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POLYMORPHISMS AND LINEARIZATION OF NONLINEAR POLYNOMIALS

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Abstract. We prove that if a nonconstant holomorphic function f satisfies a non-discrete family of functional equations of the form $f(az+b) = P(f(z))$ where $a \neq 0$ and b are constants and P is a polynomial of degree $k \geq 2$, then f must be an exponential function.

Let D be a domain of the complex plane $\mathbf{C}$, and let $\operatorname{Aff}(D)$ be the group of all holomorphic polynomials of degree one mapping D onto itself. Every element $\phi \in \operatorname{Aff}(D)$ is an affine conformal automorphism of D, and $\operatorname{Aff}(D)$ is a closed subgroup of the Lie group $\operatorname{Aff}(\mathbf{C})$.

Let $f\colon D \to \mathbf{C}$ be a holomorphic function, and let $\phi \in \operatorname{Aff}(D)$. We say that ϕ is a *polymorphism of* f if there exists a polynomial P such that $f \circ \phi = P \circ f$. If f is nonconstant, then P is uniquely determined by ϕ; we say that P is the polynomial *associated* with the polymorphism ϕ, and the degree $\deg P$ of P is the f*-degree* of ϕ.

Let $\Pi(f)$ be the set of polymorphisms of f. Then $\Pi(f)$ is a topological semigroup: if $f \circ \phi_j = P_j \circ f$ for $j = 1,\ 2$, then $f \circ (\phi_1 \circ \phi_2) = (P_1 \circ P_2) \circ f$. The identity element of $\Pi(f)$ is the *trivial* polymorphism of f.

Many elementary functions have nontrivial polymorphisms. For example, nonconstant homogeneous polynomials are associated with polymorphisms of e^z, and Chebyshev polynomials are associated with polymorphisms of $\cos z$. The power function $f(z) = z^\kappa$ has a non-discrete group of polymorphisms of f-degree one for each $\kappa \neq 0$. In general, if f admits nontrivial polymorphisms of f-degree one, then f is a polymorphic function in the sense of Pommerenke [7].

Each polymorphism of a nonconstant holomorphic function f gives rise to a linearization of the associated polynomial. The linearizing map f linearizes simultaneously all polynomials associated with elements of $\Pi(f)$. For entire functions the problem of simultaneous linearization has been studied already by Fatou [4] and Julia [6].

We say that two holomorphic functions $f\colon D \to \mathbf{C}$ and $g\colon D \to \mathbf{C}$ are *conjugate* if there exist $\sigma \in \operatorname{Aff}(D)$ and $\tau \in \operatorname{Aff}(\mathbf{C})$ such that $f \circ \sigma = \tau \circ g$. If f and g are conjugate, then $\Pi(f)$ and $\Pi(g)$ are isomorphic.

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Let $\Pi_k(f)$ be the subset of $\Pi(f)$ of polymorphisms of f-degree k. In this article we show that $\Pi_k(f)$ is always discrete for $k \ge 2$ unless f is either constant or conjugate to the exponential function. This result answers affirmatively a question raised in [3] where a similar theorem was proved in the case $k = 2$.

Theorem 1. *Let $f\colon D \to \mathbf{C}$ be a nonconstant holomorphic function, and suppose that $\Pi_k(f)$ is not discrete for some $k \ge 2$. Then D is either the whole plane $\mathbf{C}$ or a half plane and f is conjugate to the exponential function e^z.*

The idea of proof of Theorem 1 differs slightly from the approach used in [3] where we obtained f as a solution of a differential equation. Now we show, by using analytic continuation, that under the hypotheses of Theorem 1 f must be the restriction of an entire function.

For entire functions Theorem 1 can be proved under weaker assumptions. It suffices to assume that f is locally one-to-one and that there exist $k \ge 2$ and $\phi \in \Pi_k(f)$ such that ϕ has a fixed point in $\mathbf{C}$. By using Picard's theorem as in [3] we can then show that the polynomial P associated with ϕ is conjugate to a homogeneous polynomial and that f satisfies the same differential equation as the exponential function e^z.

Before proving Theorem 1 we introduce some additional terminology. We say that an element $\phi \in \mathrm{Aff}(\mathbf{C})$ is a *pseudopolymorphism* of $f\colon D \to \mathbf{C}$ if there is a subdomain $D' \subset D$ and a polynomial P such that $\phi(D') \subset D$ and $f\big(\phi(z)\big) = P\big(f(z)\big)$ for each $z \in D'$. If I is an open interval of $\mathbf{R}$ and $\alpha \in \mathbf{R}$, we define a *half-strip*

$$S(\alpha, I) = \{x + iy \mid x > \alpha \text{ and } y \in I\}$$

and a corresponding *strip*

$$S(I) = \{x + iy \mid y \in I\}.$$

The following lemma contains the key argument on analytic continuation needed in the proof of Theorem 1.

Lemma 1. *Let g be holomorphic and one-to-one in $S(\alpha, I)$, and suppose that there exists $\beta \in \mathbf{R}$ such that the translation $z \mapsto z + \beta$ is a pseudopolymorphism of g.*

(a) *If $\beta < 0$, then g has a holomorphic extension to $S(I)$.*
(b) *If $\beta > 0$, then I can be divided into finitely many subintervals $I_1, \dots, I_m$ such that for each $j \in \{1, \dots, m\}$ the restriction of g to $S(\alpha, I_j)$ has a holomorphic extension to $S(I_j)$.*

Proof. By hypothesis there exists a polynomial P such that

$$g(z + \beta) = P\big(g(z)\big) \tag{1}$$

for each z in a subdomain of $S(\alpha, I)$.

(a) If $\beta < 0$, we can define an extension of g to $S(\alpha+\beta, I)$ so that

$$g(z) = P\bigl(g(z-\beta)\bigr) \tag{2}$$

for each $z \in S(\alpha+\beta, I)$. This follows from the fact that the right hand side of (2) is defined and holomorphic in $S(\alpha+\beta, I)$ and agrees with g in a subdomain of $S(\alpha, I)$. Since $z \mapsto z+\beta$ is a pseudopolymorphism of this extension of g, we may proceed by induction and obtain an extension of g to $S(\alpha+n\beta, I)$ for each $n > 0$.

(b) The forward orbit of a point $w \in \mathbf{C}$ under P consists of all points of the form $P^{\circ n}(w)$ where $P^{\circ n}$ is the nth iterate of P. Let Y be the set of all $y \in I$ such that $g(x+iy)$ is contained in the forward orbit of a critical point of P for some $x > \alpha$. Suppose that $g(x_1+iy_1)$ and $g(x_2+iy_2)$ are contained in the same orbit so that e.g. $g(x_1+iy_1) = P^{\circ n}g(x_2+iy_2)$ for some $n > 0$. Then by iteration of (1) it follows that $g(x_1+iy_1) = g(x_2+n\beta+iy_2)$, and hence $y_1 = y_2$ because g is one-to-one. Thus different points of Y correspond to different critical points of P, and we conclude that Y is finite.

Let J be any open subinterval of I contained in the complement of Y. It remains to prove that the restriction of g to $S(\alpha, J)$ has a holomorphic extension to $S(J)$.

Let D be the set of all $w \in \mathbf{C}$ such that the forward orbit of w under P contains points of $g\bigl(S(\alpha, J)\bigr)$. There is an obvious equivalence relation in D: points w_1, $w_2 \in D$ are equivalent if their forward orbits intersect each other. The quotient space under this relation is a Riemann surface X, and the canonical projection ψ: $D \to X$ is a covering map; this follows from the fact that P is one-to-one in $g\bigl(S(\alpha, J)\bigr)$ and orbits of all critical points of P lie outside of D.

A similar equivalence relation can be defined in $S(\alpha, J)$ and in $S(J)$: two points are equivalent if their difference is a multiple of β. These relations in $S(\alpha, J)$ and in $S(J)$ have the same quotient space E, and g maps equivalent points of $S(\alpha, J)$ to equivalent points of D. Hence g induces a holomorphic quotient map g_*: $E \to X$, and the restriction of g to $S(\alpha, J)$ is actually a lifting of the composite map $S(\alpha, J) \overset{\pi_\alpha}{\to} E \overset{g_*}{\to} X$ where π_α is the natural projection. This projection extends to the canonical projection π: $S(J) \to E$. Since $S(J)$ is simply connected and ψ: $D \to X$ is a covering map, the map $g_* \circ \pi$ has a lifting $\tilde{g}$: $S(J) \to D$ which agrees with g in $S(\alpha, J)$ and so is the desired holomorphic extension of g. This completes the proof of Lemma 1.

Lemma 2. *If $k \geq 2$ and f is nonconstant, then each element of $\Pi_k(f)$ is of infinite order. Moreover, the intersection of $\Pi_k(f)$ with every abelian subgroup of* Aff (D) *is discrete.*

Proof. If $\phi \in \Pi_k(f)$, there is a polynomial P of degree k such that $f \circ \phi = P \circ f$. Then $f \circ \phi^{\circ n} = P^{\circ n} \circ f$ for each positive integer n, and we conclude that $\phi^{\circ n} \in \Pi_{kn}(f)$, because $\deg P^{\circ n} = kn$. Thus the polymorphisms $\phi^{\circ n}$ are all distinct, because they have different f-degrees.

To prove the second assertion of the lemma let Γ be an abelian subgroup of $\mathrm{Aff}(D)$. Let $\mathscr{P}$ be the set of all polynomials associated with an element of $\Gamma \cap \Pi_k(f)$. If $P_j \in \mathscr{P}$ for $j = 1,\ 2$, there exist $\phi_j \in \Gamma \cap \Pi_k(f)$ such that $f \circ \phi_j = P_j \circ f$. Since $f \circ (\phi_i \circ \phi_j) = (P_i \circ P_j) \circ f$ for $i,\ j = 1,\ 2$ and since Γ is abelian, it follows that $P_1 \circ P_2 = P_2 \circ P_1$. Therefore all elements of $\mathscr{P}$ commute with each other. Thus $\mathscr{P}$ is finite, because the number of polynomials of degree k commuting with a given polynomial of degree k is finite [5].

It might happen that a polynomial $P \in \mathscr{P}$ is associated with several polymorphisms in the set $\Gamma \cap \Pi_k(f)$. However, the set of such polymorphisms is always a discrete subset of $\mathrm{Aff}(D)$ [3, Lemma 4]. Since $\mathscr{P}$ is finite, we conclude that $\Gamma \cap \Pi_k(f)$ is discrete.

In the proof of Theorem 1 we shall also need the following result which deals with contracting polymorphisms of a function defined in the upper half plane $H = \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$. We say that a polymorphism $\phi \in \Pi(f)$ is *contracting* if $|\phi'| < 1$.

Theorem 2. *Suppose that $f\colon H \to \mathbf{C}$ is holomorphic and that $\Pi(f)$ contains two contracting polymorphisms which do not commute. Then f has a holomorphic extension to the boundary of H only if f is a polynomial of degree ≤ 1.*

Proof. Let $\phi_1, \phi_2 \in \Pi(f)$ be contracting polymorphisms which do not commute. Then there exist polynomials P_1, P_2 such that

$$f \circ \phi_j = P_j \circ f \qquad (j = 1,\ 2). \tag{3}$$

Suppose that f has a holomorphic extension to the boundary of H, and let ξ_j be the attracting fixed point of ϕ_j. Note that $\xi_1 \ne \xi_2$ because ϕ_1 and ϕ_2 do not commute. Since ϕ_1 and ϕ_2 are contracting, both ξ_1 and ξ_2 lie on the real axis; we may assume that $\xi_1 < \xi_2$.

We prove first that the holomorphic extension of f to any domain containing ξ_1 and ξ_2 is either constant or one-to-one. If f is not one-to-one, there exist z_1 and z_2 such that $z_1 \ne z_2$ and $f(z_1) = f(z_2)$. Then iteration of (3) shows that $f\bigl(\phi_2^{\circ n}(z_1)\bigr) = f\bigl(\phi_2^{\circ n}(z_2)\bigr)$ for each $n \ge 1$, and it follows that f is not locally one-to-one at $\xi_2 = \lim_{n\to\infty} \phi_2^{\circ n}(z_1) = \lim_{n\to\infty} \phi_2^{\circ n}(z_2)$. Thus $f'(\xi_2) = 0$, and by differentiation of $f \circ \phi_1^{\circ n} = P_1^{\circ n} \circ f$ we conclude that $f'\bigl(\phi_1^{\circ n}(\xi_2)\bigr) = 0$ for each $n \ge 1$. But then the zeros of f' accumulate at $\xi_1 = \lim_{n\to\infty} \phi_1^{\circ n}(\xi_2)$ and we see that f must be constant.

Next assume that the extension of f is one-to-one. It suffices to prove that f is the restriction of an entire function, because every holomorphic injection of $\mathbf{C}$ is an element of $\mathrm{Aff}(\mathbf{C})$.

The function g defined by

$$g(w) = f(e^{-w} + \xi_2) \tag{4}$$

satisfies the hypotheses of Lemma 1 (b) for $\beta = \log 1/\phi_2'$. In fact, since f has a holomorphic extension to a neighborhood V of ξ_2, g is defined and holomorphic in $S\big(\alpha, (0,\pi)\big)$ if α is sufficiently large. Moreover, because (3) holds in V by analytic continuation and because $\phi_2(z) = (z-\xi_2)\phi_2' + \xi_2$ for each $z \in \mathbf{C}$, we have

$$g(w+\beta) = f\big(e^{-(w+\beta)} + \xi_2\big) = f\big(\phi_2(e^{-w}+\xi_2)\big) = P_2\big(f(e^{-w}+\xi_2)\big) = P_2 g(w) \tag{5}$$

for each $w \in S\big(\alpha, (0,\pi)\big)$.

Let Y be the set of all $y \in (0,\pi)$ such that the restriction of g to $S\big(\alpha,\, (0,y)\big)$ has a holomorphic extension to $S\big((0,y)\big)$. By Lemma 1 (b) Y is not empty, and it is evident that Y is closed in $(0,\pi)$. We show next that Y is also open.

The domain of the equations (3), (4) and (5) can be enlarged by analytic continuation. In fact, g can be defined by (4) not only in $S\big(\alpha,(0,\pi)\big)$ but also in $S\big((-\pi,0)\big)$ and on the real axis. By analytic continuation (5) holds also in this larger domain. For $j=1$ equation (3) is equivalent to

$$g\Big(\log\frac{1}{\phi_1(z)-\xi_2}\Big) = P_1 g\Big(\log\frac{1}{z-\xi_2}\Big) \tag{6}$$

where $z \in H$. If $y \in Y$, the right side of (6) is defined and holomorphic in the domain $\Delta_y = \{z \in \mathbf{C}\setminus\{\xi_2\} \mid -y < \arg(z-\xi_2) < \pi\}$, and therefore (6) defines a holomorphic extension of g to the domain

$$\Big\{\log\frac{1}{\phi_1(z)-\xi_2}\ \Big|\ z \in \Delta_y\Big\}. \tag{7}$$

Since $\xi_1 < \xi_2$ and ϕ_1 is contracting, Δ_y contains all points of the form $\phi_1^{-1}(z)$ where $\arg(z-\xi_2) = -y$. Consequently the domain (7) contains all points w with $\operatorname{Im} w = y$. Hence Y is open.

We conclude that $Y = (0,\pi)$, because Y is nonempty, open and closed in $(0,\pi)$. Thus g is holomorphic in $S\big((-\pi,\pi)\big)$, and it follows that f has a holomorphic extension to the whole complex plane $\mathbf{C}$. Note that branch points cannot occur, because by assumption f has a single-valued holomorphic extension to the real axis. The proof of Theorem 2 is now complete.

Proof of Theorem 1. Let $k \geq 2$ be such that $\Pi_k(f)$ is not discrete. Then there exists a sequence of distinct polymorphisms $\phi_j \in \Pi_k(f)$ such that $\lim_{j\to\infty}\phi_j = \psi \in \Pi_k(f)$. Moreover, since $\Pi_k(f)$ is contained in $\mathrm{Aff}(D)$, the group $\mathrm{Aff}(D)$ is not discrete. Domains with this property have been classified in [2]. The classification shows that if D is not the whole plane $\mathbf{C}$ or a half plane, then either $\mathrm{Aff}(D)$ is abelian or D is the image of a strip under an element of $\mathrm{Aff}(\mathbf{C})$.

If D is affinely equivalent to a strip, then each element of $\Pi_k(f)$ is either a translation or an involution of order two. However, in view of Lemma 2 no involutions can occur, so that each element of $\Pi_k(f)$ belongs to the abelian group of translations of D. By Lemma 2 this is impossible, because $\Pi_k(f)$ is not discrete. It also follows from Lemma 2 that $\mathrm{Aff}(D)$ cannot be abelian if $\Pi_k(f)$ is not discrete. We conclude that D is either the whole plane $\mathbf{C}$ or a half plane.

Next we show that $f'(z) \neq 0$ for each $z \in D$. For each j choose a polynomial P_j such that $\deg P_j = k$ and

$$f \circ \phi_j = P_j \circ f, \tag{8}$$

and let $X = \{z \in D \mid f'(z) = 0\}$. Differentiation of (8) shows that $\phi_j(X) \subset X$ for each j. Since $\{\phi_j\}$ is not discrete in $\mathrm{Aff}(D)$, Lemma 3 of [3] implies that there exists at most one point $z_0 \in D$ such that the set of points $\phi_j(z_0)$ is discrete in D. On the other hand X is discrete in D because f is nonconstant. It follows that X contains at most one point. However, such a point would be a fixed point of each ϕ_j, because $\phi_j(X) \subset X$ for each j. This contradicts Lemma 2, because the stabilizer of a point of D is an abelian subgroup of $\mathrm{Aff}(D)$. We conclude that X must be empty.

To examine the dependence of P_j on ϕ_j we write

$$P_j(w) = a_{j0} + a_{j1}w + \cdots + a_{jk}w^k$$

and try to express the coefficients $a_{j0}, \ldots, a_{jk}$ of P_j in terms of ϕ_j. To this end we differentiate (8) k times with respect to the independent variable $z \in D$. Since ϕ_j' is a nonzero constant, this differentiation yields

$$\begin{aligned} (f' \circ \phi_j)\phi_j' &= (P_j' \circ f)f' = Q_{j1}(f, f') \\ (f'' \circ \phi_j)(\phi_j')^2 &= (P_j'' \circ f)(f')^2 + (P_j' \circ f)f'' = Q_{j2}(f, f', f'') \\ &\ \ \vdots \\ (f^{(k)} \circ \phi_j)(\phi_j')^k &= Q_{jk}(f, f', f'', \ldots, f^{(k)}) \end{aligned} \tag{9}$$

where each $Q_{j\nu}$ is a polynomial of $\nu + 1$ variables whose coefficients are linear expressions of the coefficients of P_j. Therefore, together with (8) the equations (9) constitute a system which is linear with respect to $a_{j0}, a_{j1}, \ldots, a_{jk}$ and admits a unique solution. More precisely, there is a unique set of constants $a_{j0}, a_{j1}, \ldots, a_{jk}$ such that (8) and (9) are satisfied at each $z \in D$.

We show next that these constants depend holomorphically on ϕ_j and z. The proof will depend on the following lemma.

Lemma 3. *The Wronskian determinant* $W(1,f,f^2,\dots,f^k)$ *is equal to*

$$k!!\,(f')^{k(k+1)/2}$$

where $k!! = 1!\cdot 2!\cdots k!$.

Proof. For the proof of this lemma I am indebted to Kari Katajamäki and Ilpo Laine.

By definition,

$$W(1,f,f^2,\dots,f^k) = \det\left(\frac{d^i f^j}{dz^i}\right) \qquad (0 \le i,j \le k)$$

so that $W(1,f,f^2,\dots,f^k)$ is also the determinant of the linear system (8)–(9).

Successive expansion of the determinant in terms of the elements of the first column shows that

(10)
$$W(1,f,f^2,\dots,f^k) = \nu!!(\nu!)^{(k-\nu)}(f')^{\nu(2k-\nu+1)/2} \times W\left(\binom{\nu+0}{0}f^0, \binom{\nu+1}{1}f^1, \dots, \binom{\nu+(k-\nu)}{k-\nu}f^{k-\nu}\right)$$

for each $\nu = 1,\dots,k$. The assertion follows from (10) when $\nu = k$.

Lemma 3 implies that the determinant of the linear system (8)–(9) is nonzero at each $z \in D$. In particular, the system admits a unique solution of the form

$$a_{j0} = a_0(\phi_j, z), \dots, a_{jk} = a_k(\phi_j, z)$$

where $a_0,\dots,a_k$ are holomorphic functions defined in a subdomain of $\mathrm{Aff}(\mathbf{C})\times\mathbf{D}$. Moreover, since the expressions $a_0(\phi_j,z),\dots,a_k(\phi_j,z)$ are constant with respect to z for each j, we have

$$\frac{\partial^\mu}{\partial z^\mu} a_\nu(\phi, z_0) = 0 \qquad (0 \le \nu \le k;\ \mu = 1,\ 2,\dots) \tag{11}$$

whenever $z_0 \in D$ and $\phi = \phi_j$ for some j. As in [3] we conclude that given $z_0 \in D$ there is a subdomain N of $\mathrm{Aff}(\mathbf{C})$ containing ψ such that the set M of all $\phi \in N$ satisfying (11) is a non-discrete complex analytic subset of N. Since the complex dimension of M at ψ is positive, it follows from the structure theory of one-dimensional complex analytic sets [1, p. 68] that there exists a nonconstant holomorphic map η from the open unit disk U into M such that $\eta(0) = \psi$.

From the definition of M it follows that for each $t \in U$ the polynomial P_t with coefficients $a_0\big(\eta(t), z_0\big),\dots,a_k\big(\eta(t), z_0\big)$ satisfies

$$f\big(\eta(t)(z)\big) = P_t\big(f(z)\big) \tag{12}$$

for each z in a neighborhood of z_0. Moreover, because the function $t \mapsto \deg P_t$ is semicontinuous and $\deg P_0 = k$, we may assume that $\deg P_t \geq k$ for each $t \in U$.

From (12) we see that each element of $\eta(U)$ is a pseudopolymorphism of f and, if we choose N small enough as in [3] the composite of two elements of $\eta(U)$ is again a pseudopolymorphism of f. Since each abelian subgroup of $\mathrm{Aff}(\mathbf{C})$ contains at most countably many of such pseudopolymorphisms corresponding to nonlinear polynomials [3, Lemma 6], by repeating the argument presented in [3, p. 76–77] we conclude that there exists a constant $a \neq 1$ and a nonconstant holomorphic function $b: U \to \mathbf{C}$ such that

$$\eta(t)(z) = az + b(t)$$

for each $(t,z) \in U \times \mathbf{C}$.

Let us first consider the case when D is the whole complex plane $\mathbf{C}$. Then (12) holds by analytic continuation for each $(t,z) \in U \times \mathbf{C}$, and we see that $\Pi_k(f)$ is uncountable. Theorem 2 of [3] implies that f is conjugate to the exponential function e^z.

In the remaining case D is a half plane, and by conjugation we may assume that D is the upper half plane H. Even in this case it suffices to prove that f is the restriction of an entire function.

Since $\phi_j(H) = H$ for each j, $\phi_j(0)$ is real and $\phi_j' > 0$ for each j. Thus by continuity $b(0) = \lim_{j\to\infty} \phi_j(0)$ is real and

$$a = \frac{d}{dz}\eta(0) = \psi' = \lim_{j\to\infty} \phi_j'$$

is real and positive.

Since b is holomorphic and nonconstant and since $b(0)$ is real, there is $t_0 \in U$ such that $\operatorname{Im} b(t_0) < 0$. Then (12) defines a holomorphic extension of f to the domain $\eta(t_0)(H)$. In particular, f has a holomorphic extension to the real axis, because $\operatorname{Im} b(t_0) < 0$.

There are two cases to consider, according as $a < 1$ or $a > 1$.

The case $a > 1$ is easy. In fact, iteration of (12) yields

$$f \circ \eta(t_0)^{\circ n} = P_{t_0}{}^{\circ n} \circ f$$

in a subdomain of H, and this equation defines a holomorphic extension of f to the domain $\eta(t_0)^{\circ n}(H)$ for each $n \geq 1$. Thus f is the restriction of an entire function, and by Theorem 2 of [3] f is conjugate to the exponential function e^z.

If $a < 1$, there exist t_1, $t_2 \in U$ such that $b(t_1)$ and $b(t_2)$ are different real numbers. Then $\eta(t_1)$ and $\eta(t_2)$ are contracting polymorphisms of f which do not commute, and f has a holomorphic extension to the boundary of H. By Theorem 2 f must be the restriction of a linear polynomial. Hence f is the restriction of an entire function. This completes the proof of Theorem 1.

For a given domain D it may be of interest to determine the semigroup $\mathrm{Pol}(D)$ of all holomorphic polynomials mapping D one-to-one onto itself. If D is a strip, a disk or a half plane, it is known that $\mathrm{Pol}(D) = \mathrm{Aff}(D)$. In these simple cases $\mathrm{Pol}(D)$ does not contain any nonlinear polynomials. On the other hand, if D is a Siegel disk of a nonlinear polynomial, then $\mathrm{Pol}(D)$ contains a non-discrete subset of nonlinear polynomials, all of which are irrational rotations of the Siegel disk. We do not know any examples of a situation where $\mathrm{Pol}(D)$ would contain two nonlinear polynomials which are not permutable. However, by using Theorem 1 we can prove the following general result.

Theorem 3. *For each $k \geq 2$ the set $\{P \in \mathrm{Pol}(D) \mid \deg P = k\}$ is a discrete subset of the Lie group of conformal automorphisms of D.*

In particular, it follows from Theorem 3 that the number of nonlinear polynomials in the group of conformal automorphisms of D is at most countable. A proof will appear in a different article.

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