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UNIFORM DOMAINS OF HIGHER ORDER II

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Abstract. We continue the investigation of (p,c)-uniform domains in a normed space. Special emphasis is on the basic properties of homological (p,c)-uniformity.

1. Introduction

1.1. This paper is a continuation to [Al], in which the first author introduced the class of (p,c)-uniform domains, where $p \ge 0$ is an integer and $c \ge 1$. We recall the definition. Suppose that E is a real normed space with $\dim E \ge 2$. A domain $G \subset E$ is *homotopically* (p,c)*-uniform* if every continuous map $f\colon S^p \to G$ has a continuous extension $g\colon \overline{B}^{p+1} \to G$ satisfying the *lens condition*

$$d(x, |f|) \le c d(x, \partial G) \qquad \text{for all } x \in |g|, \tag{1.2}$$

and the *turning condition*

$$d(|g|) \le c d(|f|). \tag{1.3}$$

Here $|f| = \operatorname{im} f$, and $d(A)$ is the diameter of a set A. Together these conditions are called the *uniformity conditions.* For $p = 0$, we obtain the c-uniform domains in the ordinary distance sense. The class defined above has been independently considered by J. Heinonen and S. Yang [HY].

A homological version of the definition was also given in [Al]. The domain G is *homologically* (p,c)*-uniform* if each reduced singular p-cycle f in G bounds a chain g such that (1.2) and (1.3) are true; now $|f|$ is the carrier of f.

To abbreviate terminology we shall often replace the word ‘homotopically’ and ‘homologically’ by ‘htop’ and ‘hlog’, respectively.

The main emphasis in this paper will be on the basic properties of homological uniformity. Several results on hlog (p,c)-uniformity were given in [Al] only for $p = 1$. Our first task is to extend these for arbitrary p. This is done in Section 3. Before that, we develop in Section 2 a technique to construct singular homologies, called the *prismatic method.* In the rest of the paper we shall consider relations

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between htop and hlog uniformity, some related topological properties, and p-dimensional *plumpness*, which was introduced in [Al] in the homotopical case; a homological version will be given in Section 6.

Part III of this investigation is under preparation. It will contain, for example, results on cartesian products and planar sections of null-sets for htop and hlog (p,c)-uniform domains.

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1.4. *Terminology and notation.* The basic notation will be the same as in the first part; see [Al, p. 6]. Throughout the paper, E will denote a real normed space, and we shall usually assume that $\dim E \geq 2$. However, results dealing with porosity are relevant also in the one-dimensional case. We let $|a-b|$ denote the distance between points a,b in any metric space X. Open and closed balls in X are written as $B(a,r)$ and $\overline{B}(a,r)$. More generally, if $A \subset X$, we let $B(A,r)$ and $\overline{B}(A,r)$ denote the open and closed r-inflations of A. By a map we mean a continuous function. The image set of a map $f\colon A \to B$ will be written as $|f| = fA$. If $f,g\colon A \to X$ are two maps, we set $\|f-g\| = \sup\{|fa-ga| : a \in A\}$.

Our main reference on homology theory is the book [Ro] of J.J. Rotman. We shall use *reduced* singular homology with integral coefficients. Let X be a topological space and let p be an integer. The group $S_p(X)$ of singular p-chains is the free abelian group generated by all singular p-simplexes, that is, maps $\sigma\colon \Delta^p \to X$ of the standard p-simplex Δ^p. Thus each $\gamma \in S_p(X)$ has a *normal representation* $\gamma = \sum_{j\in J} n_j\sigma_j$, where J is a finite set, the singular simplexes σ_j are all distinct, and the integers n_j nonzero. We write $\sigma < \gamma$ if σ is a singular simplex appearing in the normal representation of the chain γ. The normal representation can then be written in the form $\gamma = \sum_{\sigma<\gamma} n_\sigma\sigma$. The carrier of γ is the set

$$|\gamma| = \cup\{|\sigma| : \sigma < \gamma\}.$$

For $p=-1$ there is a unique (empty) map $\sigma\colon \Delta^{-1} = \varnothing \to X$, and we identify $S_{-1}(X) = \mathbf{Z}$. For $p \leq -2$ we have $S_p(X) = 0$. The boundary homomorphism $\partial = \partial_p\colon S_p(X) \to S_{p-1}(X)$ is defined in the well-known way. For a 0-chain $\gamma = \sum_{\sigma<\gamma} n_\sigma\sigma$ we have $\partial\gamma = \sum_{\sigma<\gamma} n_\sigma$. The kernel of ∂_p is the group $Z_p(X)$ of p-cycles, and the p^{th} homology group of X is $H_p(X) = Z_p(X)/\operatorname{im}\partial_{p+1}$. In the literature, the group $H_0(X)$ is often written as $\widetilde{H}_0(X)$. It is a free abelian group of rank $m-1$ where m is the number of the path components of X. The group $H_{-1}(X)$ is trivial unless $X = \varnothing$, in which case $H_{-1}(X) = \mathbf{Z}$. We write $z \sim z'$ if the cycle z is homologous to z'.

Suppose that $Q(p)$ is a property of a set involving an integer $p \geq 0$. We say that a set A has the property $Q(p)$ *completely* if A has the property $Q(k)$ for all $0 \le k \le p$. For example, a domain G is completely hlog (p,c)-uniform if it is hlog (k,c)-uniform for all $0 \le k \le p$.

A domain is hlog (p)-uniform if it is hlog (p,c)-uniform for some $c \ge 1$. A similar convention is applied to any property involving a pair (p,c) with $p \in \mathbf{Z}$ and $c \ge 1$.

To illustrate the difference between complete and noncomplete uniformity, we consider the infinite cylinder $Z = B^{n-1} \times \mathbf{R} \subset R^n$, $n \ge 3$. It is not difficult to show that Z is htop and hlog (p)-uniform for all $1 \le p \le n-2$, in fact, $(p, \sqrt{2}\,)$-uniform. However, Z is not (0)-uniform and hence not completely htop or hlog (p)-uniform for any p.

It is possible to consider (p,c)-uniform domains G in the extended space $\dot{E} = E \cup \{\infty\}$ with $\infty \in G$. However, this essentially means that $G \setminus \{\infty\}$ is (p,c)-uniform; see [Vä$_3$, 5.4]. In this paper we consider only the case $G \subset E$. All closures and boundaries are taken in E, except when we are using the compactness method in $\dot{R}^n$.

2. Prisms

2.1. *Summary of Section* 2. We develop the *prismatic method*, which produces homologies of a given singular cycle. It will be applied several times in later sections. The method can be regarded as an elaboration of the proof of the homotopy axiom in singular homology.

2.2. *Basic concepts*. Let $\Delta = [v_0, \dots, v_p]$ be a p-simplex in R^n with the given order of vertices. We let $\tilde{\Delta}$ denote the simplicial complex consisting of all faces of Δ, including Δ. For $I = [0,1]$ we consider the prism $\Delta \times I \subset R^{n+1}$. We identify $\Delta = \Delta \times \{0\}$ and write $v'_j = (v_j, 1)$ for $0 \le j \le p$. The vertices of $\Delta \times I$ are ordered as $(v_0, \dots, v_p, v'_0, \dots, v'_p)$. Let $P\Delta$ be the standard triangulation of $\Delta \times I$. Its $(p+1)$-simplexes are

$$s_j = [v_0, \dots, v_j, v'_j, \dots, v'_p],$$

$0 \le j \le p$. The ordering of the vertices of $P\Delta$ defines an orientation of $P\Delta$.

For any oriented simplicial complex K, we let $C_*(K) = \big(C_p(K)\big)_{p \in \mathbf{Z}}$ denote the associated augmented chain complex with integral coefficients; see [Ro, p. 147].

We consider the case where Δ is the standard p-simplex $\Delta^p = [e_0, \dots, e_p]$. Let $\alpha_0\colon C_*(\tilde{\Delta}^p) \to C_*(P\Delta^p)$ be the chain map induced by the inclusion $\tilde{\Delta}^p \subset P\Delta^p$. Let X be a topological space and let $\sigma\colon \Delta^p \to X$ be a singular p-simplex in X. By a *prismoid* of σ we mean a chain map

$$\varphi\colon C_*(P\Delta^p) \to S_*(X)$$

satisfying the condition

$$\sigma = \varphi\alpha_0\Delta^p; \tag{2.3}$$

here Δ^p is considered as an element (generator) of $C_p(\tilde{\Delta}^p)$. Figure 1 illustrates two prismoids φ_1, φ_2 of a singular 1-simplex σ. The chain map φ_1 maps all elementary chains to elementary chains; the prismoid φ_2 is somewhat more complicated.

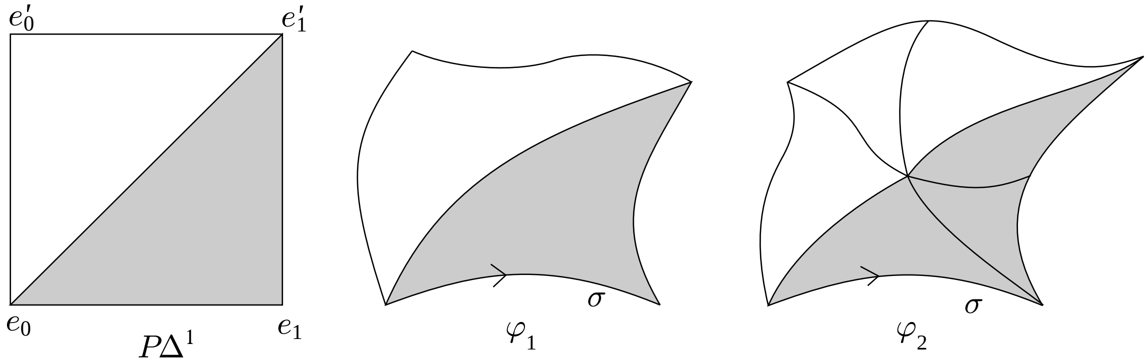


Figure 1

Let $0 \le q \le p-1$ and let $\varepsilon\colon \Delta^q \to \Delta^p$ be an order-preserving isometry onto a q-face of Δ^p. Then ε induces a chain map $\varepsilon_\#\colon C_*(\tilde{\Delta}^q) \to C_*(\tilde{\Delta}^p)$. Moreover, the map $\varepsilon \times \mathrm{id}\colon \Delta^q \times I \to \Delta^p \times I$ defines a simplicial map $\pi = \pi_\varepsilon\colon P\Delta^q \to P\Delta^p$ and a chain map $\pi_\#\colon C_*(P\Delta^q) \to C_*(P\Delta^p)$. The following functorial property is easily verified: If $\Delta^r \xrightarrow{\zeta} \Delta^q \xrightarrow{\varepsilon} \Delta^p$ are order-preserving isometries onto faces, then

$$\pi_{\varepsilon\zeta} = \pi_\varepsilon \pi_\zeta, \quad (\pi_{\varepsilon\zeta})_\# = \pi_{\varepsilon\#}\pi_{\zeta\#}. \tag{2.4}$$

Suppose that φ and ψ are prismoids of singular p-simplexes σ and τ, respectively. We say that these prismoids are *compatible* if $\varphi\pi_{\varepsilon\#} = \psi\pi_{\zeta\#}$ whenever $0 \le q \le p-1$ and $\varepsilon, \zeta\colon \Delta^q \to \Delta^p$ are order-preserving isometries onto faces such that $\sigma\varepsilon = \tau\zeta$. Observe that this is a relevant property also in the case $\sigma = \tau$, $\varphi = \psi$; we may then say that φ is self-compatible. See Figure 2.

Let γ be a singular p-chain in X with normal representation $\gamma = \sum_{\sigma<\gamma} n_\sigma \sigma$. A *prism* of γ is a chain map

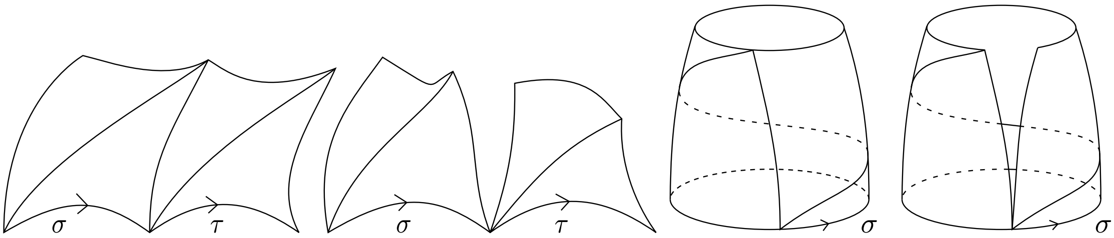
$$\varphi = \sum_{\sigma<\gamma} n_\sigma \varphi_\sigma\colon C_*(P\Delta^p) \to S_*(X)$$

such that

(1) φ_σ is a prismoid of σ for each $\sigma < \gamma$,

(2) φ_σ and φ_τ are compatible for all pairs $\sigma, \tau < \gamma$, including the case $\sigma = \tau$.

By a prism of the zero chain we mean the zero map. We see that a prismoid of a singular simplex is a prism if and only if it is self-compatible.



compatible not compatible self-compatible not self-compatible

Figure 2

2.5. *Construction of prisms.* We shall make use of prisms several times in this paper to obtain homologies of given cycles and sometimes also prisms of more general chains. The prisms will be constructed by a process where we start with the 0-chains of $P\Delta^p$ and proceed by induction to higher-dimensional skeletons of $P\Delta^p$. We shall next explain this process.

Let X be a topological space and let

$$\gamma = \sum_{\sigma<\gamma} n_\sigma \sigma$$

be a p-chain in X. We want to construct a prism $\varphi = \sum_{\sigma<\gamma} n_\sigma \varphi_\sigma$ of γ. The aim of the construction usually is to get a chain $\gamma_1 = \varphi\alpha_1\Delta^p$, where the chain map $\alpha_1\colon C_*(\tilde{\Delta}^p) \to C_*(P\Delta^p)$ is induced by the natural embedding of Δ^p onto $\Delta^p \times \{1\}$, such that γ_1 is in some sense better situated than γ.

Starting with 0-chains, let v be a vertex of $P\Delta^p$. If $v = e_j$ for some $0 \le j \le p$, we let $\varphi_\sigma v$ be the elementary 0-chain $\sigma v \in X$. If $v = e'_j = (e_j, 1)$, we can choose $\varphi_\sigma v$ to be a rather arbitrary point in X. However, to ensure the compatibility of the various φ_σ, we choose these points in a *compatible way*, which means that $\varphi_\sigma e'_j = \varphi_\tau e'_k$ whenever $\sigma e_j = \tau e_k$. The maps φ_σ are then extended linearly to homomorphisms $\varphi_\sigma\colon C_0(P\Delta^p) \to S_0(X)$.

Assume that $0 \le q \le p$ and that we have defined the chain maps $\varphi_\sigma\colon C_q(P\Delta^p) \to S_q(X)$ for all $\sigma < \gamma$. Let s be a $(q+1)$-simplex of $P\Delta^p$ and let $\sigma < \gamma$. If $s \in \tilde{\Delta}^p$, we set $\varphi_\sigma s = \sigma\varepsilon_s \in S_{q+1}(X)$, where $\varepsilon_s\colon \Delta^{q+1} \to \Delta^p$ is the order-preserving isometry onto s. Assume that s is not in $\tilde{\Delta}^p$. Then $\partial s \in C_q(P\Delta^p)$, and thus $\varphi_\sigma \partial s$ is defined. The chain $\varphi_\sigma s \in C_{q+1}(X)$ must satisfy the condition $\partial\varphi_\sigma s = \varphi_\sigma\partial s$. In the applications this will be possible by some conditions which are valid in the situation. Otherwise we can choose $\varphi_\sigma s$ rather arbitrarily, but to ensure compatibility, we choose $\varphi_\sigma s = \varphi_\tau t$ whenever $\varphi_\sigma\partial s = \varphi_\tau\partial t$. We again express this by saying that the chains $\varphi_\sigma s$ are chosen in a *compatible way.* The maps φ_σ are extended linearly to homomorphisms $\varphi_\sigma\colon C_{q+1}(P\Delta^p) \to S_{q+1}(X)$.

After the case $q = p$ we clearly have chain maps $\varphi_\sigma\colon C_*(P\Delta^p) \to S_*(X)$, which are prismoids of σ. It is a routine exercise to show that $\varphi = \sum_{\sigma<\gamma} n_\sigma\varphi_\sigma$ is a prism of γ.

In the applications, at the step $q=0$ of the induction, the 1-chains $\varphi_\sigma s$ can usually be chosen to be singular 1-simplexes, but for $q \geq 1$ we must in general choose a more general chain.

2.6. *Simple prisms.* Every continuous map $h\colon \Delta^p \times I \to X$ induces a chain map $\overline{h}\colon C_*(P\Delta^p) \to S_*(X)$ in the following way: Let s be a q-simplex of $P\Delta^p$, and let $\varepsilon_s\colon \Delta^q \to s$ be the order-preserving affine homeomorphism. Then $\overline{h}s$ is the singular simplex $h\varepsilon_s\colon \Delta^q \to X$.

Suppose that $\sigma\colon \Delta^p \to X$ is a singular p-simplex and that $h\colon \Delta^p \times I$ is a continuous extension of σ. Then clearly $\overline{h}$ is a prismoid of σ, but $\overline{h}$ need not be self-compatible. For example, if $p=1$, then $\overline{h}$ is self-compatible if and only if either $\sigma(e_0) \neq \sigma(e_1)$ or $h(e_0,t) = h(e_1,t)$ for all $t \in I$.

We say that a prism $\varphi = \sum_{\sigma<\gamma} n_\sigma \varphi_\sigma$ of a p-chain γ is *simple* if each φ_σ is of the form $\overline{h}_\sigma$ for some $h_\sigma\colon \Delta^p \times I \to X$.

To construct a simple prism of a chain $\gamma = \sum_{\sigma<\gamma} n_\sigma \sigma$ we can apply the method of 2.5. Now the maps h_σ are defined by skeletonwise induction. At the first step, the points $h_\sigma e_i'$ are chosen in a compatible way, by which we mean that $h_\sigma e_i' = h_\tau e_j'$ whenever $\sigma e_i = \tau e_j$. Assume that the maps h_σ are defined in the q-skeleton of $P\Delta^p$ and that s is a $(q+1)$-simplex in $P\Delta^p \setminus \tilde{\Delta}^p$. Then $h_\sigma|\partial s$ is defined, and we must extend this map to s. These extensions are chosen in a compatible way, that is, $h_\sigma|s = h_\tau|t$ whenever $h_\sigma|\partial s = h_\tau|\partial t$.

We remark that the standard direct proof of the homotopy axiom of singular homology makes use of simple prisms. Indeed, suppose that $h\colon X \times I \to Y$ is a homotopy connecting the maps $f_0, f_1\colon X \to Y$. Let $z = \sum_{\sigma<z} n_\sigma \sigma$ be a p-cycle in X. Setting $h_\sigma = h \circ (\sigma \times \mathrm{id})\colon \Delta^p \times I \to Y$ we obtain a simple prism of $f_{0\#}z$. It follows from Theorem 2.10 below that $f_{0\#}z$ is homologous to $f_{1\#}z$.

2.7. *Preparations.* We introduce some concepts and notation in order to state and prove the main result 2.10 on prisms. Let $\Delta = [v_0, \ldots, v_p]$ be an ordered p-simplex in R^n. As above, we let $\alpha_0, \alpha_1\colon C_*(\tilde{\Delta}) \to C_*(P\Delta)$ be the chain maps induced by the vertex maps $v_j \mapsto v_j$ and $v_j \mapsto v_j' = (v_j, 1)$, respectively. It is well known that α_0 and α_1 are connected by a chain homotopy $D = D_\Delta$ defined as follows: For each q-face $s = [u_0, \ldots, u_q]$ of Δ, considered as an element of $C_q(\tilde{\Delta})$, Ds is the $(q+1)$-chain

$$Ds = \sum_{j=0}^{q} (-1)^j [u_0, \ldots, u_j, u_j', \ldots, u_q']$$

in $C_{q+1}(P\Delta)$. This map is extended linearly to a homomorphism $D\colon C_q(\tilde{\Delta}) \to C_{q+1}(P\Delta)$ for each $0 \leq q \leq p$. The *chain homotopy law* states that

$$\partial D + D\partial = \alpha_1 - \alpha_0. \tag{2.8}$$

The chain homotopy D has the following functorial property: Suppose that $\Gamma = [u_0, \dots, u_q]$ is another ordered simplex with $q \le p$ and that $\varepsilon\colon \Gamma \to \Delta$ is an affine order-preserving embedding onto a q-face of Δ. Then ε induces a chain map $\varepsilon_\#\colon C_*(\widetilde{\Gamma}) \to C_*(\tilde{\Delta})$. Moreover, the affine embedding $\varepsilon \times \mathrm{id}\colon \Gamma \times I \to \Delta \times I$ induces a simplicial map $\pi_\varepsilon\colon P\Gamma \to P\Delta$ and a chain map $\pi_{\varepsilon\#}\colon C_*(P\Gamma) \to C_*(P\Delta)$. Then for each $j \in \mathbf{Z}$ we have the commutative diagram

$$\begin{array}{ccc} C_j(\widetilde{\Gamma}) & \xrightarrow{\varepsilon_\#} & C_j(\tilde{\Delta}) \\ \downarrow D_\Gamma & & \downarrow D_\Delta \\ C_{j+1}(P\Gamma) & \xrightarrow[\pi_{\varepsilon\#}]{} & C_{j+1}(P\Delta). \end{array}$$

Furthermore, if $\Gamma_1 \xrightarrow{\zeta} \Gamma_2 \xrightarrow{\varepsilon} \Delta$ are order-preserving affine embeddings, then obviously

$$\pi_{(\varepsilon\zeta)\#} = \pi_{\varepsilon\#}\pi_{\zeta\#}. \tag{2.9}$$

Suppose that γ is a singular p-chain in a topological space X and that φ is a prism of γ. Let $D^p\colon C_*(\tilde{\Delta}^p) \to C_*(P\Delta^p)$ be the above chain homotopy connecting the chain maps $\alpha_0, \alpha_1\colon C_*(\tilde{\Delta}^p) \to C_*(P\Delta^p)$. We again consider Δ^p as an element of $C_p(\tilde{\Delta}^p)$. The chain

$$g(\varphi) = \varphi D^p \Delta^p \in S_{p+1}(X)$$

will play a central role in the theory.

For $0 \le j \le p$ we let $\varepsilon_j\colon \Delta^{p-1} \to \Delta^p$ denote the standard face map, which embeds Δ^{p-1} onto the face of Δ^p opposite to e_j. Then $\varepsilon_j \times \mathrm{id}$ induces a simplicial map $\pi_j\colon P\Delta^{p-1} \to P\Delta^p$ as above, and we can define the chain maps

$$\varepsilon = \sum_{j=0}^{p} (-1)^j \varepsilon_{j\#}\colon C_*(\tilde{\Delta}^{p-1}) \to C_*(\tilde{\Delta}^p),$$
$$\pi = \sum_{j=0}^{p} (-1)^j \pi_{j\#}\colon C_*(P\Delta^{p-1}) \to C_*(P\Delta^p).$$

Observe that $\varepsilon\Delta^{p-1} = \partial\Delta^p$.

With this notation we give the basic result of prism theory:

2.10. **Theorem.** *Suppose that φ is a prism of a singular chain $\gamma \in S_p(X)$. Then $\psi = \varphi\pi$ is a prism of $\partial\gamma$. Setting $\gamma_1 = \varphi\alpha_1\Delta^p$ we have*

$$\partial g(\varphi) + g(\psi) = \gamma_1 - \gamma.$$

If γ is a cycle, then γ_1 is also a cycle, and $\partial g(\varphi) = \gamma_1 - \gamma$.

Proof. Let $\gamma = \sum_{i=1}^m n_i\sigma_i$ be the normal representation. We define an equivalence relation in the set $\{1,\dots,m\}\times\{0,\dots,p\}$ by calling (i,j) equivalent to (k,l) if $\sigma_i\varepsilon_j = \sigma_k\varepsilon_l$. Let G be the set of all equivalence classes and let $H\in G$. Set

$$m_H = \sum_{(i,j)\in H} (-1)^j n_i,$$

and let τ_H denote the map $\sigma_i\varepsilon_j\colon \Delta^{p-1}\to X$ for $(i,j)\in H$. Then

$$\partial\gamma = \sum_{i=1}^m\sum_{j=0}^p (-1)^j n_i\sigma_i\varepsilon_j = \sum_{H\in G}\sum_{(i,j)\in H}(-1)^j n_i\sigma_i\varepsilon_j = \sum_{H\in G} m_H\tau_H,$$

where the singular $(p-1)$-simplexes τ_H are all distinct. However, this is usually not a normal representation, since some numbers m_H may be zero. In particular, if $\partial\gamma = 0$, then $m_H = 0$ for all $H\in G$.

By the compatibility condition of the prism $\varphi = \sum_{i=1}^m n_i\varphi_i$, the chain map $\varphi_i\pi_{j\#}\colon C_*(P\Delta^{p-1})\to S_*(X)$ does not change if (i,j) is replaced by an equivalent pair. Given $H\in G$, we can therefore write $\psi_H = \varphi_i\pi_{j\#}$ for all $(i,j)\in H$. Clearly ψ_H is a prismoid of τ_H. Moreover,

$$\psi = \varphi\pi = \sum_{H\in G}\sum_{(i,j)\in H}(-1)^j n_i\varphi_i\pi_{j\#} = \sum_{H\in G} m_H\psi_H.$$

To prove that ψ is a prism of $\partial\gamma$ it suffices to show that if $H,K\in G$, then the prismoids ψ_H and ψ_K are compatible. Assume that $0\le q\le p-1$ and that $\varepsilon_H,\varepsilon_K\colon \Delta^q\to\Delta^{p-1}$ are order-preserving isometries onto faces such that $\tau_H\varepsilon_H = \tau_K\varepsilon_K$. Choose $(i,j)\in H$ and $(k,l)\in K$. Writing $\pi_H = \pi_{\varepsilon_H}$ and $\pi_K = \pi_{\varepsilon_K}$ we must show that $\psi_H\pi_{H\#} = \psi_K\pi_{K\#}$, that is,

$$\varphi_i\pi_{j\#}\pi_{H\#} = \varphi_k\pi_{l\#}\pi_{K\#}. \tag{2.11}$$

Setting $\zeta = \varepsilon_j\varepsilon_H$ and $\eta = \varepsilon_l\varepsilon_K$ we have $\sigma_i\zeta = \tau_H\varepsilon_H = \tau_K\varepsilon_K = \sigma_k\eta$. The maps $\zeta,\eta = \Delta^q\to\Delta^p$ are order-preserving isometries. By the compatibility of the prismoids φ_i and φ_k we have $\varphi_i\pi_{\zeta\#} = \varphi_k\pi_{\eta\#}$. This implies (2.11) by (2.9), and we have proved that ψ is a prism of $\partial\gamma$.

The chain homotopy law (2.8) and the chain map law $\partial g = g\partial$ give

$$\partial g(\varphi) = \partial\varphi D^p\Delta^p = \varphi\partial D^p\Delta^p = \varphi\alpha_1\Delta^p - \varphi\alpha_0\Delta^p - \varphi D^p\partial\Delta^p = \gamma_1 - \gamma - \varphi D^p\partial\Delta^p.$$

On the other hand, the functorial property of D implies

$$g(\psi) = \varphi\pi D^{p-1}\Delta^{p-1} = \varphi D^p\varepsilon\Delta^{p-1} = \varphi D^p\partial\Delta^p,$$

and we obtain the formula of the theorem. The last statement is a direct consequence. □

2.12. *Metric conditions.* In applications, the space X will always be metric, and we want that the chains $\gamma_1, g(\varphi)$ and $g(\psi)$ of 2.10 satisfy some metric conditions. In particular, we want that $g(\varphi)$ and $g(\psi)$ are not too far from γ and $\partial\gamma$, respectively. Let $\mathrm{pr}_1\colon \Delta^p \times I \to \Delta^p$ be the projection and let s be a simplex of $P\Delta^p$. Let $\gamma = \sum_{i=1}^m n_i\sigma_i$ be the normal representation. The construction of 2.5 will be carried out in such a way that

$$|\varphi_i s| \subset \overline{B}(\sigma_i \,\mathrm{pr}_1 s, r) \tag{2.13}$$

for some $r > 0$ and for all s and i. It is then clear that $|g(\varphi)| \subset \overline{B}(|\gamma|, r)$. We show that also $|g(\psi)| \subset \overline{B}(|\partial\gamma|, r)$. Let $x \in |g(\psi)|$. With the notation of the proof of 2.10, we have $x \in |\psi_H t|$ for some p-simplex $t \in P\Delta^{p-1}$ and for some $H \in G$ with $m_H \neq 0$. Choose $(i,j) \in H$. Then $\psi_H = \varphi_i\pi_{j\#}$ and $\tau_H = \sigma_i\varepsilon_j$. Moreover, $\pi_j = \varepsilon_j \times \mathrm{id}$ maps t onto a p-simplex $s \in P\Delta^p$ with $\mathrm{pr}_1 s = \varepsilon_j \,\mathrm{pr}_1 t$, and thus $\sigma_i \,\mathrm{pr}_1 s = \sigma_i\varepsilon_j \,\mathrm{pr}_1 t \subset |\tau_H|$. By (2.13) this implies

$$|\psi_H t| = |\varphi_i s| \subset \overline{B}(|\tau_H|, r).$$

Since $m_H \neq 0$, we have $|\tau_H| \subset |\partial\gamma|$, and hence $x \in \overline{B}(|\partial\gamma|, r)$. We have proved the following result:

2.14. **Theorem**. *Let X be a metric space and let $\gamma \in S_p(X)$ be a chain with normal representation $\gamma = \sum_{i=1}^m n_i\sigma_i$. Suppose that $r > 0$ and that $\varphi = \sum_{i=1}^m n_i\varphi_i$ is a prism of γ such that* (2.13) *holds for all simplexes $s \in P\Delta^p$ and for all $1 \le i \le m$. Then, with the notation of* 2.10, *we have*

$$|g(\varphi)| \subset \overline{B}(|\gamma|, r), \qquad |g(\psi)| \subset \overline{B}(|\partial\gamma|, r). \ \square$$

3. Homological uniformity

3.1. *Summary of Section* 3. We extend the results of [Al] on hlog $(1,c)$-uniformity to hlog uniformity of all orders. The central result is Theorem 3.10, which states that complete hlog (p,c)-uniformity follows from lower-order uniformity and an apparently weaker condition, which involves only 'nicely' situated p-cycles. We apply this result to characterize the complements of completely hlog (p,c)-uniform domains in R^n in terms of compact families of sets.

3.2. *Remark.* It is possible to characterize the hlog (p,c)-uniform domains in terms of homology groups without mentioning cycles and chains. Let A be a compact subset of a domain $G \subset E$, and let $c \ge 1$. Set

$$W(A,c,G) = \{x \in G : d(x,A) \le c\big(d(A) \wedge d(x,\partial G)\big)\}.$$

Then the following conditions are quantitatively equivalent:

(1) G is a hlog (p,c)-uniform domain.

(2) The homomorphism $H_p(A) \to H_p\big(W(A,c,G)\big)$ is zero for all compact sets $A \subset G$.

More precisely, (1) implies (2) with the same constant c, and (2) implies (1) with $c \mapsto 2c+1$. The proof is straightforward.

We start with an elementary lemma, which is often useful when working with uniform domains.

3.3. **Lemma.** *Let $U \subset E$ be open and let $A \subset B$ be subsets of U satisfying the lens condition*

$$d(x,A) \le cd(x,\partial U)$$

for some $c \ge 1$ and for all $x \in B$. Then $d(A,\partial U) \le 2cd(B,\partial U)$.

Proof. We may assume that $A \ne \varnothing$ and $U \ne E$. Write $r = d(A,\partial U)$ and let $x \in B$. If $d(x,A) \le r/2$, then $d(x,\partial U) \ge r/2 \ge r/2c$. If $d(x,A) \ge r/2$, then $d(x,\partial U) \ge d(x,A)/c \ge r/2c$. Hence $d(x,\partial U) \ge r/2c$. □

3.4. *Properties* HT *and* HT*. Let $G \subset E$ be a domain and let $p \in \mathbf{N}$, $c \ge 1$, $c' \ge 1$. We recall from [Al, 4.3] that G is said to have property HT(p,c',c) if for each (reduced) singular p-cycle z in G with $d(|z|) \le c'd(|z|,\partial G)$ there is $g \in S_{p+1}(G)$ such that

$$\partial g = z, \quad d(|g|) \le cd(|z|), \quad d(x,|z|) \le cd(x,\partial G) \tag{3.5}$$

for all $x \in |g|$. This was proved in [Al, 4.5] to be quantitatively equivalent to another condition HT$'(p,c',c)$. For our present needs, it is convenient to introduce a third condition, still quantitatively equivalent to HT(p,c',c). We say that G *has property* HT$^*(p,c',c)$ if for each p-cycle z in G with $d(|z|) \le c'd(|z|,\partial G)$ there is $g \in S_{p+1}(G)$ such that

$$\partial g = z, \quad d(|g|) \le cd(|z|), \quad d(|z|,\partial G) \le cd(|g|,\partial G). \tag{3.6}$$

3.7. **Lemma**. *For a domain $G \subset E$, property* HT(p,c',c) *implies* HT$^*(p,c',2c)$, *and* HT(p,c',c) *implies* HT$(p,c',c'c^2)$.

Proof. The first part of the lemma follows from 3.3. Suppose that G has property HT$^*(p,c',c)$. Let z be a p-cycle with $d(|z|) \le c'd(|z|,\partial G)$. Choose g satisfying (3.6), and let $x \in |g|$. Then

$$d(x,|z|) \le d(|g|) \le cd(|z|) \le c'cd(|z|,\partial G) \le c'c^2d(|g|,\partial G) \le c'c^2d(x,\partial G).$$

Hence (3.5) is true with c replaced by $c'c^2$. □

3.8. **Lemma**. *Suppose that $p \ge 1$ is an integer, that $c, c_1 \ge 1$ and that G is a completely hlog $(p-1,c)$-uniform domain in E. Write $c' = c'(c,p) = 8(2c)^{2p+1}(1+2(4c)^p)$. Suppose also that G has property $\mathrm{HT}^*(p,c',c_1)$ and that z is a p-cycle in G such that $d(|z|,\partial G) \ge r > 0$ and $d(|z|) > c'r$. Then there is a chain $g \in S_{p+1}(G)$ such that writing $z_1 = \partial g + z$, $c_0 = 4(4c)^p(2c)^{p+1} = c_0(c,p)$ and $c_2 = c_0 + 3c_0c_1 = c_2(c,c_1,p)$ we have*

(1) $|z_1| \subset \overline{B}(|z|, c_0 r)$,
(2) $d(|z_1|,\partial G) \ge 2r$,
(3) $|g| \subset \overline{B}(|z|, c_2 r)$,
(4) $d(|g|,\partial G) \ge r/c_2$.

Proof. Let $z = \sum_{\sigma<z} n_\sigma \sigma$ be the normal representation. Let $\mathrm{Sd}z$ be the barycentric subdivision of z; see [Ro, p. 114]. Then $|\mathrm{Sd}z| \subset |z|$ and $z - \mathrm{Sd}z$ bounds in $|z|$. Hence it suffices to prove the lemma with z replaced by $\mathrm{Sd}z$. Iterating this argument we see that we may assume that $d(|\sigma|) \le r' = 4(2c)^{p+1}r$ for all $\sigma < z$.

Following the method described in 2.5 we construct a prism $\varphi = \sum_{\sigma<z} n_\sigma \varphi_\sigma$ of z. Starting with 0-chains, assume that v is a vertex of $P\Delta^p$. If $v \in \Delta^p$, we set of course $\varphi_\sigma v = \sigma v$. Suppose that $v = e_j' = (e_j, 1)$. Since G is $(0,c)$-uniform, it is $4c$-plump by [Vä$_1$, 2.15]. Since $r' < c'r < d(|z|) < d(G)$, we can choose the point $\varphi_\sigma v \in G$ such that

$$|\varphi_\sigma v - \sigma e_j| \le r', \qquad d(\varphi_\sigma v, \partial G) \ge r'/4c.$$

Moreover, the points $\varphi_\sigma e_j'$ are chosen in a compatible way; see 2.5.

Proceeding inductively, assume that $0 \le q \le p-1$ and that the homomorphisms $\varphi_\sigma \colon C_q(P\Delta^p) \to S_q(G)$ have been defined in such a way that the following three conditions are satisfied for every q-simplex $s \in P\Delta^p$ and for every $\sigma < z$:

(i) $|\varphi_\sigma s| \subset \overline{B}(|\sigma|, (4c)^q r')$,
(ii) $d(|\varphi_\sigma s|, \partial G) \ge r/(2c)^q$,
(iii) $d(|\varphi_\sigma s|, \partial G) \ge r'/2(2c)^{q+1}$ whenever $s \subset \Delta^p \times \{1\}$.

The conditions are easily verified for $q = 0$.

Let $\sigma < z$. To define $\varphi_\sigma \colon C_{q+1}(P\Delta^p) \to S_{q+1}(G)$ assume that s is a $(q+1)$-simplex of $P\Delta^p$. If $s \subset \Delta^p$, we set of course $\varphi_\sigma s = \sigma\varepsilon_s$; see 2.5. The conditions (i), (ii), (iii) are easily verified for such s. Assume that s is not in Δ^p. Now $\varphi_\sigma \partial s$ is defined, and it is a q-cycle in G. If $\varphi_\sigma \partial s = 0$, we set $\varphi_\sigma s = 0$. If $\varphi_\sigma \partial s \ne 0$, we apply the hlog (q,c)-uniformity of G and Lemma 3.3 to find a chain $\varphi_\sigma s \in S_q(G)$ such that

$$\partial\varphi_\sigma s = \varphi_\sigma \partial s, \quad d(|\varphi_\sigma s|) \le c\, d(|\varphi_\sigma \partial s|), \quad d(|\varphi_\sigma \partial s|, \partial G) \le 2c\, d(|\varphi_\sigma s|, \partial G).$$

Moreover, the chains $\varphi_\sigma s$ are chosen in a compatible way; see 2.5.

We show that the conditions (i)–(iii) hold for each $(q+1)$-simplex s. We may assume that $\varphi\partial s \neq 0$. By the inductive hypothesis we have $|\varphi_\sigma \partial s| \subset \overline{B}(|\sigma|, (4c)^q r')$, which implies that $|\varphi_\sigma s| \subset \overline{B}(|\sigma|, r'')$ where

$$r'' = (4c)^q r' + cd(|\varphi_\sigma \partial s|).$$

Since $d(|\sigma|) \leq r'$, we have $d(|\varphi_\sigma \partial s|) \leq r' + 2(4c)^q r'$, and hence $r'' \leq (4c)^{q+1} r'$, which proves (i).

By the inductive hypothesis we have $d(|\varphi_\sigma \partial s|, \partial G) \geq r/(2c)^q$, which implies that

$$d(|\varphi_\sigma s|, \partial G) \geq r/(2c)^{q+1},$$

and hence (ii) is true. Finally, if $s \subset \Delta^p \times \{1\}$, we apply (iii) of the inductive hypothesis and obtain $d(|\varphi_\sigma \partial s|, \partial G) \geq r'/2(2c)^{q+1}$, which yields

$$d(|\varphi_\sigma s|, \partial G) \geq r'/2(2c)^{q+2},$$

and hence (iii) is true for s.

As the last step of the induction we obtain chain maps $\varphi_\sigma\colon C_p(P\Delta^p) \to S_p(G)$, $\sigma < z$. We shall continue by one step to $\varphi_\sigma\colon C_{p+1}(P\Delta^p) \to S_{p+1}(G)$, but instead of uniformity, we shall apply the property $\mathrm{HT}^*(p, c', c)$.

Let s be a $(p+1)$-simplex of $P\Delta^p$ and let $\sigma < z$. We again have the p-cycle $\varphi_\sigma \partial s$ in G. Assuming that $\varphi_\sigma \partial s \neq 0$ we infer from (i) and (ii) that

$$\frac{d(|\varphi_\sigma \partial s|)}{d(|\varphi_\sigma \partial s|, \partial G)} \leq \frac{\bigl(1 + 2(4c)^p\bigr) r'}{r/(2c)^p} = c'/2 < c'.$$

Hence property $\mathrm{HT}^*(p, c', c_1)$ gives a $(p+1)$-chain $\varphi_\sigma s$ such that $\partial \varphi_\sigma s = \varphi_\sigma \partial s$ and such that

$$\begin{aligned} d(|\varphi_\sigma s|) &\leq c_1 d(|\varphi_\sigma \partial s|) \leq c_1 \bigl(1 + 2(4c)^p\bigr) r', \\ d(|\varphi_\sigma s|, \partial G) &\geq d(|\varphi_\sigma \partial s|, \partial G)/c_1 \geq r/c_1(2c)^p. \end{aligned} \tag{3.9}$$

These chains are also chosen in a compatible way. The construction of the prism $\varphi = \sum_{\sigma < z} n_\sigma \varphi_\sigma$ has now been completed.

Setting $z_1 = \varphi \alpha_1 \Delta^p$ and $g = g(\varphi)$ we have $\partial g = z_1 - z$ by 2.10. We show that the conditions (1)–(4) of the lemma are true. Since $z_1 = \varphi s$ with $s = \Delta^p \times \{1\}$, (1) follows from (i). The condition (2) follows from (iii), since $r'/2(2c)^{p+1} = 2r$. To prove (3) let $x \in |g|$. Then $x \in |\varphi_\sigma t|$ for some $(p+1)$-simplex $t \in P\Delta^p$ and some $\sigma < z$. Now (i) implies that

$$|\varphi_\sigma \partial t| \subset \overline{B}\bigl(|\sigma|, (4c)^p r'\bigr) = \overline{B}(|\sigma|, c_0 r).$$

Since $d(|\sigma|) \leq r' < c_0 r$, this proves $d(|\varphi_\sigma \partial t)| \leq 3c_0 r$. By (3.9) we obtain $d(|\varphi_\sigma t|) \leq 3c_0 c_1 r$. It follows that $x \in \overline{B}(|z|, c_2 r)$, and (3) is proved.

Finally, (4) follows from (3.9), since $d(|g|, \partial G) \geq r/c_1(2c)^p > r/c_2$. □

We next extend [Al, 4.13] to uniform domains of all orders.

3.10. **Theorem.** *Suppose that $p \ge 1$ and that $G \subset E$ is a completely hlog $(p-1, c)$-uniform domain with property* $\mathrm{HT}^*(p, c', c_1)$*, where $c' = c'(c, p)$ is given in* 3.8*. Then G is hlog (p, c_2)-uniform with $c_2 = c_2(c, c_1, p)$.*

The theorem is valid also in the case $p = 0$; the $(-1, c)$-uniformity is interpreted as c-plumpness.

Proof. The case $p = 0$ was given in [Vä$_1$, 2.15]. Assume that $p \ge 1$, and let z_0 be a p-cycle in G. We look for a chain $g \in S_{p+1}(G)$ with $\partial g = z_0$ and satisfying the uniformity conditions. Write

$$d_0 = d(|z_0|), \qquad r_0 = d(|z_0|, \partial G).$$

If $d_0 \le c' r_0$, the desired g is given by $\mathrm{HT}^*(p, c', c)$ and Lemma 3.7. Suppose that $d_0 > c' r_0$. By successive applications of 3.8 we get sequences

$$z_0, \dots, z_k \in Z_p(G), \qquad g_1, \dots, g_k \in S_{p+1}(G)$$

such that $\partial g_i = z_i - z_{i-1}$ and such that writing $d_i = d(|z_i|)$, $r_i = d(|z_i|, \partial G)$ we have for all $1 \le i \le k$

(1) $|z_i| \subset \overline{B}(|z_{i-1}|, c_0 r_{i-1})$,
(2) $r_i \ge 2 r_{i-1}$,
(3) $|g_i| \subset \overline{B}(|z_{i-1}|, c_2 r_{i-1})$,
(4) $d(|g_i|, \partial G) \ge r_{i-1}/c_2$,

where the constants $c_0 = c_0(c, p)$ and $c_2 = c_2(c, c_1, p)$ are as in 3.8. We proceed with the construction as long as we get an index k such that $d_k \le c' r_k$. We first show that such an index exists. Assume that $d_i > c' r_i$ for $0 \le i \le k-1$. We show that

$$d_i \le d_0 + 4 c_0 r_{i-1} \le 2 d_0 \tag{3.11}$$

for all $i = 1, \dots, k$.

By (1) we obtain $d_i \le d_{i-1} + 2 c_0 r_{i-1}$. By iteration and (2) this gives

$$d_i \le d_0 + 2c_0(r_{i-1} + \dots + r_0) \le d_0 + 2c_0 r_{i-1}(1 + 2^{-1} + \dots + 2^{1-i}) \le d_0 + 4 c_0 r_{i-1},$$

which is the first inequality of (3.11). The second inequality follows by induction, since assuming $d_{i-1} \le 2 d_0$ and observing that $c' \ge 8 c_0$ we obtain

$$4 c_0 r_{i-1} < 4 c_0 d_{i-1}/c' \le d_{i-1}/2 \le d_0.$$

If $d_k < c' r_k$, then (3.11) and (2) give

$$8 c_0 \le c' < d_k / r_k \le d_0 / r_k + 4 c_0 r_{k-1}/r_k \le 2^{-k} d_0 / r_0 + 2 c_0.$$

Since this gives a contradiction for large k, we can choose the smallest $k \ge 1$ for which $d_k \le c' r_k$. Then (3.11) holds for $1 \le i \le k$.

By property $\mathrm{HT}^*(p, c', c_1)$ there is $g_{k+1} \in S_{p+1}(G)$ such that

$$\partial g_{k+1} = -z_k, \quad d(|g_{k+1}|) \le c_1 d_k, \quad d(|g_{k+1}|, \partial G) \ge r_k / c_1. \tag{3.12}$$

Setting

$$g = -\sum_{i=1}^{k+1} g_i$$

we have $\partial g = z_0$. We show that g satisfies the uniformity conditions in G with some constant $c_3 = c_3(c, c_1, p)$.

To prove the turning condition let $x \in |g|$. Then $x \in |g_i|$ for some $1 \le i \le k+1$. If $i \le k$, then (1), (2) and (3) yield

$$d(x, |z_0|) \le c_2 r_{i-1} + c_0(r_0 + \cdots + r_{i-2}) \le (c_2 + c_0) r_{i-1}. \tag{3.13}$$

Since $r_{i-1} < d_{i-1}/c' < d_{i-1} \le 2d_0$ by (3.11) and since $c_2 \ge c_0$, we get $d(x, |z_0|) \le 4c_2 d_0$. If $i = k+1$, then (3.12) and (3.11) give

$$d(x, |z_k|) \le d(|g_{k+1}|) \le c_1 d_k \le 2c_1 d_0.$$

Since $|z_k| \subset |g_k| \subset \overline{B}(|z_0|, 4c_2 d_0)$ by the first case, we have $d(x, |z_0|) \le (2c_1 + 4c_2) d_0$ in both cases. This gives the turning condition $d(|g|) \le (1 + 4c_1 + 8c_2) d_0$.

To prove the lens condition let again $x \in |g_i|$ for some $1 \le i \le k+1$. If $i \le k$, then (4) implies $d(x, \partial G) \ge r_{i-1}/c_2$. By (3.13) we obtain

$$d(x, |z_0|) \le c_2(c_2 + c_0) d(x, \partial G). \tag{3.14}$$

Assume that $i = k+1$. Now (3.12) implies that

$$d(x, \partial G) \ge d(|g_{k+1}|, \partial G) \ge r_k / c_1. \tag{3.15}$$

Choose a point $y \in |z_k|$. Then (1) and (2) yield

$$d(y, |z_0|) \le c_0(r_0 + \cdots + r_{k-1}) \le c_0 r_k.$$

By (3.12) and (3.15) this gives

$$\begin{aligned} d(x, |z_0|) &\le |x - y| + d(y, z_0) \le d(|g_{k+1}|) + c_0 r_k \\ &\le c_1 d_k + c_0 r_k \le (c_1 c' + c_0) r_k. \\ &\le c_1 (c_1 c' + c_0) d(x, \partial G). \end{aligned}$$

This and (3.14) prove the lens condition. □

3.16. *Uniformity and compactness.* With the aid of Theorem 3.10, we shall generalize the results of [Al, §6] on hlog (1)-uniform domains to hlog (p)-uniform domains with arbitrary p. For $p = 0$ these results were known earlier.

We recall that an open set $U \subset E$ is *c-plump* if for each $x \in U$ and $0 < r < d(U)$ there is $z \in \overline{B}(x,r)$ with $B(z, r/c) \subset U$. If, in addition, $\text{int}(E \setminus U) = \varnothing$, the closed set $E \setminus U$ is said to be *c-porous.*

We recall some notation and terminology of [Al, §6]. Let K^n_∞ be the family of all compact sets A with $\infty \in A \subset \dot{R}^n = R^n \cup \{\infty\}$. Each closed subset A of R^n will be identified with $A \cup \{\infty\}$ and considered as an element of K^n_∞. With the Hausdorff metric induced by the spherical metric, K^n_∞ is a compact metric space. For $0 \le p \le n-2$, let $\text{HU}(p,c)$ denote the family of all $A \in K^n_\infty$ such that $R^n \setminus A$ is a completely hlog (p,c)-uniform domain, and let $\text{HU}(-1,c)$ be the family of all sets $A \in K^n_\infty$ such that $R^n \setminus A$ is c-plump. We set $\text{HU}(p) = \cup\{\text{HU}(p,c) : c \ge 1\}$.

We fix a unit vector $e_1 \in R^n$. A family $H \subset K^n_\infty$ is *stable* if H is invariant under similarities and if $H^2 = \{A \in H : \{0, e_1\} \subset \partial A\}$ is compact in K^n_∞. To each $H \subset K^n_\infty$ we associate the subfamily

$$\sigma(H) = \cup\{L \subset H : L \text{ stable}\}.$$

A *filtration* of $\sigma(H)$ is a function $c \mapsto M_c$, defined for $c \in [1, \infty)$, such that

(1) $c < d$ implies $M_c \subset M_d$,
(2) each M_c is contained in a stable subfamily of H,
(3) each stable subfamily of H is contained in some M_c.

For $-1 \le p \le n-2$ let D_p be the family of all $A \in K^n_\infty$ such that $H_k(R^n \setminus A) = 0$ for $k \le p$. By Alexander duality, a set $A \in K^n_\infty$ is in D_p if and only if $H^q(A) = 0$ for $n-p-1 \le q \le n$, where $H^q(A)$ denotes the q^{th} reduced Čech cohomology group of A. Observe that $D_{-1} = K^n_\infty \setminus \{\dot{R}^n\}$.

3.17. **Theorem**. *For every $-1 \le p \le n-2$ we have $\sigma(D_p) = \text{HU}(p)$, and the function $c \mapsto \text{HU}(p,c)$ is a filtration of $\sigma(D_p)$.*

Proof. The case $p = -1$ is [Vä$_1$, 3.4]. Let $p \ge 0$. The inclusion $\text{HU}(p) \subset \sigma(D_p)$ was proved in [Al, 6.6]. In fact, the proof shows that each $\text{HU}(p,c)$ is contained in a stable subfamily of D_p. Conversely, assume that $M \subset D_p$ is a stable family. Using inductively [Al, 6.4], 3.7 and 3.10 we see that M is contained in some $\text{HU}(p,c)$. □

3.18. **Corollary.** *A domain $G \subset R^n$ is not completely hlog (p)-uniform if and only if there is a sequence (α_j) of similarities of R^n such that*

(1) $\{0, e_1\} \subset \alpha_j \partial G$,
(2) $\dot{R}^n \setminus \alpha_j G \to A \in K^n_\infty$, *and either* $0 \in \text{int}A$ *or* $H_k(R^n \setminus A) \ne 0$ *for some* $0 \le k \le p$. □

3.19. NUD *sets.* We recall that a closed set $A \subset R^n$ is hlog (p,c)-NUD (null-set for uniform domains) if $\text{int}A = \varnothing$ and if $R^n \setminus A$ is a hlog (p,c)-uniform domain. For $0 \le p \le n-2$ let $\text{HN}(p,c)$ denote the family of all completely hlog (p,c)-NUD sets in R^n, and let $\text{HN}(-1,c)$ be the family of all c-porous sets. We set $\text{HN}(p) = \cup\{\text{HN}(p,c) : c \ge 1\}$. Let L_p be the family of all closed sets in R^n with topological dimension at most p.

3.20. **Theorem**. *Let $p \ge -1$ and $q \ge 0$ be integers with $p+q = n-2$. Then $\sigma(L_q) = \text{HN}(p)$, and $c \mapsto \text{HN}(p,c)$ is a filtration of $\sigma(L_q)$.*

Proof. The case $p = -1$, $q = n-1$, is [Vä$_2$, 5.4]. Let $p \ge 0$. In [Al, 6.10] it was proved that the closure of $\text{HN}(p,c)$ is a stable subfamily of L_q. Conversely, let $M \subset L_q$ be stable. By [HW, VIII.4F, p. 137], we have $L_q \subset D_p$. Hence $M \subset \text{HU}(p,c)$ for some c by 3.17, which implies that $M \subset \text{HN}(p,c)$. □

3.21. **Theorem**. *Suppose that $A \in \text{HU}(p,c)$ and that $f\colon A \to \dot{R}^n$ is a θ-quasimöbius map with $\infty \in fA$. Then $fA \in \text{HU}(p,c_1)$ with $c_1 = c_1(c,\theta)$.*

Proof. See [Al, 6.11]. The condition $\infty \in fA$ is unnecessary if we allow the possibility $\infty \in G$ for uniform domains; see [Vä$_3$, 5.4]. □

3.22. **Theorem**. *Let E be a normed space of dimension at least n with $n \ge 3$. Let $\gamma \subset E$ be an arc of c-bounded turning. Then γ is completely hlog $(n-3,c_1)$-NUD with $c_1 = c_1(c,n)$.*

Proof. This follows inductively from 3.7, 3.10, and [Al, 5.4]. □

3.23. *Remark.* Theorem 3.22 is valid in the case $\dim E = \infty$. However, one can show that in this case, every compact set in E is htop (p,c_0)-NUD for every $p \ge 0$ with a universal constant c_0. This will be proved in Part III.

4. Relations between homotopical and homological uniformity

4.1. *Summary of Section* 4. Roughly speaking, homotopical properties imply the corresponding homological properties. For example, if a space X is p-connected, it is also p-acyclic, that is, $H_k(X) = 0$ for $0 \le k \le p$. The converse is not true, but it is true for 1-connected spaces. This follows directly from the Hurewicz theorem [Sp, 7.5.5].

In this section we consider quantitative versions of these results. We show in 4.2 that a completely htop (p,c)-uniform domain is completely hlog (p,c_1)-uniform with $c_1 = c_1(c,p)$. The case $p = 0$ was given in [Al, p. 7] and the case $p = 1$ in [Al, 1.7], both with $c_1 = 2c+1$. The converse is true for $p = 0$ [Al, p. 7] but not for $p \ge 1$, since for $n \ge 4$, the complement of a BT arc in R^n is completely hlog $(n-3)$-uniform by 3.22, but it need not be simply connected and hence not htop (1)-uniform; see [Al, pp. 32–33].

We conjecture that the converse of 4.2 is true for htop $(1,c)$-uniform domains. In 4.5 we prove this conjecture in R^n, but the proof makes use of compact families of sets, and hence the bound for the htop uniformity depends also on n.

4.2. **Theorem**. *If a domain $G \subset E$ is completely htop (p,c)-uniform, it is completely hlog (p,c_1)-uniform with $c_1 = c_1(c,p)$.*

Proof. The easy case $p = 0$ was given in [Al, p. 7] with $c_1 = 2c+1$. Proceeding inductively, we see that it suffices to prove that if G is completely htop (p,c)-uniform, it is hlog (p,c_1)-uniform with $c_1 = c_1(c,p)$. By 3.10, it suffices to show that G has property $\mathrm{HT}^*(p,c',c_1)$ where $c' = c'(c,p)$ is given in 3.8.

Let $z \in Z_p(G)$ be such that $d \le c'r$ where $d = d(|z|)$ and $r = d(|z|,\partial G)$. We must show that $z = \partial g$ for some g such that

$$d(|g|) \le c_1 d, \qquad d(|g|,\partial G) \ge r/c_1. \tag{4.3}$$

Let $z = \sum_{\sigma<z} n_\sigma \sigma$ be the normal representation. Fix a point $a \in |z|$. We use the method described in 2.5 and 2.6 to construct maps $h_\sigma\colon \Delta^p \times I \to G$ such that $\varphi = \sum_{\sigma<z} n_\sigma \overline{h}_\sigma$ is a simple prism of z. For $0 \le q \le p+1$, let K_q be the subcomplex of $P\Delta^p$ consisting of the simplexes $s \in P\Delta^p$ such that either $s \subset \Delta^p \times \{0,1\}$ or $\dim s \le q$. For $\sigma < z$ and $y \in |K_0|$, define $h_\sigma(y)$ by $h_\sigma(x,0) = \sigma(x)$ and $h(x,1) = a$. The compatibility condition of 2.6 is clearly satisfied for the 0-simplexes in K_0. Proceeding inductively, assume that for some $0 \le q \le p$, the maps h_σ have been defined in $|K_q|$ in such a way that setting $A_q = \cup\{h_\sigma|K_q| : \sigma < z\}$ we have

$$d(A_q) \le (2c+1)^q d, \qquad d(A_q,\partial G) \ge r/(2c)^q. \tag{4.4}$$

Since $A_0 = |z|$, (4.4) holds for $q = 0$. We extend these maps to $|K_{q+1}|$ as follows: Let $s \in K_{q+1} \setminus K_q$ and let $\sigma < z$. Then the map $w_s^\sigma = h_\sigma|\partial s$ is defined. By (4.4) we have $d(|w_s^\sigma|) \le (2c+1)^q d$ and $d(|w_s^\sigma|,\partial G) \ge r/(2c)^q$. Since G is htop (q,c)-uniform, we can extend w_s^σ to a map $u_s^\sigma\colon s \to G$ such that

$$d(|u_s^\sigma|) \le c d(|w_s^\sigma|) \le c d(A_q),$$

$$d(|u_s^\sigma|,\partial G) \ge r/(2c)^{q+1},$$

where the second inequality follows from 3.3. These maps are chosen for all $s \in K_{q+1} \setminus K_q$ and for all $\sigma < z$ in a compatible way, that is, $u_s^\sigma = u_t^\tau$ whenever $w_s^\sigma = w_t^\tau$. It follows that

$$d(A_{q+1}) \le d(A_q) + 2c d(A_q) \le (2c+1)^{q+1} d,$$

$$d(A_{q+1}, \partial G) \geq r/(2c)^{q+1},$$

and hence (4.4) holds for $q \mapsto q+1$.

After the step $q = p$ we have the maps $h_\sigma\colon \Delta^p \times I \to X$, which give a simple prism $\varphi = \sum_{\sigma<z} n_\sigma \overline{h}_\sigma$ of z. By 2.10 we have $\partial g(\varphi) = z_1 - z$ where z_1 is the cycle $\varphi\alpha_1\Delta^p$. Thus $|z_1| \subset \{a\}$, and hence $z_1 = \partial g_1$ for a chain g_1 with $|g_1| \subset \{a\}$. Setting $g = g_1 - g(\varphi)$ we have $\partial g = z$ and $|g| \subset A_{p+1}$. From (4.4) it follows that g satisfies (4.3) with $c_1 = (2c+1)^{p+1}$. □

4.5. **Theorem**. *If a domain $G \subset R^n$ is completely hlog (p,c)-uniform and htop $(1,c)$-uniform, then G is completely htop (p,c_1)-uniform with $c_1 = c_1(c,n)$.*

Proof. We use the notation of [Al, §6], partially recalled in 3.16. The complement of G belongs to the family $H = \mathrm{HU}(p,c) \cap U(1,c)$. By 3.17 and by [Al, 6.8], there are stable families $L \subset D_p$ and $L' \subset C_1$ such that H is contained in $L \cap L'$, which clearly is a stable subfamily of $D_p \cap C_1$. Since $D_p \cap C_1 = C_p$ by the Hurewicz theorem, the theorem follows from [Al, 6.8]. □

5. Turning conditions

5.1. *Summary of Section* 5. A metric space X is of *c-bounded turning* or briefly c-BT if each pair a, b of points in X can be joined by an arc γ with $d(\gamma) \leq c|a-b|$. More generally, let $p \geq 0$ be an integer. The space X is htop (p,c)-BT if each map $f\colon S^p \to X$ extends to a map $g\colon \overline{B}^{p+1} \to X$ such that $d(|g|) \leq cd(|f|)$. Thus c-BT is equivalent to htop $(0,c)$-BT.

In this section we give some improvements on the results of [Al] concerning htop (p,c)-BT, and also study the corresponding homological property.

5.2. *Basic concepts.* We say that a metric space X is hlog (p,c)-BT if each p-cycle z in X bounds a chain g with $d(|g|) \leq cd(|z|)$. This implies that $H_p(X) = 0$. Furthermore, let $a \in X$ and $r > 0$. It is easy to see that the homomorphism $H_p\bigl(B(a,r)\bigr) \to H_p\bigl(B(a,c_1 r)\bigr)$ is zero for $c_1 = 2c+1$. Conversely, if this is true for all a and r, then X is hlog (p,c)-BT for any $c > 2c_1$. Consequently, the property hlog (p,c)-BT is quantitatively equivalent to the property hlog outer (p,c)-joinable [Vä$_3$] together with the condition $H_p(X) = 0$.

In some cases these quantitative properties are unnecessarily strong, and they can be replaced by the corresponding topological conditions. Let G be an open set in E, and let $p \geq 0$ be an integer. We say that G is p-LC rel E if each neighborhood U of each point $a \in \partial G$ contains a neighborhood V such that every map $f\colon S^p \to V \cap G$ is null-homotopic in $U \cap G$. If this holds for all $0 \leq p \leq m$, G is said to be LC^m rel E.

Replacing homotopy by homology, we obtain the concepts p-lc rel E and lc^m rel E. For example, G is p-lc rel E if for each $a \in \partial G$ and for each neighborhood

U of a there is a neighborhood V of a such that $H_p(V \cap G) \to H_p(U \cap G)$ is zero.

These and related concepts have been extensively studied in the topological literature since the thirties. See, for example, [EW] and [Wi].

5.3. *Remarks.* 1. The quantitative properties imply the topological properties in the obvious way. For example, if $G \subset E$ is completely hlog (p,c)-BT, G is lc^p rel E.

2. If X is hlog $(0,c)$-BT, X is c-BT. Conversely, a c-BT space is hlog $(0, 2c+1)$-BT. The proofs are elementary.

3. If X is completely htop (p,c)-BT, then X is completely hlog (p,c_1)-BT with $c_1 = c_1(c,p)$. This can be proved by using similar techniques as in 4.2.

4. Let $F \subset R^n$ be closed with $\mathrm{int} F = \varnothing$, and let $0 \le p \le n-2$ and $c \ge 1$. Set $G = R^n \setminus F$. Then the following conditions are equivalent:

(1) $\dim F \le n - p - 2$,
(2) G is completely hlog $(p, \sqrt{2}\,)$-BT,
(3) G is completely hlog (p)-BT,
(4) G is lc^p rel R^n.

The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial, and (4) $\Rightarrow$ (1) $\Rightarrow$ (2) follows from [Ku, Th. 2′, p. 9]. Indeed, assume that (1) is true and that z is a k-cycle in G with $k \le p$. By Jung's theorem [Fe, 2.10.41], $|z|$ is contained in an open ball of radius $d(|z|)/\sqrt{2}$. Since $k < n - \mathrm{dim} F - 1$, it follows from [Ku, Th. 2′, p. 9] that $z = \partial g$ for some chain g in $B \setminus F$. Since $d(|g|) < d(|z|)\sqrt{2}$, (2) is true.

5.4. **Lemma**. *Suppose that U is open in E and* LC^p rel E. *Suppose also that X is a compact polyhedron with $\dim X \le p+1$. Then for each map $f\colon X \to \overline{U}$ and for each $\varepsilon > 0$ there is a map $g\colon X \to U$ with $\|g - f\| < \varepsilon$.*

Proof. Using the compactness of fX and the LC^p-condition it is easy to show that for every $s > 0$ there is $t > 0$ such that if $y \in fX$ and $0 \le q \le p$, then each map $\varphi\colon S^q \to U \cap B(y,t)$ is null-homotopic in $U \cap B(y,s)$. The map g is found by choosing a sufficiently fine triangulation of X, starting with the vertices and proceeding by induction to skeletons of higher dimension. □

5.5. *Plumpness and porosity conditions.* The definition of a c-plump open set $U \subset E$ was recalled in 3.16. This property could also be called $(0,c)$-plumpness, since it is a special case of (p,c)-plumpness defined in [Al, 3.1]. In this paper we shall call this property htop (p,c)-plumpness, since a hlog version will be given in §6. We recall the definition. Let $p \ge 0$ be an integer and let $c \ge 1$. An open set $U \subset E$ is *htop (p,c)-plump* if for each map $f\colon \Delta^p \to \overline{U}$ and for each $0 < r < d(U)$ there is a map $g\colon \Delta^p \to U$ such that $\|g - f\| \le r$ and $d(|g|, \partial U) \ge r/c$.

It is essential that we consider maps $f\colon \Delta^p \to \overline{U}$ and not only maps into U. If the condition above is valid for every map $f\colon \Delta^p \to U$, we say that U is *htop* *inner* (p,c)*-plump.* This is an absolute concept, which makes sense in any metric space.

A closed set $F\subset E$ is *htop* (p,c)*-porous* or *htop inner* (p,c)*-porous* if $\operatorname{int}F=\varnothing$ and if $U=R^n\setminus F$ satisfies the appropriate plumpness condition.

Trivially, htop (p,c)-plumpness and (p,c)-porosity imply the corresponding inner condition. The converse is true for $p=0$ with an arbitrary small change of c, but not for $p\ge 1$. For example, let F be a line segment in R^3. Then F is htop (1)-porous and htop inner (2)-porous but not htop (2)-porous.

We next give a result in the converse direction:

5.6. **Theorem**. *Suppose that* $U\subset E$ *is open,* $\mathrm{LC}^{p-1}\operatorname{rel}E$ *and htop inner* (p,c)*-plump. Then* U *is htop* (p,c_1)*-plump for every* $c_1>c$.

Proof. Let $f\colon \Delta^p\to\overline{U}$ be continuous and let $0<r<d(U)$. Given $c_1>c$, we set $\varepsilon=(1-c/c_1)r$. By 5.4 there is a map $f'\colon\Delta^p\to U$ such that $\|f'-f\|\le\varepsilon$. Since U is htop inner (p,c)-plump, there is a map $g\colon\Delta^p\to U$ such that $\|g-f'\|\le r-\varepsilon$ and $d(|g|,\partial U)\ge(r-\varepsilon)/c=r/c_1$. Since $\|g-f\|\le r$, the theorem follows. □

We apply Theorem 5.6 to improve 4.8 and 3.10 of [Al]:

5.7. **Theorem**. *Let* $G\subset E$ *be a domain and let* $1\le p<\dim E$*. Then the following conditions are quantitatively equivalent*:

(1) *G is htop (p,c)-plump and completely htop $(p-1,c)$*-BT.
(2) *G is htop inner (p,c)-plump and completely htop $(p-1,c)$*-BT.
(3) *G is completely htop $(p-1,c)$-uniform.*

Proof. The implication (1) ⇒ (2) is trivial, and (1) ⇔ (3) was proved in [Al, 4.8]. Since htop $(p-1,c)$-BT implies $(p-1)$-LC rel E, (2) implies (1) by 5.6. □

5.8. **Theorem**. *Let* $D\subset E$ *be a domain and let* F *be closed in* E *with* $\operatorname{int}F=\varnothing$*. Suppose that* $0\le p<\dim E$*, that* D *is htop* (p,c)-BT *and that* $E\setminus F$ *is* LC^p rel E*. Then* $G=D\setminus F$ *is htop* (p,c_1)-BT *for every* $c_1>c$.

Proof. Let $\varepsilon>0$. We show that G is htop $(p,c+\varepsilon)$-BT. Let $f\colon S^p\to G$ be continuous and nonconstant. Define $f'\colon S^p(1/2)\to D$ by $f'(x)=f(2x)$. Since D is htop (p,c)-BT, there is an extension $g'\colon\overline{B}^{p+1}(1/2)\to D$ of f' such that $d(|g'|)\le cd(|f'|)=cd(|f|)$. Write $4\delta=\varepsilon d(|f|)\wedge d(|g'|,\partial D)\wedge d(|f|,F)$. By 5.4, there is a map $g_1\colon\overline{B}^{p+1}(1/2)\to E\setminus F$ such that $\|g_1-g'\|\le\delta$. We extend g_1 to a map $g\colon\overline{B}^{p+1}\to E$ by setting $g\bigl((1-t)e/2+te\bigr)=(1-t)g_1(e/2)+tf(e)$ for $e\in S^p$ and $0\le t\le 1$. By the choice of δ we have $|g|\subset D\cap(E\setminus F)=G$. Moreover, $g\mid S^p=f$, and

$$d(|g|)\le d(|g_1|)+2\delta\le d(|g'|)+4\delta\le cd(|f|)+4\delta\le(c+\varepsilon)d(|f|).$$

Thus g is the required extension of f. □

5.9. *The homological case.* We shall give a homological version of 5.8 in 5.11. The proof is based on Lemma 5.10 below, which can be regarded as a homological version of Lemma 5.4. A homological version of 5.7 will be given in Section 6, where we discuss homological plumpness.

5.10. **Lemma**. *Suppose that U is open in E and* lc^{p-1} rel E*. Let $\gamma \in S_p(\overline{U})$ and let $\varepsilon > 0$. Then there are $g \in S_{p+1}(E)$ and $\gamma_1 \in S_p(U)$ such that*

(1) $|g| \subset \overline{B}(|\gamma|, \varepsilon)$,

(2) $|\gamma - \gamma_1 - \partial g| \subset \overline{B}(|\partial\gamma|, \varepsilon)$.

If γ is a cycle, (2) *means that $\partial g = \gamma - \gamma_1$, and hence γ_1 is also a cycle.*

Proof. Let $\gamma = \sum_{\sigma<\gamma} n_\sigma \sigma$ be the normal representation. Since $|\gamma|$ is compact, it follows from the lc^{p-1}-condition that for each $r > 0$ there is $0 < t(r) < r$ such that the map $H_k\big(B\big(a, t(r)\big) \cap U\big) \to H_k\big(B(a, r) \cap U\big)$ is zero whenever $a \in |\gamma|$ and $0 \le k \le p-1$. Set $r_{p+1} = \varepsilon$ and define inductively the numbers $r_p, \dots, r_0$ by $r_q = t(r_{q+1})/2$.

We first assume that $d(|\sigma|) \le r_0$ for all $\sigma < \gamma$. We shall construct a prism $\varphi = \sum_{\sigma<\gamma} n_\sigma \varphi_\sigma$ of γ by the method explained in 2.5. Thus we start with the 0-chains of $P\Delta^p$ and set $\varphi_\sigma v = \sigma v$ for the vertices $v = e_j$, $0 \le j \le p$. For $v = e'_j = (e_j, 1)$ we let $\varphi_\sigma v$ be a point in U with $|\varphi_\sigma v - \sigma e_j| \le r_0$. These points are chosen in a compatible way; see 2.5. Proceeding inductively, assume that $0 \le q \le p$ and that the chain maps $\varphi_\sigma\colon C_k(P\Delta^p) \to S_k(E)$ have been defined for all $0 \le k \le q$ in such a way that for each k-simplex s of $P\Delta^p$ and for all $\sigma < \gamma$ we have

(i) $|\varphi_\sigma s| \subset \overline{B}(\sigma \mathrm{pr}_1 s, r_k)$,

(ii) $|\varphi_\sigma s| \subset U$ whenever $s \subset \Delta^p \times \{1\}$.

Let s be a $(q+1)$-simplex of $P\Delta^p$ and let $\sigma < \gamma$. If $s \subset \Delta^p$, we follow the procedure of 2.5 and set $\varphi_\sigma s = \sigma\varepsilon_s$, where $\varepsilon_s\colon \Delta^{q+1} \to \Delta^p$ is the order-preserving isometry onto s. Assume that $s \not\subset \Delta^p$. Now $\varphi_\sigma \partial s$ is defined, and it is a cycle in E. By (i) we have $|\varphi_\sigma \partial s| \subset \overline{B}(\sigma \mathrm{pr}_1 s, r_q)$. We consider two cases.

Case 1. $s \subset \Delta^p \times \{1\}$. Now $|\varphi_\sigma \partial s| \subset U$ by (ii). Choose a point $a \in \sigma \mathrm{pr}_1 s$. Then (i) implies that $|\varphi_\sigma \partial s| \subset \overline{B}(a, 2r_q)$, since $r_q + d(|\sigma|) \le r_q + r_0 \le 2r_q$. Since $2r_q = t(r_{q+1})$, there is a chain $\varphi_\sigma s \in S_{q+1}(U)$ such that $\partial \varphi_\sigma s = \varphi_\sigma \partial s$ and $|\varphi_\sigma s| \subset \overline{B}(a, r_{q+1})$. Clearly $\varphi_\sigma s$ satisfies (i) and (ii) for $k = q+1$.

Case 2. $s \not\subset \Delta^p \times \{1\}$. Now we use the cone construction to define the chain $\varphi_\sigma s \in S_{q+1}(E)$ such that $\partial \varphi_\sigma s = \varphi_\sigma \partial s$ and such that $|\varphi_\sigma s|$ is contained in the convex hull of $|\varphi_\sigma \partial s|$. Since $|\varphi_\sigma \partial s| \subset \overline{B}(\sigma \mathrm{pr}_1 s, r_q)$, the condition (i) holds for $k = q+1$.

The choices of the chains $\varphi_\sigma s$ for $\sigma < \gamma$ are made in a compatible way; see 2.5. The construction gives a prism $\varphi = \sum_{\sigma<\gamma} n_\sigma \varphi_\sigma$ of γ. With the notation of 2.7 and 2.10 we then have $\partial g(\varphi) + g(\psi) = \gamma_1 - \gamma$ by 2.10. Moreover, (i) and

2.14 imply that $|g(\varphi)| \subset \overline{B}(|\gamma|,\varepsilon)$ and $|g(\psi)| \subset \overline{B}(|\partial\gamma|,\varepsilon)$. Hence $g = -g(\varphi)$ and γ_1 are the desired chains, and we have proved the lemma in the case where $d(|\sigma|) \le r_0$ for $\sigma < \gamma$.

The general case reduces to the case above by barycentric subdivision. Let $\gamma \in S_p(\overline{U})$. It suffices to show that if the lemma holds with γ replaced by $\mathrm{Sd}\gamma$, it holds also for γ. Thus we may assume that there are $g' \in S_{p+1}(E)$ and $\gamma_1 \in S_p(U)$ such that

$$|g'| \subset \overline{B}(|\mathrm{Sd}\gamma|,\varepsilon), \qquad |\mathrm{Sd}\gamma - \gamma_1 - \partial g'| \subset \overline{B}(|\partial \mathrm{Sd}\gamma|,\varepsilon).$$

Let $T\colon S_*(E) \to S_*(E)$ be the chain homotopy given in [Ro, 6.13] satisfying $\partial T + T\partial = \mathrm{id} - \mathrm{Sd}$ and $|T\beta| \subset |\beta|$ for every singular chain β. Setting $g = g' + T\gamma$ we show that g and γ_1 satisfy (1) and (2). First,

$$|g| \subset |g'| \cup |T\gamma| \subset \overline{B}(|\mathrm{Sd}\gamma|,\varepsilon) \cup |\gamma| \subset \overline{B}(|\gamma|,\varepsilon),$$

which gives (1). Next, we have $T\partial\gamma = \gamma - \partial T\gamma - \mathrm{Sd}\gamma$, and hence

$$\begin{aligned} |\gamma - \gamma_1 - \partial g| &= |\gamma - \gamma_1 - \partial g' - \partial T\gamma - \mathrm{Sd}\gamma + \mathrm{Sd}\gamma| \\ &\subset |\gamma - \partial T\gamma - \mathrm{Sd}\gamma| \cup |\mathrm{Sd}\,\gamma - \gamma_1 - \partial g'| \\ &\subset |T\partial\gamma| \cup \overline{B}(|\partial \mathrm{Sd}\gamma|,\varepsilon). \end{aligned}$$

Since $|T\partial\gamma| \subset |\partial\gamma|$ and $|\partial \mathrm{Sd}\gamma| = |\mathrm{Sd}\partial\gamma| \subset |\partial\gamma|$, this gives (2). □

5.11. **Theorem**. *Let $D \subset E$ be a domain and let F be closed in E with $\mathrm{int} F = \varnothing$. Suppose that $0 \le p < \dim E$, that D is hlog (p,c)-BT and that $E \setminus F$ is $\mathrm{lc}^p \operatorname{rel} E$. Then $G = D \setminus F$ is hlog (p,c_1)-BT for every $c_1 > c$.*

Proof. Let $\varepsilon > 0$. We show that G is hlog (p,c_1)-BT for $c_1 = c + \varepsilon$. Let $z \in Z_p(G)$, and assume that $d(|z|) > 0$. Since D is hlog (p,c)-BT, there is $g' \in S_{p+1}(D)$ such that $\partial g' = z$ and $d(|g'|) \le c d(|z|)$. Choose δ with $0 < 2\delta < \varepsilon d(|z|) \wedge d(|z|,F) \wedge d(|g'|,\partial D)$. By 5.10 there are $h \in S_{p+2}(E)$ and $g_1 \in S_{p+1}(E \setminus F)$ such that $|h| \subset \overline{B}(|g'|,\delta)$ and $|g' - g_1 - \partial h| \subset \overline{B}(|z|,\delta)$. Write $\beta = g' - g_1 - \partial h$ and $g = g_1 + \beta = g' - \partial h$. Then $\partial g = \partial g' = z$. Moreover, $|g| \subset |g'| \cup |\partial h| \subset \overline{B}(|g'|,\delta) \subset D$ and $|g| \subset |g_1| \cup |\beta| \subset |g_1| \cup \overline{B}(|z|,\delta) \subset E \setminus F$. Since $d(|g|) \le d(|g'|) + 2\delta \le (c+\varepsilon) d(|z|)$, the theorem follows. □

6. Homological plumpness

6.1. *Summary of Section* 6. We shall define a homological version of (p,c)-plumpness. It turns out that most results of [Al] and §5 on htop (p,c)-plumpness have homological versions.

6.2. *Definitions.* Let $0 \le p < \dim E$ be an integer and let $c \ge 1$. An open set $U \subset E$ is *hlog* (p,c)*-plump* if for each chain $\gamma \in S_p(\overline{U})$ and for each $0 < r < d(U)$ there is $\gamma_1 \in S_p(U)$ such that

(1) $|\gamma_1| \subset \overline{B}(|\gamma|, r)$,
(2) $d(|\gamma_1|, \partial U) \ge r/c$,
(3) $\partial\gamma_1 \sim \partial\gamma$ in $\overline{B}(|\partial\gamma|, r)$.

If the condition holds for all $\gamma \in S_p(U)$, U is *hlog inner* (p,c)*-plump.* If $F \subset E$ is closed with $\mathrm{int} F = \varnothing$ and if $E \setminus F$ is hlog (p,c)-plump, F is *hlog* (p,c)*-porous. Hlog inner* (p,c)*-porosity* is defined in the obvious way.

A slightly stronger condition will be considered in 6.8.

6.3. *Remarks.* 1. The condition in 6.2 is relevant only if $\partial\gamma \ne 0$. Indeed, if γ is a cycle, the condition holds with $c = 1$ and $\gamma_1 = 0$.

2. We remind the reader that we are using the reduced theory. For $p = 0$, the condition (3) of 6.2 should be understood as $\partial\gamma_1 = \partial\gamma \in \mathbf{Z}$. It is easy to see that hlog $(0,c)$-plumpness is equivalent to htop $(0,c)$-plumpness, that is, ordinary c-plumpness.

3. In [Al, 3.3.3] it was proved by an easy argument that htop (p,c)-plumpness implies htop (q,c)-plumpness for all $0 \le q \le p$. We do not know whether the corresponding hlog result is true.

We first give homological versions of [Al, 3.4, 3.8, 3.9].

6.4. **Lemma**. *Let $U \subset E$ be open and hlog (p,c)-plump, let $0 < r < d(U)$, and let $\gamma \in S_p(\overline{U})$ be a chain such that $d(|\partial\gamma|, \partial U) \ge (c+1)r/c$. Then there is $\gamma_1 \in S_p(U)$ satisfying $\partial\gamma_1 = \partial\gamma$ and the conditions* (1) *and* (2) *of* 6.2.

Proof. Let γ_1' be the chain given by 6.2. By 6.2(3), there is $g \in S_p(E)$ such that $\partial g = \partial\gamma - \partial\gamma_1'$ and $|g| \subset \overline{B}(|\partial\gamma|, r)$. We show that $\gamma_1 = \gamma_1' + g$ is the desired chain. Clearly $|\gamma_1| \subset \overline{B}(|\gamma|, r)$ and $\partial\gamma_1 = \partial\gamma$. Moreover, for every $x \in |g|$ we have

$$d(x, \partial U) \ge d(|\partial\gamma|, \partial U) - d(x, |\partial\gamma|) \ge (c+1)r/c - r = r/c. \ \square$$

6.5. **Theorem**. *Let $U \subset E$ be hlog (p, c_1)-plump and let $F \subset E$ be hlog (p, c_2)-porous. Then $V = U \setminus F$ is hlog $(p, 3c_1c_2)$-plump.*

Proof. Let $0 < r < d(V)$ and let $\gamma \in S_p(\overline{V}) = S_p(\overline{U})$. Then there is $\gamma_2 \in S_p(U)$ such that $|\gamma_2| \subset \overline{B}(|\gamma|, 2r/3)$, $\partial\gamma_2 \sim \partial\gamma$ in $\overline{B}(|\partial\gamma|, 2r/3)$, and $d(|\gamma_2|, \partial U) \ge 2r/3c_1$. By the hlog (p, c_2)-porosity of F, there is $\gamma_1 \in S_p(E \setminus F)$ such that $|\gamma_1| \subset \overline{B}(|\gamma_2|, r/3c_1)$, $\partial\gamma_1 \sim \partial\gamma_2$ in $\overline{B}(|\partial\gamma_2|, r/3c_1)$, and $d(|\gamma_1|, F) \ge r/3c_1c_2$.

We show that γ_1 is the desired chain. First, we have $|\gamma_1| \subset \overline{B}(|\gamma|, 2r/3 + r/3c_1) \subset \overline{B}(|\gamma|, r)$, which proves (1) of 6.2. Next,

$$d(|\gamma_1|, \partial U) \ge d(|\gamma_2|, \partial U) - r/3c_1 \ge 2r/3c_1 - r/3c_1 = r/3c_1.$$

Since $d(|\gamma_1|, F) \ge r/3c_1c_2$, we obtain $d(|\gamma_1|, \partial V) \ge r/3c_1c_2$. Finally, $\partial\gamma \sim \partial\gamma_2 \sim \partial\gamma_1$ in $\overline{B}(|\partial\gamma|, 2r/3 + r/3c_1) \subset \overline{B}(|\partial\gamma|, r)$. □

6.6. **Corollary**. *The union of a hlog* (p, c_1)*-porous set and a hlog* (p, c_2)*-porous set is hlog* $(p, 3c_1c_2)$*-porous.*

6.7. *Plumpness and uniformity.* We recall that htop (p, c)-uniformity and htop $(p+1, c)$-plumpness are closely connected; see 5.7. In particular, a closed set $F \subset E$ is completely htop (p, c)-NUD if and only if, quantitatively, it is htop $(p+1, c)$-porous [Al, 4.9]. We shall give homological versions of these and some related results. More precisely, the homological versions of [Al, 4.2], 5.6, 5.7, [Al, 4.9], [Al, 4.10] and [Al, 4.11] will be given as 6.9, 6.12, 6.13, 6.15, 6.16 and 6.17, respectively.

6.8. *Strong hlog plumpness.* We shall also consider the following variation of hlog plumpness: An open set $U \subset E$ is said to be *strongly hlog* (p, c)*-plump* if for each $\gamma \in S_p(\overline{U})$ and for each $0 < r < d(U)$ there are $\gamma_1 \in S_p(U)$ and $g \in S_{p+1}(E)$ such that

(1) $|g| \subset \overline{B}(|\gamma|, r)$,
(2) $d(|\gamma_1|, \partial U) \ge r/c$,
(3) $|\partial g - \gamma + \gamma_1| \subset \overline{B}(|\partial\gamma|, r)$.

It is easy to show that this property implies hlog (p, c)-plumpness, defined in 6.2. Indeed, (2) above is the same as 6.2(2). Setting $h = \partial g - \gamma + \gamma_1$ we have $\partial h = \partial\gamma_1 - \partial\gamma$, and hence $\partial\gamma_1 \sim \partial\gamma$ in $\overline{B}(|\partial\gamma|, r)$. Finally, $|\gamma_1| \subset |h| \cup |\gamma| \cup |g| \subset \overline{B}(|\gamma|, r)$.

We do not know whether the converse is true. In 6.15 we show that it is true in the special case $\operatorname{int}(E \setminus U) = \varnothing$.

6.9. **Theorem**. *If* $G \subset E$ *is a completely hlog* (p, c)*-uniform domain, then* G *is strongly hlog* $(p+1, c_1)$*-plump and hence hlog* $(p+1, c_1)$*-plump with* $c_1 = c_1(c, p)$.

Proof. The proof is a variation of the proof of 5.10. Let $\gamma \in S_p(\overline{U})$ with normal representation $\gamma = \sum_{\sigma<\gamma} n_\sigma \sigma$. Let $0 < r < d(G)$ and set $r' = r/2(4c)^{p+1}$. As in 5.10, we may assume that $d(|\sigma|) \le r'$ for all $\sigma < \gamma$, using barycentric subdivision. We apply the method of 2.5 to construct a prism $\varphi = \sum_{\sigma<\gamma} n_\sigma \varphi_\sigma$ of γ. Thus we set $\varphi_\sigma v = \sigma v$ if $v = e_j$ is a vertex of Δ^p. Since G is hlog $(0, c)$-uniform, it is $4c$-plump; cf. [Vä$_1$, 2.15]. If $v = e'_j = (e_j, 1)$, we can therefore choose the point $\varphi_\sigma v \in G$ such that $|\varphi_\sigma v - \sigma e_j| \le 2r'$ and $d(\varphi_\sigma v, \partial G) \ge r'/2c$. These points are chosen in a compatible way; see 2.5.

Proceeding inductively, assume that $0 \le q \le p$ and that the chain maps $\varphi_\sigma \colon C_k(P\Delta^p) \to S_k(E)$ have been defined for $0 \le k \le q$ in such a way that for each k-simplex $s \in P\Delta^p$ and for each $\sigma < \gamma$ we have

(i) $|\varphi_\sigma s| \subset \overline{B}\big(\sigma \mathrm{pr}_1 s, 2(4c)^k r'\big)$,
(ii) $d(|\varphi_\sigma s|, \partial G) \ge r'/(2c)^{k+1}$ whenever $s \subset \Delta^p \times \{1\}$.

These conditions are clearly true for $k = 0$. Let $s \in P\Delta^p$ be a $(q+1)$-simplex and let $\sigma < \gamma$. If $s \subset \Delta^p$, we set $\varphi_\sigma s = \sigma \varepsilon_s$ according to 2.5. Assume that $s \not\subset \Delta^p$. Now $\varphi_\sigma \partial s$ is defined, and it is a q-cycle in E. By (i) we have

$$|\varphi_\sigma \partial s| \subset \overline{B}\big(\sigma \mathrm{pr}_1 s, 2(4c)^q r'\big). \tag{6.10}$$

We consider two cases.

Case 1. $s \subset \Delta^p \times \{1\}$. Now $|\varphi_\sigma \partial s| \subset G$. Since G is hlog (q,c)-uniform, $\varphi_\sigma \partial s = \partial \varphi_\sigma s$ for some chain $\varphi_\sigma s \in S_{q+1}(G)$ satisfying the uniformity conditions

$$d(|\varphi_\sigma s|) \le c d(|\varphi_\sigma \partial s|), \qquad d(|\varphi_\sigma \partial s|, \partial G) \le 2c d(|\varphi_\sigma s|, \partial G), \tag{6.11}$$

where the second inequality follows from 3.3. We show that $\varphi_\sigma s$ satisfies (i) and (ii) with $k = q+1$.

From (6.10) and (6.11) it follows that $|\varphi_\sigma s| \subset \overline{B}(\sigma \mathrm{pr}_1 s, r'')$, where

$$r'' = 2(4c)^q r' + c d(|\varphi_\sigma \partial s|) \le 2(4c)^q r' + c(r' + 4(4c)^q r') < 2(4c)^{q+1} r'.$$

This gives (i) for $k = q+1$. By (ii) of the inductive hypothesis we have $d(|\varphi_\sigma \partial s|, \partial G) \ge r'/(2c)^{q+1}$, which implies (ii) for $k = q+1$ by (6.11).

Case 2. $s \not\subset \Delta^p \times \{1\}$. As in 5.10, we use the cone construction to get a chain $\varphi_\sigma s \in S_{q+1}(E)$ such that $\partial \varphi_\sigma s = \varphi_\sigma \partial s$ and such that $|\varphi_\sigma s|$ is contained in the convex hull of $|\varphi_\sigma \partial s|$. Since $d(|\sigma|) \le r'$, (6.10) implies that

$$d(|\varphi_\sigma s|) \le d(|\varphi_\sigma \partial s|) \le \big(1 + 4(4c)^q\big) r' < 2(4c)^{q+1} r',$$

and hence (i) is true for $k = q+1$.

The choices of the chains $\varphi_\sigma s$ for $\sigma < \gamma$ are made in a compatible way. The construction gives a prism $\varphi = \sum_{\sigma < \gamma} n_\sigma \varphi_\sigma$ of γ. With the notation of 2.7 and 2.10 we have $\partial g(\varphi) + g(\psi) = \gamma_1 - \gamma$ by 2.10. Since $r = 2(4c)^{p+1} r'$, it follows from (i) and 2.14 that $|g(\varphi)| \subset \overline{B}(|\gamma|, r)$ and $|g(\psi)| \subset \overline{B}(|\partial\gamma|, r)$. Furthermore, (ii) implies that $d(|\gamma_1|, \partial G) \ge r/c_1$ with $c_1 = 2(2c)^{p+1}(4c)^{p+1}$. Since $-\partial g(\varphi) - \gamma + \gamma_1 = g(\psi)$, the strong hlog $(p+1, c_1)$-plumpness condition holds with $g = -g(\varphi)$. □

6.12. **Theorem**. *Suppose that $U \subset E$ is open, $\mathrm{lc}^{p-1} \mathrm{rel}\, E$ and hlog inner (p,c)-plump. Then U is hlog (p, c_1)-plump for every $c_1 > c$.*

Proof. This is a homological version of 5.6. Assume that $\gamma' \in S_p(\overline{U})$ and that $0 < r < d(U)$. Given $c_1 > c$, we set $\varepsilon = (1 - c/c_1) r$. By 5.10, there are $g \in S_{p+1}(E)$ and $\gamma_1' \in S_p(U)$ such that

$$|g| \subset \overline{B}(|\gamma|, \varepsilon), \qquad |\gamma - \gamma_1' - \partial g| \subset \overline{B}(|\partial\gamma|, \varepsilon).$$

Hence $\partial\gamma \sim \partial\gamma_1'$ in $\overline{B}(|\partial\gamma|, \varepsilon)$ and $|\gamma_1'| \subset \overline{B}(|\gamma|, \varepsilon)$. Since U is hlog inner (p,c)-plump, there is $\gamma_1 \in S_p(U)$ such that $\gamma_1 \subset \overline{B}(|\gamma_1'|, r - \varepsilon)$, $d(|\gamma_1|, \partial U) \ge (r - \varepsilon)/c = r/c_1$, and $\partial\gamma_1 \sim \partial\gamma_1'$ in $\overline{B}(|\partial\gamma_1'|, r - \varepsilon)$. It follows that $|\gamma_1| \subset \overline{B}(|\gamma|, r)$ and $\partial\gamma \sim \partial\gamma_1$ in $\overline{B}(|\partial\gamma|, r)$. □

6.13. **Theorem**. *Let $G \subset E$ be a domain and let $1 \le p < \dim E$. Then the following conditions are quantitatively equivalent*:

(1) *G is completely hlog (p, c)-plump and completely hlog $(p-1, c)$*-BT.

(2) *G is completely hlog inner (p, c)-plump and completely hlog $(p-1, c)$*-BT.

(3) *G is completely hlog $(p-1, c)$-uniform.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial. Since hlog (q, c)-BT implies q-lc rel E, (2) implies (1) by 6.12. The implication (3) $\Rightarrow$ (1) follows from 6.9. It remains to show that (1) implies (3).

We use induction on p. Assume that $m \ge 1$ and that the theorem holds for $1 \le p \le m-1$. Suppose that (1) is true for $p = m$. We show that G is hlog $(m-1, c_1)$-uniform with $c_1 = c_1(c, p)$. The argument will be valid also in the case $m = 1$ without any inductive hypothesis.

By 3.10 and the inductive hypothesis, it suffices to show that for every $c' \ge 1$, G has property $\mathrm{HT}^*(m-1, c', c_2)$ of 3.4 with some $c_2 = c_2(c, c')$. Let z be an $(m-1)$-cycle in G with $d \le c'r$ where $d = d(|z|)$ and $r = d(|z|, \partial G)$. Since G is hlog $(m-1, c)$-BT, $z = \partial g$ for some $g \in S_m(G)$ with $d(|g|) \le cd(|z|)$. Since $(c+1)/c \le 2$, we can apply 6.4 with $r \mapsto r/2$, $p \mapsto m$, $\gamma \mapsto g$ to get a chain $g_1 \in S_m(G)$ such that

$$\partial g_1 = z, \quad |g_1| \subset \overline{B}(|g|, r/2), \quad d(|g_1|, \partial G) \ge r/2c.$$

Then

$$\begin{aligned} d(|g_1|) &\le d(|g|) + r \le cd + r \le (cc' + 1)r, \\ d(|z|, \partial G) &= r \le 2cd(|g_1|, \partial G). \end{aligned}$$

These inequalities give $\mathrm{HT}^*(m-1, c', c_2)$ with $c_2 = 2cc' + 1$. □

6.14. **Lemma**. *Suppose that $U \subset E$ is hlog $(p-1, c)$*-BT *and that $F \subset E$ is hlog (p)-porous. Then $V = U \setminus F$ is hlog $(p-1, c_1)$*-BT *for all $c_1 \ge c$.*

Proof. Let $c_1 > c$ and let $z \in Z_{p-1}(V)$. Then there is $g \in S_p(U)$ such that $\partial g = z$ and $d(|g|) \le cd(|z|)$. Applying 6.4 with the substitution $U \mapsto E \setminus F$, $\gamma \mapsto g$ and with a sufficiently small r we find a p-chain g_1 in V with $\partial g_1 = z$ and $d(|g_1|) \le c_1 d(|z|)$. □

6.15. **Theorem**. *Let $F \subset E$ be closed and let $1 \le p < \dim E$. Then the following conditions are quantitatively equivalent*:

(1) *F is completely hlog $(p-1, c)$*-NUD.

(2) *F is completely hlog (p, c)-porous.*

(3) *F is completely strongly hlog (p, c)-porous.*

Proof. The implication (3) $\Rightarrow$ (2) was explained in 6.8. The theorem follows from 6.9, 6.13 and 6.14. □

6.16. **Theorem**. *Suppose that $D \subset E$ is a completely hlog (p,c)-uniform domain and that $F \subset E$ is completely* hlog (p,c)-NUD. *Then $D \setminus F$ is a completely hlog (p,c_1)-uniform domain with $c_1 = c_1(c,p)$.*

Proof. By 6.13, the open sets D and $E \setminus F$ are completely hlog $(p+1,c_0)$-plump and completely hlog (p,c_0)-BT with $c_0 = c_0(c,p)$. Hence $G = D \setminus F$ is completely hlog $(p+1, 3c_0^2)$-plump by 6.5. Moreover, G is completely hlog (p,c_0)-BT by 6.14. From 6.13 it follows that G is completely hlog (p,c_1)-uniform with $c_1 = c_1(c,p)$. □

6.17. **Corollary**. *If F_1 and F_2 are completely hlog* (p,c)-NUD, *then $F_1 \cup F_2$ is completely hlog (p,c_1)*-NUD *with $c_1 = c_1(c,p)$.* □

6.18. *Relations between htop and hlog plumpness.* In 4.2 we proved that complete htop (p,c)-uniformity quantitatively implies complete hlog (p,c)-uniformity. We do not know whether the corresponding result for plumpness is true. However, it follows from [Al, 4.9], 4.2 and 6.15 that a completely htop (p,c)-porous set $F \subset E$ is completely hlog (p,c_1)-porous with $c_1 = c_1(c,p)$.

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