}, x_j]$ and $[0, x_j]$) is bounded uniformly away from $\pi/2$. Since $[x_{j-1}, x_j]$ meets A_j at angle $\pi/2$, we deduce that the angle between $[0, x_j]$ and A_j is uniformly bounded away from zero, as desired.

We now construct an infinite path γ connecting 0 to Λ which satisfies the conditions of Lemma 2.1 and which diverges from $G(0)$ (i.e., its projection leaves every compact set in $M = \mathbf{B}^3/G$). Lemma 2.1 implies this path is at bounded hyperbolic distance from a geodesic ray and the endpoint of this ray on S^2 will be the desired point.

We will define γ as a limit of finite piecewise geodesic arcs $\{\gamma_n\}$ where γ_n connects 0 to some point in the orbit $G(x_n)$. We begin by setting $\gamma_1 = L_1$. Suppose that we have defined γ_n with the properties that

(1) γ_n is a piecewise geodesic path connecting 0 to some point of $G(x_n)$.
(2) Each geodesic piece of γ_n has hyperbolic length at least M (the constant in Lemma 2.1 with $\theta = \pi/2$).
(3) Adjacent geodesic pieces of γ make an angle larger than $\pi/2$ at their common endpoint.
(4) The last geodesic segment in γ_n lies in $G(L_n)$.

By definition, γ_n ends at some point $\tilde{x}_n = g(x_n) \in \tilde{A}_n = g(A_n)$. As noted above, the angle between γ_n and $\tilde{A}_n$ is $\pi/2$. The geodesic arc $\tilde{L}_{n+1} = g(L_n)$ also meets $\tilde{A}_n$ at $\tilde{x}_n$. We do not know what angle it makes with the set $\tilde{A}_n$, but there are two possible ways to orient $\tilde{A}_n$ as a path and we choose an orientation so that the angle is $\geq \pi/2$. Follow $\tilde{A}_n$ in this orientation until we reach a point $y_n = h(x_n) \in \tilde{A}_n \cap G(x_n)$ such that

$$\rho(y_n, \tilde{x}_n) \geq M.$$

Let S_n be the arc of $\tilde{A}_n$ joining $\tilde{x}_n$ to y_n. By our choice of direction along $\tilde{A}_n$, S_n and $h(L_{n+1})$ meet at y_n and have angle greater than $\pi/2$. Define γ_{n+1} by adjoining first S_n and then $h(L_n)$ to γ_n. Clearly we have attained each of the inductive assumptions.

The construction is easier to follow on $M = \mathbf{B}^3/G$; choose a sequence of closed geodesics C_j which leave every compact set, fix a point $x_i \in C_1$ and let L_j be the shortest geodesic connecting C_{j+1} to C_j and let x_j be its landing point on C_{j+1}. By passing to a subsequence we may assume the arcs L_j are as long as we want. We would like the path γ to just be the concatenation of the arcs $\{L_j\}$, but then we would have no control over the angles at which they meet. Therefore between each L_j and L_{j+1} we traverse the closed geodesic some integral number of times. Making many turns gives us an arc as long as we wish and choosing one of the two possible directions gives us angle $\leq \pi/2$ with L_{j+1} (the angle with L_j is automatically $\pi/2$ by the minimality of L_j).

By Lemma 2.1 our path $\gamma \subset \mathbf{B}^3$ terminates at a well defined point $x \in S^2$. Since γ hits infinitely many loxodromic axes, its easy to see that x is a limit of loxodromic fixed points and hence is in the limit set.

Next we want to check that our path diverges to infinity in $M = \mathbf{B}^3/G$. The parts of the path made up of the $\{S_n\}$ are easy to handle, since

$$\operatorname*{dist}_{\rho}\bigl(S_n, G(0)\bigr) \geq \operatorname*{dist}_{\rho}(A_n, 0) \to \infty,$$

as $n \to \infty$.

The rest of the path is made of translates of the segments $\{L_n\}$. We would like to say that each L_n is a subset of the fundamental domain F. This would be true by convexity if its endpoints $\{x_{n-1}, x_n\}$ were in F. This is true (by definition) for x_1 since it was defined to be the closest point on A_1 to 0. In general, the point x_n is the closest point to x_{n-1}, not to 0. However, by (2.2), x_n lies within some bounded distance C of F and hence L_n must lie in a C-neighborhood of F also. This implies

$$\operatorname*{dist}_{\rho}\bigl(L_n, G(0)\bigr) \geq \operatorname{dist}(L_n, 0) - C.$$

Since the arcs A_n are assumed to be converging to a single point on S^2 both $\operatorname{diam}(L_n)$ and $\operatorname{dist}(L_n, S^2)$ (both measured in the Euclidean metric) tend to zero. This implies $\operatorname{dist}_\rho(L_n, 0) \to \infty$, as desired.

The fact that γ/G diverges to infinity in M immediately implies that its endpoint is not a conical limit point. To see that the endpoint is not a parabolic point we have two options. First, we note that we have many choices for the segments S_n in each stage of the construction of γ_n. Suppose γ, γ' are two possible paths that agree until the choice of y_n. Then the (doubly infinite) path we obtain by $(\gamma \cup \gamma') \setminus (\gamma \cap \gamma')$ satisfies the hypotheses of Lemma 2.1 (if the two choices of y_n are far enough apart). Thus it is within a bounded distance of a hyperbolic geodesic and hence the endpoints of γ and γ' on S^2 are distinct. Therefore we can construct uncountably many possible γ's with distinct endpoints and hence uncountably many points in $\Lambda \setminus \Lambda_c$. Since the parabolic fixed points are only countable, there must be a point of $\Lambda \setminus \Lambda_c$ which is not such a point. This also proves Corollary 1.2.

A second way to see that γ does not terminate at a parabolic fixed point x is to note that since γ approaches x through a non-tangential cone, it would eventually be contained in any horoball at x. Since the projection of γ to $M = \mathbf{B}/G$ contains closed geodesics, this would mean the parabolic thin part corresponding to x intersects closed geodesics, which contradicts the Margulis lemma, e.g., [3]. □

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