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ON THE ZEROS OF THE WRONSKIAN OF AN ENTIRE OR MEROMORPHIC FUNCTION AND ITS DERIVATIVES

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Abstract. Let g be a meromorphic function in the complex plane, and define the homogeneous differential polynomial ψ by $\psi = W(g, g^{(k_1)}, g^{(k_2)}, \dots, g^{(k_{n-1})})$ where W denotes the Wronskian and $k_1, k_2, \dots, k_{n-1}$ are pairwise distinct positive integers.

In the case of an entire function g, we give sharp upper and lower bounds for the Nevanlinna counting function $N(r, 1/\psi)$ of the zeros of ψ in terms of $N(r, 1/g)$. In particular, we show that if g is not an exponential sum then ψ has few zeros in the sense that $N(r, 1/\psi) = S(r, g)$ if and only if $N(r, 1/g) = S(r, g)$. One of the main tools is a new result on the proximity function of quotients of certain Wronskians which might be of independent interest.

For meromorphic functions g, we present two methods to obtain lower bounds for $N(r, 1/\psi)$ in terms of $N(r, 1/g)$ and $\overline{N}(r, g)$. As a tool, we give formulas for the coefficients of the greatest common divisor of two linear differential operators.

1. Introduction

In this paper the term "meromorphic" will always mean meromorphic in the complex plane $\mathbf{C}$. We use the standard notations and results of the Nevanlinna theory (see [9], [5], or [3] for example). In particular, $S(r, f)$ plays the role of an error term. $\mathbf{N}_0 = \{0, 1, 2, \dots\}$ is the set of non-negative integers.

Let f be a non-constant meromorphic function. If a is a meromorphic function satisfying $T(r, a) = S(r, f)$ and if $k_0, k_1, \dots, k_n \in \mathbf{N}_0$ then

$$M[f] = a f^{k_0} (f')^{k_1} \cdots (f^{(n)})^{k_n}$$

is a *differential monomial* (in f). The degree γ_M and the weight Γ_M of M are defined by

$$\gamma_M = k_0 + k_1 + \cdots + k_n, \qquad \Gamma_M = k_0 + 2k_1 + \cdots + (n+1)k_n.$$

A finite sum $P[f] = \sum_{j=1}^{m} M_j[f]$ of differential monomials is a *differential polynomial* (in f). Degree γ_P and weight Γ_P of P are defined by $\gamma_P = \max_{j=1,\dots,m} \gamma_{M_j}$ and $\Gamma_P = \max_{j=1,\dots,m} \Gamma_{M_j}$. P is called *homogeneous* if $\gamma_{M_j} = \gamma_P$ for $j = 1, \dots, m$.

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Starting with the fundamental work of Hayman [4], many papers have been written on the following problem: Decide whether a given differential polynomial in an entire or meromorphic function has (infinitely many) zeros, and if not, determine the exceptional cases.

This article is concerned with homogeneous differential polynomials. The work was motivated by the following result of Mues [8, Satz 1].

Theorem A. *Let* g *be an entire function and* $\psi = gg'' - ag'^2$ *where* $a \in \mathbf{C} \setminus \{1\}$. *If* ψ *has no zeros then one of the following cases must occur*:

(i) $g(z) = e^{Az+B}$ *where* $A, B \in \mathbf{C}$, $A \neq 0$;
(ii) $a \neq 0$ *and* $g(z) = Az + B$ *where* $A, B \in \mathbf{C}$, $A \neq 0$;
(iii) $a = \frac{1}{2}$ *and* $g(z) = Az^2 + Bz + C$ *where* $A, B, C \in \mathbf{C}$, $4AC - B^2 \neq 0$.

Results for similar homogeneous differential polynomials were obtained by Ozawa [11] and G. Lehners [7].

Mues pointed out that Theorem A fails for $a = 1$. He gave the following example. If $g = \exp(Q)$ with an entire function Q then $\psi = gg'' - g'^2 = Q''e^{2Q}$, thus ψ has no zeros if $Q = \iint \exp(h)$ with an entire function h. Even for an arbitrary entire function Q we have

$$N\Big(r, \frac{1}{\psi}\Big) = N\Big(r, \frac{1}{Q''}\Big) \leq T(r, Q'') + O(1) = S(r, g)$$

by the lemma of the logarithmic derivative. Hence ψ has "few" zeros if g has no zeros. Therefore, it seems sensible to compare the Nevanlinna counting function of the zeros of ψ and the counting function of the zeros of g.

The aim of this paper is to give estimates for the counting function of the zeros not only for $W(g, g') = gg'' - g'^2$ but more generally for Wronskians of the form

$$\psi = W\big(g, g^{(k_1)}, g^{(k_2)}, \dots, g^{(k_{n-1})}\big) \tag{1.1}$$

where $k_1, k_2, \dots, k_{n-1} \in \mathbf{N}$ are pairwise distinct. ψ is a homogeneous differential polynomial where all monomials not only have the same degree but also the same weight.

It is necessary to distinguish whether g is or is not an exponential sum. We call g an *exponential sum* if

$$g(z) = c_1(z)e^{\gamma_1 z} + \dots + c_p(z)e^{\gamma_p z} \tag{1.2}$$

where $p \in \mathbf{N}$, $c_1, \dots, c_p$ are polynomials (not identically zero) and $\gamma_1, \dots, \gamma_p \in \mathbf{C}$ are pairwise distinct. The number $\sum_{j=1}^{p}(1 + \deg c_j)$ will be called the *order* of g. Note that if ψ in (1.1) is identically zero then g must be an exponential sum.

For entire functions g and the Wronskians

$$\psi_n = W\big(g, g', \dots, g^{(n-1)}\big), \qquad n \in \mathbf{N}, \tag{1.3}$$

sharp upper and lower bounds were obtained in [13, Theorem 1 and Theorem 2]:

Theorem B. *Let g be an entire function which is not an exponential sum and define ψ_n by* (1.3)*. Then for every $n \in \mathbf{N}$ and every $\varepsilon > 0$*

$$(1.4) \qquad (1-\varepsilon)N\Big(r, \frac{1}{g}\Big) + S(r,g) \le N\Big(r, \frac{1}{\psi_n}\Big) \le nN\Big(r, \frac{1}{g}\Big) + S(r,g).$$

The factor $(1-\varepsilon)$ cannot be replaced by any factor greater than 1 and the factor n cannot be replaced by any smaller factor.

Theorem C. *Let g be an exponential sum of order m and define ψ_n by* (1.3)*. Then for $n = 1, \dots, m-1$*

$$(1.5) \qquad N\Big(r, \frac{1}{\psi_n}\Big) = N\Big(r, \frac{1}{\psi_{m-n}}\Big) + O(\log r)$$

and

$$(1.6) \qquad N\Big(r, \frac{1}{g}\Big) + O(\log r) \le N\Big(r, \frac{1}{\psi_n}\Big) \le \min\{n, m-n\}N\Big(r, \frac{1}{g}\Big) + O(\log r).$$

The bounds in (1.6) *are sharp.*

Tohge [16] investigated the differential polynomial $\psi = gg'' - ag'^2$, $a \in \mathbf{C}$, for meromorphic functions g. In the case $a = 1$, i.e. $\psi = W(g, g')$, he obtained the following result ([16, Theorem 2] in a slightly different notation).

Theorem D. *Let g be a non-constant meromorphic function and assume that $\psi = gg'' - g'^2 \not\equiv 0$. Then either*

$$\overline{N}\Big(r, \frac{1}{g}\Big) \le 5\overline{N}\Big(r, \frac{1}{\psi}\Big) + 4\overline{N}(r,g) + S(r,g),$$

or $g(z) = c_1 \exp(\lambda_1 z) + c_2 \exp(\lambda_2 z)$ where $\lambda_1, \lambda_2, c_1, c_2 \in \mathbf{C}$, $\lambda_1 \ne \lambda_2$ and $c_1, c_2 \ne 0$.

In Section 2, we first collect some basic facts about Wronskians. Then we define *generalized Wronskians* and prove new results on the proximity functions of quotients of generalized Wronskians.

In Section 3, we give sharp estimates for the counting function of the zeros of the Wronskians (1.1) where g is an entire function and $k_1, \dots, k_{n-1} \in \mathbf{N}$ are pairwise distinct. Most results in this section are valid only if certain Wronskians are not identically zero or if g is not an exponential sum.

Two examples in Section 4 show that the results of Section 3 do not hold for exponential sums. Only if the order of the exponential sum is large enough, the same estimates can be obtained.

In the sections 5–7, we present two methods to obtain lower estimates for the zeros of the Wronskians (1.1) where the entire function g is now replaced by a meromorphic function f. The first method is to generalize the methods from Section 3 directly, this is done in Section 5. We will consider only the special case $k_j = j$ in this section.

In Section 7, we use a completely different approach. The function f is written as the common solution of two linear differential equations. Elimination of the higher derivatives in these two equations gives a representation of f'/f as a rational function of the coefficients of the differential equations. The formulas for this elimination process are given in Section 6.

This article is based on the author's *Habilitationsschrift* [14].

2. Wronskians and linear differential equations

We need the following generalization of the error term $S(r,f)$. Let $f_1,\dots,f_n$ be meromorphic functions. By $S(r,f_1,\dots,f_n)$ we denote every function ϕ: $(0,\infty) \mapsto \mathbf{R}$ satisfying $\phi(r) = o\big(\sum_{k=1}^n T(r,f_k)\big)$ for $r\to\infty$, $r\notin E$, where $E\subset(0,\infty)$ has finite Lebesgue measure.

We recall some basic facts about Wronskians (see [2, §1], for example).

Lemma 2.1. *Let $u_1,\dots,u_n$, $v_1,\dots,v_m$ and φ be meromorphic functions.*

(i) $W(u_1,\dots,u_n)\equiv 0$ *if and only if the functions $u_1,\dots,u_n$ are linearly dependent.*
(ii) $W(\varphi u_1,\dots,\varphi u_n) = \varphi^n W(u_1,\dots,u_n)$.
(iii) $\big(W(u_1,\dots,u_n)\big)' = \sum_{j=1}^n W(u_1,\dots,u_{j-1},u_j',u_{j+1},\dots,u_n)$.
(iv) $W(u_1,\dots,u_n)^{m-1}W(u_1,\dots,u_n,v_1,\dots,v_m) = W(w_1,\dots,w_m)$, *where the functions $w_1,\dots,w_m$ are defined by $w_j = W(u_1,\dots,u_n,v_j)$, $j=1,\dots,m$.*

An nth order homogeneous linear differential equation

$$w^{(n)} + a_{n-1}(z)w^{(n-1)} + \cdots + a_1(z)w' + a_0(z)w = 0 \tag{2.1}$$

with meromorphic coefficients $a_0,\dots,a_n$ has in general no meromorphic solutions. If there are meromorphic solutions, at most n of them are linearly independent and we call a system $(f_1,\dots,f_n)$ of n linearly independent solutions a *meromorphic fundamental system* of (2.1).

If $f_1,\dots,f_n$ are linearly independent meromorphic functions then there is exactly one equation of the form (2.1) which has $(f_1,\dots,f_n)$ as a fundamental system. This equation can be written as

$$\frac{W(f_1,\dots,f_n,w)}{W(f_1,\dots,f_n)} = 0.$$

The following estimates for the proximity function of the coefficients of a linear differential equation is essential for our results. They are essentially due to Frei [1] but Frei formulated the estimates only for entire functions and not explicitly in the form we need here. For the sake of completeness we give a proof.

Lemma 2.2. *Let* $a_0, \dots, a_{n-1}$ *be meromorphic functions. If* $(f_1, \dots, f_n)$ *is a meromorphic fundamental system of the differential equation*

$$w^{(n)} + a_{n-1}(z)w^{(n-1)} + \cdots + a_1(z)w' + a_0(z)w = 0 \tag{2.2}$$

then we have

$$m(r, a_k) = O\Big(\sum_{j=1}^{n} m\Big(r, \frac{f_j'}{f_j}\Big)\Big) + \sum_{j=1}^{n} S\Big(r, \frac{f_j'}{f_j}\Big), \tag{2.3}$$

and, in particular, $m(r, a_k) = S(r, f_1, \dots, f_n)$ *for* $k = 1, \dots, n-1$.

Proof. We prove the lemma using induction on n. For $n = 1$ the differential equation (2.2) reads $w' + a_0(z)w = 0$ and (2.3) follows by the lemma of the logarithmic derivative. Now let $n \geq 2$. The substitution $w = f_1 \int v$ in (2.2) gives

$$v^{(n-1)} + b_{n-2}(z)v^{(n-2)} + \cdots + b_1(z)v' + b_0(z)v = 0 \tag{2.4}$$

where

$$b_k = a_{k+1} + \sum_{j=k+2}^{n-1} \binom{j}{k+1} a_j \frac{f_1^{(j-k-1)}}{f_1} + \binom{n}{k+1} \frac{f_1^{(n-k-1)}}{f_1} \tag{2.5}$$

for $k = 0, \dots, n-2$. The reduced differential equation (2.4) has the fundamental system

$$v_1 := \Big(\frac{f_2}{f_1}\Big)', \dots, v_{n-1} := \Big(\frac{f_n}{f_1}\Big)'.$$

The induction hypotheses gives

$$m(r, b_k) = O\Big(\sum_{j=1}^{n-1} m\Big(r, \frac{v_j'}{v_j}\Big)\Big) + \sum_{j=1}^{n-1} S\Big(r, \frac{v_j'}{v_j}\Big) \tag{2.6}$$

for $k = 0, \dots, n-2$. Using

$$\frac{v_j'}{v_j} = \frac{\Big(\frac{f_{j+1}}{f_1}\Big)''}{\Big(\frac{f_{j+1}}{f_1}\Big)'} = \frac{\Big(\frac{f_{j+1}'}{f_{j+1}} - \frac{f_1'}{f_1}\Big)'}{\frac{f_{j+1}'}{f_{j+1}} - \frac{f_1'}{f_1}} + \frac{f_{j+1}'}{f_{j+1}} - \frac{f_1'}{f_1}$$

and the lemma of the logarithmic derivative we conclude from (2.6)

$$m(r, b_k) = O\Big(\sum_{j=1}^{n} m\Big(r, \frac{f_j'}{f_j}\Big)\Big) + \sum_{j=1}^{n} S\Big(r, \frac{f_j'}{f_j}\Big). \tag{2.7}$$

From (2.7) and (2.5) the assertion (2.3) follows successively for $k = n-1, n-2, \dots, 1$. Finally, the assertion (2.3) for $k = 0$ follows by substituting $w = f_1$ in the differential equation (2.2). □

Definition 2.1. Let $f_1,\dots,f_n$ be meromorphic functions and $j_1,\dots,j_n\in\mathbf{N}_0$. The function $W_{j_1,\dots,j_n}(f_1,\dots,f_n)$ defined by

$$W_{j_1,\dots,j_n}(f_1,\dots,f_n)=\begin{vmatrix} f_1^{(j_1)} & f_2^{(j_1)} & \dots & f_n^{(j_1)}\\ f_1^{(j_2)} & f_2^{(j_2)} & \dots & f_n^{(j_2)}\\ \vdots & \vdots & & \vdots\\ f_1^{(j_n)} & f_2^{(j_n)} & \dots & f_n^{(j_n)}\end{vmatrix}$$

is called a *generalized Wronskian* of $f_1,\dots,f_n$.

Let $f_1,\dots,f_n$ be linearly independent functions and let

$$w^{(n)}+a_{n-1}(z)w^{(n-1)}+\dots+a_1(z)w'+a_0(z)w=0$$

be the differential equation with fundamental system $(f_1,\dots,f_n)$. Then

$$a_j=(-1)^{n-j}\frac{W_{0,\dots,j-1,j+1,\dots,n}(f_1,\dots,f_n)}{W(f_1,\dots,f_n)}$$

for $j=0,\dots,n-1$ and Lemma 2.2 shows that $m(r,a_j)=S(r,f_1,\dots,f_n)$. We now prove a corresponding result for quotients of generalized Wronskians.

Lemma 2.3. *Let $f_1,\dots,f_n$ be linearly independent meromorphic functions and $j_1,\dots,j_n\in\mathbf{N}_0$. Then*

$$m\Big(r,\frac{W_{j_1,\dots,j_n}(f_1,\dots,f_n)}{W(f_1,\dots,f_n)}\Big)=S(r,f_1,\dots,f_n).$$

Proof. Let

$$w^{(n)}+a_{n-1}(z)w^{(n-1)}+\dots+a_1(z)w'+a_0(z)w=0$$

be the linear differential equation with fundamental system $(f_1,\dots,f_n)$. Lemma 2.2 gives $m(r,a_j)=S(r,f_1,\dots,f_n)$ for $j=0,\dots,n-1$. Differentiating the equation

$$f_j^{(n)}=-a_0f_j-a_1f_j'-\dots-a_{n-1}f_j^{(n-1)}$$

successively yields

$$f_j^{(k)}=A_{k,0}f_j+A_{k,1}f_j'+\dots+A_{k,n-1}f_j^{(n-1)}$$

for $k\in\mathbf{N}_0$ and $j=1,\dots,n$, where $A_{k,l}$ are meromorphic functions satisfying $m(r,A_{k,l})=S(r,f_1,\dots,f_n)$. Now the assertion follows directly from the matrix equation

$$\begin{pmatrix} f_1^{(j_1)} & f_2^{(j_1)} & \cdots & f_n^{(j_1)} \\ f_1^{(j_2)} & f_2^{(j_2)} & \cdots & f_n^{(j_2)} \\ \vdots & \vdots & & \vdots \\ f_1^{(j_n)} & f_2^{(j_n)} & \cdots & f_n^{(j_n)} \end{pmatrix} = \begin{pmatrix} A_{j_1,0} & A_{j_1,1} & \cdots & A_{j_1,n-1} \\ A_{j_2,0} & A_{j_2,1} & \cdots & A_{j_2,n-1} \\ \vdots & \vdots & & \vdots \\ A_{j_n,0} & A_{j_n,1} & \cdots & A_{j_n,n-1} \end{pmatrix} \times \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix}. \square$$

Lemma 2.4. *Let $f_1, \ldots, f_n$ be linearly independent meromorphic functions and $k \in \mathbf{N}$. Then*

$$m\left(r, \frac{\sum_{j=1}^n W(f_1, \ldots, f_{j-1}, f_j^{(k)}, f_{j+1}, \ldots, f_n)}{W(f_1, \ldots, f_n)}\right) = S(r, f_1, \ldots, f_n).$$

Proof. Let

$$\psi := \sum_{j=1}^n W(f_1, \ldots, f_{j-1}, f_j^{(k)}, f_{j+1}, \ldots, f_n).$$

We want to show that ψ can be written as a linear combination of some generalized Wronskians $W_{j_1,\ldots,j_n}(f_1, \ldots, f_n)$ with suitably chosen $(j_1, \ldots, j_n)$. To this aim let $m := n + k$. We define a function $G : \underbrace{\mathbf{C}^m \times \cdots \times \mathbf{C}^m}_{n} \mapsto \mathbf{C}$ by

$$G(x_1, \ldots, x_n) = \sum_{j=1}^n \begin{vmatrix} x_{1,1} & x_{2,1} & \cdots & x_{j-1,1} & x_{j,k+1} & x_{j+1,1} & \cdots & x_{n,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{j-1,2} & x_{j,k+2} & x_{j+1,2} & \cdots & x_{n,2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_{1,n} & x_{2,n} & \cdots & x_{j-1,n} & x_{j,m} & x_{j+1,n} & \cdots & x_{n,n} \end{vmatrix}$$

where $x_l = (x_{l,1}, x_{l,2}, \ldots, x_{l,m})^T$ for $l = 1, \ldots, n$. G is an alternating multilinear form. A basis for the vector space of all alternating multilinear forms on $(\mathbf{C}^m)^n$ are the functions

$$F_{j_1,\ldots,j_n}(x_1, \ldots, x_n) = \begin{vmatrix} x_{1,j_1} & x_{2,j_1} & \cdots & x_{n,j_1} \\ x_{1,j_2} & x_{2,j_2} & \cdots & x_{n,j_2} \\ \vdots & \vdots & & \vdots \\ x_{1,j_n} & x_{2,j_n} & \cdots & x_{n,j_n} \end{vmatrix}, \quad 1 \le j_1 < j_2 < \cdots < j_n \le m$$

(see [15, Theorem 4–5 and Problem 4–1], for example). Hence there are numbers $c_{j_1,\dots,j_n} \in \mathbf{C}$ such that

$$G(x_1,\dots,x_n) = \sum_{1\le j_1<j_2<\cdots<j_n\le m} c_{j_1,\dots,j_n} F_{j_1,\dots,j_n}(x_1,\dots,x_n)$$

for arbitrary vectors $x_1,\dots,x_n \in \mathbf{C}^m$. Setting $x_l = \big(f_l(z), f_l'(z),\dots,f_l^{(m-1)}(z)\big)^T$ for $l = 1,\dots,n$ gives

$$\psi = \sum_{1\le j_1<j_2<\cdots<j_n\le m} c_{j_1,\dots,j_n} W_{j_1-1,j_2-1,\dots,j_n-1}(f_1,\dots,f_n).$$

The assertion follows using Lemma 2.3. ▫

3. Results for entire functions

First we give two examples which will show the sharpness of our estimates.

3.1. Examples. Let h be an entire function which is not a polynomial of degree at most one. For every $j \in \mathbf{N}_0$ we have $(e^h)^{(j)} = \tau_j[h']e^h$ where $\tau_j[h']$ is a differential polynomial in h'. In particular, $T(r,\tau_j[h']) = S(r,e^h)$.

Example 3.1. Let $g = \exp(h) + \exp(-h)$ and define ψ by (1.1) where $1 \le k_1 < k_2 < \cdots < k_{n-1}$. Then

$$N\Big(r,\frac{1}{\psi}\Big) = nN\Big(r,\frac{1}{g}\Big) + S(r,g).$$

Proof. We have

$$\begin{aligned}\psi &= W\big(e^h + e^{-h}, \tau_{k_1}[h']e^h + \tau_{k_1}[-h']e^{-h},\dots,\tau_{k_{n-1}}[h']e^h + \tau_{k_{n-1}}[-h']e^{-h}\big)\\ &= A_n e^{nh} + A_{n-2}e^{(n-2)h} + \cdots + A_{-n+2}e^{(-n+2)h} + A_{-n}e^{-nh},\end{aligned}$$

where $A_n, A_{n-2},\dots,A_{-n+2},A_{-n}$ are differential polynomials in h'. Hence $T(r,A_j) = S(r,e^h)$ for $j = -n,-n+2,\dots,n-2,n$. In particular,

$$A_n = W\big(1,\tau_{k_1}[h'],\dots,\tau_{k_{n-1}}[h']\big) \quad \text{and} \quad A_{-n} = W\big(1,\tau_{k_1}[-h'],\dots,\tau_{k_{n-1}}[-h']\big).$$

Assume that $A_n \equiv 0$. Then the functions

$$e^h, (e^h)^{k_1},\dots,(e^h)^{(k_{n-1})}$$

are linearly dependent, thus $w = e^h$ is the solution of a homogeneous linear differential equation with constant coefficients. It follows that e^h is an exponential sum and hence the order of e^h is less than or equal to one. The sharp estimate of the proximity function of the logarithmic derivative (see [10], for example) gives $m(r,h') = m\big(r,(e^h)'/e^h\big) = o(\log r)$. This is only possible if h' is constant in contradiction to our assumption.

Thus $A_n \not\equiv 0$ and, analogously, $A_{-n} \not\equiv 0$. Using [13, Lemma 2] we conclude that

$$N\Big(r,\frac{1}{\psi}\Big) = 2nT(r,e^h) + S(r,e^h) = nN\Big(r,\frac{1}{g}\Big) + S(r,g). \ \square$$

Example 3.2. Let $g = 1 + \exp(h)$ and define ψ by (1.1) where $1 \le k_1 < k_2 < \cdots < k_{n-1}$. Then

$$N\Big(r,\frac{1}{\psi}\Big) = N\Big(r,\frac{1}{g}\Big) + S(r,g).$$

Proof. Now we have

$$\psi = W\big(1+e^h, \tau_{k_1}[h']e^h, \tau_{k_2}[h']e^h, \ldots, \tau_{k_{n-1}}[h']e^h\big) = A_{n-1}e^{(n-1)h} + A_n e^{nh}$$

where

$$A_{n-1} = W\big(\tau_{k_1+1}[h'], \tau_{k_2+1}[h'], \ldots, \tau_{k_{n-1}+1}[h']\big)$$

and

$$A_n = W\big(1, \tau_{k_1}[h'], \tau_{k_2}[h'], \ldots, \tau_{k_{n-1}}[h']\big).$$

Arguing as in the proof of Example 3.1, we get

$$N\Big(r,\frac{1}{\psi}\Big) = T(r,e^h) + S(r,e^h) = N\Big(r,\frac{1}{g}\Big) + S(r,g). \ \square$$

3.2. An estimate from above. For an entire function g we can estimate the counting function of the zeros of the Wronskians (1.1) from above in terms of the counting function of the zeros of g.

Theorem 3.1. *Let g be an entire function and*

$$\psi = W\big(g, g^{(k_1)}, g^{(k_2)}, \ldots, g^{(k_{n-1})}\big)$$

where $1 \le k_1 < k_2 < \cdots < k_{n-1}$. If $\psi \not\equiv 0$ then

$$N\Big(r,\frac{1}{\psi}\Big) \le nN\Big(r,\frac{1}{g}\Big) + S(r,g).$$

The factor n is best possible.

Proof. From

$$\psi = g^n W\Big(1, \frac{g^{(k_1)}}{g}, \frac{g^{(k_2)}}{g}, \ldots, \frac{g^{(k_{n-1})}}{g}\Big)$$

and the lemma of the logarithmic derivative we conclude that $m(r,\psi/g^n) = S(r,g)$. Hence

$$\begin{aligned} N\Big(r,\frac{1}{\psi}\Big) - nN\Big(r,\frac{1}{g}\Big) &= N\Big(r,\frac{g^n}{\psi}\Big) - N\Big(r,\frac{\psi}{g^n}\Big) \\ &= m\Big(r,\frac{\psi}{g^n}\Big) - m\Big(r,\frac{g^n}{\psi}\Big) + O(1) \le S(r,g) \end{aligned}$$

and the assertion follows. Example 3.1 shows that the factor n is best possible. $\square$

Remark 3.1. In Theorem 3.2, the Wronskian $W\big(g, g^{(k_1)}, g^{(k_2)}, \dots, g^{(k_{n-1})}\big)$ can be replaced by an arbitrary homogeneous differential polynomial in g with degree n.

3.3. Estimates from below. Now we estimate the counting function of the zeros of the Wronskian (1.1) from below in terms of the counting function of the zeros of g. The main tools are Lemma 2.4 and the following lemma.

Lemma 3.1. *Let g be an entire function, $k_0, k_1, \dots, k_n \in \mathbf{N}_0$ pairwise distinct and assume that*

$$W\big(g^{(k_0)}, g^{(k_1)}, \dots, g^{(k_n)}\big) \not\equiv 0. \tag{3.1}$$

Then

$$\begin{aligned}
&N\Big(r, \frac{1}{W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}\big)}\Big) + N\Big(r, \frac{1}{W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_{n-1})}, g^{(k_n)}\big)}\Big) \\
&\le N\Big(r, \frac{1}{W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_{n-1})}\big)}\Big) + N\Big(r, \frac{1}{W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_n)}\big)}\Big) + S(r,g).
\end{aligned}$$

Proof. Using Lemma 2.1 (iv) we get

$$\begin{aligned}
&W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}\big) \cdot W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_{n-1})}, g^{(k_n)}\big) \\
&\quad = W\Big(W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_{n-1})}\big), W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_n)}\big)\Big) \\
&\quad = W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_{n-1})}\big) \cdot W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_n)}\big) \cdot q
\end{aligned} \tag{3.2}$$

where we have set

$$q := \frac{W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_n)}\big)'}{W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_n)}\big)} - \frac{W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_{n-1})}\big)'}{W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_{n-1})}\big)}.$$

The assumption (3.1) guarantees that all the Wronskians are not identically zero. Applying the first main theorem to (3.2) gives

$$\begin{aligned}
N\Big(r, &\frac{1}{W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}\big)}\Big) + N\Big(r, \frac{1}{W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_{n-1})}, g^{(k_n)}\big)}\Big) \\
&- N\Big(r, \frac{1}{W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_{n-1})}\big)}\Big) \\
&- N\Big(r, \frac{1}{W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_n)}\big)}\Big) \\
&\quad = m(r,q) - m\Big(r, \frac{1}{q}\Big) + O(1) \le S(r,g).
\end{aligned}$$

The last inequality follows from the lemma of the logarithmic derivative. □

Theorem 3.2. *Let g be an entire function and $k \in \mathbf{N}$. Define functions ψ_n, $n \in \mathbf{N}_0$, by $\psi_0 = 1$ and*

$$\psi_n = W\big(g, g^{(k)}, g^{(2k)}, \dots, g^{(nk-k)}\big) \qquad \text{for } n \in \mathbf{N}.$$

If $m \in \mathbf{N}$ and $\psi_n \not\equiv 0$ for $n = 1, \dots, m$ then

$$N\Big(r, \frac{1}{\psi_b}\Big) \geq \frac{c-b}{c-a} N\Big(r, \frac{1}{\psi_a}\Big) + \frac{b-a}{c-a} N\Big(r, \frac{1}{\psi_c}\Big) + S(r, g) \tag{3.3}$$

for $0 \leq a < b < c \leq m$.

Proof. Setting $f_j = g^{(jk-k)}$, $j = 1, \dots, n$, in Lemma 2.4 gives

$$m\Big(r, \frac{W\big(g, g^{(k)}, \dots, g^{(nk-2k)}, g^{(nk)}\big)}{W\big(g, g^{(k)}, \dots, g^{(nk-2k)}, g^{(nk-k)}\big)}\Big) = S(r, g)$$

for $n = 1, \dots, m-1$. It follows that

$$N\Big(r, \frac{1}{W\big(g, g^{(k)}, \dots, g^{(nk-2k)}, g^{(nk)}\big)}\Big) \leq N\Big(r, \frac{1}{\psi_n}\Big) + S(r, g).$$

Together with Lemma 3.1 (with $k_j = jk$, $j = 0, \dots, n$) we get the estimate

$$N\Big(r, \frac{1}{\psi_{n-1}}\Big) + N\Big(r, \frac{1}{\psi_{n+1}}\Big) \leq 2N\Big(r, \frac{1}{\psi_n}\Big) + S(r, g). \tag{3.4}$$

Now let $0 \leq a < b < c \leq m$. From (3.4) we conclude that

(3.5)

$$\begin{aligned} S(r,g) &\geq \sum_{\mu=b+1}^{c} \sum_{\nu=b+1}^{\mu-1} \Big\{\Big(N\Big(r, \frac{1}{\psi_{\nu+1}}\Big) - N\Big(r, \frac{1}{\psi_\nu}\Big)\Big) - \Big(N\Big(r, \frac{1}{\psi_\nu}\Big) - N\Big(r, \frac{1}{\psi_{\nu-1}}\Big)\Big)\Big\} \\ &= \sum_{\mu=b+1}^{c} \Big\{\Big(N\Big(r, \frac{1}{\psi_\mu}\Big) - N\Big(r, \frac{1}{\psi_{\mu-1}}\Big)\Big) - \Big(N\Big(r, \frac{1}{\psi_{b+1}}\Big) - N\Big(r, \frac{1}{\psi_b}\Big)\Big)\Big\} \\ &= N\Big(r, \frac{1}{\psi_c}\Big) - N\Big(r, \frac{1}{\psi_b}\Big) - (c-b)\Big(N\Big(r, \frac{1}{\psi_{b+1}}\Big) - N\Big(r, \frac{1}{\psi_b}\Big)\Big) \end{aligned}$$

and that

(3.6)

$$\begin{aligned} S(r,g) &\geq \sum_{\mu=a}^{b-1} \sum_{\nu=\mu+1}^{b} \Big\{\Big(N\Big(r, \frac{1}{\psi_{\nu+1}}\Big) - N\Big(r, \frac{1}{\psi_\nu}\Big)\Big) - \Big(N\Big(r, \frac{1}{\psi_\nu}\Big) - N\Big(r, \frac{1}{\psi_{\nu-1}}\Big)\Big)\Big\} \\ &= \sum_{\mu=a}^{b-1} \Big\{\Big(N\Big(r, \frac{1}{\psi_{b+1}}\Big) - N\Big(r, \frac{1}{\psi_b}\Big)\Big) - \Big(N\Big(r, \frac{1}{\psi_{\mu+1}}\Big) - N\Big(r, \frac{1}{\psi_\mu}\Big)\Big)\Big\} \\ &= (b-a)\Big(N\Big(r, \frac{1}{\psi_{b+1}}\Big) - N\Big(r, \frac{1}{\psi_b}\Big)\Big) - N\Big(r, \frac{1}{\psi_b}\Big) + N\Big(r, \frac{1}{\psi_a}\Big). \end{aligned}$$

Elimination of $N(r, 1/\psi_{b+1})$ from (3.5) and (3.6) gives exactly the assertion (3.3). □

Corollary 3.1. *Let g, k and ψ_n be as in Theorem 3.2. If g is not an exponential sum then*

$$N\Big(r,\frac{1}{\psi_n}\Big)\geq(1-\varepsilon)N\Big(r,\frac{1}{\psi_{n-1}}\Big)+S(r,g) \tag{3.7}$$

for all $n\in\mathbf{N}$ and all $\varepsilon>0$. In particular,

$$N\Big(r,\frac{1}{\psi_n}\Big)\geq(1-\varepsilon)N\Big(r,\frac{1}{g}\Big)+S(r,g) \tag{3.8}$$

for all $n\in\mathbf{N}$ and all $\varepsilon>0$. The factor $1-\varepsilon$ in (3.7) and (3.8) cannot be replaced by any factor greater than 1.

Proof. Theorem 3.2 with $a=n-1$, $b=n$ gives

$$N\Big(r,\frac{1}{\psi_n}\Big)\geq\frac{c-n}{c-n+1}N\Big(r,\frac{1}{\psi_{n-1}}\Big)+S(r,g).$$

By letting $c\to\infty$ the assertion (3.7) follows. Example 3.2 shows that the factor $(1-\varepsilon)$ is best possible in the sense stated in the corollary. □

Remark 3.2. If we neglect the term $S(r,g)$ in equation (3.3), this equation means that for every fixed $r>0$ the function $n\mapsto N(r,1/\psi_n)$ is concave on $\{0,1,\dots,m\}$.

Remark 3.3. Setting $a=0$ in (3.3) and using $N(r,1/\psi_0)=0$ gives

$$\frac{1}{c}N\Big(r,\frac{1}{\psi_c}\Big)\leq\frac{1}{b}N\Big(r,\frac{1}{\psi_b}\Big)+S(r,g)$$

for $0<b<c\leq m$. This is a more precise estimate for the functions ψ_n than the estimate of the more general Theorem 3.1.

Combining Lemma 3.1 and Corollary 3.1 we now prove that the counting function of the zeros of a Wronskian of g and arbitrary derivatives of g can be estimated from below in terms of the counting function of the zeros of g.

Theorem 3.3. *Let g be an entire function which is not an exponential sum. Let $k_0,k_1,\dots,k_{n-1}\in\mathbf{N}_0$ be pairwise distinct. Then we have for every $\varepsilon>0$*

$$N\Big(r,\frac{1}{W\big(g^{(k_0)},\dots,g^{(k_{n-2})},g^{(k_{n-1})}\big)}\Big)\geq(1-\varepsilon)N\Big(r,\frac{1}{W\big(g^{(k_0)},\dots,g^{(k_{n-2})}\big)}\Big)+S(r,g). \tag{3.9}$$

If $\min\{k_0,\dots,k_{n-1}\}=0$ then, in particular, for every $\varepsilon>0$

$$N\Big(r,\frac{1}{W\big(g^{(k_0)},\dots,g^{(k_{n-1})}\big)}\Big)\geq(1-\varepsilon)N\Big(r,\frac{1}{g}\Big)+S(r,g). \tag{3.10}$$

The factor $(1-\varepsilon)$ in (3.9) and (3.10) cannot be replaced by any factor greater than 1.

Proof. We may assume that $\min\{k_0,\dots,k_{n-1}\}=0$. First we prove that

$$N\Big(r,\frac{1}{W\big(g^{(k_0)},\dots,g^{(k_{n-2})},g^{(k_{n-1})}\big)}\Big)\geq N\Big(r,\frac{1}{W\big(g^{(k_0)},\dots,g^{(k_{n-2})}\big)}\Big) - \varepsilon N\Big(r,\frac{1}{g}\Big)+S(r,g) \tag{3.11}$$

for every $\varepsilon>0$. Applying (3.11) repeatedly gives (3.10). Finally (3.9) follows from (3.11) and (3.10).

We prove (3.11) using induction on

$$d:=d(k_0,\dots,k_{n-1}):=\max\{k_0,\dots,k_{n-1}\}-n.$$

Note that $d\geq -1$.

First we consider the case $d=-1$. Then $\{k_0,\dots,k_{n-1}\}$ is a permutation of $\{0,1,\dots,n-1\}$. We set $j:=k_{n-1}$ and define functions ψ_l, $l\in\mathbf{N}_0$, by $\psi_0=1$ and

$$\psi_l:=W\big(g,g',\dots,g^{(l-1)}\big)\qquad\text{for } l\in\mathbf{N}.$$

There exist $s_1,s_2\in\{-1,1\}$ such that

$$W\big(g^{(k_0)},\dots,g^{(k_{n-1})}\big)=s_1\psi_n$$

and

$$\begin{aligned}W\big(g^{(k_0)},\dots,g^{(k_{n-2})}\big)&=s_2W\big(g,\dots,g^{(j-1)},g^{(j+1)},\dots,g^{(n-1)}\big)\\&=s_2W_{0,\dots,j-1,j+1,\dots,n-1}\big(g,g',\dots,g^{(n-2)}\big)\\&=s_2\frac{W_{0,\dots,j-1,j+1,\dots,n-1}\big(g,g',\dots,g^{(n-2)}\big)}{W\big(g,g',\dots,g^{(n-2)}\big)}\psi_{n-1}.\end{aligned}$$

Using Lemma 2.3 and the first main theorem of the Nevanlinna theory it follows that

$$N\Big(r,\frac{1}{W\big(g^{(k_0)},\dots,g^{(k_{n-2})}\big)}\Big)-N\Big(r,\frac{1}{W\big(g^{(k_0)},\dots,g^{(k_{n-2})},g^{(k_{n-1})}\big)}\Big) \leq N\Big(r,\frac{1}{\psi_{n-1}}\Big)-N\Big(r,\frac{1}{\psi_n}\Big)+S(r,g). \tag{3.12}$$

Because of Corollary 3.1 and Theorem 3.1, the right hand side of (3.12) can be estimated from above by

$$\varepsilon N\Big(r,\frac{1}{\psi_n}\Big)+S(r,g)\leq \varepsilon n N\Big(r,\frac{1}{g}\Big)+S(r,g)$$

for every $\varepsilon > 0$. This proves (3.11) in the case $d = -1$.

Now let $d \geq 0$. There exists $k_n \in \mathbf{N}_0$ such that

$$k_n \leq \max\{k_0, \dots, k_{n-1}\} \qquad \text{and} \qquad k_n \notin \{k_0, \dots, k_{n-1}\}.$$

Applying Lemma 3.1 gives

$$\begin{aligned} N\Big(r, \frac{1}{W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}\big)}\Big) - N\Big(r, \frac{1}{W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_{n-1})}\big)}\Big) \\ \leq N\Big(r, \frac{1}{W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_n)}\big)}\Big) \\ - N\Big(r, \frac{1}{W\big(g^{(k_0)}, \dots, g^{(k_{n-2})}, g^{(k_{n-1})}, g^{(k_n)}\big)}\Big) + S(r, g). \end{aligned}$$

Since

$$d(k_0, \dots, k_n) = \max\{k_0, \dots, k_n\} - (n+1) = d - 1,$$

the assertion (3.11) follows using the induction hypothesis. □

From Theorem 3.1 and Theorem 3.3 we conclude

Corollary 3.2. *Let g be an entire function which is not an exponential sum. If $k_1, \dots, k_{n-1} \in \mathbf{N}$ are pairwise distinct then*

$$N\Big(r, \frac{1}{W\big(g, g^{(k_1)}, \dots, g^{(k_{n-1})}\big)}\Big) = S(r, g) \quad \text{if and only if} \quad N\Big(r, \frac{1}{g}\Big) = S(r, g).$$

4. Results for exponential polynomials

The counting function of the zeros of an exponential sum g is determined by the exponents $\gamma_1, \dots, \gamma_p$ in the representation (1.2).

Lemma 4.1 ([12, §2]). *Let g be the exponential sum (1.2). Then*

$$N\Big(r, \frac{1}{g}\Big) = \frac{L}{2\pi} r + O(\log r) \qquad \text{for } r \to \infty$$

where L is the length of the convex hull of the points $\gamma_1, \dots, \gamma_p$ in $\mathbf{C}$.

For the Wronskians $\psi_n := W\big(g, g', \dots, g^{(n-1)}\big)$, $n \in \mathbf{N}$, Theorem C gives sharp upper and lower bounds for the counting function of the zeros.

Now we give two examples to show that, in general, it is not possible to estimate the zeros of $W\big(g, g^{(k_1)}, \dots, g^{(k_{n-1})}\big)$ from below in terms of the zeros of g, if g is an exponential sum.

Example 4.1. Let $0 < \gamma < \delta$ and $g(z) = e^{-\gamma z} + e^{\gamma z} + e^{\delta z}$. Then

$$W(g, g'') = (\delta^2 - \gamma^2)\big((\delta+\gamma)e^{(\delta-\gamma)z} + (\delta-\gamma)e^{(\delta+\gamma)z}\big).$$

Lemma 4.1 gives

$$N\Big(r, \frac{1}{g}\Big) = \frac{\delta+\gamma}{\pi} r + O(\log r) \quad \text{and} \quad N\Big(r, \frac{1}{W(g, g'')}\Big) = \frac{2\gamma}{\pi} r + O(\log r),$$

hence

$$N\Big(r, \frac{1}{W(g, g'')}\Big) = \frac{2\gamma}{\delta+\gamma} N\Big(r, \frac{1}{g}\Big) + O(\log r).$$

Example 4.2. Let $\delta > 0$ and $g(z) = z + e^{\delta}$. Then

$$W(g, g'') = \delta^3 z e^{\delta z}.$$

Now we have

$$N\Big(r, \frac{1}{g}\Big) = \frac{\delta}{\pi} r + O(\log r) \quad \text{and} \quad N\Big(r, \frac{1}{W(g, g'')}\Big) = O(\log r),$$

thus

$$N\Big(r, \frac{1}{W(g, g'')}\Big) = S(r, g).$$

Only if the order m of the exponential sum is large enough, the methods from the last section can be used again.

Theorem 4.1. *Let g be an exponential sum of order m and $1 \le k_1 < k_2 < \cdots < k_{n-1}$. If $m \ge 2k_{n-1} + 1$ then*

$$N\Big(r, \frac{1}{W\big(g, g^{(k_1)}, \ldots, g^{(k_{n-2})}, g^{(k_{n-1})}\big)}\Big) \ge N\Big(r, \frac{1}{W\big(g, g^{(k_1)}, \ldots, g^{(k_{n-2})}\big)}\Big) + O(\log r)$$

and, in particular,

$$N\Big(r, \frac{1}{W\big(g, g^{(k_1)}, \ldots, g^{(k_{n-1})}\big)}\Big) \ge N\Big(r, \frac{1}{g}\Big) + O(\log r).$$

Proof. Again we define for $l \in \mathbf{N}$

$$\psi_l = W\big(g, g', g'', \ldots, g^{(l-1)}\big).$$

Proceeding as in the proof of Theorem 3.3, repeated application of Lemma 3.1 gives

$$\begin{aligned} &N\Big(r, \frac{1}{W\big(g, g^{(k_1)}, \ldots, g^{(k_{n-2})}\big)}\Big) - N\Big(r, \frac{1}{W\big(g, g^{(k_1)}, \ldots, g^{(k_{n-2})}, g^{(k_{n-1})}\big)}\Big) \\ &\quad \le N\Big(r, \frac{1}{\psi_{k_{n-1}}}\Big) - N\Big(r, \frac{1}{\psi_{1+k_{n-1}}}\Big) + O(\log r). \end{aligned} \tag{4.1}$$

From Theorem C we have

$$N\Big(r, \frac{1}{\psi_{k_{n-1}}}\Big) = N\Big(r, \frac{1}{\psi_{m-k_{n-1}}}\Big) + O(\log r).$$

Since $k_{n-1} < 1 + k_{n-1} \le m - k_{n-1}$, we can use (3.3) in Theorem 3.2 with $a = k_{n-1}$, $b = 1 + k_{n-1}$ and $c = m - k_{n-1}$. It follows that the right hand side of (4.1) is less than $O(\log r)$. □

5. Estimates for meromorphic functions using the recursion formulas for Wronskians

In this section f will always be a meromorphic function which is not an exponential sum. Define the functions $\psi_n, n \in \mathbf{N}_0$, by

$$\psi_0 = 1, \qquad \psi_n = W\big(f, f', f'', \dots, f^{(n-1)}\big) \text{ for } n \in \mathbf{N}. \tag{5.1}$$

Lemma 5.1. *$N(r, \psi_n) = nN(r, f) + (n-1)n\overline{N}(r, f)$ for $n \in \mathbf{N}_0$.*

Proof. ψ_n can have a pole only where f has a pole. It remains to show that if $f(z_0) = \infty$ with multiplicity p then $\psi_n(z_0) = \infty$ with multiplicity $np + (n-1)n$. This follows from the following lemma. □

Lemma 5.2. *Let $f_1, \dots, f_n$ be meromorphic functions and $z_0 \in \mathbf{C}$. If*

$$f_j(z_0) = \infty \text{ with multiplicity } p_j, \quad j = 1, \dots, n,$$

where $1 \le p_1 < p_2 < \dots < p_n$, then $W(f_1, f_2, \dots, f_n)(z_0) = \infty$ with multiplicity

$$p_1 + \dots + p_n + \tfrac{1}{2}(n-1)n.$$

Proof. (By induction.) For $n = 1$ there is nothing to show. For $n \ge 2$ we have

$$W(f_1, \dots, f_n) = f_1^n W\Big(1, \frac{f_2}{f_1}, \dots, \frac{f_n}{f_1}\Big) = f_1^n W\Big(\Big(\frac{f_2}{f_1}\Big)', \dots, \Big(\frac{f_n}{f_1}\Big)'\Big).$$

Using the induction hypothesis, the right hand side has a pole of multiplicity

$$np_1 + (p_2 - p_1 + 1) + \dots + (p_n - p_1 + 1) + \tfrac{1}{2}(n-2)(n-1) = p_1 + \dots + p_n + \tfrac{1}{2}(n-1)n. \ □$$

Lemma 5.3. *For $n \in \mathbf{N}$ we have*

$$N\Big(r, \frac{1}{\psi_{n-1}}\Big) + N\Big(r, \frac{1}{\psi_{n+1}}\Big) \le 2N\Big(r, \frac{1}{\psi_n}\Big) + 2\overline{N}(r, f) + S(r, f).$$

Proof. As in the proof of Lemma 3.1, it follows from the recursion formulas for Wronskians (Lemma 2.1 (iv)) that

$$\psi_{n-1}\psi_{n+1} = W(\psi_n, \psi_n') = \psi_n^2 \Big(\frac{\psi_n'}{\psi_n}\Big)'. \tag{5.2}$$

Applying the first main theorem to (5.2) gives

$$\begin{aligned} &N\Big(r, \frac{1}{\psi_{n-1}}\Big) - N(r, \psi_{n-1}) + N\Big(r, \frac{1}{\psi_{n+1}}\Big) - N(r, \psi_{n+1}) \\ &\qquad = 2N\Big(r, \frac{1}{\psi_n}\Big) - 2N(r, \psi_n) + m(r, q_n') - m\Big(r, \frac{1}{q_n'}\Big) + O(1) \end{aligned} \tag{5.3}$$

with $q_n := \psi_n'/\psi_n$. The lemma of the logarithmic derivative yields $m(r, q_n') = S(r, f)$. From Lemma 5.1 we conclude that

$$N(r, \psi_{n-1}) - 2N(r, \psi_n) + N(r, \psi_{n+1}) = 2\overline{N}(r, f).$$

Combining this with (5.3) gives the assertion. □

Proceeding now as in the proof of Theorem 3.2, we conclude the following theorem from Lemma 5.3.

Theorem 5.1. *Let f be a meromorphic function which is not an exponential sum and define the functions $\psi_n, n \in \mathbf{N}_0$, by* (5.1)*. Then*

$$N\Big(r, \frac{1}{\psi_b}\Big) \geq \frac{c-b}{c-a} N\Big(r, \frac{1}{\psi_a}\Big) + \frac{b-a}{c-a} N\Big(r, \frac{1}{\psi_c}\Big) - (b-a)(c-b)\overline{N}(r,f) + S(r,f)$$

for $0 \leq a < b < c$.

Setting $a = 1$, $b = n$, $c = n+m$ and using $N(r, 1/\psi_c) \geq 0$ for $r > 1$ gives

Corollary 5.1. *Let f and ψ_n be as in Theorem* 5.1*. Then*

$$N\Big(r, \frac{1}{f}\Big) \leq \Big(1 + \frac{n-1}{m}\Big) N\Big(r, \frac{1}{\psi_n}\Big) + (n-1)(n+m-1)\overline{N}(r,f) + S(r,f) \tag{5.4}$$

for $n, m \in \mathbf{N}$.

Remark 5.1. If f is an entire function or, more generally, $\overline{N}(r,f) = S(r,f)$, one can let m tend to ∞ in (5.4). The result is again the left hand inequality of Theorem B.

Example 5.1. If f is not an exponential sum then setting $n = 2$ and $m = 1$ in (5.4) gives

$$N\Big(r, \frac{1}{f}\Big) \leq 2N\Big(r, \frac{1}{W(f,f')}\Big) + 2\overline{N}(r,f) + S(r,f).$$

6. Common solutions of two linear differential equations

Given two homogeneous linear differential equations $L_1[w] = 0$ and $L_2[w] = 0$ there always exists a homogeneous linear differential equation $M[w] = 0$ whose solutions are exactly the common solutions of the given two equations. The coefficients of M are rational functions of the coefficients of L_1 and L_2 and their derivatives.

In this section, we develop formulas to compute the coefficients of M in terms of the coefficients of L_1 and L_2.

We denote by $\mathscr{L}$ the vector space of all linear differential operators

$$L = a_n \mathbf{D}^n + a_{n-1}\mathbf{D}^{n-1} + \cdots + a_1 \mathbf{D} + a_0$$

where $\mathbf{D} = d/dz$ and the coefficients $a_0, \ldots, a_n$ are meromorphic functions. If $a_n \not\equiv 0$, $\operatorname{ord}(L) = n$ is the *order* of L, $\operatorname{ord}(0) = -\infty$.

Together with the composition as multiplication, $\mathscr{L}$ is a non-commutative ring.

Lemma 6.1 ([6, §5.4]). *Given $L_1, L_2 \in \mathscr{L} \setminus \{0\}$ there exist $P, Q \in \mathscr{L}$ such that*

$$L_1 = PL_2 + Q \qquad \text{and} \qquad \operatorname{ord}(Q) < \operatorname{ord}(L_2).$$

Using the Euclidean algorithm, the following lemma is an easy consequence of the previous one.

Lemma 6.2. *Given* $L_1, L_2 \in \mathscr{L}\setminus\{0\}$ *there exists a unique* $M \in \mathscr{L}$ *satisfying*

(i) $M = P_1L_1 + P_2L_2$ *with some* $P_1, P_2 \in \mathscr{L}$,
(ii) $L_1 = Q_1M$ *and* $L_2 = Q_2M$ *with some* $Q_1, Q_2 \in \mathscr{L}$,
(iii) *the main coefficient of* M *is* 1.

M is called the *greatest common divisor* of L_1 and L_2.

Lemma 6.3. *Let* $L_1, L_2 \in \mathscr{L}\setminus\{0\}$ *and let* M *be the greatest common divisor of* L_1 *and* L_2. *Let* $m = \operatorname{ord}(L_1)$ *and* $n = \operatorname{ord}(L_2)$. *Then for every* $k \in \mathbf{N}_0$, $\operatorname{ord}(M) \geq k$ *if and only if there exist* $R_1, R_2 \in \mathscr{L}\setminus\{0\}$ *such that*

$$\operatorname{ord}(R_1) \leq n-k, \quad \operatorname{ord}(R_2) \leq m-k \quad \text{and} \quad R_1L_1 = R_2L_2. \tag{6.1}$$

Proof. Let U be an open disk in the plane where all the coefficients of L_1, L_2 and M are holomorphic and the main coefficients have no zeros.

Suppose that there exist $R_1, R_2 \in \mathscr{L}\setminus\{0\}$ satisfying (6.1). Let $f_1, \dots, f_m$ be a fundamental system of $L_1[w] = 0$ in U. Then $R_2L_2[f_j] = 0$ for $j = 1, \dots, m$. Since $\operatorname{ord}(R_2) \leq m-k$, at most $m-k$ of the functions $L_2[f_1], \dots, L_2[f_m]$ are linearly independent. Without loss of generality we may assume that

$$L_2[f_j] = c_{j,k+1}L_2[f_{k+1}] + \cdots + c_{j,m}L_2[f_m]$$

with constants $c_{j,k+1}, \dots, c_{j,m} \in \mathbf{C}$, $j = 1, \dots, k$. Define functions $v_1, \dots, v_k$ by

$$v_j = f_j - c_{j,k+1}f_{k+1} - \cdots - c_{j,m}f_m, \qquad j = 1, \dots, k.$$

Then $L_2[v_j] = 0$ and hence $M[v_j] = 0$. Since $v_1, \dots, v_k$ are linearly independent, it follows that $\operatorname{ord}(M) \geq k$.

Now we suppose that $\operatorname{ord}(M) \geq k$. Let $(f_1, \dots, f_l)$ be a fundamental system of $M[w] = 0$ in U, $l = \operatorname{ord}(M)$. There are functions $g_1, \dots, g_{m-l}$ and $h_1, \dots, h_{n-l}$, holomorphic in U, such that

$(f_1, \dots, f_l, g_1, \dots, g_{m-l})$ is a fundamental system of $L_1[w] = 0$ in U and
$(f_1, \dots, f_l, h_1, \dots, h_{n-l})$ is a fundamental system of $L_2[w] = 0$ in U.

Define $K \in \mathscr{L}$ by

$$K[w] = W(f_1, \dots, f_l, g_1, \dots, g_{m-l}, h_1, \dots, h_{n-l}, w).$$

Every solution of $L_1[w] = 0$ is also a solution of $K[w] = 0$. Using Lemma 6.1 we see that there is a $R_1 \in \mathscr{L}$ such that $K = R_1L_1$. We have

$$\operatorname{ord}(R_1) = \operatorname{ord}(K) - \operatorname{ord}(L_1) = (m+n-l) - m = n-l \leq n-k.$$

Similarly, there exists a $R_2 \in \mathscr{L}$ satisfying $K = R_2L_2$ and $\operatorname{ord}(R_2) \leq m-k$. Then $R_1L_1 = K = R_2L_2$ and the assertion follows. □

Lemma 6.4. *Let* $L_1, L_2 \in \mathscr{L} \setminus \{0\}$ *be defined by*

$$\begin{aligned} L_1 &= a_m \mathbf{D}^m + a_{m-1}\mathbf{D}^{m-1} + \cdots + a_1\mathbf{D} + a_0, \qquad && a_m \not\equiv 0, \\ L_2 &= b_n \mathbf{D}^n + b_{n-1}\mathbf{D}^{n-1} + \cdots + b_1\mathbf{D} + b_0, && b_n \not\equiv 0. \end{aligned}$$

Let M *be the greatest common divisor of* L_1 *and* L_2, $k = \operatorname{ord}(M)$. *Then*

$$M = \mathbf{D}^k + \frac{c_{k-1}}{c_k}\mathbf{D}^{k-1} + \cdots + \frac{c_1}{c_k}\mathbf{D} + \frac{c_0}{c_k}$$

where

$$c_j = \begin{vmatrix} a_{j,0} & a_{k+1,0} & \cdots & a_{m,0} & 0 & & \cdots & & 0 \\ a_{j,1} & a_{k+1,1} & \cdots & & a_{m+1,1} & 0 & \cdots & & 0 \\ \vdots & \vdots & & & & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & & & & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & & & & \ddots & 0 \\ a_{j,n-k-1} & a_{k+1,n-k-1} & & & \cdots & & & & a_{m+n-k-1,n-k-1} \\ b_{j,0} & b_{k+1,0} & \cdots & & b_{n,0} & 0 & \cdots & & 0 \\ b_{j,1} & b_{k+1,1} & & \cdots & & b_{n+1,1} & 0 & \cdots & 0 \\ \vdots & \vdots & & & & & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & & & & \ddots & 0 \\ b_{j,m-k-1} & b_{k+1,m-k-1} & & & \cdots & & & & b_{m+n-k-1,m-k-1} \end{vmatrix}$$

for $j = 0, \ldots, k$. *The meromorphic functions* $a_{l,j}$ *and* $b_{l,j}$ *are defined by*

$$\begin{aligned} \mathbf{D}^j L_1 &= \sum_{l=0}^{m+j} a_{l,j}\mathbf{D}^l \qquad && \text{for } j = 0, \ldots, n-k-1, \\ \mathbf{D}^j L_2 &= \sum_{l=0}^{n+j} b_{l,j}\mathbf{D}^l && \text{for } j = 0, \ldots, m-k-1. \end{aligned}$$

Proof. Let

$$\begin{aligned} P_1 &= \alpha_{n-k-1}\mathbf{D}^{n-k-1} + \cdots + \alpha_1\mathbf{D} + \alpha_0, \\ P_2 &= \beta_{m-k-1}\mathbf{D}^{m-k-1} + \cdots + \beta_1\mathbf{D} + \beta_0 \end{aligned}$$

and $\widetilde{M} = P_1L_1 + P_2L_2$ with meromorphic coefficients $\alpha_0, \ldots, \alpha_{n-k-1}$ and $\beta_0, \ldots, \beta_{m-k-1}$. Setting

$$a_{l,j} = 0 \qquad \text{for} \quad l > m+j \quad \text{and} \quad b_{l,j} = 0 \qquad \text{for} \quad l > n+j,$$

gives

$$\begin{aligned}\widetilde{M} &= \sum_{j=0}^{n-k-1} \alpha_j \sum_{l=0}^{m+j} a_{l,j}\mathbf{D}^l + \sum_{j=0}^{m-k-1} \beta_j \sum_{l=0}^{n+j} b_{l,j}\mathbf{D}^l \\ &= \sum_{l=0}^{m+n-k-1} \Big(\sum_{j=0}^{n-k-1} a_{l,j}\alpha_j + \sum_{j=0}^{m-k-1} b_{l,j}\beta_j \Big)\mathbf{D}^l.\end{aligned}$$

We try to determine the functions $\alpha_0, \dots, \alpha_{n-k-1}$ and $\beta_0, \dots, \beta_{m-k-1}$ in a way that $\widetilde{M}$ has the exact order k and the main coefficient 1, hence

$$\sum_{j=0}^{n-k-1} a_{l,j}\alpha_j + \sum_{j=0}^{m-k-1} b_{l,j}\beta_j = \begin{cases} 1 & \text{for } l = k, \\ 0 & \text{for } l = k+1, \dots, m+n-k-1. \end{cases} \tag{6.2}$$

This is a linear system of $m+n-2k$ equations for the same number of variables.

Let us first assume that the determinant of the coefficients is identically zero. In this case we can choose P_1 and P_2 (not both equal to zero) in such a way that $\operatorname{ord}(\widetilde{M}) < k$. On the other hand, using Lemma 6.2 (ii) we see that $\widetilde{M} = RM$ with some $R \in \mathscr{L}$, hence $\widetilde{M} = 0$. It follows that

$$P_1L_1 = -P_2L_2, \quad \operatorname{ord}(P_1) \le n-k-1, \quad \operatorname{ord}(P_2) \le m-k-1.$$

This is a contradiction to Lemma 6.3.

Thus the determinant of the coefficients of (6.2) is not identically zero and there exists a (unique) solution. Because of $\widetilde{M} = RM$ with some $R \in \mathscr{L}$ and since M and $\widetilde{M}$ have the same order and the same main coefficient, it follows that $\widetilde{M} = M$.

Applying Cramer's rule to (6.2) gives for the coefficients of M

$$\sum_{j=0}^{n-k-1} a_{l,j}\alpha_j + \sum_{j=0}^{m-k-1} b_{l,j}\beta_j = \frac{c_l}{c_k}, \qquad l = 0, \dots, k. \ \square$$

7. An estimate using the "method of the greatest common divisor"

Theorem 7.1. *Let f be a meromorphic function and define φ by*

$$\varphi = W\big(f, f^{(k_1)}, f^{(k_2)}, \dots, f^{(k_{n-1})}\big)$$

where $1 \le k_1 < k_2 < \dots < k_{n-1}$. Assume that

$$W\big(f, f^{(k_1)}, f^{(k_2)}, \dots, f^{(k_{n-1})}, f^{(k_{n-1}+k_1)}, f^{(k_{n-1}+k_2)}, \dots, f^{(k_{n-1}+k_{n-1})}\big) \not\equiv 0. \tag{7.1}$$

Then

$$\overline{N}\Big(r, \frac{1}{f}\Big) \le (C+1)\overline{N}\Big(r, \frac{1}{\varphi}\Big) + C\overline{N}(r, f)$$

where $C = (n-1+k_{n-1})(n-1)$.

Remark 7.1. Note that (7.1) is satisfied if f is not an exponential sum, in particular if f has at least one pole.

Example 7.1. (Compare Theorem D.) Let f be a meromorphic function. If $W(f,f',f'')\not\equiv 0$ then

$$\overline{N}\Big(r,\frac{1}{f}\Big)\le 3\,\overline{N}\Big(r,\frac{1}{W(f,f')}\Big)+2\,\overline{N}(r,f)+S(r,f).$$

Proof of Theorem 7.1. Let $p:=k_{n-1}$. We define linear differential operators $L_1,L_2\in\mathscr{L}$ by

$$L_1[w]=\frac{W\big(f,f^{(k_1)},f^{(k_2)},\dots,f^{(k_{n-1})},w\big)}{W\big(f,f^{(k_1)},f^{(k_2)},\dots,f^{(k_{n-1})}\big)}\quad\text{and}\quad L_2[w]=L_1[w^{(p)}].$$

Let M be the greatest common divisor of L_1 and L_2. Since $L_1[f]=L_2[f]=0$ we have

$$M[f]=0. \tag{7.2}$$

We want to show that $\operatorname{ord}(M)=1$. To this aim, let v be an arbitrary local solution of $M[w]=0$. Then $L_1[v]=L_2[v]=0$ and hence

$$v=\alpha_0 f+\alpha_1 f^{(k_1)}+\dots+\alpha_{n-1}f^{(k_{n-1})}\quad\text{where }\alpha_0,\dots,\alpha_{n-1}\in\mathbf{C}$$

and

$$v^{(p)}=\beta_0 f+\beta_1 f^{(k_1)}+\dots+\beta_{n-1}f^{(k_{n-1})}\quad\text{where }\beta_0,\dots,\beta_{n-1}\in\mathbf{C}.$$

Thus

$$\alpha_0 f^{(k_{n-1})}+\alpha_1 f^{(k_{n-1}+k_1)}+\dots+\alpha_{n-1}f^{(k_{n-1}+k_{n-1})}=\beta_0 f+\beta_1 f^{(k_1)}+\dots+\beta_{n-1}f^{(k_{n-1})}.$$

Using the assumption (7.1) it follows that $\alpha_1=\alpha_2=\dots=\alpha_{n-1}=0$. Hence we have $v=\alpha_0 f$ with $\alpha_0\in\mathbf{C}$. It follows that the order of M is equal to 1.

Now we write L_1 in the form

$$L_1=\mathbf{D}^n+a_{n-1}\mathbf{D}^{n-1}+\dots+a_1\mathbf{D}+a_0.$$

Then

$$L_2=\mathbf{D}^{p+n}+a_{n-1}\mathbf{D}^{p+n-1}+\dots+a_1\mathbf{D}^{p+1}+a_0\mathbf{D}^p$$

and $M=\mathbf{D}+c_0/c_1$ where c_0 and c_1 are given in Lemma 6.4. It follows from (7.2) that $f'/f=-c_0/c_1$ and hence

$$\begin{aligned}\overline{N}\Big(r,\frac{1}{f}\Big)+\overline{N}(r,f)&\le\overline{N}(r,c_0)+\overline{N}\Big(r,\frac{1}{c_1}\Big)\\&\le\overline{N}(r,c_0)+N(r,c_1)+m(r,c_1)+O(1).\end{aligned}\tag{7.3}$$

We will estimate the terms on the right hand side of this inequality separately. Using Lemma 6.4 gives

(7.4)

$$c_j = \begin{vmatrix}
a_{j,0} & a_{2,0} & \cdots & a_{n-1,0} & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
a_{j,1} & a_{2,1} & \cdots & \cdots & a_{n,1} & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & & \ddots & \ddots & \ddots & & & & \vdots \\
\vdots & \vdots & & & & & \ddots & \ddots & \ddots & & & \vdots \\
\vdots & \vdots & & & & & & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & & & & & & & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & & & & & & & & \ddots & \ddots & 0 \\
a_{j,n+p-2} & a_{2,n+p-2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{2n+p-3,n+p-2} & 1 \\
0 & \cdots & 0 & a_{0,0} & \cdots & \cdots & \cdots & a_{n-1,0} & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & a_{0,1} & \cdots & \cdots & \cdots & \cdots & a_{n,1} & 1 & \ddots & \vdots \\
\vdots & & \vdots & \vdots & & & & & & \ddots & \ddots & 0 \\
0 & \cdots & 0 & a_{0,n-2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{2n-3,n-2} & 1
\end{vmatrix}$$

for $j = 0$ and $j = 1$. Here the meromorphic functions $a_{l,j}$ are defined by

$$\mathbf{D}^j L = \mathbf{D}^{n+j} + \sum_{l=0}^{n+j-1} a_{l,j} \mathbf{D}^l \quad \text{for } j = 0, \dots, n+p-2.$$

The $a_{l,j}$ are polynomials in the coefficients $a_0, \dots, a_{n-1}$ of L_1 and their derivatives. It follows from Lemma 2.2 that

$$m(r, a_{l,j}) = S(r, f) \quad \text{for } j = 0, \dots, n+p-2 \quad \text{and} \quad l = 0, \dots, n+j-1.$$

Hence

$$m(r, c_1) = S(r, f). \tag{7.5}$$

Poles of c_0 and c_1 can only occur where at least one of the functions $a_{l,j}$ and hence at least one of the functions $a_0, \dots, a_{n-1}$ has a pole. Since

$$a_l = (-1)^{n-l} W_{0,\dots,l-1,l+1,\dots,n}\big(f, f^{(k_1)}, f^{(k_2)}, \dots, f^{(k_{n-1})}\big) \frac{1}{\varphi}$$

for $l = 0, \dots, n-1$, we have

$$\overline{N}(r, c_0) \le \overline{N}\Big(r, \frac{1}{\varphi}\Big) + \overline{N}(r, f). \tag{7.6}$$

Now let z_0 be a pole of c_1. The Fuchsian theory (see [6, §15.3], for example) applied to the equation $L_1[w] = 0$ gives

$$a_l(z_0) = \infty \quad \text{with multiplicity at most} \quad n - l$$

for $l = 0, \ldots, n-1$. It follows that

$$a_{l,j}(z_0) = \infty \quad \text{with multiplicity at most} \quad n + j - l$$

for $j = 0, \ldots, n+p-2$ and $l = 0, \ldots, n+j-1$. Using (7.4) with $j = 1$ we get

$$c_1(z_0) = \infty \quad \text{with multiplicity at most} \quad (n+p-1)(n-1)$$

and thus

$$N(r, c_1) \le (n+p-1)(n-1)\left(\overline{N}\Big(r, \frac{1}{\varphi}\Big) + \overline{N}(r, f)\right). \tag{7.7}$$

Substitution of (7.5), (7.6) and (7.7) in (7.3) gives exactly the assertion of the theorem. □

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