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DEFORMING REAL PROJECTIVE STRUCTURES

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Abstract. Let S be a compact Riemann surface of hyperbolic type and $T(S)$ the Teichmüller space of S. We show that the Maskit grafting map is a homeomorphism from $T(S)$ onto $T(S)$. It follows from this result that S has a projective structure which is different from the canonical Fuchsian uniformization, and has discrete, injective holonomy contained in $\mathrm{PSL}(2,\mathbf{R})$.

1. Introduction

We are concerned in this paper with the grafting operation along a simple, closed geodesic on a closed, hyperbolic Riemann surface (see Section 3 for definitions). This construction was first considered by Maskit [7] in the context of projective structures on surfaces; he used it to provide examples of distinct structures with identical holonomy maps. Later, Goldman [6] showed that, with the exception of the canonical Fuchsian structure, all real projective structures with discrete, injective holonomy are obtained by grafting.

It is natural to identify homotopy classes of simple, closed geodesics on distinct surfaces of the same genus via the marked equivalence classes of these surfaces in Teichmüller space. Using this identification, we will show (Theorem 3) that, given a hyperbolic Riemann surface S and a simple, closed geodesic on that surface, there exists a (unique) surface S' such that S can be obtained from S' by grafting along the corresponding geodesic in S'. As a corollary to this result (Corollary 4), it follows that every surface has a real projective structure which is distinct from the canonical Fuchsian uniformizing structure, and has discrete, injective holonomy contained in $\mathrm{PSL}(2,\mathbf{R})$; this solves a problem posed by Maskit in [7].

Section 2 contains basic definitions and theorems from the theories of Teichmüller spaces and projective structures. In Section 3, we define the Maskit grafting map for a simple closed geodesic and study the behavior of this map under the assumption that the geodesic has small hyperbolic length.

To analyze the grafting map for geodesics of arbitrarily large hyperbolic length, we need to consider the extremal length properties of families of curves homotopic to these geodesics; this is done in Section 4. The main result of the paper, Theorem 3, is proved in Section 3.

The constructions of Section 4 leading up to the proof of Lemma 4.2 are obtained with the use of an auxiliary metric; the author is grateful to J. Velling for

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pointing out the usefulness of this metric. We note that similar results have been obtained by H. Tanigawa (oral communication) using harmonic maps. Finally, the author would like to thank the referee for several helpful suggestions for bringing the paper into its final form.

2. Teichmüller spaces and projective structures

2.1. Throughout, Γ will denote a Fuchsian covering group acting on the upper half plane U and covering a compact surface of genus $g > 1$. The space $M(\Gamma)$ consists of all measurable functions $\mu\colon U \to \mathbf{C}$, with $\|\mu\|_\infty < 1$, satisfying $\mu\big(A(z)\big)\overline{A'(z)} = \mu(z)A'(z)$, for all $A \in \Gamma$.

For each $\mu \in M(\Gamma)$ there is a unique quasiconformal map $w^\mu\colon U \to U$ which fixes 0, 1, ∞ and satisfies $w^\mu_{\overline{z}} = \mu w^\mu_z$ (here the derivatives are taken in the generalized sense). The quantity $(1+\|\mu\|_\infty)/(1-\|\mu\|_\infty) \geq 1$ is called the *dilatation* of w^μ. Further, $w^\mu \circ A = A^\mu \circ w^\mu$ for some $A^\mu \in \mathrm{PSL}(2,\mathbf{R})$. It follows that μ defines an isomorphism θ_μ from Γ onto a Fuchsian group Γ^μ with $\theta_\mu(A) = A^\mu$ (see [1]).

Two elements μ, $\nu \in M(\Gamma)$ are *equivalent* (denoted by $\mu \sim \nu$) if they induce identical isomorphisms. The *Teichmüller space* $T(\Gamma)$ is the space of equivalence classes (denoted by $[\mu] \in T(\Gamma)$) of elements in $M(\Gamma)$.

For $\mu \in M(\Gamma)$, let $k = \inf_{\nu\sim\mu} \|\nu\|_\infty$, and $K = (1+k)/(1-k)$. The *Teichmüller distance* from $[0]$ to $[\mu]$ is given by

$$d_\Gamma([0],[\mu]) = \tfrac{1}{2}\log K.$$

The *right translation mapping* $R_\mu^{-1}\colon M(\Gamma) \to M(\Gamma^\mu)$ is defined by letting

$$R_\mu^{-1}(\nu) = \frac{\big(w^\nu \circ (w^\mu)^{-1}\big)_{\overline{z}}}{\big(w^\nu \circ (w^\mu)^{-1}\big)_z}$$

(the *Beltrami coefficient* of $w^\nu \circ (w^\mu)^{-1}$). Now, the *Teichmüller metric* on $T(\Gamma)$ is defined by

$$d_\Gamma([\mu],[\nu]) = d_{\Gamma^\mu}\big([0],[R_\mu^{-1}(\nu)]\big). \tag{1}$$

It is well known that $T(\Gamma)$ has the natural structure of a real analytic manifold and that it is a cell of real dimension $6g-6$.

2.2. Let $B_2(\Gamma)$ be the $6g-6$ real analytic space of holomorphic quadratic differentials defined on U for Γ. There exists a real analytic vector bundle $Q(\Gamma)$ over $T(\Gamma)$, with projection map $p\colon Q(\Gamma) \to T(\Gamma)$ and fibers equal to $B_2(\Gamma^\mu)$. Given $([\mu],\varphi) \in Q(\Gamma)$ (here $\varphi \in B_2(\Gamma^\mu)$), let f_φ be a solution of the Schwarzian differential equation $S(f) = \varphi$. The differential operator $S(f)$ is defined by

$$S(f) = (f''/f')' - \tfrac{1}{2}(f''/f')^2.$$

The function f_φ determines a homomorphism $\Theta^{f_\varphi}\colon \Gamma^\mu \to \mathrm{PSL}(2,\mathbf{C})$ (called the *holonomy* map), with $f_\varphi \circ \gamma^\mu = \Theta^{f_\varphi}(\gamma^\mu) \circ f_\varphi$, for all $\gamma^\mu \in \Gamma^\mu$, and the pair $(f_\varphi, \Theta^{f_\varphi})$ is called a *projective structure* on U/Γ^μ. We say that the projective structure has *real* holonomy if the image of Γ^μ under Θ^{f_φ} is contained in $\mathrm{PSL}(2,\mathbf{R})$.

There is also the associated homomorphism $\Theta_{f_\varphi}\colon \Gamma \to \mathrm{PSL}(2,\mathbf{C})$ defined by $\Theta_{f_\varphi}(\gamma) = \Theta^{f_\varphi}(\gamma^\mu)$. For $T \in \mathrm{PSL}(2,\mathbf{C})$, $T \circ f_\varphi$ is also a solution of $S(f) = \varphi$, and this determines the homomorphism $\gamma^\mu \to T \circ \Theta^{f_\varphi}(\gamma^\mu) \circ T^{-1}$. One is thus led to consider the equivalence class $[\Theta_\varphi]$ of Θ_{f_φ} under conjugation by elements of $\mathrm{PSL}(2,\mathbf{C})$.

Letting $\mathrm{Hom}\big(\Gamma, \mathrm{PSL}(2,\mathbf{C})\big) = \mathrm{Hom}\,\Gamma$ denote the space of equivalence classes of homomorphisms of Γ into $\mathrm{PSL}(2,\mathbf{C})$, one defines $\Phi\colon Q(\Gamma) \to \mathrm{Hom}\,\Gamma$, the *monodromy* map, by

$$\Phi([\mu], \varphi) = [\Theta_\varphi].$$

It is well known that there exists a real analytic manifold $\mathrm{Hom}'\,\Gamma \subset \mathrm{Hom}\,\Gamma$ such that $\Phi\big(Q(\Gamma)\big) \subset \mathrm{Hom}'\,\Gamma$ and that $\Phi\colon Q(\Gamma) \to \mathrm{Hom}'\,\Gamma$ is an analytic local diffeomorphism (see [3]). The manifold $\mathrm{Hom}'\big(\Gamma, \mathrm{PSL}(2,\mathbf{R})\big) \subset \mathrm{Hom}'\,\Gamma$ is defined similarly.

The following is proved in [4, p. 261]:

Theorem 1 (Faltings). *The space* $R(\Gamma) = \Phi^{-1}\big(\mathrm{Hom}'\big(\Gamma, \mathrm{PSL}(2,\mathbf{R})\big)\big)$ *is a real analytic submanifold of* $Q(\Gamma)$ *of dimension* $6g-6$*. Moreover,* $R(\Gamma)$ *is transverse to the fibers of* $Q(\Gamma)$*. The projection map* $p\colon Q(\Gamma) \to T(\Gamma)$ *is a local diffeomorphism when restricted to* $R(\Gamma)$*.*

3. The Maskit grafting construction

3.1. We normalize Γ so that the positive imaginary axis $i\mathbf{R}^+$ projects onto a simple, closed geodesic l on U/Γ. Let $\gamma \in \Gamma$, with $\gamma(z) = r_* z$, $r_* > 1$, be a generator for the cyclic subgroup of Γ which stabilizes $i\mathbf{R}^+$. (The group Γ will remain fixed from now on, with this normalization in force.) Choose $0 < \alpha < \frac{1}{2}\pi$ so that

$$B_\alpha = \{z : \tfrac{1}{2}\pi + \alpha > \arg z > \tfrac{1}{2}\pi - \alpha\}$$

projects onto a collar about l in U/Γ (see [2]).

Define $I_\alpha = [\frac{1}{2}\pi - \alpha, \frac{1}{2}\pi + \alpha]$, $I_{\alpha+2\pi} = [\frac{1}{2}\pi - \alpha, \frac{5}{2}\pi + \alpha]$ and let $v\colon I_\alpha \to I_{\alpha+2\pi}$ be a C^1 homeomorphism, with $v(\frac{1}{2}\pi - \alpha) = \frac{1}{2}\pi - \alpha$ and $v(\frac{1}{2}\pi + \alpha) = \frac{5}{2}\pi + \alpha$. We assume that v has derivative equal to 1 at both endpoints and that $v'(\theta) \geq 1$ for all $\theta \in I_\alpha$. The map v will be called an *allowable* map.

A local homeomorphism $f_{\alpha,v}\colon U \to \widehat{\mathbf{C}}$ is defined as follows: Let $\langle\gamma\rangle$ be the cyclic subgroup generated by γ and set $D_\alpha = \bigcup_{A \in \Gamma/\langle\gamma\rangle} A(B_\alpha)$. Let $f_{\alpha,v}(z) = z$ for $z \in U - D_\alpha$, and $f_{\alpha,v}(z) = re^{iv(\theta)}$, for $z = re^{i\theta} \in B_\alpha$. For $w = A(z)$, with $z \in B_\alpha$ and $A \in \Gamma$, let $f_{\alpha,v}(w) = A\big(f_{\alpha,v}(z)\big)$.

One computes easily that

$$\mu_{\alpha,v}(z) = \frac{z}{\bar{z}}\frac{1 - v'(\theta)}{1 + v'(\theta)},$$

for $z = re^{i\theta} \in B_\alpha$; hence $\|\mu_{\alpha,v}\|_\infty < 1$. Clearly, $f_{\alpha,v} \circ A = A \circ f_{\alpha,v}$ for all $A \in \Gamma$, and consequently,

$$\mu_{\alpha,v} = (f_{\alpha,v})_{\bar{z}}/(f_{\alpha,v})_z \in M(\Gamma). \tag{2}$$

Denote the quasiconformal map $w^{\mu_{\alpha,v}}$ (defined as in Section 2) by $w^{\alpha,v}$. Then $f_{\alpha,v} \circ (w^{\alpha,v})^{-1} = g_{\alpha,v}$ is meromorphic in U and $\varphi = S(g_{\alpha,v}) \in B_2(\Gamma^{\mu_{\alpha,v}})$. Note that $(g_{\alpha,v}, \Theta^{f_\varphi})$ defines a projective structure on $\Gamma^{\mu_{\alpha,v}}$, where $\Theta^{f_\varphi}\colon \Gamma^{\mu_{\alpha,v}} \to \Gamma$ is an isomorphism for the solution $f_\varphi = g_{\alpha,v}$, and $\Theta^{f_\varphi}(\gamma^{\mu_{\alpha,v}}) = \gamma$.

Lemma 3.1. *The element* $\big([\mu_{\alpha,v}], S(g_{\alpha,v})\big) \in Q(\Gamma)$ *is independent of the choices of* α *and* v.

Proof. Fix $0 < \alpha < \frac{1}{2}\pi$ and let u, v be allowable maps. Choose f_t, $0 \le t \le 1$, a C^1 homotopy of allowable maps with $f_0 = u$, $f_1 = v$ and define $h\colon [0,1] \to Q(\Gamma)$ by $h(t) = \big([\mu_{\alpha,f_t}], S(g_{\alpha,f_t})\big)$. (Note that convex combinations of allowable maps are allowable so that $f_t = tv + (1-t)u$ could be used.)

We first show that h is continuous. Suppose $t_n \to t \in [0,1]$. One verifies easily that $f_{\alpha,f_{t_n}} \to f_{\alpha,f_t}$ and $\mu_{\alpha,f_{t_n}} \to \mu_{\alpha,f_t}$ uniformly on compact subsets of U. From basic properties of the solution to the Beltrami equation (see [1]), one has that $w^{\alpha,f_{t_n}} \to w^{\alpha,f_t}$. (Note that $\|\mu_{\alpha,f_{t_n}}\|_\infty$ is uniformly bounded away from 1.) Hence, $g_{\alpha,f_{t_n}} \to g_{\alpha,f_t}$ and $S(g_{\alpha,f_{t_n}}) \to S(g_{\alpha,f_t})$.

By its construction, $\Phi \circ h(t)$ is constant and, since Φ is locally injective, it follows that $h(t)$ is constant.

To show that $[\mu_{\alpha,v}]$ is independent of the choice of α, let $0 < \alpha' < \alpha$ and choose $v\colon I_{\alpha'} \to I_{\alpha'+2\pi}$ an allowable map. Extend v to an allowable map $\hat{v}\colon I_\alpha \to I_{\alpha+2\pi}$, letting

$$\hat{v}(\theta) = \begin{cases} \theta, & \theta \in [\frac{1}{2}\pi - \alpha, \frac{1}{2}\pi - \alpha'], \\ v(\theta), & \theta \in [\frac{1}{2}\pi - \alpha', \frac{5}{2}\pi + \alpha'], \\ \theta + 2\pi, & \theta \in [\frac{5}{2}\pi + \alpha', \frac{5}{2}\pi + \alpha]. \end{cases}$$

Observe that for $\alpha' < s < \alpha$, $\hat{v}_s = \hat{v} \mid I_s$ is allowable. As in the preceding argument, the map $h(s) = \big([\mu_{s,\hat{v}_s}], S(g_{s,\hat{v}_s})\big) \in Q(\Gamma)$ is continuous, and by the same reasoning, it must be constant. The result follows. □

3.2. We will now write $[\mu^l] = [\mu_{\alpha,v}] \in T(\Gamma)$ and $\varphi_l = S(g_{\mu_{\alpha,v}}) \in B_2(\Gamma^{\mu^l})$.

Lemma 3.2. *The Teichmüller distance*

$$d_\Gamma([0],[\mu^l]) \le \frac{1}{2}\log\left(1+\frac{\pi}{\frac{1}{2}\pi-\tan^{-1}\bigl(\sinh(\hat{l}/2)\bigr)}\right),$$

where $\hat{l}$ *is the hyperbolic length of* l.

Proof. Fix $0<\alpha<\frac{1}{2}\pi$ and, for an allowable map $v\colon I_\alpha \to I_{\alpha+2\pi}$, let $r_v=\sup_{x\in I_\alpha} v'(x)<\infty$. Then $\inf\{r_v : v \text{ allowable}\}=M_\alpha$, where $M_\alpha = 1+(\pi/\alpha)$ is the slope of the linear map $L\colon I_\alpha\to I_{\alpha+2\pi}$, with $L(\frac{1}{2}\pi-\alpha)=\frac{1}{2}\pi-\alpha$, $L(\frac{1}{2}\pi+\alpha)=\frac{5}{2}\pi+\alpha$.

It follows that for $\varepsilon>0$ we may choose v_ε an allowable map such that

$$\|\mu_{\alpha,v_\varepsilon}\|_\infty = \left\|\frac{1-v_\varepsilon'}{1+v_\varepsilon'}\right\|_\infty \le k_\varepsilon, \tag{3}$$

where $M_\alpha+\varepsilon=(k_\varepsilon+1)/(1-k_\varepsilon)$. Here, $\|(1-v_\varepsilon')/(1+v_\varepsilon')\|_\infty$ is taken over I_α and the equality in (3) is obtained by a straightforward computation from (2).

Applying Lemma 3.1, we have $d_\Gamma([0],[\mu^l])\le\frac{1}{2}\log(M_\alpha+\varepsilon)$, and letting $\varepsilon\to 0$,

$$d_\Gamma([0],[\mu^l])\le\tfrac{1}{2}\log M_\alpha. \tag{4}$$

There is a collar lemma (see Buser [2]) which guarantees that B_{α_l} projects onto a collar about l, where $\alpha_l\ge\frac{1}{2}\pi-\tan^{-1}\bigl(\sinh(\hat{l}/2)\bigr)$. Applying Lemma 3.1 and the inequality (4), we have

$$d_\Gamma([0],[\mu^l])\le\tfrac{1}{2}\log M_{\alpha_l}\le\tfrac{1}{2}\log\left(1+\frac{\pi}{\frac{1}{2}\pi-\tan^{-1}\bigl(\sinh(\hat{l}/2)\bigr)}\right). \quad\square$$

3.3. We will now allow $[\nu]$ to vary over $T(\Gamma)$ and perform Maskit's grafting construction on U/Γ^ν in the following manner: The element γ^ν stabilizes $i\mathbf{R}^+$, and $i\mathbf{R}^+$ projects onto a simple, closed geodesic l_ν in U/Γ^ν. Using the previous notation, we obtain $[\mu^{l_\nu}]\in T(\Gamma^\nu)$ and $([\mu^{l_\nu}],\varphi_{l_\nu})\in Q(\Gamma^\nu)$.

Define the *grafting* map $\Psi\colon T(\Gamma)\to T(\Gamma)$ by $\Psi([\nu])=[\nu']$, where ν' is the Beltrami coefficient of $w^{\mu_{l_\nu}}\circ w^\nu$. Note that $w^{\nu'}\circ\Gamma\circ(w^{\nu'})^{-1}=w^{\mu_{l_\nu}}\circ\Gamma^\nu\circ(w^{\mu_{l_\nu}})^{-1}$ and that φ_{l_ν} represents a projective structure on $U/\Gamma^{\nu'}$. Here, $\Theta^{f_{\varphi_{l_\nu}}}\colon\Gamma^{\nu'}\to\Gamma^\nu$, and $\theta_\nu=\Theta_{f_{\varphi_{l_\nu}}}=\Theta^{f_{\varphi_{l_\nu}}}\circ\theta_{\nu'}\colon\Gamma\to\Gamma^\nu$ are isomorphisms.

We also define the map $\hat\Psi\colon T(\Gamma)\to Q(\Gamma)$ given by $\hat\Psi([\nu])=([\nu'],\varphi_{l_\nu})$.

Lemma 3.3. $\widehat\Psi$ *is a homeomorphism onto its image in* $R(\Gamma)\subset Q(\Gamma)$.

Proof. We first show that the map $\widehat{\Psi}$ is continuous. Suppose $[\nu_j] \to [\nu]$. It is well known that $\hat{l}_{\nu_j} \to \hat{l}_\nu$. Hence, we may choose $0 < \alpha < \frac{1}{2}\pi$ such that B_α projects onto a collar about l_{ν_j} in U/Γ^{ν_j}, for all j sufficiently large. Let $v\colon I_\alpha \to I_{\alpha+2\pi}$ be an allowable map and define (using the group Γ^{ν_j}), for each j, the local homeomorphism $f^{\nu_j}_{\alpha,v}\colon U \to \widehat{\mathbf{C}}$, as in Section 3.1. Now, for $A \in \Gamma$, the corresponding elements $A^{\nu_j} \in \Gamma^{\nu_j}$ converge to $A^\nu \in \Gamma^\nu$. Hence, $f^{\nu_j}_{\alpha,v} \to f^{\nu}_{\alpha,v}$ uniformly on compact subsets of U.

Let μ_j (respectively μ_ν) be the Beltrami coefficient of $f^{\nu_j}_{\alpha,v}$ (respectively $f^{\nu}_{\alpha,v}$). From elementary properties of the solutions w^{μ_j}, we have that $w^{\mu_j} \to w^{\mu_\nu}$. It follows that $[\nu'_j] \to [\nu']$, $g_{\mu_j} \to g_{\mu_\nu}$, and $S(g_{\mu_j}) = \varphi_{l_{\nu_j}} \to S(g_{\mu_\nu}) = \varphi_{l_\nu}$. Hence, $\widehat{\Psi}$ is continuous.

Now let $[\nu_1], [\nu_2] \in T(\Gamma)$ be distinct elements and suppose $[\nu'_1] = [\nu'_2]$, so that $\widehat{\Psi}([\nu_1]) = ([\nu'_1], \varphi_1)$, $\widehat{\Psi}([\nu_2]) = ([\nu'_2], \varphi_2)$, where $\varphi_1 = \varphi_{l_{\nu_1}}$, $\varphi_2 = \varphi_{l_{\nu_2}}$ are quadratic differentials for the same group $\Gamma^{\nu'_1} = \Gamma^{\nu'_2}$. As we have noted, there are solutions, f_{φ_1} and f_{φ_2}, with $S(f_{\varphi_1}) = \varphi_1$, $S(f_{\varphi_2}) = \varphi_2$, so that $\Theta^{f_{\varphi_1}}\colon \Gamma^{\nu'_1} \to \Gamma^{\nu_1}$ and $\Theta^{f_{\varphi_2}}\colon \Gamma^{\nu'_1} \to \Gamma^{\nu_2}$. Moreover, the Teichmüller isomorphisms $\theta_{\nu_1} = \Theta_{f_{\varphi_1}} = \Theta^{f_{\varphi_1}} \circ \theta_{\nu'_1}\colon \Gamma \to \Gamma^{\nu_1}$ and $\theta_{\nu_2} = \Theta_{f_{\varphi_2}} = \Theta^{f_{\varphi_2}} \circ \theta_{\nu'_1}\colon \Gamma \to \Gamma^{\nu_2}$ are not conjugate in $\mathrm{PSL}(2,\mathbf{C})$, and it follows that $\Theta^{f_{\varphi_1}}$ and $\Theta^{f_{\varphi_2}}$ are not conjugate, either. Hence, the quadratic differentials φ_1, φ_2 are distinct, and $\widehat{\Psi}$ is an injective map. □

Corollary 2. *The map* Ψ *is a local homeomorphism.*

Proof. Since $\widehat{\Psi}\big(T(\Gamma)\big) \subset R(\Gamma)$, we now apply Lemma 3.3 and Theorem 1 to deduce that $p \circ \widehat{\Psi} = \Psi$ is locally injective. But $T(\Gamma)$ is a finite dimensional space, and it follows that Ψ is a local homeomorphism. □

Remark. It is easy to see that $\widehat{\Psi}\big(T(\Gamma)\big)$ is a closed subset of $R(\Gamma)$; hence, it also follows from Lemma 3.3 that $\widehat{\Psi}\big(T(\Gamma)\big)$ is a connected component of $R(\Gamma)$.

Lemma 3.4. *Let* $[\nu_n] \in T(\Gamma)$, *with* $d_\Gamma([0],[\nu_n]) \to \infty$, *and let* l *be a simple, closed geodesic in* U/Γ; *suppose* $\hat{l}_{\nu_n} \le N$ *for some positive constant* N. *Then* $d_\Gamma\big([0], \Psi([\nu_n])\big) \to \infty$, *also.*

Proof. We obtain, applying Lemma 3.2,

$$d_{\Gamma^{\nu_n}}\big([0],[\mu^{l_{\nu_n}}]\big) \le \frac{1}{2}\log\left(1 + \frac{\pi}{\frac{1}{2}\pi - \tan^{-1}\big(\sinh(\hat{l}_{\nu_n}/2)\big)}\right)$$
$$\le \frac{1}{2}\log\left(1 + \frac{\pi}{\frac{1}{2}\pi - \tan^{-1}\big(\sinh(N/2)\big)}\right) = c_0.$$

Using the triangle inequality and the definition (1), we have

$$d_\Gamma([0],[\nu'_n]) \ge d_\Gamma([0],[\nu_n]) - d_{\Gamma^{\nu_n}}([0],[\mu^{l_{\nu_n}}]) \ge d_\Gamma([0],[\nu_n]) - c_0.$$

Hence, $d_\Gamma([0],[\nu'_n]) \to \infty$. □

Lemma 3.5. *Let* $[\nu_n] \in T(\Gamma)$, *with* $d_\Gamma([0],[\nu_n]) \to \infty$, *and let* l *be a simple, closed geodesic in* U/Γ; *suppose* $\hat{l}_{\nu_n} \to \infty$. *Then* $d_\Gamma\big([0],\Psi([\nu_n])\big) \to \infty$, *also.*

The proof of Lemma 3.5 will be given in Section 4.

Theorem 3. Ψ: $T(\Gamma) \to T(\Gamma)$ *is a surjective homeomorphism.*

Proof. Suppose that $\Psi([\nu_n]) \to [\mu] \in T(\Gamma)$. By Lemmas 3.4 and 3.5, it follows that there exist $[\nu] \in T(\Gamma)$ and a subsequence $[\nu_n] \to [\nu]$; by continuity, $\Psi([\nu]) = [\mu]$. Hence, $\Psi\big(T(\Gamma)\big)$ is closed. Since Ψ is an open map (Lemma 3.3), we must have $\Psi\big(T(\Gamma)\big) = T(\Gamma)$.

Now let $[\nu] \in T(\Gamma)$. In view of Corollary 2 and Lemmas 3.4, 3.5, we conclude that $\Psi^{-1}([\nu])$ is finite. Hence, Ψ is a covering map; but $T(\Gamma)$ is simply connected, and it must be a homeomorphism. □

Since φ_{l_ν} defines a quadratic differential for the group $\Gamma^{\nu'}$, where $\Psi([\nu]) = [\nu']$, the proof of the following corollary is clear.

Corollary 4. *Every compact, hyperbolic Riemann surface has a projective structure where the holonomy is real, injective, discrete, and different from the canonical Fuchsian uniformizing structure.*

4. Extremal length properties

Consider a family $\mathscr{F}$ of simple, closed curves[1] in U/Γ. The *extremal length* $\Lambda(\mathscr{F})$ is defined by $\Lambda(\mathscr{F}) = \sup_\rho\big(L(\rho)^2/A(\rho)\big)$, where $\rho|\mathrm{dz}|$ is a measurable, nonnegative metric on U/Γ; $L(\rho) = \inf \int_\beta \rho|\mathrm{dz}|$, where the infimum is over all β in $\mathscr{F}$ (here, $\int_\beta \rho|\mathrm{dz}| = \infty$ if ρ is not measurable over β); and $A(\rho) = \iint_{U/\Gamma} \rho^2\,\mathrm{dx}\,\mathrm{dy} \neq 0, \infty$. (See [1] or [5].)

Let s be a simple, closed geodesic in U/Γ. An annulus $A \subset U/\Gamma$ is *homotopic* to s, if it contains a nontrivial, simple, closed curve homotopic (in U/Γ) to s. Denote by $\mathscr{F}_{\mathrm{s}}$ the family of all simple, closed curves in U/Γ which are homotopic to s. We have the basic observation:

Lemma 4.1. *Let* s *be a simple, closed geodesic in* U/Γ. *Then* $\Lambda(\mathscr{F}_{\mathrm{s}}) < \infty$.

Proof. Choose $A \subset U/\Gamma$ an annulus homotopic to s, and let $\mathscr{F}_A \subset \mathscr{F}_{\mathrm{s}}$ be the set of all curves in $\mathscr{F}_{\mathrm{s}}$ which are contained in A. It is well known that $\Lambda(\mathscr{F}_A) < \infty$ (see [1]). (The same proof given in [1] remains valid when the area integral taken over $\mathbf{C}$ is replaced by an area integral over U/Γ.) Since $\mathscr{F}_A \subset \mathscr{F}_{\mathrm{s}}$, we also have $\Lambda(\mathscr{F}_{\mathrm{s}}) \leq \Lambda(\mathscr{F}_A)$ (see [1]). □

The idea of the proof of Lemma 3.5 is to show that $\Lambda(\mathscr{F}_{l_{\nu_n'}}) \to \infty$. This will imply that $[\nu_n']$ cannot have a convergent subsequence. First, an inequality relating $\Lambda(\mathscr{F}_{l_{\nu_n'}})$ and $\hat{l}_{\nu_n}$ is needed (Lemma 4.2).

[1] In order to integrate, we assume in this section that all curves are piecewise smooth.

We continue using the notations of Section 3 and assume that B_α projects onto a collar about the geodesic l in U/Γ. Let $v\colon I_\alpha \to I_{\alpha+2\pi}$ be an allowable map. In order to get an estimate for $\Lambda(\mathscr{F}_{l_{\nu_{\alpha,v}}})$ we will construct a non-negative metric on $U/\Gamma^{\mu_{\alpha,v}}$. For this, we need to establish complex local coordinates on $U/\Gamma^{\mu_{\alpha,v}}$.

Let $G = \{(z, f_{\alpha,v}(z)) : z \in U\} \subset U \times \widehat{\mathbf{C}}$ be the graph of $f_{\alpha,v}\colon U \to \widehat{\mathbf{C}}$, and let $p_2\colon G \to \widehat{\mathbf{C}}$ be the projection on the second factor. Since p_2 is a local homeomorphism, we use it to obtain complex local coordinates on G.

The group Γ acts discontinuously on G as a group of biholomorphic self maps via the action $\gamma(z, f_{\alpha,v}(z)) = (\gamma(z), \gamma\circ f_{\alpha,v}(z))$. Thus, G/Γ has a natural complex structure which it inherits from G. The map $\hat f\colon U/\Gamma \to G/\Gamma$, given by $\hat f([z]) = [(z, f_{\alpha,v}(z))]$, lifts to a quasiconformal homeomorphism $f\colon U \to G$, with Beltrami coefficient $\mu_{\alpha,v}$; hence, G/Γ is conformally equivalent to $U/\Gamma^{\mu_{\alpha,v}}$ via the conformal map $\widehat{w}^{\mu_{\alpha,v}} \circ \hat f^{-1}$. (Here $\widehat{w}^{\mu_{\alpha,v}}$ is the homeomorphism from U/Γ onto $U/\Gamma^{\mu_{\alpha,v}}$ induced by $w^{\mu_{\alpha,v}}$.)

Let $\theta_0 = v^{-1}(\frac12\pi)$ and $\theta_1 = v^{-1}(\frac52\pi)$, and define

$$B'_\alpha = \{re^{i\theta} \in B_\alpha : \theta \in [\theta_0, \theta_1]\},$$
$$D'_\alpha = \{z \in D_\alpha : z = \gamma(w), w \in B'_\alpha, \gamma \in \Gamma\}.$$

Set

$$S = \{[(re^{i\theta}, re^{iv(\theta)})] \in G/\Gamma :\ re^{i\theta} \in B'_\alpha\} \subset G/\Gamma.$$

One sees easily that $(U/\Gamma) - l$ is conformally equivalent to $(G/\Gamma) - S$. Indeed, the map $g\colon (U/\Gamma) - l \longrightarrow (G/\Gamma) - S$, given by

$$g([re^{i\theta}]) = [(re^{i\theta}, re^{i\theta})], \qquad \text{for } re^{i\theta} \in U - D_\alpha;$$
$$g([re^{i\theta}]) = [(re^{iv^{-1}(\theta)}, re^{i\theta})], \qquad \text{for } re^{i\theta} \in B_\alpha,\ \text{with } \theta < \tfrac12\pi;$$
$$g([re^{i\theta}]) = [(re^{iv^{-1}(\theta+2\pi)}, re^{i(\theta+2\pi)})], \qquad \text{for } re^{i\theta} \in B_\alpha,\ \text{with } \theta > \tfrac12\pi,$$

is a conformal homeomorphism from $(U/\Gamma) - l$ onto $(G/\Gamma) - S$.

We will now construct a non-negative, continuous metric on G/Γ, and use it to obtain a lower bound on $\Lambda(\mathscr{F}_{\mathrm{s}})$, where s is the unique geodesic in G/Γ homotopic to $\hat f(l) = [(re^{i\pi/2}, re^{iv(\pi/2)})] \subset G/\Gamma$.

We consider the Poincaré metric on U/Γ. When restricted to $(G/\Gamma) - S$, this metric is given locally by

$$\rho(re^{i\vartheta})|\mathrm{d}re^{i\vartheta}| = \frac{\sqrt{dr^2 + r^2 d\vartheta^2}}{r\sin\vartheta},$$

for the local coordinates $re^{i\vartheta} = f_{\alpha,v}(se^{i\theta})$ on $[(se^{i\theta}, f_{\alpha,v}(se^{i\theta}))] \in (G/\Gamma) - S$, $se^{i\theta} \in U - D'_\alpha$.

We extend $\rho(re^{i\vartheta})|\mathrm{d}re^{i\vartheta}|$ to a metric $\rho(re^{i\vartheta})|\mathrm{d}re^{i\vartheta}|$ on all of G/Γ, defining

$$\rho(re^{i\vartheta})|\mathrm{d}re^{i\vartheta}| = \frac{\sqrt{dr^2 + r^2 d\vartheta^2}}{r},$$

for the local coordinates $re^{i\vartheta} = re^{iv(\theta)}$ on $[(re^{i\theta}, re^{iv(\theta)})] \in S$, $re^{i\theta} \in B'_\alpha$.

Note that $\rho > 0$ yields a continuous metric on G/Γ.

For $\beta(t)$, $0 \le t \le 1$, a simple, closed curve in $\mathscr{F}_s$, define the simple, closed curve on U/Γ, $\hat{\beta}(t) = \hat{f}^{-1} \circ \beta(t)$. Denote the Poincaré metric on U/Γ by $\rho_1(se^{i\theta})|\mathrm{d}se^{i\theta}|$, and observe that, since $\hat{\beta}$ is homotopic to l,

$$\int_{\hat{\beta}} \rho_1(se^{i\theta})\,|\mathrm{d}se^{i\theta}| \ge \hat{l}.$$

Define $c_1 = \sin(\frac{1}{2}\pi + \alpha) < 1$.

Now, $\int_\beta \rho(re^{i\vartheta})\,|\mathrm{d}re^{i\vartheta}| = \int_0^1 R(t)\,\mathrm{d}t$, and $\int_{\hat{\beta}} \rho_1(se^{i\theta})\,|\mathrm{d}se^{i\theta}| = \int_0^1 S(t)\,\mathrm{d}t$, where $R(t)$ and $S(t)$ are defined as follows:

For $t \in [0,1]$, with

$$\beta(t) = \left[\left(r(t)e^{iv^{-1}(\vartheta(t))}, r(t)e^{i\vartheta(t)}\right)\right] \in S,$$

$r(t)e^{iv^{-1}(\vartheta(t))} \in B'_\alpha$, we have $\hat{\beta}(t) = [r(t)e^{iv^{-1}(\vartheta(t))}]$, and

$$R(t) = \frac{\sqrt{r'(t)^2 + r(t)^2\vartheta'(t)^2}}{r(t)} \ge c_1 \frac{\sqrt{r'(t)^2 + r(t)^2\left(v^{-1\prime}\left(\vartheta(t)\right)^2\vartheta'(t)\right)^2}}{r(t)\sin v^{-1}\left(\vartheta(t)\right)} = c_1 S(t)$$

(recall that $v' \ge 1$, and $c_1 < \sin v^{-1}(\vartheta(t))$.

For $t \in [0,1]$, with

$$\beta(t) = \left[\left(r(t)e^{iv^{-1}(\vartheta(t))}, r(t)e^{i\vartheta(t)}\right)\right] \in (G/\Gamma) - S,$$

and $r(t)e^{iv^{-1}(\vartheta(t))} \in B_\alpha - B'_\alpha$, we have $\hat{\beta}(t) = \left[r(t)e^{iv^{-1}(\vartheta(t))}\right]$, and

$$R(t) = \frac{\sqrt{r'(t)^2 + r(t)^2\vartheta'(t)^2}}{r(t)\sin\vartheta} \ge c_1 \frac{\sqrt{r'(t)^2 + r(t)^2\left(v^{-1\prime}\left(\vartheta(t)\right)^2\vartheta'(t)\right)^2}}{r(t)\sin v^{-1}\left(\vartheta(t)\right)} = c_1 S(t)$$

(again, we have $v' \ge 1$, and $c_1 \sin\left(\vartheta(t)\right) < \sin v^{-1}\left(\vartheta(t)\right)$).

For $t \in [0,1]$, with

$$\beta(t) = \left[\left(r(t)e^{i\theta(t)}, r(t)e^{i\vartheta(t)}\right)\right] \in (G/\Gamma) - S,$$

$r(t)e^{i\theta(t)} \in U - D_\alpha$, we have $\theta(t) = \vartheta(t)$, and $\hat{\beta}(t) = [r(t)e^{i\vartheta(t)}]$, so that $R(t) = S(t)$.

Hence, in general, for $t \in [0,1]$, $R(t) \geq c_1 S(t)$, and it follows that

$$\int_\beta \rho\,|\mathrm{d}re^{i\vartheta}| = \int_0^1 R(t)\,\mathrm{d}t \geq c_1 \int_0^1 S(t)\,\mathrm{d}t = c_1 \int_{\hat{\beta}} \rho_1\,|\mathrm{d}se^{i\theta}| \geq c_1\hat{l}. \tag{5}$$

Since β was an arbitrary curve in $\Lambda(\mathscr{F}_{\mathrm{s}})$, we conclude from (5) that

$$L(\rho) \geq c_1\hat{l}. \tag{6}$$

On the other hand,

$$\begin{aligned} A(\rho) &= \iint_{(G/\Gamma)-S} \rho^2\, r\mathrm{d}\,r\mathrm{d}\vartheta + \int_0^{2\pi}\int_1^{r_*} \frac{r\mathrm{d}r\mathrm{d}\vartheta}{r^2} \\ &= 2\pi(2g-2) + 2\pi\log r_* = 2\pi(2g-2) + 2\pi\hat{l}. \end{aligned}$$

(Recall that $\gamma(z) = r_* z$ generates the stabilizer of B_α in Γ.)

Hence,

$$\Lambda(\mathscr{F}_{\mathrm{s}}) \geq \frac{L(\rho)^2}{A(\rho)} = \frac{c_1^2\hat{l}^2}{2\pi(2g-2)+2\pi\hat{l}}\,. \tag{7}$$

We can now prove the following:

Lemma 4.2. *For the simple, closed geodesic l in U/Γ and $[\nu] \in T(\Gamma)$, we have*

$$\Lambda(\mathscr{F}_{l_{\nu'}}) \geq \frac{\hat{l}_\nu^2}{2\pi(2g-2)+2\pi\hat{l}_\nu}. \tag{8}$$

Proof. Choose $0 < \alpha < \frac{1}{2}\pi$ so that B_α projects onto a collar about l_ν in U/Γ^ν and let v be an allowable map. Define $f_{\alpha,v}\colon U \to \widehat{\mathbf{C}}$ as in Section 3.1 and let G be the graph of $f_{\alpha,v}$. Define $\hat{f}\colon U/\Gamma^\nu \to G/\Gamma^\nu$ by $\hat{f}([re^{i\theta}]) = \big[\big(re^{i\theta}, f_{\alpha,v}(re^{i\theta})\big)\big]$. Hence, G/Γ^ν is conformally equivalent to $U/\Gamma^{\nu'}$ via the map

$$\hat{h} = \widehat{w}^{\mu_{\alpha,v}} \circ \hat{f}^{-1}\colon G/\Gamma^\nu \to U/\Gamma^{\nu'}.$$

Let s be the unique geodesic in G/Γ^ν homotopic to $\hat{f}(l_\nu)$. It follows from the inequality (7) that

$$\Lambda(\mathscr{F}_{\mathrm{s}}) \geq \frac{c_1^2\hat{l}_\nu^2}{2\pi(2g-2)+2\pi\hat{l}_\nu}. \tag{9}$$

Clearly, $\hat{h}(\mathscr{F}_{\mathrm{s}}) = \mathscr{F}_{l_{\nu'}}$, and since $\hat{h}$ is holomorphic, $\Lambda(\mathscr{F}_{\mathrm{s}}) = \Lambda(\mathscr{F}_{l_{\nu'}})$ (see [1]). Hence,

$$\Lambda(\mathscr{F}_{l_{\nu'}}) \geq \frac{c_1^2 \hat{l}_\nu^2}{2\pi(2g-2) + 2\pi \hat{l}_\nu}. \tag{10}$$

Observe, however, that c_1 depends on the choice of α, while $\mathscr{F}_{l_{\nu'}}$ does not (Lemma 3.1). Thus, choosing α arbitrarily small, and noting that $c_1 \to 1$ as $\alpha \to 0$, we obtain from (10)

$$\Lambda(\mathscr{F}_{l_{\nu'}}) \geq \frac{\hat{l}_\nu^2}{2\pi(2g-2) + 2\pi \hat{l}_\nu}, \tag{11}$$

which is the desired result. □

Proof of Lemma 3.5. Suppose that there exists a convergent subsequence $\Psi([\nu_n]) = [\nu'_n] \to [\nu] \in T(\Gamma)$.

We can choose a constant K and quasiconformal maps w_n of dilatation $K_n \leq K$ (where $K_n \geq 1$) such that $w_n(U/\Gamma^{\nu'_n}) = U/\Gamma^\nu$, with $w_n(l_{\nu'_n})$ homotopic to l_ν. Further, we can assume w_n is C^1 on the complement of a finite set (the minimal Teichmüller map has this property [5]), so that $w_n(\mathscr{F}_{l_{\nu'_n}}) = \mathscr{F}_{l_\nu}$. It is shown in [5] (see also [1]) that, in this situation,

$$\Lambda(\mathscr{F}_{l_\nu}) \geq \frac{1}{K_n} \Lambda(\mathscr{F}_{l_{\nu'_n}}) \geq \frac{1}{K} \Lambda(\mathscr{F}_{l_{\nu'_n}}). \tag{12}$$

Applying Lemma 4.2, we have $\Lambda(\mathscr{F}_{l_{\nu'_n}}) \geq \hat{l}_{\nu_n}^2 / \big(2\pi(2g-2) + 2\pi \hat{l}_{\nu_n}\big)$, and, since $\hat{l}_{\nu_n} \to \infty$, we deduce $\Lambda(\mathscr{F}_{l_{\nu'_n}}) \to \infty$.

From the inequality (12), we we now obtain $\Lambda(\mathscr{F}_{l_\nu}) = \infty$; this contradicts Lemma 4.1, and we conclude that

$$\mathrm{d}_\Gamma\big([0], \Psi([\nu_n])\big) = \mathrm{d}_\Gamma([0], [\nu'_n]) \to \infty. \ \square$$

Remarks. It is clear from the construction of the grafting map that allowable maps of the form $v \colon I_\alpha \to I_{\alpha+2\pi n} = [\frac{1}{2}\pi - \alpha, \frac{1}{2}\pi + 2\pi n + \alpha]$, $n \in \mathbf{Z}^+$, could have been used. Thus, for every such n, there is a map $\Psi_{(l,n)} \colon T(\Gamma) \to T(\Gamma)$. The proof of the corresponding version of Theorem 2 remains valid, and $\Psi_{(l,n)}$ is a homeomorphism onto $T(\Gamma)$. Similarly, the restriction to one simple, closed geodesic was unnecessary; any finite collection $l_1, \ldots, l_m$ of simple, closed, disjoint geodesics could have been used.

References

[1] Ahlfors, L.V.: Lectures on Quasiconformal Mappings. - Van Nostrand, Princeton, NJ, 1966.

[2] Buser, P.: Geometry and Spectra of Compact Riemann Surfaces. - Birkhäuser, Boston, MA, 1992.

[3] Earle, C.J.: On variation of projective structures. - In: Riemann Surfaces and Related Topics. - Ann. of Math. Stud. 97, Princeton University Press, Princeton, N.J., 1981, 87–99.

[4] Faltings, G.: Real projective structures on compact surfaces. - Compositio Math. 48, 1983, 223–269.

[5] Gardiner, F.P.: Teichmüller Theory and Quadratic Differentials. - John Wiley & Sons, New York, N.Y., 1987.

[6] Goldman, W.M.: Fuchsian structures with Fuchsian holonomy. - J. Differential Geom. 25, 1987, 297–326.

[7] Maskit, B.: On a class of Kleinian groups. - Ann. Acad. Sci. Fenn. Ser. A I Math. 442, 1969, 1–8.

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