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# ZEROS OF DERIVATIVES OF STRICTLY NON-REAL MEROMORPHIC FUNCTIONS

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**Abstract.** Let $f$ be a meromorphic non-entire function in the plane, and suppose that for every $n \geq 0$, the derivative $f^{(n)}$ has only real zeros. We have proved that then $f$ is rational and of a special form, and that all possibilities can be listed. In this paper we prove that part of this result which is related to properties of strictly non-real functions.

## 1. Introduction and results

**1.1.** Let $f$ be a function meromorphic in the complex plane $\mathbf{C}$. We consider the question of under what circumstances all the derivatives of $f$, including $f$ itself, can have only real zeros. We may and will assume that $f$ is not a polynomial so that none of the derivatives $f^{(n)}$ vanishes identically. We say that $f$ is *real* if $f(z)$ is real or $f(z) = \infty$ whenever $z$ is real. If $f$ is not a constant multiple of a real function, then $f$ is called *strictly non-real*. We have proved the following result. The proof is given partly in this paper and partly in the two companion papers [3], [4].

**Theorem A.** *Let $f$ be a non-entire meromorphic function in the complex plane, and suppose that for every integer $k \geq 0$, the derivative $f^{(k)}$ has only real zeros. Then there are real numbers $a$ and $b$ where $a \neq 0$, and a polynomial $P$ with only real zeros, such that*

(i) $f(az+b) = P(z)/Q(z)$, *where* $Q(z) = z^n$ *or* $Q(z) = (z^2+1)^n$, $n$ *is a positive integer, and* $\deg P \leq \deg Q + 1$; *or*
(ii) $f(az+b) = C(z-i)^{-n}$ *where* $C$ *is a non-zero complex constant*; *or*
(iii) $f(az+b) = C(z-\alpha)/(z-i)$, *where* $\alpha$ *is a real number and* $C$ *is a non-zero complex constant.*

*Conversely, if $f$ is as in* (i) *with* $\deg P \leq \deg Q$, *or if $f$ is as in* (ii) *or* (iii), *then $f^{(k)}$ has only real zeros for all $k \geq 0$. If $f$ is as in* (i) *with* $\deg P = \deg Q + 1$ *then $f^{(k)}$ has only real zeros for all $k \geq 0$ if, and only if, $f'$ (or, equivalently, $zP'(z) - nP(z)$ or $(z^2+1)P'(z) - 2nzP(z)$) has only real zeros.*

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If $f$ is as in (i) of Theorem A and $\deg P = \deg Q + 1$ then there are polynomials $P$ for which $f'$ has only real zeros, and other polynomials $P$ for which $f'$ has at least two non-real zeros (cf. [3]).

The complete proof of Theorem A is long, and is divided into three papers (this paper and [3], [4]). The structure of the proof of Theorem A, showing how the different pieces are put together, is explained in [3]. In this paper we prove the following result that basically deals with strictly non-real functions (referred to in [3] as [3, Theorem 2]).

**Theorem 1.** *Let $f$ be given by*

$$f(z) = \frac{g(z)}{(z+i)^m (z-i)^n} \tag{1.1}$$

*where $g$ is an entire function with only real zeros with $g(i)g(-i) \neq 0$ and $m$ and $n$ are integers, not necessarily equal, with $m \geq 0$ and $n \geq 1$. Suppose that $f'$ and $f''$ have only real zeros.*

*If $m = 0$ then there is a non-zero complex constant $C$ such that*

(i) $g(z) \equiv C$; *or*
(ii) $g(z) \equiv C(z-\alpha)$ *for some real $\alpha$, and then $n = 1$; or*
(iii) $g(z) \equiv Ce^{inz}$.

*If $g$ is as in* (i) *or* (ii) *then $f^{(k)}$ has only real zeros for all $k \geq 0$. If $g$ is as in* (iii) *then $f'''$ has at least one non-real zero.*

*If $m \geq 1$ then $m = n$ and $g$ is a constant multiple of a real entire function.*

We define a class of functions. We say that $f \in U_0$ if $f$ is of the form

$$f(z) = g(z)\exp\{-az^2\}$$

where $a \geq 0$ and $g$ is a constant multiple of a real entire function with genus not exceeding 1 and with only real zeros. The class $U_0$ is the so-called Laguerre–Pólya class. We have $f \in U_0$ if, and only if, there are real polynomials $P_n$ with only real zeros such that $P_n \to f$ locally uniformly in $\mathbf{C}$. Also, $f \in U_0$ if, and only if, we may write

$$f(z) = cz^m e^{-az^2+bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{z/z_n}$$

where $c$ is a non-zero complex constant, $m$ is a non-negative integer, $a \geq 0$, $b$ is a real number, $z_n \in \mathbf{R} \setminus \{0\}$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} z_n^{-2} < \infty$. Here $\mathbf{R}$ denotes the real axis. Thus if $f \in U_0$ then $f^{(n)} \in U_0$ and so $f^{(n)}$ has only real zeros for all $n \geq 0$.

Hellerstein, Shen and Williamson [2] developed a method for dealing with the reality of zeros of the first two derivatives of a strictly non-real meromorphic function with only real poles. We will follow their approach to the extent possible. Differences start to appear after the first few formulas have been obtained. They are due to the presence of non-real poles, even if there are at most two poles. One of the principal changes is that in some places, functions such as $\cos(az+b)+i\sin(az+b)$ are replaced by functions of the form $\cos(az+b)+z\sin(az+b)$. The latter functions can no longer be expressed as exponentials and have infinitely many zeros. Also we cannot take fractional powers of such functions and obtain single-valued functions in the plane. Therefore certain arguments that take up a few pages in [2] are replaced by much longer ones that amount to a careful analysis of what can be said about the new functions that we end up with. The case $m \geq 1$ of Theorem 1 is still reasonably close to [2] but in the case $m=0$ the situation changes more radically.

## 2. Proof of Theorem 1 when $m \geq 1$

**2.1.** Suppose that $f$ is as in (1.1) where $g$ is an entire function with only real zeros with $g(i)g(-i) \neq 0$ and $m$ and $n$ are positive integers. We first consider the case $m \neq n$. Then $f$ is strictly non-real since $m \neq n$ (for if $f$ were a constant multiple of a real function then $f$ would have a pole of the same order at $i$ and at $-i$). Write

$$f(z) = \overline{f(\overline{z})}H(z), \tag{2.1}$$

$$f'(z) = \overline{f'(\overline{z})}G(z). \tag{2.2}$$

Then $H$ and $G$ are meromorphic in the plane and have modulus 1 on $\mathbf{R}$. Suppose that $f$, $f'$, and $f''$ have only real zeros. Then $H$ and $G$ have no zeros or poles at points other than $\pm i$.

We denote various non-zero complex constants by $A_1, A_2, \ldots$. Close to $z=i$ we have

$$f(z) \sim \frac{A_1}{(z-i)^n}, \qquad \overline{f(\overline{z})} \sim \frac{A_2}{(z-i)^m}$$

so that $H(z) \sim A_3(z-i)^{m-n}$. Similarly, $G(z) \sim A_4(z-i)^{m-n}$ as $z \to i$. As $z \to -i$ we similarly get $H(z) \sim A_5(z+i)^{n-m}$ and $G(z) \sim A_6(z+i)^{n-m}$. Hence there is an entire function $h$ such that

$$H(z) = \left(\frac{z-i}{z+i}\right)^{m-n} e^{ih(z)}$$

where $|e^{ih(z)}| = 1$ when $z$ is real. Thus $h$ is a real entire function. Similarly,

$$G(z) = \left(\frac{z-i}{z+i}\right)^{m-n} e^{ig_0(z)}$$

where $g_0$ is a real entire function. From (2.1) we get

$$\frac{f'}{f}(z) = \overline{\frac{f'}{f}(\overline{z})} + \frac{H'}{H}(z) = \overline{\frac{f'}{f}(\overline{z})} + ih'(z) + \frac{2i(m-n)}{z^2+1}. \tag{2.3}$$

Following Hellerstein, Shen and Williamson [2, p. 321], we recall the functions from the Tsuji value distribution theory in a half plane defined by $T_0(r,f) = m_0(r,f) + N_0(r,f)$,

$$m_0(r,f) = \frac{1}{2\pi}\int_{\arcsin(r^{-1})}^{\pi-\arcsin(r^{-1})} \log^+\bigl|f\bigl(r(\sin\theta)e^{i\theta}\bigr)\bigr| \frac{d\theta}{r\sin^2\theta}$$

and

$$N_0(r,f) = \int_1^r \frac{n(t,\infty)}{t^2}\,dt = \sum_{1\le r_k<r\sin\theta_k}\left(\frac{\sin\theta_k}{r_k} - \frac{1}{r}\right).$$

Here the points $z_k = r_k e^{i\theta_k}$ are the poles of $f$ with $\operatorname{Im} z_k > 0$, with due count of multiplicity, and $n(t,\infty)$ denotes the number of poles of $f$ in $\{z : |z - \frac{1}{2}it| \le \frac{1}{2}t,\ |z| \ge 1\}$, also with due count of multiplicity. As in [2, Lemma D], we get

$$\begin{aligned} T_0(r,h') &= m_0(r,h') \\ &\le m_0(r,f'/f) + m_0\bigl(r,\overline{(f'/f)(\overline{z})}\bigr) + m_0\bigl(r,2(m-n)/(z^2+1)\bigr) + \log 3 \\ &= O(\log r). \end{aligned}$$

The same with $z$ replaced by $-z$ gives

$$T_0\bigl(r,h'(-z)\bigr) = m_0\bigl(r,h'(-z)\bigr) = O(\log r).$$

As in [2, (2.11–12)], this gives $T(r,h') = O(r\log r)$ so that the order of $h'$, and hence of $h$, is at most $1$. Write

$$h_1(z) = h'(z) + \frac{2(m-n)}{z^2+1}.$$

Then $h_1$ is meromorphic with poles at $\pm i$ only and the order of $h_1$ is the same as that of $h$, hence at most $1$.

Since

$$\frac{f'}{f}(z) - \overline{\frac{f'}{f}(\overline{z})} = ih_1(z) \tag{2.4}$$

and all the zeros of $f'$, hence all the zeros of $f'/f$, are real, it follows that all the zeros of $f'/f$ are also zeros of $h_1$. Let $\pi_1$ be the canonical product of the zeros of $f'/f$. Then the exponent of convergence of $\pi_1$ does not exceed 1 and

$$\gamma(z) \equiv \pi_1(z)\frac{f}{f'}(z) \tag{2.5}$$

is entire. Since the order of the real entire function $\pi_1$ does not exceed $1$, we see as in [2, p. 323] that

$$T_0(r,\pi_1)=m_0(r,\pi_1)=O(r^\varepsilon) \qquad \text{for each } \varepsilon>0.$$

Hence

$$\begin{aligned} m_0(r,\gamma)=T_0(r,\gamma)&\le T_0(r,f/f')+T_0(r,\pi_1)+\log 2\\ &\le T_0(r,f'/f)+T_0(r,\pi_1)+O(1)=O(\log r)+O(r^\varepsilon)=O(r^\varepsilon)\end{aligned}$$

and similarly $m_0\big(r,\gamma(-z)\big)=T_0\big(r,\gamma(-z)\big)=O(r^\varepsilon)$ for all $\varepsilon>0$. (To see that $T_0(r,f'/f)=O(\log r)$, we apply [2, Lemma D, p. 322], noting that $ff''$ has only real zeros and $f$ has only finitely many non-real poles.) Thus the order of $\gamma$ is at most $1$. Let $\pi_2$ be the canonical product of the zeros of $\gamma$. Then

$$\gamma(z)=\pi_2(z)e^{-(pz+q)} \tag{2.6}$$

for some complex constants $p$ and $q$. Substituting (2.6) into (2.5) we get

$$\frac{f'}{f}(z)=\frac{\pi_1}{\pi_2}(z)e^{pz+q}. \tag{2.7}$$

All the zeros of $f'/f$, hence all of $\pi_1$, are real. The zeros of $\pi_2$ are the poles of $f'/f$, that is, the zeros and poles of $f$, which are all real, apart from $\pm i$, which are simple zeros of $\pi_2$. Hence in any case $\pi_2$ is real if $m\ge 1$. Thus

$$\frac{\pi_1}{\pi_2}(z)=\overline{\frac{\pi_1}{\pi_2}(\overline{z})} \qquad \text{if } m\ge 1. \tag{2.8}$$

Solving for $\pi_1/\pi_2$ using (2.4), (2.7), and (2.8) and substituting into (2.7) we get

$$\frac{f'}{f}(z)=\frac{ih_1(z)e^{pz+q}}{e^{pz+q}-e^{\overline{p}z+\overline{q}}}=\frac{h_1(z)e^{i(az+b)}}{2\sin(az+b)} \qquad \text{if } m\ge 1 \tag{2.9}$$

where $a=\operatorname{Im}p$, $b=\operatorname{Im}q$. Note that $\sin(az+b)\not\equiv 0$ since $h_1\not\equiv 0$ (since $h_1(\pm i)=\infty$). So (2.9) makes sense.

Replacing $f$ and $f'$ by $f'$ and $f''$ above, we get a similar conclusion provided that $T_0(r,f''/f')=O(\log r)$. The conclusion is that

$$\frac{f''}{f'}(z)=\frac{g_1(z)e^{i(Az+B)}}{2\sin(Az+B)} \qquad \text{if } m\ge 1 \tag{2.10}$$

where

$$\frac{G'}{G}(z)=ig_1(z)=i\left(g_0(z)+\frac{2(m-n)}{z^2+1}\right).$$

To prove that indeed $T_0(r,f''/f')=O(\log r)$, take the logarithmic derivative of (2.9) to get

$$\frac{f''}{f'}(z)=\frac{f'}{f}(z)+\frac{h_1'}{h_1}(z)+ia-a\cot(az+b). \tag{2.11}$$

Since $T_0(r,f'/f)=O(\log r)$, we obtain $T_0(r,f''/f')=O(\log r)$ provided that $T_0(r,h_1'/h_1)=O(\log r)$ and $T_0(r,\alpha'/\alpha)=O(\log r)$ where $\alpha(z)=\sin(az+b)$. By [2, Lemma A] we have

$$m_0(r,\alpha'/\alpha)=O\bigl(\log r+\log T_0(r,\alpha)\bigr)$$

for all $r$ outside a set of finite linear measure, while $T_0(r,\alpha)=O(r^{\varepsilon})$ for all $\varepsilon>0$ (in the same way as for $T_0(r,\gamma)$ above) since $\alpha$ has order $1$. Thus $m_0(r,\alpha'/\alpha)=O(\log r)$. Now $T_0(r,\alpha'/\alpha)=m_0(r,\alpha'/\alpha)=O(\log r)$ since $\alpha$ has only real zeros. (If $a=0$, we need not consider $\alpha$ at all.) The same applies to $h_1$ if we can show that $h_1$ has only finitely many non-real zeros and poles (recall that the order of $h_1$ is at most $1$). From (2.9) we see that if $h_1(z)=0$ then $\sin(az+b)=0$ or $(f'/f)(z)=0$. All such zeros are real, so all zeros of $h_1$ are real and hence $T_0(r,h_1'/h_1)=O(\log r)$ as required. This completes the proof of (2.10). Note that the function $g_1$ in (2.10) has only real zeros, has poles at $\pm i$ only, both of them simple poles, and that $g_1$ is real and of order at most $1$.

Now by (2.10) and (2.11) we get

$$\begin{aligned}\frac{h_1'}{h_1}(z)+\Bigl(\frac{h_1(z)}{2}-a\Bigr)\cot(az+b)+i\Bigl(\frac{h_1(z)}{2}+a\Bigr)\\ =\frac{g_1(z)}{2}\cot(Az+B)+i\frac{g_1(z)}{2}.\end{aligned} \tag{2.12}$$

For $z\in\mathbf{R}$, equate the real and imaginary parts in (2.12) to get

$$h_1(z)+2a=g_1(z), \tag{2.13}$$

$$\frac{h_1'}{h_1}(z)+\Bigl(\frac{h_1(z)}{2}-a\Bigr)\cot(az+b)=\frac{g_1(z)}{2}\cot(Az+B), \tag{2.14}$$

which remain valid for all $z\in\mathbf{C}$ by analytic continuation. Using (2.13) in (2.14) to eliminate $g_1$ we obtain

$$\frac{h_1'}{h_1}(z)+\frac{h_1(z)}{2}\bigl(\cot(az+b)-\cot(Az+B)\bigr)=a\bigl(\cot(az+b)+\cot(Az+B)\bigr). \tag{2.15}$$

Now $f'/f$ has residue $-m$ at $-i$ and $-n$ at $i$. For each of $g_1$ and $h_1$ these residues are $i(m-n)$ and $-i(m-n)$. For $f''/f'$ they are $-(m+1)$ and $-(n+1)$. So by (2.9),

$$-m = \frac{i(m-n)e^{i(-ai+b)}}{2\sin(-ai+b)}, \tag{2.16}$$

$$-n = \frac{-i(m-n)e^{i(ai+b)}}{2\sin(ai+b)}, \tag{2.17}$$

and by (2.10),

$$-(m+1) = \frac{i(m-n)e^{i(-Ai+B)}}{2\sin(-Ai+B)}, \tag{2.18}$$

$$-(n+1) = \frac{-i(m-n)e^{i(Ai+B)}}{2\sin(Ai+B)}. \tag{2.19}$$

Dividing (2.16) by (2.17) and (2.18) by (2.19) we obtain

$$\frac{m}{n} = -\frac{\sin(ai+b)}{\sin(-ai+b)}e^{2a}, \qquad \frac{m+1}{n+1} = -\frac{\sin(Ai+B)}{\sin(-Ai+B)}e^{2A} \tag{2.20}$$

so that by taking absolute values we get

$$2a = \log\frac{m}{n}, \qquad 2A = \log\frac{m+1}{n+1}. \tag{2.21}$$

We have $a \neq 0 \neq A$ since $m \neq n$. Furthermore, it follows that

$$\sin(ai+b) = -\sin(-ai+b) = \sin(ai-b),$$

hence $\cos ai \sin b = 0$, so $\sin b = 0$ and hence $b = \nu\pi$ for some integer $\nu$.

If $\nu$ is even then $\sin(az+b) = \sin az$ and $e^{i(az+b)} = e^{iaz}$ so that we may take $\nu = 0$. If $\nu$ is odd then

$$\frac{e^{i(az+b)}}{\sin(az+b)} = \frac{-e^{iaz}}{-\sin az} = \frac{e^{iaz}}{\sin az}$$

so that we may always take $b = 0$ and similarly $B = 0$. Now (2.15) reads

$$\frac{h_1'}{h_1}(z) + \frac{h_1(z)}{2}\big(\cot az - \cot Az\big) = a\big(\cot az + \cot Az\big). \tag{2.22}$$

Dividing (2.22) by $-h_1$ we get

$$\left(\frac{1}{h_1}\right)' + a(\cot az + \cot Az)\frac{1}{h_1} = \frac{1}{2}(\cot az - \cot Az).$$

Thus for $0 < \varepsilon \le Az < \pi$ we have

$$\frac{1}{h_1} = e^{-(\log\sin az + (a/A)\log\sin Az)} \int^z \bigl(\sin at \sin^{a/A}(At)\bigr)\frac{1}{2}\bigl(\cot at - \cot At\bigr)\,dt$$

with a suitable implied constant of integration, so that if

$$h_2(z) \equiv \frac{\sin az \sin^{a/A} Az}{h_1(z)} \qquad \text{for } 0 < \varepsilon \le Az < \pi$$

then

$$h_2(z) - h_2(\varepsilon/A) = \frac{1}{2}\int_{\varepsilon/A}^{z} \sin^{(a/A)-1} At \, \sin(A-a)t\,dt.$$

Since $m \ne n$, we have

$$\beta \equiv \frac{a}{A} = \frac{\log(m/n)}{\log\bigl((m+1)/(n+1)\bigr)} > 0$$

so that the integral on the right has a finite limit as $\varepsilon \to 0+$ or $Az \to \pi-$ and so $h_2$ extends continuously to $[0, A\pi]$ or $[A\pi, 0]$. Write $h_3(z) = h_2(z/A)$. Then for $0 \le z \le \pi$,

$$h_3(z) - h_3(\varepsilon) = \frac{1}{2A}\int_{\varepsilon}^{z} \sin^{\beta-1} t \sin(1-\beta)t\,dt$$

while $h_3(z) = \bigl(\sin\beta z \, \sin^{\beta} z\bigr)/h_1(z/A)$.

If $\beta$ is not an integer, then, since $h_1$ is entire and $h_3$ is continuous on $[0,\pi]$, we see that $h_3(0) = h_3(\pi) = 0$. Hence

$$0 = \int_0^{\pi} \sin(\beta-1)t \, \sin^{\beta-1} t\,dt.$$

Proceeding in the same way with $\tilde h_1(z) = h_1(z/A)$ we get

$$\frac{1}{2A}(\cot\beta z - \cot z) = \frac{1}{\tilde h_1(z)}\beta(\cot\beta z + \cot z) + \left(\frac{1}{\tilde h_1(z)}\right)'.$$

Set $\zeta = z - \pi$, where $\pi < z < 2\pi$, and write $\hat h_1(\zeta) = \tilde h_1(\zeta + \pi)$. Then

$$\left(\frac{1}{\hat h_1'}\right)' + \frac{1}{\hat h_1}\beta\bigl[\cot(\beta\zeta + \beta\pi) + \cot\zeta\bigr] = \frac{1}{2A}\bigl[\cot(\beta\zeta + \beta\pi) - \cot\zeta\bigr].$$

Now $0 < \zeta < \pi$ and with

$$\tilde h_2(\zeta) = \frac{\sin(\beta\zeta + \beta\pi)\sin^{\beta}(\zeta)}{\hat h_1(\zeta)}$$

we obtain

$$\tilde{h}_2(\zeta) - \tilde{h}_2(\varepsilon) = \frac{1}{2A} \int_\varepsilon^\zeta \sin^{\beta-1} \xi \sin\big((\beta-1)\xi + \beta\pi\big)\, d\xi \qquad \text{for } 0 < \varepsilon < \zeta < \pi.$$

As above we find that if $\beta$ is not an integer then

$$0 = \int_0^\pi \sin^{\beta-1} t \, \sin\big((\beta-1)t + \beta\pi\big)\, dt.$$

As in [2, pp. 326–327] we now see that $\beta$ must be a positive integer, and then, as in [2, Lemma 5] we see that $\beta \in \{1,2,3\}$. Each of the choices $\beta = 1,2,3$ leads, as in [2, pp. 328–330], to an explicitly given function $h_1$. But none of these functions $h_1$ has any non-real poles, which gives a contradiction since our $h_1$ must have a pole at $\pm i$.

We conclude that if $f$ is of the form (1.1) with $m \geq 1$, $n \geq 1$, and $m \neq n$ then at least one of $f$, $f'$, and $f''$ has at least one non-real zero. This completes the proof of Theorem 1 when $m \geq 1$ and $m \neq n$.

Suppose now that $m = n \geq 1$ and that $f$ (or, equivalently, $g$) is strictly non-real. Then we may proceed as above, starting at the beginning of §2.1 and continuing up to and including (2.9), apart from the possibility that $h_1 \equiv 0$. Note that (2.11) is still valid. But now $h_1 = h'$, and if $h_1 \equiv 0$ then by (2.3), we have $H'/H \equiv 0$ so that $H$ is constant. By (2.1) this means that $f$ is a constant multiple of a real function, which is a contradiction. Hence $h_1 \equiv h' \not\equiv 0$, and so $\sin(az+b) \not\equiv 0$ in (2.9).

If now $f'$ is a constant multiple of a real function then $f''/f'$ is real and by (2.11), the function $f'/f + ia$ is real. By (2.3), $H'/H = -2ia$ so that $H = e^{-2iaz} = e^{ih(z)}$ and again by (2.9), $f'/f = \frac{1}{2}(-2a)\big(i + \cot(az+b)\big)$ so that $f(z) = e^{-aiz+C}/\sin(az+b)$, which is a contradiction both if $a = 0$ or if $a \neq 0$, since then $f$ has no poles at $\pm i$. It follows that $f'$ is strictly non-real and we see that the previous argument is valid up to (2.15). Now (2.16)–(2.21) are still valid so that we get $a = A = 0$. Then if $\sin b \neq 0$, by (2.20) we obtain

$$1 = \frac{m}{n} = -\frac{\sin b}{\sin b} = -1,$$

which is a contradiction. Thus $a = \sin b = 0$ so that $\sin(az+b) \equiv 0$, which is also a contradiction. We conclude that if $m = n \geq 1$ then $f$ is a constant multiple of a real function, as asserted. This completes the proof of Theorem 1 when $m \geq 1$. We remark that it has been proved in [3, Theorems 4, 5] that when $m = n \geq 1$ (and $f$ and $g$ are real), the functions $g$ and $f$ are of finite order.

## 3. Preliminaries for the proof of Theorem 1 when $m = 0$

**3.1.** Suppose that $f$ is as in Theorem 1 with $m = 0$ and $n \geq 1$, and with $g(i) \neq 0$. Then $f$ is strictly non-real. Again we denote various non-zero complex constants by $A_1, A_2, \ldots$. Close to $z = i$ we have

$$f(z) \sim \frac{A_1}{(z-i)^n}, \qquad f'(z) \sim \frac{A_2}{(z-i)^{n+1}}, \qquad \overline{f(\overline{z})} \sim A_3, \qquad \overline{f'(\overline{z})} \sim A_4.$$

We define $H$ and $G$ as in (2.1) and (2.2). Hence

$$H(z) \sim A_5(z-i)^{-n} \qquad \text{and} \qquad G(z) \sim A_7(z-i)^{-(n+1)} \qquad \text{as } z \to -i,$$

while

$$H(z) \sim A_6(z+i)^{n} \qquad \text{and} \qquad G(z) \sim A_8(z+i)^{n+1} \qquad \text{as } z \to -i.$$

Thus there are real entire functions $h$ and $g_0$ such that

$$H(z) = \left(\frac{z+i}{z-i}\right)^n e^{ih(z)}, \qquad G(z) = \left(\frac{z+i}{z-i}\right)^{n+1} e^{ig_0(z)}. \tag{3.1}$$

From (2.1) and (3.1) we get

$$\frac{f'}{f}(z) - \overline{\frac{f'}{f}(\overline{z})} = \frac{H'}{H}(z) = ih'(z) - \frac{2in}{z^2+1} \equiv ih_1(z). \tag{3.2}$$

Again $h_1$ is real, and $h'$ and $h_1$ have the same order, which does not exceed 1. Define the functions $\pi_1$, $\gamma$ and $\pi_2$ as before. Thus (2.5) holds, the functions $\pi_1$ and $\gamma$ have order at most 1, and (2.6) and (2.7) hold.

Now the zeros of $\pi_2$ are real apart from a simple zero at $z = i$. Thus we may choose

$$\pi_2(z) = (z-i)\pi_3(z)$$

where $\pi_3$ is real. Hence

$$\frac{\pi_1}{\pi_2}(z) = \overline{\frac{\pi_1}{\pi_2}(\overline{z})}\frac{z+i}{z-i}. \tag{3.3}$$

Substituting (2.7) and (3.3) into (2.4) (which still holds) we get

$$\frac{\pi_1}{\pi_2}(z)e^{pz+q} = \frac{\pi_1}{\pi_2}(z)e^{\overline{p}z+\overline{q}}\frac{z-i}{z+i} + ih_1(z).$$

Solving for $\pi_1/\pi_2$ and substituting into (2.7) we obtain, with $a = \operatorname{Im} p$ and $b = \operatorname{Im} q$ that

$$\begin{aligned} \frac{f'}{f}(z) &= \frac{ih_1(z)e^{i(az+b)}}{e^{i(az+b)} - e^{-i(az+b)}(z-i)/(z+i)} \\ &= \frac{ih_1(z)(z+i)e^{i(az+b)}}{(z+i)e^{i(az+b)} - (z-i)e^{-i(az+b)}} \\ &= \frac{h_1(z)(z+i)e^{i(az+b)}}{2\big(\cos(az+b) + z\sin(az+b)\big)}. \end{aligned} \tag{3.4}$$

The counterpart of (2.11) is

$$\begin{aligned} \frac{f''}{f'}(z) = \frac{f'}{f}(z) + \frac{h_1'}{h_1}(z) + \frac{1}{z+i} + ia \\ - \frac{\sin(az+b) + a\big(z\cos(az+b) - \sin(az+b)\big)}{\cos(az+b) + z\sin(az+b)}. \end{aligned}$$

If $a = 0$ then clearly $T_0(r, f''/f') = O(\log r)$. Suppose that $a \neq 0$. We can deduce that $T_0(r, f''/f') = O(\log r)$ provided that $\cos(az+b) + z\sin(az+b)$ has only finitely many non-real zeros. This last function vanishes at a point $z$ if and only if $\tan(az+b) = -1/z$, that is, if $\tan w = 1/(cw+d)$ for suitable real $c, d$ with $c \neq 0$, where $w = az+b$. It is easily seen that this last equation has only finitely many (in fact, at most two) non-real solutions (using the method of solution to [7, Problem V.172, pp. 65, 244]). Thus $T_0(r, f''/f') = O(\log r)$ as required.

It now also follows from (3.4) that every zero of $h_1$ is real, apart from the at most two non-real zeros of $\cos(az+b) + z\sin(az+b)$, at which $h_1$ must vanish. These non-real zeros are distinct from the points $\pm i$ since

$$\cos(az+b) + z\sin(az+b) = e^{i(ai\pm b)} \neq 0 \qquad \text{at} \qquad z = \pm i.$$

The same analysis for $f'$ and $f''$ instead of $f$ and $f'$ gives

$$\frac{f''}{f'}(z) = \frac{g_1(z)(z+i)e^{i(Az+B)}}{2\big(\cos(Az+B) + z\sin(Az+B)\big)}. \tag{3.5}$$

Let us write from now on

$$\begin{aligned} c = \cos(az+b), \qquad & s = \sin(az+b), \\ C = \cos(Az+B), \qquad \text{and} \qquad & S = \sin(Az+B). \end{aligned} \tag{3.6}$$

The counterpart of (2.12) is

$$\frac{g_1(z+i)(C+iS)}{2(C+zS)} = \frac{h_1(z+i)(c+is)}{2(c+zs)} + \frac{h_1'}{h_1} + \frac{1}{z+i} + ia - \frac{s + a(zc-s)}{c+zs}. \tag{3.7}$$

Considering the real and imaginary parts of (3.7) for $z \in \mathbf{R}$ and noting that $h_1$ and $g_1$ are real, we get

$$\frac{g_1}{2} = \frac{h_1}{2} + a - \frac{1}{z^2+1} \tag{3.8}$$

from the imaginary parts and then, after a calculation,

$$\begin{aligned} \frac{g_1}{2}\frac{zC-S}{zS+C} &= \frac{h_1}{2}\frac{zc-s}{zs+c} + \frac{h_1'}{h_1} - \frac{zc-s}{zs+c}\left(a - \frac{1}{z^2+1}\right) \\ &= \frac{h_1}{2}\frac{zc-s}{zs+c} + \frac{h_1'}{h_1} + \frac{z}{z^2+1} - \frac{(1-a)s+azc}{c+zs} \end{aligned} \tag{3.9}$$

from the real parts. Substituting (3.8) into (3.9) we obtain

$$\left\{\frac{h_1}{2} + a - \frac{1}{z^2+1}\right\}\frac{zC-S}{zS+C} = \frac{h_1'}{h_1} + \frac{z}{z^2+1} + \frac{h_1}{2}\frac{zc-s}{c+zs} - \frac{(1-a)s+azc}{c+zs}. \tag{3.10}$$

Note that by (3.8), the formulas (3.4) and (3.5) read

$$\frac{f'}{f} = \frac{h_1}{2}\frac{(z+i)(c+is)}{c+zs}, \tag{3.11}$$

$$\frac{f''}{f'} = \left(\frac{h_1}{2} + a - \frac{1}{z^2+1}\right)\frac{(z+i)(C+iS)}{C+zS}. \tag{3.12}$$

We have now developed the formalism along the lines of Hellerstein, Shen and Williamson [2]. The rest of the proof can be characterized as a struggle to find a contradiction, showing that no function $f$ satisfying the assumptions of Theorem 1 can arise after all, except in three very special cases.

We define

$$h_2 = \frac{f}{f'}\frac{e^{i(az+b)}(C+zS)}{z-i} = \frac{2}{h_1}\frac{(c+zs)(C+zS)}{z^2+1} \tag{3.13}$$

and

$$h_3 = \frac{h_2}{C+zS} = \frac{f}{f'}\frac{e^{i(az+b)}}{z-i} = \frac{2}{h_1}\frac{(c+zs)}{z^2+1}. \tag{3.14}$$

We write

$$S_0 = Sc - Cs = \sin\bigl((A-a)z + (B-b)\bigr), \qquad C_0 = Cc + Ss = \cos\bigl((A-a)z + (B-b)\bigr).$$

We note that

$$h_2' = S_0 + (A-a)\frac{zC-S}{C+zS}h_2 \tag{3.15}$$

and

$$\frac{h_2'}{h_2} = \frac{S_0}{h_2} + (A-a)\frac{zC-S}{C+zS}. \tag{3.16}$$

Consequently, since $h_3 = h_2/(C+zS)$, we have

$$h_3' = \frac{S_0}{C+zS} - \frac{(1-a)S+azC}{C+zS}h_3,$$

that is,

$$(C+zS)h_3' = S_0 - \big((1-a)S+azC\big)h_3. \tag{3.17}$$

We prove (3.15). Dividing by $-h_1/2$ we obtain from (3.10) that

$$\begin{aligned}\left(\frac{2}{h_1}\right)' + \frac{2}{h_1}\left\{\left(a - \frac{1}{z^2+1}\right)\frac{zC-S}{zS+C} - \frac{z}{z^2+1} + \frac{(1-a)s+azc}{c+zs}\right\}\\ = \frac{-zC+S}{C+zS} + \frac{zc-s}{c+zs}.\end{aligned} \tag{3.18}$$

Since

$$\frac{-zC+S}{C+zS} + \frac{zc-s}{c+zs} = \frac{(z^2+1)S_0}{(C+zS)(c+zs)}$$

and

$$\frac{1}{z^2+1}\frac{zC-S}{C+zS} = \frac{z}{z^2+1} - \frac{S}{C+zS},$$

we can write (3.18) as

$$\begin{aligned}\left(\frac{2}{h_1}\right)' + \frac{2}{h_1}\left\{\left(\frac{(C+zS)(c+zs)}{z^2+1}\right)' \Big/ \left(\frac{(C+zS)(c+zs)}{z^2+1}\right) + (a-A)\frac{zC-S}{C+zS}\right\}\\ = \frac{(z^2+1)S_0}{(C+zS)(c+zs)}.\end{aligned}$$

A calculation based on the definition (3.13) of $h_2$ in terms of $h_1$ now yields (3.15). Note that in all cases, neither one of the meromorphic functions $h_2$ and $h_3$ can vanish identically. Note further that each of $h_2$ and $h_3$ is real and is of order at most $1$. Since the pole of $f$ at $z=i$ is cancelled out in $h_2$ and $h_3$, and since $f$ and $f'$ have only real zeros and no poles other than $z=i$, it follows that $h_3$ has only real zeros and poles, and $h_2$ has only real poles and only real zeros with the possible exception of the at most two non-real zeros of $C+zS$. (We saw earlier that $c+zs$ has at most two non-real zeros, and the same applies to $C+zS$.)

We consider a number of cases:

*Case* I: $A=a$ (so that $S_0$ is constant) and $S_0 \neq 0$;
*Case* II: $A=a$ and $S_0 = 0$;
*Case* III: $a \neq A = 0$;
*Case* IV: $A \neq a = 0$;
*Case* V: $A \neq a$ and $aA \neq 0$.

We shall need later on the fact that a function that differs from a function in $U_0$ by a polynomial factor, behaves in certain ways like a function in $U_0$ when $|x|$ is large enough. We formulate this result as a lemma, which will be proved in the last Section 11 of this paper.

**Lemma 1.** *Let $P$ be a non-constant real rational function, and suppose that the real entire function $\Phi$ belongs to $U_0$ and has no common zeros with $P$. Define $\Psi=\Phi/P$. Then there is a positive number $R$ such that on each of the intervals $(-\infty,-R)$ and $(R,\infty)$, the zeros of $\Psi$ and $\Psi'$ are interlaced. That is, if $x_1$ and $x_2$ are consecutive zeros of $\Psi$ with $R<x_1<x_2$ or with $x_1<x_2<-R$, then $\Psi'$ has exactly one zero on $(x_1,x_2)$, and this point is a simple zero of $\Psi'$. Further, if $x_1$ and $x_2$ are consecutive zeros of $\Psi'$ with $R<x_1<x_2$ or with $x_1<x_2<-R$, then $\Psi$ has exactly one zero on $(x_1,x_2)$, and this point is a simple zero of $\Psi$, unless it is the case that exactly one of $x_1$ and $x_2$ is also a zero of $\Psi$, in which case $\Psi$ has no zeros on $(x_1,x_2)$.*

## 4. Case I

Here $A=a$ and the constant $S_0=\sin(B-b)\neq 0$. Now (3.15) reads $h_2'=S_0$ so that $h_2(z)=S_0z+\alpha$ for some real $\alpha$. By (3.13),

$$\frac{f'}{f}=\frac{e^{i(az+b)}(C+zS)}{(z-i)(S_0z+\alpha)},\qquad \frac{h_1}{2}=\frac{(C+zS)(c+zs)}{(z^2+1)(S_0z+\alpha)}. \tag{4.1}$$

Considering residues of both sides of (4.1) at $z=i$, we obtain

$$-n=\frac{e^{-2A}e^{i(b+B)}}{S_0i+\alpha}=\frac{e^{-2A}e^{i(b+B)}(-S_0i+\alpha)}{S_0^2+\alpha^2},$$

which gives after a calculation that $\alpha=S_0\cot(b+B)$, so that

$$h_2=\big(z+\cot(b+B)\big)S_0,\qquad\text{and}\qquad -nS_0=e^{-2A}\sin(b+B)\neq 0.$$

Thus

$$\frac{f'}{f}=\frac{e^{i(az+b)}(C+zS)}{(z-i)\big(z+\cot(b+B)\big)S_0}.$$

Since $f'/f$ has only real zeros, also $C+zS$ has only real zeros. If $A=0$ then $C+zS$ is a linear real polynomial, which has only real zeros. If $A\neq 0$ then it follows that

$$\tan\big(Az+B-\tfrac12\pi\big)-z=\tan\big(Az+B-\tfrac12\pi\big)\big(Az+B-\tfrac12\pi\big)/A+\big(B-\tfrac12\pi\big)/A$$

has only real zeros. By [5], this is possible only if $1/A \geq 1$, that is, $0 < A \leq 1$ (further, $B$ must satisfy a more complicated condition given in [5], but we shall not make use of that).

Suppose first that $a = A = 0 \neq S_0$. Then $C = \cos B$ and $S = \sin B$ are constants and

$$\frac{f'}{f} = \frac{e^{ib}(C + zS)}{(z-i)\big(z + \cot(b+B)\big)S_0} = \frac{-n}{z-i} + \frac{\beta}{z + \cot(b+B)}$$

for some non-negative integer $\beta$ while $-nS_0 = \sin(b+B) \neq 0$. A calculation shows that

$$\beta = -\frac{e^{-iB}\sin b}{\sin(B-b)}.$$

Since $\beta$ is real, we have $\sin B = 0$ so that $B = k\pi$ and $C = \cos B = (-1)^k$ for some integer $k$. This gives $\beta = 1$ and $n = 1$. Hence $f$ is a non-zero complex constant multiple of $\big(z + \cot(b+B)\big)/(z-i)$ and is of the form (ii) in Theorem 1. The remaining statements in Theorem 1 concerning part (ii) are clear.

Suppose then that $0 < A \leq 1$. By (3.13), we have

$$\frac{h_1}{2} = \frac{(C+zS)(c+zs)}{(z^2+1)\big(z+\cot(b+B)\big)S_0}.$$

The equation (3.5) still holds. Since $f''$ and $C + zS$ have only real zeros, it follows from (3.5) and (3.8) that $g_1$ and hence

$$\begin{aligned} X(z) &\equiv (z^2+1)\big(z+\cot(b+B)\big)S_0 g_1/2 \\ &= (z^2+1)\big(z+\cot(b+B)\big)S_0 h_1/2 + \big(a(z^2+1)-1\big)\big(z+\cot(b+B)\big)S_0 \\ &= (C+zS)(c+zs) + a(z^2+1)\big(z+\cot(b+B)\big)S_0 - \big(z+\cot(b+B)\big)S_0 \end{aligned}$$

have only real zeros. Hence $X \in U_0$. We have $a \neq 0$ since $a = A \in (0,1]$. Since $(C+zS)(c+zs) = O(z^2)$ as $z \to \pm\infty$ along the real axis, it follows that $X(z) \neq 0$ for all real $z$ such that $|z|$ is sufficiently large. Hence $X$ has only finitely many zeros so that $X(z) = P(z)e^{\gamma z}$, where $\gamma \in \mathbf{R}$ and $P$ is a polynomial with only real zeros. Differentiating four times we find that $\big((C+zS)(c+zs)\big)^{(4)} = P_1(z)e^{\gamma z}$, where $P_1$ is a polynomial with only real zeros. But since $(C+zS)(c+zs)$ is a real function with infinitely many real zeros, it follows from Rolle's theorem that $\big((C+zS)(c+zs)\big)^{(4)}$ has infinitely many zeros, which gives a contradiction. Thus no function $f$ satisfying the assumptions of Theorem 1 arises in this case. This completes our treatment of Case I.

## 5. Case II

Here $A = a$ and the constant $S_0 = \sin(B-b) = 0$. Then $B = b + k\pi$ for some integer $k$, and $C = (-1)^k c$, $S = (-1)^k s$ (recall the notation (3.6)). Now (3.15) reads $h_2' = 0$ so that $h_2(z) = \alpha$ for some non-zero real $\alpha$. By (3.13),

$$\frac{f'}{f} = \frac{(-1)^k e^{i(Az+B)}(C+zS)}{(z-i)\alpha} = \frac{(-1)^k e^{i(az+b)}(c+zs)}{(z-i)\alpha},$$

so that $C+zS$ and $c+zs$ have only real zeros. Considering residues at $z=i$ we get $-n = (-1)^k e^{-2A}e^{2iB}/\alpha$ so that $\alpha = (-1)^{k+1}e^{-2A}e^{2iB}/n$ and

$$\frac{f'}{f} = \frac{-ne^{2A}e^{-2iB}(c+is)(c+zs)}{z-i}. \tag{5.1}$$

Hence by (3.11),

$$\frac{h_1}{2} = \frac{f'}{f}\frac{c+zs}{(c+is)(z+i)} = \frac{-ne^{2A}e^{-2iB}(c+zs)^2}{z^2+1}. \tag{5.2}$$

Since $f''$ and $C+zS$ have only real zeros, it follows from (3.5) and (3.8) that $g_1$ and hence

$$\frac{g_1}{2} = \frac{h_1}{2} + a - \frac{1}{z^2+1} = \frac{-ne^{2A}e^{-2iB}(c+zs)^2}{z^2+1} + a - \frac{1}{z^2+1}$$

have only real zeros. Also, since $g_1$ is real, we have $e^{-2iB} = e^{-2ib} = \pm 1$, and the function

$$X(z) = K(c+zs)^2 + a(z^2+1) - 1 \tag{5.3}$$

has only real zeros, where $K = -e^{-2ib}ne^{2a}$. Hence $X \in U_0$. Since $e^{-2ib} = \pm 1$, the number $b$ is an integral multiple of $\pi/2$. Since replacing $b$ and $B$ by $b+l\pi$ and $B+l\pi$, respectively, for some integer $l$ does not change the problem, we may assume that $b \in \{0, \pi/2\}$.

Since $c+zs$ has only real zeros, we have $c+zs \in U_0$, so that $(1-a)s + azc = (c+zs)' \in U_0$. Suppose that $a \neq 1$ and $a \neq 0$. Then

$$\tan(az+b) + (1-a)^{-1}(az+b) - b/(1-a)$$

has only real zeros. By [5], this implies that $-(1-a)^{-1} \geq 1$, that is, $1 < a \leq 2$. Since $c+zs$ has only real zeros, the function $\cos z + (\sin z)(z-b)/a$, which is $c+zs$ evaluated at $(z-b)/a$, has only real zeros, and therefore so does the function $\tan z + a/(z-b)$. If $b=0$, then the equation $\tan z = -a/z$, where $1 < a \leq 2$, has only real solutions. Taking $z = it$, where $t \in \mathbf{R} \setminus \{0\}$, we see that then $\tanh t = a/t$ has no real solutions, which is not true. Thus $b \neq 0$, and so $b = \pi/2$. Hence $e^{2ib} = -1$ and so $K = ne^{2a}$.

Since $K > 0$ and $a > 0$, it follows that $X(x) > 0$ when $x$ is real and $|x|$ is large enough. Hence $X$ has only finitely many zeros so that $X(z) = P(z)e^{\gamma z}$, where $\gamma \in \mathbf{R}$ and $P$ is a polynomial with only real zeros. Differentiating three times we find that $\big((c+zs)^2\big)''' = X''' = P_1(z)e^{\gamma z}$, where $P_1$ is a polynomial with only real zeros. But since $c + zs \in U_0$ and $c + zs$ has infinitely many zeros, it follows that $\big((c+zs)^2\big)'''$ has infinitely many zeros, which gives a contradiction. This rules out the case $a \notin \{0, 1\}$.

Suppose then that $a = 1$. Then the function $\cos z + (z-b)\sin z$, which is $c+zs$ evaluated at $z-b$, has only real zeros, and therefore so does $\tan z + 1/(z-b)$, where $b \in \{0, \pi/2\}$. If $b = 0$, we get a contradiction as above, considering $\tanh t = 1/t$. Hence $b = \pi/2$ so that $e^{2ib} = -1$ and $K = ne^{2a} = ne^2 > 0$. Now we get a final contradiction as in the previous paragraph, considering $X'''$ for $X$ defined by (5.3). This completes our treatment of the case when $A = a \neq 0$ and $S_0 = 0$.

Suppose that $A = a = S_0 = 0$. Then by (5.1),

$$\frac{f'}{f} = \frac{-ne^{-2ib}e^{ib}(c+zs)}{z-i} = \frac{-ne^{-ib}(c+zs)}{z-i} = \frac{-n}{z-i} - nse^{-ib}.$$

Since $h_1/2 = -ne^{-2ib}(c+zs)^2/(z^2+1)$, as given by (5.2), is real, we have $e^{-2ib} = \pm 1$. Again, we may assume that $b \in \{0, \pi/2\}$. If $b = \pi/2$ then $c = 0$ and $-se^{-ib} = i$. Hence

$$\frac{f'}{f} = \frac{-n}{z-i} + in,$$

which gives $f(z) = A_0 e^{inz}/(z-i)^n$ for some non-zero complex constant $A_0$. Thus $f$ is as in (iii) of Theorem 1. Now $f' = A_0 e^{inz} inz/(z-i)^n$ and $f''(z) = -A_0 ne^{inz}(nz^2-1)/(z-i)^{n+2}$ so that $f$, $f'$, and $f''$ have only real zeros. But $f'''(z) = -iA_0 n^3 e^{inz}\big(z^3 - (3/n)z - 2in^{-2}\big)/(z-i)^{n+3}$. Since $z^3 - (3/n)z - 2in^{-2}$ is not a constant multiple of a real polynomial, it follows that $f'''$ has at least one non-real zero. In fact, all the three zeros of $f'''$ are non-real.

If $b = 0$ then $s = 0$ and $f'/f = -n/(z-i)$. Hence $f(z) = A_0/(z-i)^n$ for some non-zero complex constant $A_0$. Thus $f$ is as in (i) of Theorem 1. In this case $f^{(k)}$ has no zeros and hence has only real zeros for all $k \geq 0$. This completes our treatment of Case II.

## 6. Sign analysis: Case IV

We write $\operatorname{sgn} x = x/|x|$ when $x \neq 0$ and $\operatorname{sgn} 0 = 0$. As in [2, p. 325], we note that $(h_1'/h_1)(iy) = o(y)$ as $y \to \infty$. Further, since $(z^2+1)h_1$ has order at most 1 and belongs to $U_0$, we see that $|h_1(iy)| \geq By$ for some fixed positive constant $B$ as $y \to \infty$ provided that $h_1$ has at least three zeros with due count of multiplicity.

Suppose that $A \neq 0$. As $y \to \infty$, we have $(S/C)(iy) = i \operatorname{sgn} A + O(e^{-2|A|y})$. Now we may write (3.10) for $z = iy$ as

$$(6.1) \qquad (i/2)h_1(iy)(\operatorname{sgn} A + ic/s)\big(1 + o(1)\big) = o(y) \qquad \text{if } a = 0 \neq s,$$

$$(6.2) \qquad (i/2)h_1(iy)(\operatorname{sgn} A - y)\big(1 + o(1)\big) = o(y) \qquad \text{if } a = s = 0,$$

$$(6.3) \qquad (i/2)h_1(iy)(\operatorname{sgn} A - \operatorname{sgn} a)\big(1 + o(1)\big) = o(y) \qquad \text{if } a \neq 0.$$

Consider first the case $a \neq 0$ so that (6.3) holds. This gives a contradiction as $y \to \infty$ unless $\operatorname{sgn} A - \operatorname{sgn} a = 0$ or unless $h_1(z) = e^{\alpha z}P(z)/(z^2+1)$ where $\alpha \in \mathbf{R}$ and $P$ is a polynomial of degree at most $2$. Now if $\operatorname{sgn} A - \operatorname{sgn} a = 0$ then $a/A > 0$. Suppose then that $\operatorname{sgn} A - \operatorname{sgn} a \neq 0$, so that $\operatorname{sgn} A = -\operatorname{sgn} a$, and that

$$(6.4) \qquad h_1(z) = e^{\alpha z}\frac{P(z)}{(z^2+1)}.$$

As $y \to \infty$, we have $(s/c)(iy) = i \operatorname{sgn} a + O(e^{-2|a|y}) \to i \operatorname{sgn} a = -i \operatorname{sgn} A \neq 0$. We can write (3.10) as

$$(6.5) \qquad \begin{aligned} &\left\{a - \frac{1}{z^2+1} + \frac{e^{\alpha z}P}{2(z^2+1)}\right\}\frac{(C/S) - (1/z)}{1 + (C/S)(1/z)} \\ &= \alpha + \frac{P'}{P} - \frac{z}{z^2+1} + \frac{e^{\alpha z}P}{2(z^2+1)}\,\frac{(c/s) - (1/z)}{1 + (c/s)(1/z)} - \frac{a(c/s) + \big((1-a)/z\big)}{1 + (c/s)(1/z)}. \end{aligned}$$

If $\alpha \neq 0$, we consider (6.5) for $z = x$ where $\operatorname{sgn}(\alpha x) = 1$ and $|x| \to \infty$ in a sequence such that $1/M \le |c/s| \le M$ and $1/M \le |C/S| \le M$ at all these $x$ for some fixed positive number $M$. Clearly such a sequence can be found. Since the terms involving $e^{\alpha x}$ dominate, we deduce that $(C/S) - (1/x) \sim (c/s) - (1/x)$ and hence $C/S \sim c/s$ as $|x| \to \infty$ in any such sequence. Thus $Cs \sim cS$, that is, $Cs = cS\big(1 + o(1)\big)$. Hence $\sin(az + b - Az - B) = Cs - cS = o(1)$ in any such sequence. If $a \neq A$, we get a contradiction, for then we can choose the sequence of $x = x_k$ so that for some fixed small $\varepsilon > 0$ we have $|ax + b - (q\pi/2)| > \varepsilon$, $|Ax + B - (q\pi/2)| > \varepsilon$ and $|ax + b - (Ax + B) - (q\pi/2)| > \varepsilon$ for all integers $q$ and all $x = x_k$. It follows that $a = A$, which contradicts our assumption that $\operatorname{sgn} A \neq \operatorname{sgn} a$. We conclude that $\alpha = 0$.

Since $(C/S)(iy) = i \operatorname{sgn} a + O(e^{-2|A|y})$ and $(c/s)(iy) = -i \operatorname{sgn} a + O(e^{-2|a|y})$ as $y \to \infty$, a calculation shows that we may now write (3.10) or (6.5) as

$$\begin{aligned} &\left\{a - \frac{1}{z^2+1} + \frac{P}{2(z^2+1)}\right\}\frac{i \operatorname{sgn} a - (1/z)}{1 + (i \operatorname{sgn} a)(1/z)} \\ &= \frac{P'}{P} - \frac{z}{z^2+1} + \frac{P}{2(z^2+1)}\,\frac{(-i \operatorname{sgn} a) - (1/z)}{1 - (i \operatorname{sgn} a)(1/z)} \\ &\quad - \frac{-ai \operatorname{sgn} a + ((1-a)/z)}{1 - (i \operatorname{sgn} a)(1/z)} + O(e^{-Ky}) \end{aligned}$$

and hence, as a longer calculation shows, as

$$\frac{-(z^2+1)P' + 2zP + P^2 i \operatorname{sgn} a}{(z^2+1)P} = O(e^{-Ky}) \tag{6.6}$$

valid for $z = iy$ as $y \to \infty$, with some constant $K > 0$. Now (6.6) can be valid only if $-(z^2+1)P' + 2zP + P^2 i \operatorname{sgn} a \equiv 0$. This can be written as $\big((z^2+1)/P\big)' = -i \operatorname{sgn} a$, which is impossible since $(z^2+1)/P$ is real. This contradiction shows that the assumption $\operatorname{sgn} a \neq \operatorname{sgn} A$ cannot be satisfied. This proves that when $aA \neq 0$, we have $\operatorname{sgn} a = \operatorname{sgn} A$, that is, $a/A > 0$.

Suppose then that $A \neq 0$ and $a = s = 0$, so that (6.2) holds. Now (6.2) implies that $h_1(iy) = o(1)$ as $y \to \infty$. It follows that $h_1$ is of the form (6.4) where $\deg P \leq 1$. Now (3.10) reads

$$\left\{\frac{e^{\alpha z}P(z)}{2(z^2+1)} - \frac{1}{z^2+1}\right\}\frac{zC-S}{zS+C} = \alpha + \frac{P'}{P} - \frac{z}{z^2+1} + \frac{e^{\alpha z} z P(z)}{2(z^2+1)}. \tag{6.7}$$

If $\alpha \neq 0$, we consider (6.7) in a sequence of $x$ for which $\operatorname{sgn}(\alpha x) = 1$ and $S(x)$ stays bounded away from zero, and find that $C/S \sim x$, which is impossible. Thus $\alpha = 0$. Hence $(zC - S)/(C + zS)$ is a rational function, which is impossible. So this case cannot occur.

Suppose then that $A \neq 0$ and $a = 0 \neq s$, so that (6.1) holds. Now (6.1) implies that $h_1(iy) = o(y)$ as $y \to \infty$. It follows that $h_1$ is of the form (6.4) where $\deg P \leq 2$. Now (3.10) reads

$$\left\{\frac{e^{\alpha z}P(z)}{2(z^2+1)} - \frac{1}{z^2+1}\right\}\frac{zC-S}{zS+C} = \alpha + \frac{P'}{P} - \frac{z}{z^2+1} + \frac{e^{\alpha z}P(z)}{2(z^2+1)}\frac{zc-s}{c+zs} - \frac{s}{c+zs}. \tag{6.8}$$

If $\alpha \neq 0$, we consider (6.8) in a sequence of $x$ for which $\operatorname{sgn}(\alpha x) = 1$ and $S(x)$ stays bounded away from zero, and find that $C/S \sim c/s$, which is impossible since $c/s$ is a fixed constant while the sequence of values of $x$ can be chosen so that $C/S$ has no limit. Thus $\alpha = 0$. Hence again $(zC - S)/(C + zS)$ is a rational function, which is impossible. So this case cannot occur.

We conclude that if $A \neq 0$ then $a \neq 0$ and $a/A > 0$. In particular, Case IV, which means that $A \neq 0 = a$, does not occur. The conclusion that if $A \neq 0$ then $a \neq 0$ and $a/A > 0$, will also be used when considering Case V later on.

## 7. Sign analysis: Case III

Consider Case III, so that $a \neq 0 = A$. We have $(s/c)(iy) \to i \operatorname{sgn} a$ as $y \to \infty$. As before, we note that $(h_1'/h_1)(iy) = o(y)$ as $y \to \infty$. Hence (3.10) gives, as $y \to \infty$,

$$\tfrac{1}{2}h_1(iy)\big(iy + i \operatorname{sgn} a + o(1)\big) + iay = o(y) \qquad \text{if } S = 0, \tag{7.1}$$

and

$$\tfrac{1}{2}h_1(iy)\big(C/S+i\operatorname{sgn}a+o(1)\big)=o(y)\qquad \text{if } S\neq 0. \tag{7.2}$$

Suppose that $S=0$ so that (7.1) holds. Then $h_1(iy)=-2a\big(1+o(1)\big)$ as $y\to\infty$. Hence $h_1(z)=P(z)/(z^2+1)$ where $P$ is a real polynomial of degree 2 with leading coefficient $-2a$. Now (3.10) reads $(c+zs)R_1=R_2c+R_3s$ where $R_1$, $R_2$, and $R_3$ are certain real rational functions. This implies that $R_1=R_2$ and $zR_1=R_3$. Now

$$R_1=\left\{\frac{h_1}{2}+a-\frac{1}{z^2+1}\right\}\frac{zC-S}{zS+C}-\frac{h_1'}{h_1}-\frac{z}{z^2+1},$$
$$R_2=\frac{h_1}{2}z-az,\qquad\text{and}\qquad R_3=-\frac{h_1}{2}-(1-a).$$

We have $zR_2=zR_1=R_3$, which yields

$$(z+1)h_1/2=-1+a(1+z^2).$$

Thus

$$P(z)=(z^2+1)h_1(z)=2(z^2+1)\big(-1+a(1+z^2)\big)/(z+1),$$

which is not a polynomial of degree 2 for any non-zero value of $a$. This contradiction shows that the case $S=0$ is impossible.

Suppose that $S\neq 0$ so that (7.2) holds. Then necessarily $h_1(iy)=o(y)$, and thus $h_1(z)/2=e^{\alpha z}P(z)/(z^2+1)$ where $\alpha$ is real and $P$ is a real polynomial of degree at most 2. Now (3.10) reads

$$(e^{\alpha z}R_4+R_5)(c+zs)=(e^{\alpha z}R_6+R_7)c+(e^{\alpha z}R_8+R_9)s, \tag{7.3}$$

where the $R_j$ are certain real rational functions for $4\le j\le 9$.

If $\alpha\neq 0$, we consider both sides of (7.3) for $z=x_k$ for various sequences $x_k$ of real numbers such that $\operatorname{sgn}(\alpha x_k)=1$, $|x_k|\to\infty$ and either $s(x_k)=0$ for all $k$ or $c(x_k)=0$ for all $k$. This shows that $R_4=R_6$, $R_5=R_7$, $zR_4=R_8$, and $zR_5=R_9$. Calculating the expressions for the $R_j$ we deduce that

$$0=zR_6-R_8=\frac{z^2P(z)}{z^2+1}+\frac{P(z)}{z^2+1},$$

hence $P\equiv 0$, which is impossible.

Thus $\alpha=0$, and we may adjust notation so that $R_4=R_6=R_8=0$. Now $R_5=R_7$ and $zR_5=R_9$. Thus

$$0=zR_7-R_9=\frac{z^2P(z)}{z^2+1}-az^2+\frac{P(z)}{z^2+1}+(1-a)=P(z)+1-a(z^2+1).$$

This yields $h_1/2 = a - (z^2+1)^{-1}$. Next,

$$
\begin{aligned}
0 = R_5 - R_7 &= 2\left\{a - \frac{1}{z^2+1}\right\}\frac{zC-S}{zS+C} - \left(\frac{-2az}{1-a(z^2+1)} - \frac{2z}{z^2+1}\right)\\
&\quad - \frac{z}{z^2+1} - \frac{z(a(z^2+1)-1)}{z^2+1} + az\\
&= 2\left\{a - \frac{1}{z^2+1}\right\}\frac{zC-S}{zS+C} + \frac{-2az}{1-a(z^2+1)} + \frac{2z}{z^2+1}.
\end{aligned} \tag{7.4}
$$

Suppose that $a \neq 1$. Since $a \neq 0$, the function $1 - a(z^2+1)$ has two distinct zeros, neither one of which is $\pm i$. At least one of these zeros is different from the zero of $zS + C$. At this zero of $1 - a(z^2+1)$, the right hand side of (7.4) has a pole, which gives a contradiction. Thus $a = 1$, and (7.4) reads

$$2\left\{1 - \frac{1}{z^2+1}\right\}\frac{zC-S}{zS+C} + \frac{2}{z} + \frac{2z}{z^2+1} = 0. \tag{7.5}$$

Hence $C = 0$, for otherwise the origin is a pole of the left hand side of (7.5). Thus

$$-2\left\{1 - \frac{1}{z^2+1}\right\} + 2 + \frac{2z^2}{z^2+1} = 0.$$

This reads $2 = 0$, which is a contradiction. This completes our treatment of the Case III, showing that Case III cannot occur at all.

## 8. Case V

Consider Case V so that $A \neq a$ and $aA \neq 0$. We shall show that this leads to a contradiction so that in this case no function $f$ satisfying the assumptions of Theorem 1 arises. We have already seen in Section 6 that now $a/A > 0$.

Consider the real meromorphic function $h_3$ of order at most 1 defined by (3.14) and satisfying (3.17). By (3.14), $h_3$ has only real zeros and poles since $f/f'$ does apart from the point $z = i$, at which $h_3 \neq 0, \infty$. Also the poles of $h_3$ occur exactly at those zeros of $f'$ that are not zeros of $f$. We first show that $h_3$ can have only finitely many poles, if any. Let $x$ be a pole of $h_3$ so that $f'(x) = 0 \neq f(x)$ and so $(C + zS)(x) = 0$, by (3.12). Let $x$ be a zero of $f'$ of order $\nu \geq 1$, and hence a pole of $h_3$ of order $\nu$. Suppose that $h_2 = (C + zS)h_3 \neq 0, \infty$ at $x$. Considering (3.15), we deduce that $zC - S = 0$. Since $C = -zS$ at $x$ so that $S \neq 0$ since $C^2 + S^2 = 1$, we get $x^2 + 1 = 0$, hence $x = \pm i$. This is impossible since $x$ is real. If $h_2(x) = 0$ then by (3.14), $C + zS$ has a multiple zero at $x$. Since $C + zS = 0 = (C + zS)' = (1 - A)S + AzC$ at $z = x$, this can happen only if $A \neq 0$ and $x^2 = 1 - A^{-1}$, hence at no more than two points $x$. Suppose then that $h_2(x) = \infty$. Considering (3.15) at $x$, we see that $x$ is a simple zero of

$C + zS$ (since $(zC - S)(x) \neq 0$). Thus by (3.14), $x$ is a pole of order $\nu - 1$ of $h_2$ and hence we must have $\nu \geq 2$. Comparing residues on both sides of (3.16) at $x$, we deduce that

$$-(\nu - 1) = (A - a)\frac{-(x^2+1)}{1 - A(x^2+1)},$$

that is,

$$x^2 + 1 = \frac{\nu - 1}{\nu A - a}.$$

If $A \neq 0$, the right hand side is bounded as $\nu$ varies over integers not less than 2 so that $|x|$ is bounded. Thus there can be only finitely many poles $x$ of $h_2$ and hence of $h_3$. If $A = 0$ then $C + zS$ is a polynomial of degree at most 1 so that by (3.12), $f'$ has at most one zero (ignoring multiplicity). Hence in all cases, all zeros of $f'$ apart from finitely many at most, occur at zeros of $f$. Also there is a real polynomial $P$ with only real zeros such that if $h_4 = Ph_3$ then $h_4$ has no poles and $h_4$ has the same zeros as $h_3$. Thus the order of $h_4$ is at most 1 and $h_4 \in U_0$.

## 9. Proof that $a/A$ is an integer

We continue with Case V. Recall that $a/A > 0$. We shall next show that $a/A$ is an integer. Note that $C + zS$ has infinitely many real zeros so that there are infinitely many intervals $I = (x_1, x_2)$ between successive simple real zeros $x_1$, $x_2$ of $C + zS$ such that $C + zS > 0$ on $(x_1, x_2)$. For $x \in I$, define

$$(9.1)\qquad h_5(x) = h_2(x)[(C+zS)(x)]^{(a/A)-1}\exp\left\{-\left(\frac{a}{A} - 1\right)\int_\alpha^x \frac{S(t)\,dt}{(C+zS)(t)}\right\}$$

for some $\alpha \in I$. A calculation shows that by (3.15), we have

$$(9.2)\qquad h_5'(x) = S_0(x)[(C+zS)(x)]^{(a/A)-1}\exp\left\{-\left(\frac{a}{A} - 1\right)\int_\alpha^x \frac{S(t)\,dt}{(C+zS)(t)}\right\}.$$

As $t \to x_1$, we have

$$\frac{S(t)}{(C+zS)(t)} = \frac{1}{1 - A(1+x_1^2)}\,\frac{1}{t - x_1} + O(1).$$

An analogous equation holds with $x_1$ replaced by $x_2$ when $t \to x_2$. When $x_1$ is large enough, we have

$$(9.3)\qquad \frac{a}{A} - \frac{(a/A) - 1}{1 - A(1+x_1^2)} > 0,$$

and similarly for $x_2$. Since $h_3$ has only finitely many poles, so that by (3.14), $h_2$ vanishes (at least as fast as $z-x_1$) at all but finitely many zeros $x_1$ of $C+zS$, it follows from (9.1) and (9.3) that $h_5$ can be extended continuously from the open interval $I$ to $x_1$ and $x_2$ by setting $h_5(x_1)=h_5(x_2)=0$. Also we see that $h_5'$ is integrable over $[x_1,x_2]$ with

$$\int_{x_1}^{x_2} S_0(x)[(C+zS)(x)]^{(a/A)-1}\exp\left\{-\left(\frac{a}{A}-1\right)\int_{\alpha}^{x}\frac{S(t)\,dt}{(C+zS)(t)}\right\}=0. \tag{9.4}$$

Without affecting the validity of (9.4), we may and will choose $\alpha$ to be the unique point in $I$ with $S(\alpha)=1$. The factors in the integrand in (9.4) other than $S_0$ are positive on $(x_1,x_2)$. When $|x_1|$ is large enough then, since $S(x_1)=-C(x_1)/x_1\to 0$, there is an integer $k$ such that $|(Ax_1+B)-k\pi|\le 1/|x_1|$, thus $Ax_1+B-k\pi$ is close to zero. Since $x_2$ is the next larger zero of $C+zS$, we have $|(Ax_2+B)-(k+1)\pi|\le 1/|x_2|$ if $A>0$ (if $A<0$ we have the same with $k+1$ replaced by $k-1$). Let us assume, for simplicity, that $A>0$. In fact, when $|k|$ is large enough, there is a unique zero $x$ of $C+zS$ for which $Ax+B$ is closer to $k\pi$ than to $l\pi$ for any other integer $l$, and each zero of $C+zS$ that has a sufficiently large modulus corresponds to a unique integer $k$ in this way. Let us denote such a zero by $y_k$.

Now the sign of $S_0(t)$ can only change at a point $t=t_1$ such that $(A-a)t_1+(B-b)=l\pi$ for some integer $l$. There must be at least one such point $t_1$ on $(x_1,x_2)$.

We proceed to show that $a/A$ is a positive integer, hence $a/A\ge 2$ since $a\ne A$. Now $x_1$ above is equal to a certain $y_l$. For simplicity, let us assume that $l>0$ and that $x_1>0$ and $x_1$ is large. We can write

$$\begin{aligned}\frac{S(t)}{(C+zS)(t)}-\frac{1}{1-A(1+y_l^2)}\frac{(C+zS)'(t)}{(C+zS)(t)}&=-A\frac{y_l\big(y_lS(t)+C(t)\big)+(t-y_l)C(t)}{\big(1-A(1+y_l^2)\big)(C+zS)(t)}\\&=-A\left\{\frac{y_l}{1-A(1+y_l^2)}+\frac{t-y_l}{(C+zS)(t)}\frac{C(t)-y_lS(t)}{1-A(1+y_l^2)}\right\}\\&\equiv -A\psi(t)\equiv -A\psi_l(t),\end{aligned}$$

say, for $x_1<t<x_2$. We next prove that $|\psi(t)|\le L/|y_l|$ for a positive absolute constant $L$ provided that $|y_l|$ is large enough and $x_1<t\le\alpha$. For the first term $\big(y_l/\big(1-A(1+y_l^2)\big)\big)$ in $\psi(t)$, this is clear. Since $|t-y_l|<2\pi/A$, say, the same holds for the second term in $\psi(t)$ for those $t$ for which $|(C+zS)(t)|\ge 1/3$, say. Suppose then that $|(C+zS)(t)|<1/3$. Then $|tS(t)|<4/3$, hence

$$|y_lS(t)|=|tS(t)-(t-y_l)S(t)|<2\pi/A+4/3,$$

and $|C(t)-y_lS(t)|<2\pi/A+7/3$. By the mean value theorem, there is a point $t'$ between $y_l$ and $t$ such that

$$\begin{aligned}\frac{(C+zS)(t)}{t-y_l}&=\frac{(C+zS)(t)-(C+zS)(y_l)}{t-y_l}=(C+zS)'(t')\\&=\big((1-A)S+AzC\big)(t').\end{aligned}$$

Since $|(C+zS)(t)|<1/3$, we have $|(C+zS)(t')|<1/3$ for any such $t'$ (since $y_l<t\le\alpha$), so that $|t'S(t')|<4/3$, $|S(t')|<4/(3|t'|)$, $|C(t')|>1/2$, say, and so

$$\begin{aligned}\big|\big((1-A)S+AzC\big)(t')\big|&\ge|A|\,|y_l|\,|C(t')|-|1-A|\,|S(t')|\\&>|A|\,|y_l|/2-4|1-A|/(3|t'|)>1,\end{aligned}$$

say. Hence

$$\left|\frac{t-y_l}{(C+zS)(t)}\,\frac{C(t)-y_lS(t)}{1-A(1+y_l^2)}\right|=O(1/y_l^2).$$

This completes the proof of $|\psi(t)|\le L/|y_l|$. Similarly we see that $|\psi_{l+1}(t)|\le L/|y_{l+1}|$ for $\alpha<t<y_{l+1}$.

Now (9.4) implies, after multiplying both sides of (9.4) by $\alpha^{1-(a/A)}$, that

$$\begin{aligned}0=&\int_{y_l}^{\alpha}S_0(x)\left[\frac{(C+zS)(x)}{\alpha}\right]^{((a/A)-1)(1-1/(1-A(1+y_l^2)))}\exp\left\{\left(\frac{a}{A}-1\right)\int_\alpha^x A\psi_l(t)\right\}dx\\&+\int_\alpha^{y_{l+1}}S_0(x)\left[\frac{(C+zS)(x)}{\alpha}\right]^{((a/A)-1)(1-1/(1-A(1+y_{l+1}^2)))}\\&\times\exp\left\{\left(\frac{a}{A}-1\right)\int_\alpha^x A\psi_{l+1}(t)\right\}dx.\end{aligned}$$

Note that by our choice of $\alpha$, we have $(C+zS)(\alpha)=\alpha$.

Suppose that $0<\varepsilon<1$. Clearly there is a large positive integer $l_0$ such that if $l\ge l_0$ then, since the above integral equals zero, we have

$$\left|\int_{y_l}^{y_{l+1}}S_0(x)|S(x)|^{(a/A)-1}\,dx\right|<\varepsilon/|A|. \tag{9.5}$$

Write $Ay_l+B=l\pi+\delta_l$, where $\delta_l\to0$ as $|l|\to\infty$. (Then $A\alpha+B=(l+1/2)\pi$.) Note that $l$ is even (since $C+zS>0$ on $(y_l,y_{l+1})$ and $A>0$) and large but otherwise arbitrary. Changing variables in (9.5) we obtain

$$\left|\int_{\delta_l}^{\pi+\delta_{l+1}}\sin\left(\frac{(A-a)(x+l\pi-B)}{A}+B-b\right)|\sin x|^{(a/A)-1}\,dx\right|<\varepsilon. \tag{9.6}$$

We find that

(9.7)

$$\lim_{\substack{l\to\infty\\ l \text{ even}}} \int_0^\pi \sin\bigl(\bigl(1-(a/A)\bigr)(x-B)+\bigl(1-(a/A)\bigr)l\pi+B-b\bigr)|\sin x|^{(a/A)-1}\,dx$$
$$= \lim_{\substack{l\to\infty\\ l \text{ even}}} \int_0^\pi \bigl\{\sin\bigl(\bigl(1-(a/A)\bigr)(x-B)+B-b\bigr)\cos\bigl((a/A)l\pi\bigr)$$
$$-\cos\bigl(\bigl(1-(a/A)\bigr)(x-B)+B-b\bigr)\sin\bigl((a/A)l\pi\bigr)\bigr\}(\sin x)^{(a/A)-1}\,dx=0.$$

We may deal with the intervals between successive zeros of $C+zS$ on which $C+zS<0$ in the same way. We find that

(9.8)

$$\lim_{\substack{l\to\infty\\ l \text{ odd}}} \int_0^\pi \sin\bigl(\bigl(1-(a/A)\bigr)(x-B)+\bigl(1-(a/A)\bigr)l\pi+B-b\bigr)|\sin x|^{(a/A)-1}\,dx$$
$$= \lim_{\substack{l\to\infty\\ l \text{ odd}}} \int_0^\pi \bigl\{\sin\bigl(\bigl(1-(a/A)\bigr)(x-B)+B-b\bigr)\cos\bigl((a/A)l\pi\bigr)$$
$$-\cos\bigl(\bigl(1-(a/A)\bigr)(x-B)+B-b\bigr)\sin\bigl((a/A)l\pi\bigr)\bigr\}(\sin x)^{(a/A)-1}\,dx=0.$$

If $a/A$ is irrational then there is a subsequence of even integers $l\to\infty$ along which $\cos\bigl((a/A)l\pi\bigr)\to 1$ (so that $\sin\bigl((a/A)l\pi\bigr)\to 0$) and another subsequence along which $\sin\bigl((a/A)l\pi\bigr)\to 1$ (so that $\cos\bigl((a/A)l\pi\bigr)\to 0$). This shows that

$$\int_0^\pi \sin\bigl(\bigl(1-(a/A)\bigr)(x-B)+B-b\bigr)(\sin x)^{(a/A)-1}\,dx$$
$$(9.9)\qquad = \int_0^\pi \cos\bigl(\bigl(1-(a/A)\bigr)(x-B)+B-b\bigr)(\sin x)^{(a/A)-1}\,dx=0.$$

We now use the formula due to Cauchy ([1, pp. 41–89], [6, p. 158]), also quoted by Hellerstein, Shen and Williamson [2, (3.15), p. 326], which states that for $\gamma>0$,

$$(9.10)\quad \int_0^\pi (\sin^{\gamma-1} t)\exp\{i[(\gamma-1)t+\delta\,]\}\,dt=\frac{\pi}{2^{\gamma-1}}\exp\{i[\pi(\gamma-1)/2+\delta\,]\}\neq 0.$$

We apply this with $\gamma=a/A$ and $\delta=-(B-b)-\bigl((a/A)-1\bigr)B$ and get a contradiction since the integral ought to vanish by (9.9) but it does not, by (9.10).

If $a/A$ is rational but not an integer then there are only finitely many distinct pairs $\bigl(\cos\bigl((a/A)l\pi\bigr),\sin\bigl((a/A)l\pi\bigr)\bigr)$. Thus (9.7) and (9.8) imply that

(9.11)

$$\cos\bigl((a/A)l\pi\bigr)\int_0^\pi \sin\bigl(\bigl(1-(a/A)\bigr)(x-B)+B-b\bigr)(\sin x)^{(a/A)-1}\,dx$$
$$-\sin\bigl((a/A)l\pi\bigr)\int_0^\pi \cos\bigl(\bigl(1-(a/A)\bigr)(x-B)+B-b\bigr)(\sin x)^{(a/A)-1}\,dx=0.$$

for all $l$, both even and odd. Taking real and imaginary parts in (9.10) and combining the result with (9.11) we obtain (after dividing by $\pi/2^{(a/A)-1}$)

$$\begin{aligned} &\sin\bigl(\pi\bigl((a/A)-1\bigr)/2-(B-b)-\bigl((a/A)-1\bigr)B+(a/A)l\pi\bigr) \\ (9.12)\qquad &= \cos\bigl((a/A)l\pi\bigr)\sin\bigl(\pi\bigl((a/A)-1\bigr)/2-(B-b)-\bigl((a/A)-1\bigr)B\bigr) \\ &\quad+\sin\bigl((a/A)l\pi\bigr)\cos\bigl(\pi\bigl((a/A)-1\bigr)/2-(B-b)-\bigl((a/A)-1\bigr)B\bigr)=0 \end{aligned}$$

for all $l$. This is possible only if $(a/A)l$ is an integer for all $l$, which is not the case when $a/A$ is not an integer. So it follows that $a/A$ is an integer, as claimed. Write $p=(a/A)-1$. Since $a\neq A$, it follows that $p$ is a positive integer. Furthermore, it now follows from (9.12) that

$$(9.13)\qquad \sin\bigl(\pi p/2-(B-b)-pB\bigr)=0.$$

We have derived these results assuming that $A>0$. If $A<0$ then a similar argument yields the same results. The only difference is that if $x_1$ and $x_2$ are successive simple real zeros of $C+zS$ with $x_1<x_2$, corresponding to integers $k$ and $k+1$ via the fact that $(Ax_j+B)/\pi$ is close to an integer for $j=1,2$, then $k$ corresponds now to $x_2$, not $x_1$. We leave further details to the reader.

It follows from (9.13) that there is an integer $q$ such that $\pi p/2-(B-b)-pB=-q\pi$ and hence

$$(9.14)\qquad (B-b)+pB=\pi p/2+q\pi.$$

Thus

$$\begin{aligned} S_0(z) &= \sin\left((A-a)\frac{(Az+B)-B}{A}+B-b\right) \\ &= \sin\bigl(-p(Az+B)+(B-b)+pB\bigr) \\ &= \sin\bigl(-p(Az+B)\bigr)\cos(\pi p/2+q\pi)+\cos\bigl(-p(Az+B)\bigr)\sin(\pi p/2+q\pi). \end{aligned}$$

We obtain

$$(9.15)\qquad S_0(z)=(-1)^{q+1+(p/2)}\sin\bigl(p(Az+B)\bigr)\qquad \text{if } p \text{ is even}$$

and

$$(9.16)\qquad S_0(z)=(-1)^{q+((p-1)/2)}\cos\bigl(p(Az+B)\bigr)\qquad \text{if } p \text{ is odd.}$$

By (3.15), the function $h_6=(C+zS)^p h_2$, which has only finitely many poles, has at most two distinct non-real zeros (at the non-real zeros of $C+zS$, if any), and is real of order at most $1$, satisfies

$$(9.17)\qquad h_6'=\frac{ph_6S}{C+zS}+S_0(C+zS)^p.$$

## 10. Conclusion of the proof in Case V

We continue our study of Case V. Suppose first that $p$ is even. For simplicity and definiteness, we assume that $A > 0$. An inspection of the arguments that follow shows that a similar reasoning works if $A < 0$. Let $x_1$ and $x_2$ be successive zeros of $C + zS$, with $x_1 < x_2$, with $|x_1|$, and hence $|x_2|$, sufficiently large. Suppose that $\operatorname{sgn}(C + zS) = \sigma \in \{1, -1\}$ on $(x_1, x_2)$. We denote a sufficiently small positive quantity by $\varepsilon$, not necessarily the same at every occurrence. Suppose that $\operatorname{sgn} h_6'(x) = \tau$ on $(x_1, x_1 + \varepsilon)$. Then clearly $\operatorname{sgn} h_6(x) = \tau$. By (9.17), $\operatorname{sgn} S_0(x) = \tau$ on this interval also, since $\big|(h_6 S)/\big(h_6'(C + zS)\big)\big|$ is comparable to $1/x^2$ close to $x = x_1$.

At $x = x_1$, we have $S = -C/x$ so that writing $Ax_1 + B = l\pi + \varepsilon_1$ for some integer $l$ we find that $C(x_1) \approx (-1)^l$ and $Ax + B \approx l\pi - (1/x)$ close to $x = x_1$. If $x_1 > 0$, we then have $l\pi - (1/x) < l\pi$ at $x_1$. Let the zeros of $S_0$ on $(x_1, x_2)$ be $y_1 < y_2 < \cdots < y_p$ (all zeros of $S_0$ are simple). At these points, $p(Ax + B) = \mu\pi$ where the integer $\mu$ increases (since $A > 0$) from $pl$ to $p(l+1) - 1$ for the integer $l$ found above, and $Ax + B = \mu\pi/p$. In this case, $y_1 - x_1$ is very small and tends to zero as $x_1 \to \infty$. If $x_1 < 0$ then $x_2 - y_p > 0$ is small and tends to zero as $x_1 \to -\infty$. We shall go through the case $x_1 > 0$ (and $|x_1|$ large) in detail and then simply state what the corresponding results are for $x_1 < 0$. As one can understand from the above remark, the principal difference is that certain action moves from the $x_1$-end of the interval $(x_1, x_2)$ to the $x_2$-end when $x_1$ is taken to be negative.

The zeros of $h_6$ on $[x_1, x_2]$ coincide with the zeros of the function $h_2$ and hence with the zeros of $h_5$ on this interval. Between consecutive zeros like that, there is a zero of $h_6'$ and a zero of $h_5'$ (these might occur at the same point, of course). By (9.2), these zeros of $h_5'$ occur at the zeros of $S_0$, that is, at the points $y_j$. It follows that any interval $(y_j, y_{j+1})$ for $1 \le j \le p-1$ contains at most one zero of $h_6$. Similarly, the intervals $(x_1, y_1]$ and $[y_p, x_2)$ contain no zeros of $h_6$ since $h_5$ and $h_6$ vanish at $x_1$ and at $x_2$. In particular, $h_6(y_1) \ne 0 \ne h_6(y_p)$. If $h_6(y_j) = 0$ for some $j$ with $2 \le j \le p - 1$, then by (9.17), $h_6'(y_j) = 0$. If $h_6(y_j) = 0$ then inspection of multiplicities in (9.17) shows that $y_j$ is a zero of $h_6$ of order $2$. Further, if $h_6(y_j) = 0$ then $y_j$ is a zero of $h_5$ of order $2$, and if this happens for a certain $j$ then neither $h_5$ nor $h_6$ can have any zeros on $(y_{j-1}, y_j) \cup (y_j, y_{j+1})$. Also we cannot have $h_6(y_j) = h_6(y_{j+1}) = 0$ for any $j$ with $1 \le j \le p - 1$. Considering the various possibilities that can arise, one can see that $h_6$ has at most $p - 1$ zeros on $(x_1, x_2)$, with due count of multiplicity. We shall next show that in fact $h_6$ has $p - 1$ zeros on $(x_1, x_2)$.

Now suppose that $x_1$ is large and positive. Claim: the function $h_6$ has a zero on $(x_1, y_2)$, or otherwise a double zero at $y_2$. If $h_6 \ne 0$ on $(x_1, y_2)$, then $\operatorname{sgn} h_6 = \tau$ on $(x_1, y_2)$. At $x_1$, we have $(C + zS)' = \big(1 - A(1 + x_1^2)\big)S$ whose sign is $\sigma$ since $(C + zS)(x_1) = 0$ and $\operatorname{sgn}(C + zS) = \sigma$ on $(x_1, x_2)$. Thus, since $A > 0$ and $|x_1|$ is large, we have $\operatorname{sgn} S(x_1) = -\sigma$. Further, $C(x_1) = -x_1 S(x_1)$ so that

$\operatorname{sgn} C(x_1) = \sigma$, and $|C(x_1)| \approx 1$ since $|S(x_1)| = |C(x_1)/x_1| \approx 0$. Incidentally, this shows that $\sigma = (-1)^l$. We have $S(y_1) = 0$ since $Ay_1 + B = (pl\pi)/p = l\pi$. Thus $\operatorname{sgn} S(x) = -\sigma$ on $(x_1, y_1)$ and $\operatorname{sgn} S(x) = \sigma$ on $(y_1, x_2 + \varepsilon)$. By (9.17), we have $h_6'(y_1) = 0$ since $S_0(y_1) = S(y_1) = 0$. We may apply Lemma 1 to $\Psi = h_6$. This shows that $y_1$ is a simple zero of $h_6'$ when $|x_1|$ is large enough.

If $h_6(y_2) = 0$ then $h_6'(y_2) = 0$ by (9.17) since also $S_0(y_2) = 0$. Considering multiplicities in (9.17) we see that $y_2$ must be a zero of $h_6$ of order 2, being a simple zero of $S_0$ and not a zero of $C + zS$. In this case $h_6$ cannot have any zeros on $(x_1, y_2)$ since they would be zeros of $h_5$ also and since $y_1$ is the only zero of $h_5'$ on $(x_1, y_2)$.

Suppose that $h_6(y_2) \neq 0$ (and that $h_6$ has no zeros on $(x_1, y_2)$). Since $S_0(y_2) = 0$, we have by (9.17)

$$\frac{h_6'}{h_6}(y_2) = \frac{pS(y_2)}{(C + zS)(y_2)} > 0. \tag{10.1}$$

Thus $\operatorname{sgn} h_6'(y_2) = \tau = \operatorname{sgn} h_6'(x_1 + \varepsilon)$. So $h_6'$ must have at least two zeros on $(x_1, y_2)$, not only $y_1$ but also some $\zeta_1 \neq y_1$. But then by Lemma 1, $h_6$ must have a zero between $\zeta_1$ and $y_1$, and hence on $(x_1, y_2)$, as claimed. Considering $h_5$ as above we see that $h_6$ has exactly one zero on $(x_1, y_2)$, which furthermore is a simple zero of $h_6$ and lies on $(y_1, y_2)$. This proves the Claim.

We prove by induction on $j$ that for $1 \le j \le p - 1$, the interval $(y_j, y_{j+1})$ contains points $\xi_j, \eta_j$ with $h_6'(\xi_j) = h_6(\eta_j) = 0$ and $\xi_j \le \eta_j$. (If $h_6(y_j) = 0$ for some $j$, a simple modification is required, and we omit the details.) We have done this for $j = 1$. In this case only we consider the zero of $h_6'$ at $y_1$ to substitute for a zero on $(y_1, y_2)$. If $y_2$ is a double zero of $h_6$ then if substitutes for a zero of $h_6$ on $(y_1, y_2)$ and on $(y_2, y_3)$ and for a zero of $h_6'$ on $(y_2, y_3)$. In this case $h_6(y_2) = 0$, and therefore, we have proved our claim also for $j = 2$, and we have $\operatorname{sgn} h_6(x) = \tau$ for $x \in (y_2, y_2 + \varepsilon)$.

If $h_6'$ has two distinct zeros, say $z_1 < z_2$, on $(y_j, y_{j+1})$, then by (9.17) and the fact that each of $S$, $S_0$, and $C + zS$ is non-zero and retains its sign on $(y_j, y_{j+1})$, we see that $h_6(z_1)$ and $h_6(z_2)$ are non-zero and of the same sign. Choose $z_1$ and $z_2$ to be consecutive (distinct) zeros of $h_6'$. By Rolle's theorem there is at most one zero of $h_6$ on $(z_1, z_2)$ with due count of multiplicity. Thus we find that $h_6$ has no zeros on $(z_1, z_2)$. This contradicts Lemma 1 applied to $\Psi = h_6$ when $|x_1|$ is large enough. Also by Lemma 1, any zero $z_0$ of $h_6'$ on $(y_j, y_{j+1})$ is a simple zero of $h_6'$ when $h_6(z_0) \neq 0$, and if $h_6(z_0) = h_6'(z_0) = 0 \neq \big(S_0(C + zS)^p\big)(z_0)$ then (9.17) gives a contradiction.

By this and by earlier analysis, $(y_j, y_{j+1})$ contains at most one zero of each of $h_6$ and $h_6'$, and so there is exactly one zero. To get to the induction step, we now assume that our claim has been proved up to a certain $j$ with $1 \le j \le p - 2$ (and we assume that $p \ge 4$ for otherwise there is nothing more to prove). Thus $\operatorname{sgn} h_6(x) = \operatorname{sgn} h_6'(x) = (-1)^j \tau$ for $x \in (y_{j+1}, y_{j+1} + \varepsilon)$. Analogously to (10.1), we have $(h_6'/h_6)(y_m) > 0$ for all $m \geq 2$. Therefore either both $h_6$ and $h_6'$ have a zero on $(y_{j+1}, y_{j+2})$ or neither function has any zeros there. To get a contradiction, suppose that neither $h_6$ nor $h_6'$ vanishes on $(y_{j+1}, y_{j+2})$ so that then $\operatorname{sgn} h_6(x) = \operatorname{sgn} h_6'(x) = (-1)^j \tau$ for $x \in (y_{j+1}, y_{j+2})$. By the induction assumption, there are points $t_1, t_2$ with $h_6'(t_1) = h_6(t_2) = 0$ and with $y_j \leq t_1 \leq t_2 < y_{j+1}$. We have $\operatorname{sgn} S_0 = (-1)^j \tau$ on $(y_j, y_{j+1})$. Thus there is a unique point $w \in (y_{j+1}, y_{j+2})$ such that $S_0(w) = (-1)^{j+1}\tau$. So by (9.17), at $z = w$ we have

$$\operatorname{sgn}\big(ph_6 S + (-1)^{j+1}\tau(C+zS)^{p+1}\big) = (-1)^j \sigma\tau.$$

Hence $(-1)^j \sigma\tau p h_6 S - \sigma(C+zS)^{p+1} > 0$ at $z = w$. Note that $\operatorname{sgn} S(w) = \sigma$. By the mean value theorem there is a point $t \in (t_2, w)$ with

$$(-1)^j \tau h_6'(t) = \frac{h_6(w)(-1)^j \tau}{w - t_2}. \tag{10.2}$$

On the other hand, by (9.17),

$$h_6'(t) = \frac{pS(t)h_6(t)}{(C+zS)(t)} + S_0(t)(C+zS)^p(t). \tag{10.3}$$

The function $(-1)^j \tau h_6$ has positive derivative and hence is increasing on $(t_1, y_{j+2})$. Furthermore, this function is positive on $(t_2, y_{j+2})$.

Suppose that $t \in [y_{j+1}, w)$ so that $(-1)^j \tau S_0(t) \leq 0$. Then $0 < (-1)^j \tau h_6(t) < (-1)^j \tau h_6(w)$, and so combining (10.3) and (10.2) we obtain

$$1 < \frac{(w-t_2)pS(t)}{(C+zS)(t)}.$$

Considering the relationship between $C + zS$, $S$, and $S_0$, we see that there is a positive constant $\varepsilon_0$ depending on $p$ only such that $\sigma S(t) \geq \varepsilon_0$ and $\sigma(C+zS)(t) \geq \varepsilon_0 t$ for any $t \in [y_2, y_p]$ (recall that $x_1 > 0$). Also

$$w - t_2 \leq w - y_j \leq y_p - y_1 \leq (p-1)2\pi/(pA) < 2\pi/A.$$

Thus

$$\varepsilon_0 x_1 \leq \varepsilon_0 t \leq \sigma(C+zS)(t) \leq p\sigma S(t)(w-t_2) \leq p(w-t_2) < (2\pi p)/A,$$

which gives a contradiction when $x_1$ is large enough.

Hence we must have $t \in (t_2, y_{j+1})$ and so $(-1)^j \tau S_0(t) > 0$. Then, since we still have $0 < (-1)^j \tau h_6(t) < (-1)^j \tau h_6(w)$, we obtain by (10.2) and (9.17) that

$$\begin{aligned}\frac{1}{2}\frac{(-1)^j\tau h_6(w)}{(2\pi/A)} < \frac{1}{2}\frac{(-1)^j\tau h_6(w)}{w-y_j} &< \frac{1}{2}\frac{(-1)^j\tau h_6(w)}{w-t_2}\\ &< (-1)^j\tau h_6(w)\left(\frac{1}{w-t_2}-\frac{pS(t)}{(C+zS)(t)}\right)\\ &= (-1)^j\tau\left(h_6'(t)-\frac{pS(t)h_6(w)}{(C+zS)(t)}\right)\\ &< (-1)^j\tau\left(h_6'(t)-\frac{pS(t)h_6(t)}{(C+zS)(t)}\right)\\ &= (-1)^j\tau S_0(t)(C+zS)^p(t) < (-1)^j\tau K t^p S_0(t)\end{aligned}$$

where we note, as above, that $pS(t)/(C+zS)(t)$ is comparable to $1/x_1$. Here $K>0$ is a constant depending only on $p$. Now we get

$$(-1)^j\tau h_6(w) < (-1)^j\tau K_1 t^p S_0(t) < K_1 w^p$$

where $K_1>0$ depends only on $p$ and $A$. Furthermore, by (9.17),

$$(-1)^j\tau h_6'(w) = \frac{(-1)^j\tau pS(w)h_6(w)}{(C+zS)(w)} + (-1)^j\tau S_0(w)(C+zS)^p(w)$$

so that

$$\begin{aligned}0 &< (-1)^j\tau\sigma(C+zS)(w)h_6'(w)\\ &= (-1)^j\tau\sigma pS(w)h_6(w) + (-1)^j\tau\sigma S_0(w)(C+zS)^{p+1}(w)\\ &< K_1 p w^p - |(C+zS)(w)|^{p+1}.\end{aligned}$$

But $|(C+zS)(w)| > \varepsilon_0 w$. Hence $(-1)^j\tau h_6'(w) < 0$, which is a contradiction. This proves our claim that each of $h_6$ and $h_6'$ has a zero on $(y_{j+1}, y_{j+2})$. Hence $h_6$ has exactly $p-1$ zeros on $(x_1,x_2)$, with due count of multiplicity.

These $p-1$ zeros of $h_6$ are also zeros of $c+zs$, which has exactly $p+1$ zeros, all simple, on $(x_1,x_2)$ when $|x_1|$ is large enough (see (3.13)). These zeros occur at points where $s=-c/z$ is close to zero. By (9.14),

$$\begin{aligned}s = \sin(az+b) &= \sin\bigl((p+1)(Az+B)-(p+1)B+b\bigr)\\ &= \sin\bigl((p+1)(Az+B)-\bigl(q+(p/2)\bigr)\pi\bigr) = (-1)^{q+(p/2)}\sin\bigl((p+1)(Az+B)\bigr).\end{aligned}$$

Thus the zeros of $c+zs$ are close to the points where $Az+B = l_0\pi/(p+1)$ for some integer $l_0$. The points $y_j$ are points where $Az+B=\mu\pi/p$, and here $\mu$ varies from $lp$ to $(l+1)p-1$ for some integer $l$. Thus at $y_1$, we have $Az+B=$

$\big((p+1)l\big)\pi/(p+1)$ so that $y_1$ is close to a zero of $c+zs$. As we have seen, we have $Ax_1+B \approx l\pi - (1/x_1)$ so that

$$\begin{aligned}(c+zs)(x_1) &= (-1)^{q+(p/2)}\big((-1)^{l(p+1)} + x_1(-1)^{l(p+1)}\big(-(p+1)/x_1\big)\big) + o(1)\\ &= -p(-1)^{q+(p/2)+l(p+1)} + o(1) = -p\tau(-1)^l + o(1) = -p\tau\sigma + o(1).\end{aligned}$$

Here we have used the fact that $\sigma = (-1)^l$ and that $\tau = (-1)^{q+1+(p/2)+pl+1}$, which follows since by (9.15), we have $\operatorname{sgn} S_0(x) = (-1)^{q+1+(p/2)} \operatorname{sgn}\sin\big(p(Az+B)\big) = \tau$ on $(x_1, x_1+\varepsilon)$. The $o(1)$-term tends to zero as $|x_1| \to \infty$. We further have $(c+zs)'(x_1) = \big((1-a)s+azc\big)(x_1) = ax_1(-1)^{q+(p/2)}\cos\big((p+1)(Ax_1+B)\big)+o(1) = (-1)^{q+(p/2)+(p+1)l}+o(1) = \tau(-1)^l+o(1) = \tau\sigma+o(1)$. We have $a>0$ since $A>0$ and $a/A>0$. Thus $\operatorname{sgn}(c+zs)'(x_1) = -\operatorname{sgn}(c+zs)(x_1)$. Hence the zero of $c+zs$ which is close to both $x_1$ and $y_1$ is slightly greater than $x_1$. We have $(c+zs)(y_1) = \cos(ay_1+b) = (-1)^{q+(p/2)+(p+1)l} = \tau\sigma = -\operatorname{sgn}(c+zs)(x_1)$. We conclude that $c+zs$ has a zero on $(x_1,y_1)$. This cannot be a zero of $h_6$ as we have seen above. The next $p-1$ zeros of $c+zs$ lie on the intervals $(y_j, y_{j+1})$ for $1 \le j \le p-1$ so that they must be zeros of $h_6$. When $|x_1|$ is large enough, $c+zs$ has exactly $p+1$ zeros on $(x_1,x_2)$. There is one more zero of $c+zs$ on $(x_1,x_2)$. It lies in $(y_p, x_2)$ and is not a zero of $h_6$. We may express this by saying that if for each large $l$, the two zeros of $c+zs$ immediately smaller than the value of $x$ corresponding to $Ax+B = l\pi$ are not zeros of $h_6$ while all other sufficiently large positive zeros of $c+zs$ are also zeros of $h_6$. Let us denote the two zeros mentioned by $t_l$ and $z_l$, where $t_l < z_l$. As $l \to \infty$, we have

$$At_l + B = l\pi + O(1/l) \qquad \text{and} \qquad Az_l + B = \big(l-(p+1)^{-1}\big)\pi + O(1/l).$$

Still assuming that $A>0$ and that $p$ is even, suppose that $x_1 < x_2 < 0$ and that $|x_2|$ is large enough. Then an analysis similar to the above shows that at the zeros $y_1 < \cdots < y_p$ of $S_0$ on $(x_1,x_2)$, we have $Ay_j + B = (\pi/p)(lp+j)$ for some negative integer $l$, and we now have $S = S_0 = 0$ at $y_p$ (instead of $y_1$), and for each sufficiently large negative integer $l$, the two zeros of $c+zs$ immediately larger than the value of $x$ corresponding to $Ax+B = l\pi$ are not zeros of $h_6$ while all other sufficiently large negative zeros of $c+zs$ are also zeros of $h_6$. Let us denote the two zeros mentioned by $t_l$ and $z_l$, where $z_l < t_l$. (We have $z_l < x_1 < t_l < y_1$.) As $l \to -\infty$, we have

$$Az_l + B = l\pi + O(1/l) \qquad \text{and} \qquad At_l + B = \big(l+(p+1)^{-1}\big)\pi + O(1/l).$$

Thus

$$z_l + z_{-l} = -2B/A + O(1/l) \qquad \text{and} \qquad t_l + t_{-l} = -2B/A + O(1/l) \tag{10.4}$$

as $l \to \infty$.

Since $h_6$ is a real meromorphic function with only finitely many poles and of order at most 1 and since $h_6$ has the zeros described above, we conclude that there is a real number $\alpha$ and a real rational function $R$, not identically zero, such that

$$h_6 = e^{\alpha z}(C+zS)^{p+1}(c+zs)R/h_7 \equiv \varphi/h_7 \tag{10.5}$$

where

$$h_7(z) = \prod_{l=l_0}^{\infty}\left(\left(1-\frac{z}{z_l}\right)\left(1-\frac{z}{z_{-l}}\right)\left(1-\frac{z}{t_l}\right)\left(1-\frac{z}{t_{-l}}\right)\right). \tag{10.6}$$

It is easily seen that the infinite product for $h_7$ converges and defines a real entire function of order 1 when the factors are grouped as above (it suffices to pair $z_l$ and $z_{-l}$ together, and separately $t_l$ and $t_{-l}$ together). Now (9.17) can be written as

$$\frac{\varphi'}{h_7} - \frac{\varphi h_7'}{h_7^2} = S_0(C+zS)^p + \frac{\varphi}{h_7}\,\frac{pS}{C+zS}.$$

At a point where $S_0 = 0$, this gives

$$\begin{aligned}\frac{h_7'}{h_7} &= \frac{\varphi'}{\varphi} - \frac{pS}{C+zS}\\ &= -\frac{pS}{C+zS} + \alpha + \frac{R'}{R} + (p+1)\frac{(1-A)S+AzC}{C+zS} + \frac{(1-a)s+azc}{c+zs}.\end{aligned} \tag{10.7}$$

Let the notation be as before and consider (10.7) at $z = y_1$ so that $S_0 = 0$, as required, where $y_1$ is large and positive. Thus $Ay_1 + B = l_0\pi$ for some large integer $l_0$. Now $y_1$ is close to the point $x_1$ at which $C+zS = 0$ and $S = O(1/x_1)$, and close to the common zero $z_l$ of $h_7$ and $c+zs$. Also $S(y_1) = 0$. We obtain

$$\frac{h_7'}{h_7}(y_1) = (p+1)Ay_1 + \frac{1}{y_1 - z_l} + O(1)$$

where the $O(1)$-term remains bounded as $x_1$ (and hence $y_1$) tends to infinity. On the other hand, the definition of $h_7$ together with (10.4) allows us to estimate

$$\frac{h_7'}{h_7}(y_1) = \frac{1}{y_1 - z_l} + \frac{1}{y_1 - t_l} + O(1).$$

Now $y_1 \to \infty$ and $y_1 - t_l \to \pi/\bigl(A(p+1)\bigr)$ as $x_1 \to \infty$. Thus we clearly obtain a contradiction whenever $x_1$ is sufficiently large. This contradiction shows that the case when $p$ is even and $A > 0$ cannot occur. The case when $p$ is even and $A < 0$ is dealt with in the same way, and we omit the details.

If $p$ is odd then by (9.16), we have $S_0(z) = (-1)^{q+((p-1)/2)}\cos\big(p(Az+B)\big)$, $s(z) = (-1)^{q+((p+1)/2)}\cos\big((p+1)(Az+B)\big)$, and $c(z) = (-1)^{q+((p-1)/2)}\sin\big((p+1)(Az+B)\big)$ for some integer $q$. We argue as above on an interval $(x_1, x_2)$ where $C + zS$ retains its sign. The location of the zeros of $S_0$ in this interval is now different but the number $p$ of zeros is the same. In the particularly simple case when $p = 1$, the function $h_6$ has no zeros on $(x_1, x_2)$ and we may use $h_7 = c + zs$ in (10.5). Then $S_0 \equiv (-1)^q C$, and (10.5) and (9.17) imply first that $\alpha = 0$ in (10.5) and then yield a contradiction, considering sequences of real points tending to $\pm\infty$ at which $S = 0$ or $C = 0$.

Suppose that $p$ is odd and $p \geq 3$, $A > 0$. The argument is analogous to the one used when $p$ is even, up to (10.6). However, the zeros $y_j$ of $S_0$ on $(x_1, x_2)$ are now bounded away from $x_1$ and $x_2$, and so are the zeros of $c+zs$ and hence those of $h_6$. Therefore a consideration of $(h_7'/h_7)(y_1)$ as in the case when $p$ is even does not seem to help. Instead, we may argue as follows. When $|x_1|$ is large, the zeros of $c + zs$ and hence those of $h_6$ occur closer and closer to points where $s = 0$ since $s = -c/z$ at these points. Indeed, since $(c+zs)' = -c(1-a-az^2)/z$ when $c + zs = 0$, the mean value theorem and the monotonicity of $(c+zs)'$ between a zero of $c+zs$ and the nearest zero of $s$ (where $c+zs = c = \pm 1$, $(c+zs)' = acz$) imply that the distance between these zeros is $O(1/z)$. When $|x_1|$ is large enough, the zeros of $c + zs$ that are zeros of $h_7$ are close to those zeros of $s$ at which $Az + B = k\pi \pm \pi/\big(2(p+1)\big)$ for all integers $k$ with $|k|$ sufficiently large. Write $\beta = \sin\big(\pi/\big(2(p+1)\big)\big)$. Comparing the Hadamard product representations of $h_7$ and of $\sin\big(Az+B+\pi/\big(2(p+1)\big)\big)\sin\big(Az+B-\pi/\big(2(p+1)\big)\big) = S^2 - \beta^2$ and considering (10.5), we conclude that there are a real rational function $R_1$ and a real number $\alpha_1$ such that if we define

$$h_8 = \frac{R_1 e^{\alpha_1 z}(C+zS)^{p+1}(c+zs)}{S^2-\beta^2}, \tag{10.8}$$

then for any small $\varepsilon > 0$, we have

$$h_6(x)/h_8(x) \to 1 \qquad \text{and} \qquad (h_6'/h_6)(x) - (h_8'/h_8)(x) \to 0 \tag{10.9}$$

as $x \to \pm\infty$ outside $\varepsilon$-neighbourhoods of the zeros of $S^2 - \beta^2$.

We obtain from (9.17), (10.5), and (10.8)–(10.9) that

$$\begin{aligned}\frac{h_8'}{h_8} - \frac{h_6'}{h_6} = \alpha_1 + \frac{R_1'}{R_1} + (p+1)\frac{(1-A)S + AzC}{C+zS} + \frac{(1-a)s + azc}{c+zs} \\ - \frac{2ASC}{S^2-\beta^2} - \frac{pS}{C+zS} - \frac{h_8}{h_6}\frac{S_0(S^2-\beta^2)}{R_1 e^{\alpha_1 z}(C+zS)(c+zs)} \to 0\end{aligned} \tag{10.10}$$

as $x \to \pm\infty$ outside $\varepsilon$-neighbourhoods of the zeros of $S^2 - \beta^2$. Considering (10.10) at the points where $C = 0$, as we may, so that also $S_0 = c = 0$, we see, as $x \to \infty$, that $\alpha_1 = 0$. Next, we may take $x = y_j$ so that $S_0 = 0$, and noting that the functions $C/S, s, c$ always take the same values at $x = y_j$ (depending on $j$ but independent of $x_1$), we obtain, as $x \to \infty$, that

$$\frac{(p+1)AC}{S} + \frac{ac}{s} - \frac{2ASC}{S^2 - \beta^2} = 0 \qquad \text{at } x = y_j \tag{10.11}$$

for all $j$ with $1 \le j \le p$. This is trivially true for $j = (p+1)/2$ since then $C = c = 0$. But since $p \ge 3$, there are other values of $j$ to consider also. A calculation based on trigonometric identities and the fact that $a = (p+1)A$ shows that when $C \ne 0$, (10.11) is equivalent to

$$\frac{p+1}{p} = \frac{\sin^2\Bigl(\dfrac{\pi}{2p}(2j-1)\Bigr)}{\beta^2}, \qquad \text{for } 1 \le \Bigl|j - \frac{p+1}{2}\Bigr| \le \frac{p-1}{2}. \tag{10.12}$$

If $p = 3$ and $j = 1$, (10.12) reads $4/3 = (1/4)/\sin^2(\pi/8)$, which is false. If $p \ge 5$, (10.12) cannot be satisfied since $\sin^2\bigl(\pi(2j-1)/(2p)\bigr)$ will not take the same value for all $j$. This gives a contradiction, showing that the case when $p$ is odd and $A > 0$ cannot occur. The case when $p$ is odd and $A < 0$ is dealt with in the same way, and we omit the details. This completes our treatment of Case V.

This completes the proof of Theorem 1 when $m = 0$. Hence the proof of Theorem 1 is now complete.

## 11. Proof of Lemma 1

Let the assumptions of Lemma 1 be satisfied. If $\Phi$ has only finitely many zeros, then so does $\Psi'$, so that the conclusion of Lemma 1 is obtained trivially by taking $R$ to be sufficiently large. Hence we may assume that $\Phi$ has infinitely many zeros.

We have

$$\Psi' = \frac{P\Phi' - P'\Phi}{P^2}.$$

Choose $R_0 > 1$ so that $|a| < R_0$ whenever $P(a) = 0$ or $P(a) = \infty$. Let $x_1$ and $x_2$ be two consecutive zeros of $\Psi$ and hence of $\Phi$ such that $x_1 < x_2$ and $|x_j| > R_0$ for $j = 1, 2$. By Rolle's theorem, $\Psi'$ has at least one zero on $(x_1, x_2)$. The zeros of $\Psi'$ on $(x_1, x_2)$ coincide with those of

$$\frac{\Phi'}{\Phi} - \frac{P'}{P}$$

on $(x_1, x_2)$. If there are at least two such zeros, with due count of multiplicity, then $(x_1, x_2)$ contains a point $x_3$ at which $(\Phi'/\Phi)' - (P'/P)'$ vanishes. There is a positive constant $K$ such that $|(P'/P)'(x)| < K/|x|^2$ for $|x| > R_0$. Now $\Phi'/\Phi$ has no zeros on $(x_1, x_2)$ and tends to $\infty$ and $-\infty$ at the endpoints. We can write

$$\Phi(z) = Az^m e^{-az^2+bz} \prod_{j=1}^{\infty} \Big(1 - \frac{z}{a_j}\Big) e^{z/a_j}$$

where $A$ is a non-zero real constant, $m$ is a non-negative integer, $a \geq 0$, $b$ is real, and the $a_j$ are non-zero real numbers with $\sum_j a_j^{-2} < \infty$. Thus

$$\Big(\frac{P'}{P}\Big)'(x_3) = \Big(\frac{\Phi'}{\Phi}\Big)'(x_3) = \frac{-m}{x_3^2} - 2a - \sum_{j=1}^{\infty} \frac{1}{(x_3 - a_j)^2} < -\sum_{j=1}^{\infty} \frac{1}{(x_3 - a_j)^2} < 0.$$

We obtain a contradiction if we can show that

$$\frac{K}{x^2} < \sum_{j=1}^{\infty} \frac{1}{(x - a_j)^2} \tag{11.1}$$

whenever $|x|$ is large enough and applying this to $x = x_3$.

Suppose that there are infinitely many positive numbers $a_j$. Then (11.1) holds for a given $x > 0$ provided that the number of zeros $a_j$ on $(0, x)$ is larger than $K$. We argue similarly when $x < 0$ if $\Phi$ has infinitely many negative zeros. Suppose then that $\Phi$ has infinitely many positive zeros but only finitely many negative zeros, if any, and that $x < 0$. To prove (11.1), we may assume that all the $a_j$ are positive. Choose a positive integer $N > K$. For any $j$ with $1 \leq j \leq N$, there is $\zeta_j < 0$ such that $x^{-2} < (x - a_{2j-1})^{-2} + (x - a_{2j})^{-2}$ for all $x < \zeta_j$. Thus (11.1) holds for all $x < \min\{\zeta_j : 1 \leq j \leq N\}$. A similar argument works if $x > 0$ and if $\Phi$ has infinitely many negative zeros but only finitely many positive zeros. We have now proved that $(x_1, x_2)$ contains exactly one zero of $\Psi'$, with due count of multiplicity, provided that the consecutive zeros $x_1$ and $x_2$ of $\Phi$ are of the same sign and that $|x_1|$ and $|x_2|$ are large enough, say $|x_1| > R$ and $|x_2| > R$, where $R > R_0$.

The same argument shows that if $\Phi$ has infinitely many negative zeros but only finitely many positive zeros, then $\Psi'$ can have at most one zero on $(R_1, \infty)$ for a suitable $R_1 > 0$, and hence no zeros on $(R_2, \infty)$ for some $R_2 > 0$. Choosing $R > R_2$ in this case we do not have to consider the possible existence of zeros of $\Phi$ between consecutive positive zeros of $\Psi'$. An analogous conclusion is obtained if $\Phi$ has infinitely many positive zeros but only finitely many negative zeros.

Suppose then that $x_1$ and $x_2$ are consecutive zeros of $\Psi'$ with $|x_j| > R$ for $j = 1, 2$. Suppose first that $\Phi$ vanishes at both $x_1$ and $x_2$, or at neither $x_1$ nor $x_2$. We claim that $(x_1, x_2)$ contains exactly one zero of $\Phi$, with due count of multiplicity, provided that $|x_1|$ and $|x_2|$ are large enough. If $(x_1, x_2)$ contains at least two zeros of $\Phi$, with due count of multiplicity, then it follows from Rolle's theorem that $\Psi'$ has a zero on $(x_1, x_2)$, which is against our assumption that $x_1$ and $x_2$ are consecutive zeros of $\Psi'$. Thus $(x_1, x_2)$ contains at most one zero of $\Phi$. If $\Phi$ does not vanish anywhere on $(x_1, x_2)$, then let $x_3$ be the largest zero of $\Phi$ with $x_3 \leq x_1$, and let $x_4$ be the smallest zero of $\Phi$ with $x_2 \leq x_4$. (By the previous paragraph, we may assume that $x_3$ and $x_4$ exist with $x_4 < -R$ or $x_3 > R$.) By what we have proved already, $(x_3, x_4)$ contains exactly one zero of $\Psi'$, with due count of multiplicity. This gives an immediate contradiction if $\Phi(x_1)\Phi(x_2) \neq 0$ (so that $x_3 < x_1 < x_2 < x_4$). If $\Phi(x_1) = \Phi(x_2) = 0$ then $x_3 = x_1$ and $x_4 = x_2$. But then $\Psi'$ has a zero on $(x_1, x_2)$ by Rolle's theorem, which gives a contradiction. Thus $\Psi'$ has exactly one zero on $(x_1, x_2)$ with due count of multiplicity, as claimed. Suppose then that exactly one of $x_1$ and $x_2$ is a zero of $\Phi$. If $\Phi$ has a zero on $(x_1, x_2)$ then by Rolle's theorem, $\Psi'$ has a zero on $(x_1, x_2)$, which gives a contradiction. Hence $\Phi$ has no zeros on $(x_1, x_2)$ in this case. This completes the proof of Lemma 1.

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