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# PARAMETRIZATION OF THE MODULI SPACE OF HYPERELLIPTIC AND SYMMETRIC RIEMANN SURFACES

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**Abstract.** Let $S$ be a Riemann surface with genus $g > 1$, let $\varphi$ be the hyperelliptic involution (i.e. $\varphi$ is a conformal involution such that $S/\langle\varphi\rangle$ has genus $0$) and let $\sigma$ be a symmetry of $S$ (i.e. $\sigma$ is an anticonformal involution of $S$). We obtain a finite set of real numbers which determines a canonical fundamental region of the NEC groups uniformizing $S/\langle\varphi,\sigma\rangle$. The real numbers obtained in this way are used to parametrize the strata of equisymmetric hyperelliptic Riemann surfaces in the moduli space.

## 1. Introduction

A Riemann surface $S$ is said to be *hyperelliptic* if and only if $S$ admits an automorphism $\phi \in \mathrm{Aut}^+(S)$ such that $\phi^2$ is the identity and the quotient $S/\langle\phi\rangle$ has genus $0$. The automorphism $\phi$ is called the *hyperelliptic involution.* A Riemann surface $S$ is said to be *symmetric* if and only if $S$ admits an automorphism $\psi \in \mathrm{Aut}^-(S)$ with $\psi^2$ the identity. In this case $\psi$ is a *symmetry* of $S$ and $X = S/\langle\psi\rangle$ is a Klein surface. If the Riemann surface $S$ is both hyperelliptic and symmetric then $X$ is a hyperelliptic Klein surface.

Let $p$ be the genus of $S$. If $p \ge 2$, then $S$ and $X$ can be expressed as quotients of the hyperbolic plane $D$ over surface groups $F$ and $\Gamma$, respectively, $F$ being a Fuchsian group of genus $p$ and $\Gamma$ a non-Euclidean crystallographic group (NEC group) of algebraic genus $p$.

Hyperelliptic Klein surfaces with boundary are characterized by means of NEC groups in [BEG] and non-orientable Klein surfaces without boundary in [BBGM].

The aim of this work is to parametrize the *strata* of the moduli space of hyperelliptic and symmetric Riemann surfaces $S$ according to the different groups of automorphisms of the Klein surface $X$. It is done by means of the construction of a hyperbolic polygon $R$ with right angles (unless one of them at most). $R$ is a fundamental region for an NEC group $\Gamma'$ such that $\Gamma'/\Gamma$ is isomorphic to the group of automorphisms of $X$.

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## 2. Preliminaries on NEC groups

A non-Euclidean crystallographic group (NEC group in short) $\Gamma$ is a discrete subgroup of isometries of the hyperbolic plane $D$ with compact quotient $D/\Gamma$. Each NEC group $\Gamma$ has a signature $\sigma$ that has the following form [M]

$$\sigma : \big(g; \pm; [m_1, \dots, m_r], \{C_1, \dots, C_k\}\big), \tag{2.1}$$

where $C_i = (n_{i1}, \dots, n_{is_i})$, $C_i$ are called *cycle-periods*, $n_{ij}$ *link-periods* and $m_i$ *proper periods*.

The numbers in $\sigma$ are non-negative integers; $m_i$ and $n_{ij}$ are greater than or equal to $2$. The number $g$ is the topological genus of the surface quotient $D/\Gamma$. This surface is orientable or not according as the sign in $\sigma$ is '$+$' or '$-$' respectively. The algebraic genus of $\Gamma$ is $p = \eta g + k - 1$, where $\eta = 2$ in the orientable case or $\eta = 1$ otherwise.

If $r = 0$ or $k = 0$, we write in $\sigma[-]$ or $\{-\}$, respectively. If the number $s_i$ is zero for some $i$ we denote $C_i$ by $(-)$.

We consider a Fuchsian group $F$ as a particular case of NEC group. Since every element of $F$ is an orientable isometry of the hyperbolic plane, $F$ has the following (NEC) signature:

$$\big(g; +; [m_1, \dots, m_r], \{-\}\big).$$

The signature $\sigma$ determines a canonical presentation of the group $\Gamma$ [M], [W] which is given by the generators

$$\begin{array}{ll} x_i, & i = 1, \dots, r, \\ e_i, & i = 1, \dots, k, \\ c_{ij}, & i = 1, \dots, k,\ j = 0, \dots, s_i, \\ a_i,\ b_i, & i = 1, \dots, g,\ \text{if sign '+'}, \\ d_i, & i = 1, \dots, g,\ \text{if sign '}-\text{'}, \end{array}$$

with the relations

$$\begin{array}{rl} x_i^{m_i} = 1, & i = 1, \dots, r, \\ c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1} c_{ij})^{n_{ij}} = 1, & i = 1, \dots, k,\ j = 1, \dots, s_i, \\ e_i^{-1} c_{i0} e_i c_{is_i} = 1, & i = 1, \dots, k, \\ \prod_{i=1}^{r} x_i \prod_{i=1}^{k} e_i \prod_{i=1}^{g} [a_i, b_i] = 1 & \text{if sign } + \qquad \text{or} \\ \prod_{i=1}^{r} x_i \prod_{i=1}^{k} e_i \prod_{i=1}^{g} d_i^2 = 1 & \text{if sign } - . \end{array}$$

The area of $\Gamma$ is the area of each fundamental region of $\Gamma$ and it is calculated by

$$\mu(\Gamma) = 2\pi\Big(\eta g + k - 2 + \sum_{i=1}^{r}\Big(1 - \frac{1}{m_i}\Big) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\Big(1 - \frac{1}{\eta_{ij}}\Big)\Big).$$

A signature $\sigma$ corresponds to an NEC group $\Gamma$ if and only if $\mu(\Gamma) > 0$.

If $\Gamma$ is a subgroup of $\Gamma'$ with index $N$ then the following relation between the areas holds

$$\mu(\Gamma) = N\mu(\Gamma').$$

An NEC group $\Gamma$ is a surface group if it has the signature

$$\big(g; \pm; [-], \{(-) \cdots^{k} \cdots (-)\}\big),$$

and in this case we write the signature as

$$(g; \pm; [-], \{(-)^k\}). \tag{2.2}$$

If $k = 0$ and the sign in $\sigma$ is '$+$' then $\Gamma$ is a Fuchsian surface group.

Let $\Gamma$ be an NEC group with signature $\sigma$. A canonical fundamental region for $\Gamma$ found by Wilkie [W] is a hyperbolic polygon $W$ with the sides labelled

$$\xi_1, \xi_1', \ldots, \xi_r, \xi_r'; \varepsilon_1, \gamma_{10}, \ldots, \gamma_{1s_1}, \varepsilon_i', \ldots, \varepsilon_k, \gamma_{k0}, \ldots, \gamma_{ks_k}, \varepsilon_k';$$
$$\alpha_1, \beta_1', \alpha_1', \beta_1, \ldots, \alpha_g, \beta_g', \alpha_g', \beta_g,$$

if sign '$+$' and

$$\xi_1, \xi_1', \ldots, \xi_r, \xi_r'; \varepsilon_1, \gamma_{10}, \ldots, \gamma_{1s_1}, \varepsilon_i', \ldots, \varepsilon_k, \gamma_{k0}, \ldots, \gamma_{ks_k}, \varepsilon_k'; \delta_1, \delta_1^*, \ldots, \delta_g, \delta_g^*,$$

if sign '$-$'. The sides are identified in the following way

$$\begin{aligned}
x_i(\xi_i') &= \xi_i, && i = 1, \ldots, r,\\
e_i(\varepsilon_i') &= \varepsilon_i, && i = 1, \ldots, k,\\
c_{ij}(\gamma_{ij}) &= \gamma_{ij}, && i = 1, \ldots, k,\ j = 0, \ldots, s_i,\\
a_i(\alpha_i') &= \alpha_i, && i = 1, \ldots, g,\\
b_i(\beta_i') &= \beta_i, && i = 1, \ldots, g,\\
d_i(\delta_i^*) &= \delta_i, && i = 1, \ldots, g,
\end{aligned}$$

where '$*$' denotes a non-orientable transformation.

The angles in the vertices of $W$ are

$$\langle \xi_i, \xi_i' \rangle = 2\pi/m_i, \qquad i = 1, \ldots, r,$$
$$\langle \varepsilon_i, \gamma_{i0} \rangle + \langle \gamma_{is_i}, \varepsilon_i' \rangle = \pi, \qquad i = 1, \ldots, k,$$

and without a loss of generality we may assume that both angles are equal to $\frac{1}{2}\pi$,

$$\langle \gamma_{ij-1}, \gamma_{ij} \rangle = \pi/n_{ij}, \qquad i = 1, \ldots, k,\ j = 1, \ldots, s_i,$$

and the sum of the remaining angles is $2\pi$. We may assume $W$ as a convex polygon, i.e. there is not an angle in $W$ greater than $\pi$.

### 3. Surfaces

A Klein surface (a compact surface equipped with a dianalytic structure) of algebraic genus $p \geq 2$ can be expressed as $D/\Gamma$ where $\Gamma$ is an NEC group with signature (2.2).

A group $G$ of order $N$ is a group of automorphisms of $X$ if and only if there exists an NEC group $\Gamma'$ containing $\Gamma$ as a subgroup of order $N$ such that $\Gamma'/\Gamma \approx G$. Let $S = D/F$ a Riemann surface of genus $p \geq 2$. $S$ is hyperelliptic if and only if there exists a Fuchsian group $F_1$ containing $F$ as a subgroup of index 2 with signature

$$\big(0; +; [2^{2p+2}], \{-\}\big),$$

where $2^{2p+2}$ denotes $2p+2$ proper periods equal to 2 each one.

Let $X = D/\Gamma$ be a Klein surface where $\Gamma$ has signature (2.2). $X$ is hyperelliptic if and only if there exists an NEC group $\Gamma_1$ containing $\Gamma$ as a subgroup of index 2, $\Gamma_1$ having one of the following signatures [BEG], [BBGM]:

| | | |
|---|---|---|
| i) | $(0; +; [-], \{(2^{2k})\})$ | if $g = 0$, in this case $k \geq 3$. |
| ii) | $(0; +; [2^{g+k}], \{(-)\})$ | if $g > 0$, $k \neq 0$, sign '$+$', then $k = 1$ or $2$. |
| iii) | $(0; +; [2^g], \{(2^{2k})\})$ | if $g > 0$, $k \neq 0$, sign '$-$'. |
| iv) | $(0; +; [2^g], \{(-)\})$ | if $g \geq 3$, $k = 0$, sign '$-$'. |
| v) | $(1; -; [2^g], \{-\}$ | if $g > 3$, $k = 0$, sign '$-$' in this case $g$ must be even. |

Table 1

The case v) corresponds to hyperelliptic surfaces admitting symmetries without fixed points, i.e. these surfaces correspond to purely imaginary curves. We are interested in the other cases because they provide real algebraic curves.

Let $S = D/\Gamma$ a hyperelliptic and symmetric Riemann surface and let $\psi$ be a symmetry. Then $X = S/\langle\psi\rangle$ is a hyperelliptic Klein surface. Let $G = \Gamma'/\Gamma = \mathrm{Aut}(X)$ with order $2N$. We have the following diagram that relates the different groups.

The signatures of $\Gamma'$ appears in the following table, where $p = \eta g + k - 1$ [BBM], [BBGM]. The cases of the second column are as Table 1.

| $\sigma(\Gamma)$ | $\sigma(\Gamma_1)$ | $\sigma(\Gamma')$ | $\mathrm{Aut}(X)$ | Case |
|---|---|---|---|---|
| $g=0$,<br>$k\geq 3$ | i) | $(0;+;[N],\{(2^{2k/N})\})$,<br>$N\mid k,\ N\neq k$ | $\mathbf{Z}_N\times\mathbf{Z}_2$ | 1 |
| | | $(0;+;[-],\{(\frac{1}{2}N,2^{2k/N})\})$,<br>$N$ even, $N\mid 2k$ | $\mathbf{D}_{N/2}\times\mathbf{Z}_2$ | 2 |
| $g\neq 0$,<br>$k=1,2$<br>sign '+' | ii) | $(0;+;[2N,2^{p/N}],\{(-)\})$,<br>$N\mid p,\ N\neq p$ | $\mathbf{Z}_{2N}$ | 3 |
| | | $(0;+;[N,2^{(p+1)/N}],\{(-)\})$,<br>$N\mid p+1,\ N\neq p+1$ | $\mathbf{Z}_N\times\mathbf{Z}_2$ | 4 |
| | | $(0;+;[2^r],\{(N,2^s)\})$,<br>$s=2p/N+2-2r$, $N$ even, $N\mid 2p$ | $\mathbf{D}_N$ | 5 |
| | | $(0;+;[2^r],\{(N/2,2^s)\})$,<br>$s=2(p+1)/N+2-2r$, $N$ even, $N\mid 2(p+1)$ | $\mathbf{D}_{N/2}\times\mathbf{Z}_2$ | 6 |
| $g\neq 0$,<br>$k\neq 0$,<br>sign'−' | iii) | $(0;+;[2N,2^r],\{(2^s)\})$,<br>$s=(2g+2k-2)/N-2r$, $s\neq 0$, $N\mid g-1$, $N\mid k$ | $\mathbf{Z}_{2N}$ | 7 |
| | | $(0;+;[N,2^r],\{(2^s)\}$,<br>$s=(2g+2k)/N-2r$, $s\neq 0$, $N\mid g$, $N\mid k$ | $\mathbf{Z}_N\times\mathbf{Z}_2$ | 8 |
| | | $(0;+;[2^r],\{(N,2^s)\})$,<br>$s=(2g+2k-2)/N+2-2r$,<br>$N$ even, $N\mid 2k$, $N\mid 2(g-1)$ | $\mathbf{D}_N$ | 9 |
| | | $(0;+;[2^r],\{(N/2,2^s)\})$,<br>$s=(2g+2k)/N+2-2r$, $N$ even, $N\mid 2k$, $N\mid 2g$ | $\mathbf{D}_{N/2}\times\mathbf{Z}_2$ | 10 |
| $g>3$,<br>$k=0$,<br>$g$ odd,<br>sign '−' | iv) | $(0;+;[N,2^{g/N}],\{(-)\})$,<br>$N\mid g-1$, $N\neq g-1$, or $N\mid g$, $N\neq g$, $N$ even | $\mathbf{Z}_{2N}$ | 11 |
| | | $(0;+;[2N,2^{(g-1)/N}],\{(-)\})$,<br>$N\mid g$, $N\neq g$, $N$ even, $N\neq 2$ | $\mathbf{Z}_N\times\mathbf{Z}_2$ | 12 |
| | | $(0;+;[2^r],\{(N/2,2^s)\})$,<br>$s=2g/N-2r+2$,<br>$N\mid 2(g-1)$, $N$ even or $N\mid 2g$, $N\neq 4t$ | $\mathbf{D}_N$ | 13 |
| | | $(0;+;[2^r],\{(N,2^s)\})$,<br>$s=2(g-1)/N-2r+2$, $N\mid 2g$, $4\mid N$. | $\mathbf{D}_{N/2}\times\mathbf{Z}_2$ | 14 |

Table 2

These fourteen signatures can be put in five different classes:

A : $\big(0;+;[N],\{(2^{2k/N})\}\big)$,
B : $\big(0;+;[-],\{(N/2,2^{2k/N+2})\}\big)$,
C : $\big(0;+;[M,2^{\alpha}],\{(-)\}\big)$,
D : $\big(0;+;[2^{\alpha}],\{(M,2^{\beta})\}\big)$,
E : $\big(0;+;[M,2^{\alpha}],\{(2^{\beta})\}\big)$

which correspond to the cases from Table 2:

1,
2,

3, 4, 11 and 12,
5, 6, 9, 10, 13 and 14,
7 and 8,

respectively, and $M$, $\alpha$, and $\beta$ take different values according to the different cases.

For each one of these signatures we will construct a hyperbolic polygon $R$ that will be a fundamental region for the group $\Gamma'$ and the lengths of some sides of $R$ will parametrize the *strata* of the moduli space of the hyperelliptic and symmetric Riemann surfaces $S$.

About the general construction of a right-angled hyperbolic polygon, with $n$ sides, see [CM].

## 4. Surfaces and polygons

**Class A.** For this class $\Gamma'$ has the following signature:

$$\bigl(0;+;[N],\{(2^{2k/N})\}\bigr) \tag{4.1}$$

where $N \mid k$ and $N \neq k$.

We take as parameters the lengths of the sides $\gamma_1,\dots,\gamma_{s-1}$, where $s = 2k/N$, and we construct a right-angled polygon $P_A$

The length of the side $\lambda$ is determined by $\gamma_1,\dots,\gamma_{s-1}$. Now we construct a hyperbolic pentagon with an angle $\alpha = 2\pi/N$ and four right angles, such that the length of the opposite side to the angle $\alpha$ be $\lambda$ and the two sides of the angle $\alpha$ be equal. In this condition the pentagon is unique. We join this pentagon with the polygon $P_A$ by pasting the sides '$\lambda$' and so we obtain a polygon $R_A$ that is a fundamental region for the group $\Gamma'$ with the signature (4.1). See next figure.

**Class B.** The signature of $\Gamma'$ is in this case

$$\bigl(0;+;[-],\{(N/2,2^{2k/N+2})\}\bigr), \tag{4.2}$$

where $N$ is even and $N \mid 2k$.

We take as parameters the lengths of the sides $\gamma_2, \dots, \gamma_{s-1}$, where $s = 2k/N + 2$, and we construct the right-angled polygon $P_B$. See next figure.

Now we construct a hyperbolic quadrilateral with an angle $\alpha = 2\pi/N$ and three right angles, such that the lengths of the opposite side of $\alpha$ be $\lambda$. This quadrilateral is unique. We join the quadrilateral with $P_B$ pasting the sides labelled '$\lambda$' and we obtain the following fundamental region $R_B$ for a group $\Gamma'$ with signature (4.2).

**Class C.** Let $\Gamma'$ be an NEC group with signature

$$\big(0; +; [M, 2^r], \{(-)\}\big), \tag{4.3}$$

where $M$ and $r$ take different values according to the different cases from Table 2.

Let $W$ a fundamental region of Wilkie for $\Gamma'$ where $\sum_{i=0}^{r+1} \theta_i = 2\pi$.

We draw the orthogonal lines to $\gamma$ from the points $X_i$, $i = 0, 1, \ldots, r$. The side $\gamma$ becomes divided in different parts labelled $\gamma_0, \gamma_1, \ldots, \gamma_{r+1}$, according to the 'opposite' angle $\theta_i$. Let $\lambda_i$ be the length from $X_i$ to $\gamma$ as it is shown in the next figure

We cut by $\lambda_r$ and we transform the region $W$ in the new region

$$\big(W - (\lambda_r, \xi_r, \xi_r', \gamma_{r+1}) \cup x_{r+1}(\lambda_r, \xi_r, \xi_r', \gamma_{r+1})\big).$$

In the next step we cut the new region by $\lambda_{r-1}$ and apply $x_{r-1}$ in a similar way and so on. In the last step we obtain the polygon $P_C$ that is a fundamental region for the group $\Gamma'$

This region is a hyperbolic polygon with $2r+3$ sides, an angle $\alpha = 2\pi/M$ and the remaining angles are equal to $\frac{1}{2}\pi$. The lengths of the sides are $\lambda_0$ (two sides), $2\lambda_i$ $(i=1,\dots,r)$, $\gamma_i$ $(i=1,\dots,r)$ and $\gamma_0+\gamma_{r+1}$.

Like we saw for polygons of Class A, we may construct a hyperbolic right-angled polygon taking as parameters $2\lambda_1, \gamma_2, 2\lambda_2, \gamma_3, \dots, 2\lambda_r$, and a hyperbolic pentagon with four right angles and the fifth angle equal to $\alpha$.

**Class D.** Let $\Gamma'$ be an NEC group with signature

(4.4) $$\bigl(0; +; [2^r], \{(M, 2^s)\}\bigr),$$

where $M, r > 0$ and $s > 0$ take different values according to the respective cases from Table 2.

A Wilkie fundamental region in this case is $W$ where $\sum_{i=1}^{r+1}\theta_i = 2\pi$.

We draw the orthogonal lines to $\gamma_0$ from $X_i$, $i=1,\dots,r$. Let us denote the distance from $X_i$ to $\gamma_0$ by $\lambda_i$. The side $\gamma_0$ becomes divided in several parts labelled $\gamma_1^*, \gamma_2^*, \dots, \gamma_{r+1}^*$, according to the 'opposite' angle $\theta_i$.

In a similar way to Class C we cut by $\lambda_r$ and we transform the region $W$ in the new region

$$\big(W - (\xi'_r, \varepsilon, \gamma_1^*, \lambda_r)\big) \cup x_r(\xi'_r, \varepsilon, \gamma_1^*, \lambda_r\big).$$

Repeating this process for $x_{r-1}, x_{r-2}, \ldots, x_1$ we obtain the polygon $P_D$

This polygon has an angle $\alpha = \pi/M$ and the remaining angles are equal to $\frac{1}{2}\pi$. The lengths of the sides are $2\lambda_i$, $i = 1, \ldots, r$, $\gamma_i^*$, $i = 1, \ldots, r-1$, $\gamma_1, \gamma_2, \ldots, \gamma_{s+1}, \gamma_r^*$.

We may construct a right-angled polygon with sides

$$2\lambda_1, \gamma_2^*, \ldots, 2\lambda_r, \gamma_{s+1} + \gamma_r^*, \gamma_s, \ldots, \gamma_3, a, b, c,$$

where the lengths $a$, $b$, $c$ are determined by the lengths of the remaining angles. Now we construct a hyperbolic quadrilateral with three right angles, the fourth angle equal to $\alpha$ and the opposite side to $\alpha$ with length $b$. Joining this quadrilateral to the right-angled polygon by pasting the sides '$b$' we obtain the region $P_D$ for the group $\Gamma'$ with signature (4.4)

**Class E.** The group $\Gamma'$ has the following signature

$$\big(0; +; [M, 2^r], \{(2^s)\}\big), \tag{4.5}$$

where $M, r > 0$ and $s > 0$ take different values according to the respective cases from Table 2.

A Wilkie fundamental region for $\Gamma'$ is in this case with $\sum_{i=0}^{r+1} \theta_i = 2\pi$. See the above figure.

We draw the orthogonal lines from $X_0, X_1, \ldots, X_r$ to $\gamma_0$. This last side becomes divided in parts labelled $\gamma_0^*, \gamma_1^*, \ldots, \gamma_{r+1}^*$, according to the 'opposite' angle $\theta_i$. Let $\lambda_i$ be the segment from $X_i$ to $\gamma_0$. We cut the region $W$ by $\lambda_r$ and we transform $W$ in a new region

$$\big(W - (\lambda_r, \gamma_r^*, \varepsilon, \xi_r')\big) \cup x_r(\lambda_r, \gamma_r^*, \varepsilon, \xi_r').$$

Now we cut by $\lambda_{r-1}$ and so on. Repeating the process we finally obtain a polygon with an angle $\alpha = 2\pi/M$, the remaining angles equal to $\frac{1}{2}\pi$ and sides labelled in counterclockwise

$$\begin{gathered}\gamma_0^*, \gamma_1, \ldots, \gamma_s, x_0 x_1 \cdots x_r(\lambda_r), x_0 x_1 \cdots x_{r-1}(\lambda_r),\\ x_0 x_1 \cdots x_{r-1}(\gamma_{r-1}^*), x_0 x_1 \cdots x_{r-1}(\lambda_{r-1}),\\ x_0 x_1 \cdots x_{r-2}(\lambda_{r-2}), x_0 x_1 \cdots x_{r-2}(\gamma_{r-2}^*), \ldots, x_0 x_1(\lambda_1),\\ x_0(\lambda_1), x_0(\gamma_1^*), x_0(\lambda_0), \lambda_0.\end{gathered}$$

We may construct a hyperbolic right-angled polygon with parameters

$$\gamma_1, \gamma_2, \ldots, \gamma_s, 2\lambda_r, \gamma_{r-1}^*, 2\lambda_{r-1}, \gamma_{r-2}^*, \ldots, 2\lambda_1,$$

and the three remaining sides $a$, $b$, $c$, are determined by these parameters.

The length $b$ and the angle $\alpha$ determine a unique pentagon with the condition that the two sides of $\sigma$ be equal. We join this pentagon to the right-angled polygon by pasting the sides '$b$' and we finally obtain the region $P_E$ that is a fundamental region for the group $\Gamma'$ with signature (4.5).

## 5. Parametrization of the moduli space

Let $\Gamma$ be an abstract surface Fuchsian group of genus $p$. We consider $\Omega = \mathrm{Aut}(D)$ as a topological group. The *Weil space* of $\Gamma$ with respect to $\Omega$ is the set

$$R(\Gamma) = \{\text{group of monomorphisms } r\colon \Gamma \to \Omega \text{ such that } r(\Gamma) \text{is a surface Fuchsian group}\}.$$

The *Teichmüller space* of $\Gamma$ is the orbit space $T(\Gamma) = R(\Gamma)/\operatorname{Aut}(\Omega)$. The *modular space* of $\Gamma$ is $\operatorname{Mod}(\Gamma) = \operatorname{Aut}(\Gamma)/\operatorname{Inn}(\Gamma)$ and the *moduli space* $M(p)$ is $T(\Gamma)/\operatorname{Mod}(\Gamma)$. We may also define $M(p)$ as the set of complex structures on surfaces of genus $p$ modulo orientation preserving diffeomorphisms [SS].

In this section we want to obtain a parametrization of the equisymmetric strata of the subspace of $M(p)$ given by hyperelliptic and symmetric Riemann surfaces of genus $p$. Now we need to know in our first definition of moduli space which is the subspace to be parametrized.

We are interested in the parametrization of the Fuchsian groups $F$ such that $D/F$ is a symmetric and hyperelliptic Riemann surface modulo conjugation by elements of $\Omega$ and elements of $\operatorname{Mod}(F)$. If $F$ uniformizes a hyperelliptic and symmetric Riemann surface then $F < \Gamma'$ and $\Gamma'$ has one of the signatures from Table 2.

Given a group $\Gamma'$ in some cases it is possible to obtain several groups $F$, with $F < \Gamma'$ such that $D/F$ is a hyperelliptic and symmetric Riemann surface. In fact if $\Gamma'/F \simeq G$, each epimorphism $\theta\colon \Gamma' \to G$ such that $\theta$ has a kernel which is a surface Fuchsian group of genus $p$ and $\theta$ induces a diagram as one of Section 3 and then $D/\ker\theta$ is a symmetric and hyperelliptic Riemann surface.

Let $E(F,\Gamma')$ be the set of epimorphisms

$$\theta\colon \Gamma' \to G \simeq \Gamma'/F,$$

such that $\ker\theta$ is a hyperelliptic and symmetric Riemann surface modulo the following equivalence relation:

$$\theta \approx \theta' \qquad \Longleftrightarrow \qquad \ker\theta' = \ker\theta.$$

Each element $\theta$ of $E(F,\Gamma')$ gives us a way to construct a fundamental region $R_F$ of $F$ as union of $n$ (= index of $F$ in $\Gamma'$) copies of a fundamental region $R'$ of $\Gamma'$ and the sides of $R_F$ identified according to some rules given by $\theta$. Let us note that $E(F,\Gamma')$ is a finite set.

Let $\sigma$ be a signature from Table 2 and column $\sigma(\Gamma')$. Let $P_\sigma$ be the set of classes of congruence of polygons that are fundamental regions of a group with signature $\sigma$ and with the geometrical conditions described in Section 4. Each element of $P_\sigma$ provides a group $\Gamma'$ with signature $\sigma$ and a set of epimorphisms $E(F,\Gamma')$. Let $P_\sigma \times E_\sigma$ be the set of pairs given by the class of congruence of polygons and the set of classes of epimorphisms corresponding to the group $\Gamma'$ having as fundamental region a polygon of the given class. Each pair of $P_\sigma \times E_\sigma$ provides in a natural way a group $F$ such that $D/F$ is a hyperelliptic and symmetric Riemann surface and then an element of $M_{h,s}(p)$ (subspace of hyperelliptic and symmetric Riemann surfaces of genus $p$). We have in this way a parametrization:

$$\pi\colon P_s \times E_s \to M_{h,s}(p).$$

**Theorem.** *If the signature $\sigma$ has at most one proper period then $\pi$ is a finite-to-one map.*

*Proof.* Let $S$ be a surface representing a point of $M_{h,s}(p)$ in the image of $P_s \times E_s$ by $\pi$. Let $G$ be the subgroup of $\mathrm{Aut}(S)$ such that $S/G \simeq D/\Gamma'$ where $\Gamma'$ has signature $\sigma$ and there is $\theta \in E_\sigma$ such that $\theta\colon \Gamma' \to G$ and $\ker\theta$ uniformizes $S$. If $\Gamma'$ has no proper periods then $\Gamma'$ only admits one element of $P_\sigma$ as a fundamental region. Since $\mathrm{Aut}(S)$ is finite there is only a finite number of NEC groups with signature as $\Gamma'$ containing the Fuchsian group which uniformizes $S$. In this case $\pi$ is a finite-to-one map.

If there is a proper period in the signature $\sigma$ then the orbifold $D/\Gamma'$ has a unique conic point and the polygons of $P_\sigma$ are obtained from different ways by cutting on the orthogonal line from the conic point to the boundary of $D/\Gamma'$. Since the number of such 'cuts' is finite then the number of elements in $P_\sigma$ for a given group $\Gamma'$ is finite. As $E_\sigma$ is finite for a given group $\Gamma'$ we obtain again that $\pi$ is a finite-to-one map.

If there are more than one proper period in $\sigma$ then $\pi^{-1}$ can be an infinite set as we may see in the following example.

Let $\sigma$ be the signature $\big(0; +; [2,2]\{(2,2,2)\}\big)$. Then there are infinite elements of $P_\sigma$ corresponding to the same $\Gamma'$ and so they give by $\pi$ the same element in $M_{h,s}(p)$. To see it let us consider the element $P \in P_\sigma$ represented by the following polygon:

We cut by the orthogonal line to $\xi'$ and $\gamma_{3,3}$; by the orthogonal line from $X_1$ to $\gamma_{3,3}$ and by the orthogonal line from $X_2$ to $\xi_1$ as it is showed in the next figure. Identifying the sides properly we obtain a new element $P_1 \in P_\sigma$ that is, in general, different from $P$.

We can repeat this process as many times as we want and we obtain different elements of $P_\sigma$ providing by $\pi$ the same element of $M_{h,s}(p)$.

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