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NEWTON'S METHOD FOR SOLUTIONS OF QUASI-BESSEL DIFFERENTIAL EQUATIONS

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Abstract. Let w be a solution of the differential equation

$$w''(z) + (1 + F(z))w(z) = 0,$$

where F is rational with a zero of order at least 2 at infinity. With suitable hypotheses on the zeros of $1+F(z)$ we show that if w/w' is meromorphic in $\mathbf{C}$, then the Julia set of $f(z) = z-w(z)/w'(z)$ has zero measure. This together with a paper by W. Bergweiler and N. Terglane [4] shows that Newton's method for w converges almost everywhere to zeros of w or poles of F.

1. Introduction

We consider the differential equation

$$w''(z) + \big(1 + F(z)\big)w(z) = 0, \tag{1}$$

where F is rational with a zero at infinity of order at least 2. Let w be a solution of (1). We assume that w/w' is meromorphic in $\mathbf{C}$. Then it makes sense to consider Newton's method for w, i.e. we iterate the function

$$f(z) = z - \frac{w(z)}{w'(z)}. \tag{2}$$

In [4] W. Bergweiler and N. Terglane considered differential equations of the type (1) and gave conditions under which Newton's method converges on an open dense subset of the plane to zeros of w or poles of F.

Theorem A. *Denote by $z_1, \dots, z_N$ the zeros of $1+F(z)$. Let w be a solution of (1) such that w/w' is meromorphic in $\mathbf{C}$ and define f by (2). Suppose that $f^n(z_j)$ converges to a finite limit as $n \to \infty$ for all $j \in \{1, \dots, N\}$. If f is transcendental, then $f^n(z)$ converges to zeros of w or poles of F on an open dense subset of the plane.*

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Remark 1. Theorem A was stated in a more general form in [4]. There differential equations of the type (1) were considered where $1+F(z)$ was replaced by an arbitrary rational function R.

Theorem A leads to the question:

What measure has the set of non-convergence of Newton's method for solutions of the differential equation (1)?

We shall give an answer to this question in the context of the iteration theory of meromorphic functions.

The following notation will be used in this paper:

1. $D(z,r)=\{w\in \mathbf{C}\,;|z-w|<r\}$, $B(z,r)=\{w\in \mathbf{C}\,;|z-w|>r\}$, $\mathbf{D}=D(0,1)$,
2. $O^-(A)=\{w\in \mathbf{C}\,;f^n(w)\in A$ for some $n\in \mathbf{N}_0:=\mathbf{N}\cup\{0\}\}$,
3. $\operatorname{tr}(\gamma)$ denotes the trace of the path γ.

The paper is organized as follows. In §2 we state some results of iteration theory and formulate the basic result, Theorem 1. In §3 we give a criterion for a class of meromorphic functions to have a Julia set of zero measure. By means of this criterion we prove Theorem 1 in §5 and §6 using the lemmas of §4. In §7 we investigate the case where f in (2) is rational. Finally, in §8 we apply our results to Bessel functions.

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2. Iteration theory

Let f be a rational or transcendental meromorphic function. The Fatou set of f is defined by

$$F(f)=\{z\in \overline{\mathbf{C}}\,;(f^n)\text{ is defined and normal in a neighbourhood of }z\}$$

and the complement $J(f)=\overline{\mathbf{C}}\backslash F(f)$ is called the Julia set of f. For an introduction into iteration theory we refer to the books [1], [5], [15] and the lecture notes [10] for rational functions and to the survey article [2] for transcendental functions.

We call a point z_0 periodic, if $f^n(z_0)=z_0$ for some $n\in\mathbf{N}$. If n is minimal with this property, then $m:=(f^n)'(z_0)$ is called the multiplier of z_0. By means of m we can classify the periodic points. The point z_0 is called

1. attracting if $|m|<1$ and superattracting if $m=0$,
2. repelling if $|m|>1$,
3. rationally indifferent if $m=e^{2\pi i\alpha}$ with $\alpha\in\mathbf{Q}$ and irrationally indifferent if $m=e^{2\pi i\alpha}$ with $\alpha\in\mathbf{R}\backslash\mathbf{Q}$.

Let U be a component of $F(f)$. Then $f^n(U)$ is contained in some component U_n where we set $U = U_0$. If $U_n \neq U_m$ for all $n \neq m$, we call U wandering, otherwise we call U preperiodic. If $U = U_l$ for some l, then we call U periodic.

The limiting behaviour of the iterates in periodic components is well understood. In the case of rational functions such a component is an attractive basin, a Leau domain, a Siegel disk, or a Herman ring. These periodic components do occur in the case of transcendental functions too, but there exists a further possibility, a Baker domain. For details see [1, §7] and [2, §4]. We mention that there is a close connection between the periodic components (except Baker domains) and the singularities of the inverse function, i.e. the critical and asymptotic values. See [1, §9] and [2, §4].

We set $J_0(f) = J(f)\backslash\{\infty\}$. The Fatou set is a completely invariant set, that is $z \in F(f)$ if and only if $f(z) \in F(f)$. If f is defined as in (2), then the zeros of w are attracting fixed points of f. Thus they lie in the Fatou set and, as $F(f)$ is open, even a neighbourhood of these points lies in $F(f)$. Therefore a point $z \in J_0(f)$ cannot converge to a zero of w because then $f^m(z)$ would be in a neighbourhood $U \subset F(f)$ of w for some $m \in \mathbf{N}_0$. Because of the complete invariance, z would be in $F(f)$ too. Therefore $J_0(f)$ is part of the set of non-convergence of Newton's method.

We now formulate the main result.

Theorem 1. *Let w be a non-constant solution of* (1) *such that w/w' is meromorphic in $\mathbf{C}$. Further, let f be as in* (2) *and suppose that f is transcendental. Denote by $z_1, \ldots, z_N$ the zeros of $1 + F(z)$. If $f^n(z_j)$ converges to a finite limit for $j \in \{1, \ldots, N\}$, then $J_0(f)$ has zero measure.*

This together with Theorem A yields an answer to the above question.

Corollary 1. *Under the same hypotheses as in Theorem* 1 *$f^n(z)$ converges almost everywhere to zeros of w or to poles of F.*

Proof. In the notation of §2 the open dense subset of Theorem A is the Fatou set of f. Thus the conclusion follows from Theorem A and Theorem 1. □

For the sake of completeness, we also consider the case that f in (2) is rational.

Theorem 2. *Let F in* (1) *not be identically zero and w be a non-constant solution of* (1) *such that w/w' is rational. Then w has the form*

$$w(z) = ce^{\pm iz + \int^z R(\xi)\, d\xi},$$

where $c \neq 0$ is a constant and R is rational with $R(z) = O(1/z^2)$ as $z \to \infty$. Further, let f be as in (2)*. Then each pole of R of order at least* 2 *and the point $z = \infty$ are rationally indifferent fixed points of f with multiplier* 1*. If z_0 is a pole of order* 1 *and $R(z) = a/(z - z_0) + O(1)$ as $z \to z_0$, then z_0 is a fixed point* *of f with multiplier $1-(1/a)$. Denote by $z_1,\dots,z_N$ the zeros of $1+F(z)$. If $f^n(z_j)$ converges to a limit in $\overline{\mathbf{C}}$, then $J(f)$ has zero measure. Moreover, $f^n(z)$ converges almost everywhere to zeros of w or poles of R or to the point $z=\infty$.*

Remark 2. We have excluded the case that (1) is the sine equation $w''+w=0$. Then w/w' is rational if and only if $w=ce^{\pm iz}$. But then $f(z)=z\pm i$ which is a linear transformation. The iterative behaviour of such functions is quite simple.

3. Criteria for Julia sets to have zero measure

Let $\operatorname{sing}(f^{-1})$ be the set of the finite singularities of the inverse function. We assume that the iterates of each point $z\in\operatorname{sing}(f^{-1})$ are well defined. Then the postsingular set is defined as the closure of

$$P=\bigcup_{n=0}^{\infty} f^n\bigl(\operatorname{sing}(f^{-1})\bigr).$$

We say that a set E is thin at infinity, if there exist constants $\varepsilon>0$ and $R>0$ such that

$$\operatorname{dens}\bigl(E,D(z,r)\bigr)=\frac{\operatorname{meas}\bigl(E\cap D(z,r)\bigr)}{\operatorname{meas}\bigl(D(z,r)\bigr)}<1-\varepsilon$$

for all $z\in\mathbf{C}$ and $r\geq R$. We denote by $\operatorname{dist}(\,\cdot\,,\cdot\,)$ the Euclidean distance in $\mathbf{C}$. Further, we decompose the set $\operatorname{sing}(f^{-1})$ into three disjoint sets S_I, S_F, and S_J, where S_I denotes all points $z\in\operatorname{sing}(f^{-1})\cap F(f)$ which will be attracted to a rationally indifferent cycle, S_F denotes $\bigl(\operatorname{sing}(f^{-1})\backslash S_I\bigr)\cap F(f)$, and S_J is the set $\operatorname{sing}(f^{-1})\cap J_0(f)$. Moreover, we set $P_F=\bigcup_{n=0}^{\infty} f^n(S_F)$, $P_I=\bigcup_{n=0}^{\infty} f^n(S_I)$, and $P_J=\bigcup_{n=0}^{\infty} f^n(S_J)$.

Theorem 3. *Let f be meromorphic. Suppose that the following conditions are satisfied:*

(i) $\operatorname{dist}\bigl(\overline{P_F},J_0(f)\bigr)>0$.
(ii) *S_I is a finite set.*
(iii) *$P\cap O^-(\infty)=\emptyset$. Further, $\overline{P_J}$ is a finite set.*

Then, if $J_0(f)$ is thin at infinity, $J_0(f)$ has Lebesgue measure zero.

Remark 3. If there do not exist any Baker domains or wandering domains, then, for all $z\in S_F$, $f^n(z)$ will eventually end in a cycle of attractive basins because Siegel disks and Herman rings do not exist by hypothesis (ii) and (iii). In fact, it is well known that the boundary of these components of the Fatou set lies in $\overline{P}\cap J(f)$ which is, in our situation, a finite set.

Remark 4. The case that $\operatorname{sing}(f^{-1}) = S_F$ and $\overline{P_F}$ is a compact set is due to C. McMullen [9] for entire f. G. Stallard [12] and [13] has generalized the criterion to all meromorphic functions with $\operatorname{sing}(f^{-1}) = S_F$ such that (i) holds.

If f is rational, $S_I = \emptyset$, and (iii) holds, then f is called subhyperbolic. We shall refer to the conditions (i), (ii), (iii) as subhyperbolic conditions.

Proof. We restrict ourselves to the set $\tilde{J}(f) = J(f)\backslash O^-(\infty)$. These are the points in $J(f)$ where all iterates are defined. This is no loss of generality because $O^-(\infty)$ is a countable set, thus

$$\operatorname{meas}\bigl(J_0(f)\bigr) = \operatorname{meas}\bigl(\tilde{J}(f)\bigr). \tag{3}$$

We put $Q = \overline{P_J} \cup (\overline{P_I}\backslash P_I) = \overline{P} \cap J_0(f)$ and show that

$$\limsup_{n\to\infty} \operatorname{dist}\bigl(f^n(z), Q\bigr) > 0 \tag{4}$$

holds for all $z \in \tilde{J}(f)\backslash O^-(Q)$. As $\overline{P_J}$ is finite and $\overline{P_J} \subset \tilde{J}(f)$, each point in $\overline{P_J}$ is eventually periodic. Every irrationaly indifferent cycle in $J_0(f)$ must lie in the derived set of $\overline{P}$ [1, Theorem 9.3.4] and so it follows from (i) and (ii) that every irrationally indifferent cycle in $J_0(f)$ must lie in the derived set of $\overline{P_J}$ which we know from (iii) to be empty. Therefore each point of $\overline{P_J}$ must end in a repelling or rationally indifferent cycle. We note that, as S_I is finite, $\overline{P_I}\backslash P_I$ is the union of finitely many rationally indifferent cycles.

Suppose that a point $z \in \tilde{J}(f)$ satisfies

$$\lim_{n\to\infty} \operatorname{dist}\bigl(f^n(z), Q\bigr) = 0. \tag{5}$$

Let Q be the set $\{a_1, \ldots, a_N\}$. Clearly, $Q \subset \tilde{J}(f) \subset \mathbf{C}\backslash O^-(\infty)$ and so we can choose disks $D\bigl(a_j, r(z)\bigr)$ which are disjoint and do not contain poles of f. Without loss of generality $f^n(z)$ lies in $\bigcup_{j=1}^N D\bigl(a_j, r(z)\bigr)$ for all $n \in \mathbf{N}$. As f is continuous in $\bigcup_{j=1}^N D\bigl(a_j, r(z)\bigr)$, there must be a subsequence, say $f^{n_k}(z)$, such that $f^{n_k}(z) \in D\bigl(a_i, r(z)\bigr)$ and a_i lies in a cycle $\{b_1, \ldots, b_M\}$ with $f(b_j) = b_{j+1} \pmod M$. We choose disks $D\bigl(b_j, \alpha(z)\bigr)$ such that $f\bigl(\overline{D(b_j, \alpha(z))}\bigr) \subset D\bigl(b_{j+1}, r(z)\bigr) \pmod M$. Then for all $n, n_k \geq n_0$, $f^n(z) \in \bigcup_{j=1}^N D\bigl(a_j, \alpha(z)\bigr)$ and therefore $f^{n_k+l}(z) \in \bigcup_{j=1}^M D\bigl(b_j, \alpha(z)\bigr)$ for all $l \in \mathbf{N}$. Thus $f^n(z)$ converges to a repelling or rationally indifferent cycle in Q. In the case that the cycle is repelling it is easy to see that z belongs to $O^-(Q)$. If the cycle is rationally indifferent the same is true for $z \in J_0(f)$ [10, Corollary 7.4]. Thus $z \in O^-(Q)$. Therefore (4) holds for $z \notin O^-(Q)$.

Now let $z \in \tilde{J}(f)\backslash O^-(Q)$. We choose a sequence $f^{n_k}(z)$ such that

$$\operatorname{dist}\bigl(f^{n_k}(z), Q\bigr) \geq \alpha(z) > 0 \tag{6}$$

for all $k \in \mathbf{N}$. From (i) we have a constant $\delta > 0$ such that

$$\operatorname{dist}\big(\overline{P_F}, J_0(f)\big) \geq \delta. \tag{7}$$

Let u be in S_I. It is well known that u will be attracted to a rationally indifferent cycle which lies in Q. As S_I is finite only finitely many points of P_I do not belong to $\bigcup_{j=1}^{N} D\big(a_j, \alpha(z)\big)$ and so there exists a positive constant $\varepsilon(z) \leq \min\{\delta, \alpha(z)\}$ such that

$$\operatorname{dist}\big(f^{n_k}(z), \overline{P}\big) \geq \varepsilon(z). \tag{8}$$

for all $k \in \mathbf{N}$ and $z \in \tilde{J}(f) \backslash O^-(Q)$. The Lebesgue density theorem [14, Proposition 1, p. 12] states that almost every point x of a measurable set E must be a point of density, i.e.

$$\lim_{r \to 0+} \frac{\operatorname{meas}\big(E \cap D(x,r)\big)}{\operatorname{meas}\big(D(x,r)\big)} = 1.$$

As $O^-(Q)$ is countable we have only to prove that the points of $J_0(f)$ which satisfy (8) are not points of density of $J_0(f)$. Then $J_0(f)$ has zero measure. This is the assertion.

If we analyse the proof by G. Stallard [12], we can carry over the proof in the case of entire functions after some minor modifications using (8), condition (i), and the fact that $\overline{P_J}$ is compact. We refer for details to the paper from G. Stallard [12]. Therefore the proof is done in the case of entire functions.

Similarly, if f has at least one pole, then we can carry over the proof by G. Stallard [13] to prove that there is no point of density in $\tilde{J}(f) \backslash O^-(Q)$. Thus $\tilde{J}(f)$ has zero measure and so it follows from (3) that $J_0(f)$ has zero measure. This proves the theorem. □

Remark 5. The proof shows that a transcendental meromorphic function has a Julia set of zero measure if $\limsup_{n\to\infty} \operatorname{dist}\big(f^n(z), \overline{P}\big) > 0$ holds for almost every $z \in \tilde{J}(f)$, $\overline{P} \cap J_0(f)$ is compact, and $J_0(f)$ is thin at infinity. These conditions are more general than those given in the statement of Theorem 3.

Corollary 2. *Let f be rational of degree $d \geq 2$. Further, suppose that $J \cap \overline{P}$ is finite. If $J(f) \neq \overline{\mathbf{C}}$, then $J(f)$ has zero measure.*

Proof. By hypothesis, condition (iii) of Theorem 3 is satisfied. As f has at most $2d-2$ critical points and no asymptotic values the set $\operatorname{sing}(f^{-1})$ is finite. In particular, (ii) of Theorem 3 is satisfied. Further, f has no wandering domains by Sullivan's no wandering domain theorem for rational function [16]. As f has no Baker domains, each point in S_F must be attracted to an attracting cycle (Remark 3). Thus (i) is satisfied. By conjugation with a Möbius transformation we may assume that $\infty \notin J(f)$. This implies that $J(f)$ is thin at infinity. The conclusion follows from Theorem 3. □

4. An estimate for the width of the immediate attractive basin

Let z_0 be a superattracting fixed point of f and $A^*(z_0)$ its immediate attractive basin.

Lemma 1. *If $A^*(z_0)$ does not contain a zero or a pole of $1+F(z)$, then $A^*(z_0)$ is simply connected and $f\colon A^*(z_0) \to A^*(z_0)$ is conjugate to $z^3\colon \mathbf{D} \to \mathbf{D}$.*

Proof. As

$$f'(z) = \frac{w(z)w''(z)}{w'(z)^2} \tag{9}$$

we see that the critical points of f are the simple zeros of w and the zeros of w'' which are not zeros of w and w'. The simple zeros of w are the superattracting fixed points of f and from (1) we see that the zeros of w'' which are not zeros of w are the zeros of $1+F(z)$. Therefore, by the hypotheses, there are no critical points in the immediate attractive basin other than z_0.

Moreover, we have $w(z_0) = w''(z_0) = 0$ by (1). Thus $w'(z_0) \neq 0$, because w is not identically zero. From (9) we have $f'(z_0) = 0$. Another differentiation yields $f''(z_0) = 0$. Further, we see by differentation of (1) that $w'''(z_0) \neq 0$ as the function $1+F(z)$ has neither a zero nor a pole at z_0. Thus $f'''(z_0) = 2w'''(z_0)/w'(z_0) \neq 0$.

It follows from a theorem of Böttcher [15, p. 60] that f is locally conjugate to $z \mapsto z^3$. This means that there exists a function ψ defined in a neighbourhood of z_0 which satisfies the Böttcher functional equation, i.e.

$$\psi \circ f = (\psi)^3. \tag{10}$$

It now follows by [15, Theorem 4, §3] that ψ maps $A^*(z_0)$ conformally to $\mathbf{D}$. In particular, $A^*(z_0)$ is simply connected. □

Remark 6. Theorem 4 in [15] is stated for rational functions but holds for transcendental functions too.

We set $\phi = \psi^{-1}$ where ψ is the function in (10).

Lemma 2. *Let $\sigma \in \{-1, 1\}$. Either the angular limit $\zeta = \phi(\sigma)$ exists and ζ is a repelling fixed point of f or the angular limit is infinity.*

Proof. The conclusion follows directly from a theorem by C. Pommerenke [11, Theorem 1]. □

Lemma 3. *Let $\zeta \in \mathbf{C}$ be a repelling fixed point of f. Then ζ is the angular limit $\phi(\sigma)$ for $\sigma \in \{-1, 1\}$ for only finitely many immediate attractive basins.*

Proof. In the paper cited above [11, Theorem 3], C. Pommerenke has shown that the number of immediate attractive basins for which ζ is the angular limit $\phi(\sigma)$ for $\sigma \in \{-1,1\}$ is limited by $2\log 3/\log|f'(\zeta)|$. □

The repelling fixed points of f are all poles of w. We see from (1) that w has only finitely many poles and hence f only finitely many repelling fixed points. We shall see that if f is a transcendental function given by (2), then there exist infinitely many superattracting fixed points. It follows from Lemma 2 and Lemma 3 that, for all but finitely many values of z_0, the basin $A^*(z_0)$ contains two paths which tend to infinity. The first one is the image curve of $(0,1)$, say γ_1, the second is the image of $(-1,0)$, say γ_2.

Lemma 4. *Let $R > 0$. Moreover, suppose that*

$$|f(z) - z| > c$$

holds for all $z \in A^*(z_0) \cap B(z_0, R)$ where c is a positive constant. Then, for all $w \in \operatorname{tr}(\gamma_i) \cap B(z_0, R)$, $i = 1,2$, there exists $\delta \geq c/8$ such that the disk $D(w,\delta)$ lies in $A^*(z_0)$.

For the proof we need the following lemma which may be found in [5, p. 13].

Lemma 5. *Let D be a simply connected hyperbolic domain in $\mathbf{C}$, and for $z \in D$, let $\delta(z)$ denote the Euclidean distance of z from ∂D. Then*

$$\frac{1}{2}\frac{|dz|}{\delta(z)} \leq d\rho_D(z) \leq 2\frac{|dz|}{\delta(z)}.$$

Proof of Lemma 4. We denote hyperbolic distance in $\mathbf{D}$ by $[\,\cdot\,,\cdot\,]_{\mathbf{D}}$. First, we estimate the hyperbolic distance between a point $x \in (0,1)$ and its image under $z \mapsto z^3$:

$$[x, x^3]_{\mathbf{D}} = \int_{x^3}^{x} \frac{2dt}{1-t^2} \leq \log 3. \tag{11}$$

As the hyperbolic metric is invariant under rotations, (11) is also true for $x \in (-1,0)$. We have, via ϕ, the same estimate for $[w, f(w)]_{A^*(z_0)}$ if $w \in \operatorname{tr}(\gamma_i)$, $i = 1,2$.

Let $w \neq z_0$ be an arbitrary point in $A^*(z_0)$ and γ a path which connects w and $f(w)$ in $A^*(z_0)$. Let γ be parametrized by arc length. We set $\delta := \delta(w)$. Then for a point $\gamma(s)$ we have:

$$\delta\bigl(\gamma(s)\bigr) \leq \operatorname{dist}\bigl(\gamma(s), w\bigr) + \delta(w) \leq s + \delta.$$

Denote the length of γ by L. Then by Lemma 5 we have:

$$\int_\gamma d\rho_{A^*(z_0)}(z) \geq \frac{1}{2}\int_\gamma \frac{1}{\delta(z)}\,|dz| \geq \frac{1}{2}\int_0^L \frac{1}{\delta + s}\,ds = \frac{1}{2}\log\Bigl(1 + \frac{L}{\delta}\Bigr).$$

If $w \in \operatorname{tr}(\gamma_i) \cap B(z_0, R)$, then $L \geq |f(w) - w| \geq c$ and $\int_\gamma d\rho_{A^*(z_0)}(z) \leq \log 3$ so that

$$\delta \geq \tfrac{1}{8}c. \ \square$$

5. Proof of Theorem 1: f fulfills the subhyperbolic conditions

In this paragraph we shall use the notation of Theorem 3. It has been proved in [3] that if f has the form (2) and w is a solution of (1), then f has no wandering domains. Further, it has been shown in [3] and [4] that, if f is as before, then f has no asymptotic values and any cycle of Baker domains contains a point $z \in \operatorname{sing}(f^{-1})$, i.e. a critical value. As seen in the proof of Lemma 1, these singularities of f^{-1} are the superattracting fixed points of f and the images of the zeros of $1 + F(z)$. Suppose that there exists a cycle of Baker domains. Then there is one component of this cycle where the iterates tend to infinity. Let z be a critical value which lies in the cycle. Of course, z is the image of a zero of $1+F(z)$, say z_j. Then $f^n(z_j) \to z_0$ for some $z_0 \in \mathbf{C}$ by hypothesis. On the other hand, there exists a subsequence, say $f^{n_k}(z_j)$, which converges to infinity. This yields a contradiction. Thus there are no Baker domains. Because of Remark 3 all points in S_F will be attracted to attracting cycles, if conditions (ii) and (iii) hold.

We shall verify the conditions (i), (ii), and (iii) of Theorem 3 in this section.

Let z_j be a zero of $1 + F(z)$. By the hypotheses of Theorem 1 we have that $f^n(z_j) \to z_0$ as $n \to \infty$ where $z_0 \in \mathbf{C}$ depends on j. It is easy to see that z_0 is a fixed point of f. If $z_j \in J_0(f)$, then $z_0 \in J_0(f)$. Recently, R. Pérez-Marco has proved the conjecture that $f^n(z)$ cannot converge to an irrationally indifferent fixed point z_0, unless z lies in the backward orbit of z_0. We have seen in the proof of Theorem 3 that the same is true if the fixed point is repelling or rationally indifferent and $z_j \in J_0(f)$. Thus the forward orbit of each z_j is finite. Of course, the superattracting fixed points are in $F(f)$. As there are only finitely many zeros of $1 + F(z)$, condition (iii) is satisfied.

It is clear that (ii) holds because S_I is a subset of the images of the zeros of $1 + F(z)$.

The forward orbits of the finitely many zeros of $1+F(z)$ which converge to an attracting fixed point have positive distance from the Julia set because attracting fixed points are in $F(f)$ which is open. Further, there exist only finitely many immediate attractive basins which contain a pole of F. Again, we have positive distance of the attracting fixed points from the Julia set.

Let z_0 be a superattracting fixed point for which there is no pole or zero of $1 + F(z)$ in the immediate attractive basin. Lemma 1 yields a function $\phi\colon \mathbf{D} \to A^*(z_0)$ with $\phi(0) = z_0$ and

$$\phi(z^3) = f\bigl(\phi(z)\bigr).$$

We differentiate this equation three times and set $z = 0$:

$$\bigl(\phi'(0)\bigr)^2 = \frac{6}{f'''(z_0)}.$$

Moreover, we differentiate f three times and obtain

$$f'''(z_0) = 2\frac{w'''(z_0)}{w'(z_0)}.$$

From (1) we obtain

$$w'''(z_0) = -\big(1 + F(z_0)\big)w'(z_0).$$

For large $|z_0|$ we have

$$|f'''(z_0)| = 2|1 + F(z_0)| \approx 2.$$

Thus $|f'''(z_0)| \le 3$ for large $|z_0|$ and

$$|\phi'(0)| \ge \sqrt{\frac{6}{3}} = \sqrt{2}.$$

We apply Koebe's one quarter-Theorem [6, p. 31] to

$$h(t) = \frac{\phi(t) - \phi(0)}{\phi'(0)}$$

and obtain

$$\phi(\mathbf{D}) \supset D\big(z_0, \tfrac{1}{4}|\phi'(0)|\big) \supset D\big(z_0, \tfrac{1}{4}\sqrt{2}\,\big) = \widehat{D}.$$

$\widehat{D}$ is a disk with $\widehat{D} \subset A^*(z_0)$. Thus $\mathrm{dist}\big(z_0, J_0(f)\big) \ge \frac{1}{4}\sqrt{2}$. There are at most finitely many superattracting fixed points which do not satisfy the conditions used in the preceding paragraph and the forward orbits of the finitely many zeros of $1 + F(z)$ which converge to attracting fixed points. As all attracting fixed points belong to $F(f)$ which is open, it follows that condition (i) is satisfied. Thus all three of the subhyperbolic conditions are satisfied.

6. Proof of Theorem 1: The Julia set is thin at infinity

We use some results from the theory of differential equations which can be found in [8]. For $r \ge 0$, $\alpha > 0$, and $\varphi \in \mathbf{R}$ we define $S(r, \alpha, \varphi) := \{z \in \mathbf{C}\,; |z| > r, |\arg(z) - \varphi| < \alpha\}$. Then $S(0, \pi, 0)$ and $S(0, \pi, \pi)$ are so-called normal sectors. Let $R > 0$ be sufficiently large such that $D(0, R)$ contains all poles of F. Then in $S(R, \pi, 0)$ and in $S(R, \pi, \pi)$ there exist fundamental matrices of (1),

$$\widetilde{W}(z) = \begin{pmatrix} \tilde{w}_1(z) & \tilde{w}_2(z) \\ \tilde{w}_1'(z) & \tilde{w}_2'(z) \end{pmatrix}, \qquad \widehat{W}(z) = \begin{pmatrix} \hat{w}_1(z) & \hat{w}_2(z) \\ \hat{w}_1'(z) & \hat{w}_2'(z) \end{pmatrix},$$

respectively, which converge uniformly in each closed subsector to the formal fundamental matrix

$$H(z) = \begin{pmatrix} 1 + a/z + \cdots & i + b/z + \cdots \\ i + ia/z + \cdots & 1 - ib/z + \cdots \end{pmatrix} \begin{pmatrix} e^{iz} & 0 \\ 0 & e^{-iz} \end{pmatrix}.$$

Therefore, in $S(R, \pi - \varepsilon, 0)$ for some $\varepsilon > 0$ we have

$$\tilde{w}_1(z) = \Big(1 + \frac{a}{z} + O\Big(\frac{1}{z^2}\Big)\Big)e^{iz}, \qquad \tilde{w}_1'(z) = i\Big(1 + \frac{a}{z} + O\Big(\frac{1}{z^2}\Big)\Big)e^{iz} \tag{12}$$

and

$$\tilde{w}_2(z) = \Big(i + \frac{b}{z} + O\Big(\frac{1}{z^2}\Big)\Big)e^{-iz}, \qquad \tilde{w}_2'(z) = \Big(1 - i\frac{b}{z} + O\Big(\frac{1}{z^2}\Big)\Big)e^{-iz} \tag{13}$$

for $z \to \infty$. Analogously, we obtain a representation for $\hat{w}_1$, $\hat{w}_1'$, $\hat{w}_2$, and $\hat{w}_2'$ in $S(R, \pi - \varepsilon, \pi)$ for $w(z) = c_\pi \hat{w}_1 + d_\pi \hat{w}_2$.

Let $z \in S(R, \pi - \varepsilon, 0)$ and w be a solution of (1) defined in z. Then $w(z) = c_0 \tilde{w}_1(z) + d_0 \tilde{w}_2(z)$. We obtain with (12) and (13):

$$\begin{aligned}
\frac{w(z)}{w'(z)} &= \frac{c_0 \tilde{w}_1(z) + d_0 \tilde{w}_2(z)}{c_0 \tilde{w}_1'(z) + d_0 \tilde{w}_2'(z)} \\
&= \frac{c_0 e^{iz}\big(1 + (a/z) + O(1/z^2)\big) + d_0 e^{-iz}\big(i + (b/z) + O(1/z^2)\big)}{ic_0 e^{iz}\big(1 + (a/z) + O(1/z^2)\big) + d_0 e^{-iz}\big(1 - i(b/z) + O(1/z^2)\big)} \\
&= i\frac{c_0 e^{iz} + id_0 e^{-iz}}{-c_0 e^{iz} + id_0 e^{-iz}}\big(1 + O(1/z)\big)
\end{aligned} \tag{14}$$

for all $z = x + iy \in S(R, \pi - \varepsilon, 0)$ with $|y| > c$ where c is some positive constant. Analogously, we obtain a similar equation to (14) in $S(R, \pi - \varepsilon, \pi)$.

We shall use a theorem from E. Hille [7, p. 181]. We denote by D_0 all $z \in B(0, R)$ such that the ray $z + r$, $0 \le r < \infty$, lies in $B(0, R)$.

Theorem B. *Let $w(z) \not\equiv 0$ be a solution of* (1) *which is defined in D_0. Then there exists a solution w_0 of the sine equation*

$$u''(z) + u(z) = 0$$

such that for all $z = x + iy \in D_0$

$$|w(x + iy) - w_0(x + iy)| \le M(y)\Big\{\exp\Big[\int_x^\infty |F(s + iy)|\, ds\Big] - 1\Big\} \tag{15}$$

where $M(y) = \max_s |w_0(s + iy)|$.

Remark 7. An analogous result can be obtained for a solution w of (1) defined in D_π which denotes the set of all $z \in B(0, R)$ such that the ray $z + re^{i\pi} = z - r$, $0 \le r < \infty$, lies in $B(0, R)$. We denote the approximating function by w_π.

Solutions of the sine equation have the form $u(z) = \alpha e^{iz} + \beta e^{-iz}$. Suppose that w_0 in Theorem B has the form $w_0(z) = \alpha e^{iz}$. Then, by (15), d_0 is zero. We see from Theorem B and from (14) that $w/w' \to -i$ as $z \to \infty$ in $S(R, \pi - \varepsilon, 0)$. As $S(R, \pi - \varepsilon, 0) \cap S(R, \pi - \varepsilon, \pi) = S(R, \frac{1}{2}\pi - \varepsilon, \frac{1}{2}\pi) \cup S(R, \frac{1}{2}\pi - \varepsilon, -\frac{1}{2}\pi)$ we have $d_\pi = 0$ and $w_\pi = \tilde{\alpha} e^{iz}$. Again, $w/w' \to -i$ as $z \to \infty$ in $S(R, \pi - \varepsilon, \pi)$. If we combine the asymptotics in $S(R, \pi - \varepsilon, 0)$ and $S(R, \pi - \varepsilon, \pi)$, we obtain $w/w' \to -i$ as $z \to \infty$ which is of course only possible, if w/w' is rational. Thus f in (2) is rational. Analogously, f is rational if w_0 has the form $w_0(z) = \beta e^{-iz}$. As we only have to consider transcendental functions we may assume that w_0 is of the form

$$w_0(z) = \alpha \cos(z + \beta), \tag{16}$$

$\alpha, \beta \in \mathbf{C}$. Then the constants c_0, c_π, d_0, and d_π are not zero.

To apply Theorem 3 we still have to show that the Julia set is thin at infinity. We proceed in two steps. In the first step we show that the superattracting fixed points are close to the real axis and that we can find rectangles which contain the fixed point and which are invariant under iteration. In the second step we extend the rectangles of step one to quasi-strips. By a quasi-strip we mean a set Q such that the following two conditions are satisfied:

1. For arbitrary points $z, w \in Q$ we have

$$|\operatorname{Re}(z - w)| < c_1 \tag{17}$$

   where $c_1 > 0$ does not depend on z, w.
2. For each $y \in \mathbf{R}$ there exists $x \in \mathbf{R}$ such that $D(x + iy, \delta) \subset Q$ for some δ which is independent of x, y.

If we restrict ourselves to the half-strip $S := \{z \in \mathbf{C}; \operatorname{Re}(z) > x_0 > R + 1, |\operatorname{Im}(z)| < y_0\}$, then by Theorem B and (16) our solution w of (1) is of the form

$$w(z) = \alpha \cos(z + \beta) + o(1) \tag{18}$$

for $z \to \infty$. This shows that the zeros of w are close to the zeros of $\cos(z + \beta)$, i.e. close to $z_k = \frac{1}{2}\pi + k\pi - \beta$. For large $k > 0$ the imaginary part of z_k is nearly zero. We construct a rectangle in the following manner. The left side is $\frac{1}{2}\pi - \frac{1}{4}\pi + k\pi - \beta + it$, $-y_0 \le t \le y_0$, the right side is $\frac{1}{2}\pi + \frac{1}{4}\pi + k\pi - \beta + it$, $-y_0 \le t \le y_0$, the top side is part of $\operatorname{Im}(z) = y_0$, and the bottom side is part of $\operatorname{Im}(z) = -y_0$.

We now enlarge S to $\widetilde{S} := \{z \in \mathbf{C}; \operatorname{Re}(z) > x_0 - 1,\ |\operatorname{Im}(z)| < y_0 + 1\}$ and write (18) as $w(z) = \alpha \cos(z + \beta) + h(z)$ for $z \in \widetilde{S}$ where $h(z) = w(z) - w_0(z)$ is holomorphic in $\widetilde{S}$ because $\widetilde{S}$ does not contain poles of F. Of course, (18) remains valid for $z \in \widetilde{S}$ by Theorem B, i.e. $h(z) = o(1)$ as $z \to \infty$. Then by

Cauchy's inequalities we see that, for $z \in S$, $h'(z) = o(1)$ as $z \to \infty$ and so $w'(z) = -\alpha \sin(z+\beta) + o(1)$. On the right side of the rectangle we have

$$\begin{aligned}\frac{w(z)}{w'(z)} &= \frac{w_0(z_k + \frac{1}{4}\pi + it) + o(1)}{w_0'(z_k + \frac{1}{4}\pi + it) + o(1)} = \frac{w_0(z_k + \frac{1}{4}\pi + it)}{w_0'(z_k + \frac{1}{4}\pi + it)}\big(1 + o(1)\big) \\ &= -\frac{\cos(z_k + \frac{1}{4}\pi + it)}{\sin(z_k + \frac{1}{4}\pi + it)}\big(1 + o(1)\big) \\ &= -\cot\big(\tfrac{1}{2}\pi + \tfrac{1}{4}\pi + it\big)\big(1 + o(1)\big) \\ &= \left(\frac{1}{2}\frac{1}{\frac{1}{2} + \sinh^2(t)} + i\frac{\sinh(t)\cosh(t)}{\frac{1}{2} + \sinh^2(t)}\right)\big(1 + o(1)\big)\end{aligned}$$

and hence

$$\operatorname{Re}\Big(\frac{w}{w'}\Big) > \frac{1}{2}\frac{1}{\frac{1}{2} + \sinh^2(t)}\big(1 - |o(1)| - 2|\sinh(t)|\,|\cosh(t)|\,|o(1)|\big) > 0$$

for large k. Analogously, on the left side of the rectangle we have $\operatorname{Re}(w/w') < 0$ for large k. On the top side we have $w/w' \approx i$ and on the bottom side $w/w' \approx -i$ for large y_0. This is easily seen by decomposing the cotangent into real und imaginary parts. Further, the computation shows that $|w(z)/w'(z)| > c_0$ for all z on the boundary of the rectangle where c_0 is a positive constant independent of k. By the minimum principle and (18) we obtain a fixed point in the interior of the rectangle for large k. As f has only finitely many non-superattracting fixed points this fixed point is superattracting for large k (see (23) below). Moreover, by the above estimates, the rectangle is invariant under iteration of $f(z) = z - w/w'$. Thus step one is done. Note that the rectangle is part of the immediate attractive basin of the superattracting fixed point.

For the second step let $D = \{\operatorname{Im}(z) > c\} \cap S(0, \frac{1}{2}\pi, 0)$, where c is the constant from (14). If $z \in D$, we can write the equation (14) as follows

$$\frac{w(z)}{w'(z)} = i\big(1 + O(1/z^2)\big)\frac{1 - (ic_0/d_0)e^{2iz}\big(1 + O(1/z)\big)}{1 + (ic_0/d_0)e^{2iz}\big(1 + O(1/z)\big)} \tag{19}$$

as $z \to \infty$. We derive from (19) that, if c is sufficiently large, then

$$\left|\operatorname{Re}\Big(\frac{w(z)}{w'(z)}\Big)\right| \le c_1\left(\frac{1}{|z|^2} + e^{-2y}\right) \tag{20}$$

where c_1 is a positive constant independent of $z = x + iy$. Moreover, we see from (19) that given $\frac{1}{2} > \varepsilon > 0$, if c is sufficiently large then, for f as in (2),

$$|f(z) - (z - i)| < \varepsilon \tag{21}$$

holds for all $z \in D$. Now choose a point $u_0 \in D$. By (21) and Rouché's theorem we obtain a unique $u_1 \in D(u_0 + i, 2\varepsilon)$ such that $f(u_1) = u_0$. Further, u_1 is the only preimage of u_0 in D. If we repeat this procedure, we obtain a sequence $(u_j) \subset D$ with $f(u_j) = u_{j-1}$, $j \geq 1$. The equation (19) shows that $u_j = u_{j-1} + i + o(1)$ for $j \to \infty$. Thus we can deduce from (20) that

$$\sum_{j=1}^{\infty} |\operatorname{Re}(u_j - u_{j-1})| = \sum_{j=1}^{\infty} \left| \operatorname{Re}\left(\frac{w(u_j)}{w'(u_j)} \right) \right| \leq c_2 < \infty$$

where c_2 is a positive constant independent of u_0.

Now take a rectangle from the first step with $y_0 > c + 1 + 2\varepsilon$ and containing a superattracting fixed point z_0. Then we can find a point u_1 in the rectangle such that $f(u_1) = u_0$ lies in the rectangle and $\operatorname{Im}(u_0) > c$. We connect u_0, u_1 by a straight line γ. This curve lies in D and we may construct

$$\Gamma_1 = \bigcup_{j=1}^{\infty} f^{-j}(\gamma).$$

Γ_1 is a curve which lies entirely in the immediate attracting basin $A^*(z_0)$. Further,

$$|\operatorname{Re}(u - u_0)| < c_3 \tag{22}$$

holds for all $u \in \operatorname{tr}(\Gamma_1)$ where $c_3 > 0$ is a constant independent of u_0. Analogously, we obtain a curve Γ_2 in the lower half plane. We can connect Γ_1 and Γ_2 in $A^*(z_0)$ to give a curve $\Gamma^{(k)}$ which divides the complex plane into two parts.

Let $A^*(z_0)$ be the immediate attractive basin in which $\operatorname{tr}(\Gamma^{(k)})$ lies. By the comment preceding Lemma 4 we have two curves γ_1 and γ_2 which tend between $\Gamma^{(k-1)}$ and $\Gamma^{(k+1)}$ to infinity. We show that one curve has its tail in the upper half plane and the other in the lower half plane. Suppose not, then without loss of generality both curves tend in the upper half plane to infinity. Then $\phi^{-1} \circ \Gamma_2$ must tend to a point $v_0 \neq \pm 1$ in $\partial \mathbf{D}$ [5, Theorem 2.2]. We choose a small disk $D(v_0, \varepsilon)$ such that $\pm 1 \notin D(v_0, \varepsilon)$. Then there exists $n \in \mathbf{N}$ such that $z^{3^n} \notin D(v_0, \varepsilon)$ for all $z \in D(v_0, \varepsilon) \cap \mathbf{D}$. On the other hand, for all $l \in \mathbf{N}$, there exists $z \in \operatorname{tr}(\phi^{-1} \circ \Gamma_2) \cap D(v_0, \varepsilon)$ such that z^{3^j} lies in $\operatorname{tr}(\phi^{-1} \circ \Gamma_2) \cap D(v_0, \varepsilon)$, $0 \leq j \leq l$. This follows from the construction of Γ_2. This yields a contradiction. We may assume because of (14) that $|f(z) - z| > 0.9$ holds for all z with $|\operatorname{Im}(z)| > c$. As the horizontal width of the rectangle is bigger than 1, it follows from Lemma 2 that the immediate attractive basin $A^*(z_0)$ has a horizontal width of at least 0.225. Moreover, $A^*(z_0)$ lies between $\Gamma^{(k-1)}$ and $\Gamma^{(k+1)}$ for which (22) holds. Therefore (17) does hold for $A^*(z_0)$. Thus $A^*(z_0)$ is a quasi-strip. This is step two. We can carry out a similar procedure in the left half plane. Thus, if $|k|$ is sufficiently large, z_k belongs to a quasi-strip which is contained in the Fatou set. As $|z_{k+1} - z_k|$ is equal to π, it is easy to see that $J_0(f)$ is thin at infinity. This together with §5 proves Theorem 1. □

Remark 8. We are now able to give an elementary proof of the non-existence of Baker domains. Suppose that there exists a cycle of Baker domains of period p. Let U be an unbounded component where $f^{np}(z)$ tends to infinity for all $z \in U$. In the notation of the above proof, U must lie between two curves $\Gamma^{(k)}$ and $\Gamma^{(k')}$. Thus, if $\operatorname{Im}(z)$ is large, then $f^p(z) = z - ip + o(1)$ for $z \in U$ and if $-\operatorname{Im}(z)$ is large then $f(z) = z + ip + o(1)$. Therefore a point $z \in U$ cannot tend to infinity under f^p. This is a contradiction.

7. Proof of Theorem 2

Let w be a solution of (1) and f be defined by (2). We see from (1), (2), and (9) that f satisfies the Riccati differential equation

$$f'(z) + \big(1 + F(z)\big)\big(f(z) - z\big)^2 = 0 \tag{23}$$

where $F(z) = (c_d/z^d) + \cdots$ for $|z| > R$ with $c_d \neq 0$ for some $d \geq 2$. As f is rational, for large $|z|$, f has the form

$$f(z) = az^m\big(1 + O(1/z)\big) \tag{24}$$

for some integer m. From (23) we derive

$$\frac{f'(z)}{\big(f(z) - z\big)^2} = -1 - F(z). \tag{25}$$

If we combine (24) and (25), we see that the only possible integer is $m = 1$ and the only possible constant $a = 1$. Thus $f(z) = z + c + g(z)$ for some constant c and $g(z) = O(1/z)$ as $z \to \infty$. Therefore, by (25), we have $c = \pm i$. In fact, we see from (25) that even $g(z) = O(1/z^2)$ as $z \to \infty$ because $F(z) = O(1/z^2)$ as $z \to \infty$. Thus $f(z) = z \pm i + \widetilde{R}(z)$ or $w(z)/w'(z) = -\big(\pm i + \widetilde{R}(z)\big)$, where $\widetilde{R}(z) = O(1/z^2)$ as $z \to \infty$. We have

$$\frac{w'(z)}{w(z)} = \frac{-1}{\pm i + \widetilde{R}(z)} = \pm i + R(z) \tag{26}$$

where R is rational and $R(z) = O(1/z^2)$ as $z \to \infty$. Therefore w has the form

$$w(z) = ce^{\pm iz + \int^z R(\xi)d\xi}, \tag{27}$$

where $c \neq 0$ is a constant. Conversely, each function of the form (27) satisfies a differential equation of the type (1).

The point ∞ is a fixed point. It follows from (26) that

$$\frac{d}{dz}\left(\frac{1}{f(1/z)}\right)\bigg|_{z=0} = \frac{1+\big(R'(1/z)/\big(\pm i+R(1/z)\big)^2\big)}{\big(1-\big(z/(\pm i+R(1/z))\big)^2\big)}\Bigg|_{z=0} = 1.$$

Thus ∞ is a rationally indifferent fixed point. Because

$$f'(z) = \frac{R'(z)+\big(\pm i+R(z)\big)^2}{\big(\pm i+R(z)\big)^2}$$

every pole of R which has order at least 2 is also a rationally indifferent fixed point with multiplier 1. Further, if z_0 is a simple pole, then $f'(z_0)$ equals $1-1/a$ where a is the first non-vanishing coefficient of the Laurent series of R at z_0. We see as in §5 that, if $f^n(z_j)$ converges to a limit in $\overline{\mathbf{C}}$ for each zero z_j of $1+F(z)$, then $J(f)\cap\overline{P}$ is finite. As $z=\infty$ is a rationally indifferent fixed point the Fatou set is non-empty. Of course, f has degree at least two and so, by Corollary 2, $J(f)$ has zero measure. It remains to prove that under the condition of the convergence of the zeros of $1+F(z)$, $f^n(z)$ converges for all $z\in F(f)$ to zeros of w or poles of R or to the point $z=\infty$. As seen in the proof of Corollary 2, $f^n(z)$ lies in a cycle of attractive basins or in a cycle of Leau domains for sufficiently large n. It is well known that each of these cycles must attract a point of $\operatorname{sing}(f^{-1})$. This point is a superattracting fixed point or the image of a zero of $1+F(z)$. Because of the convergence of the zeros of $1+F(z)$ we only have cycles of period one for attractive basins. Such attractive basins result from the zeros of w or the simple poles of R which are attracting fixed points of f. Then we have convergence to zeros of w or poles of R. Further, as the limit of a zero of $1+F(z)$ is a fixed point, the boundary of each Leau domain must contain a rationally indifferent fixed point, i.e. a pole of R of order at least two or the point $z=\infty$. This proves Theorem 2. □

8. Bessel functions

We consider the Bessel differential equation

$$u''(z)+\frac{1}{z}u'(z)+\Big(1-\frac{\nu^2}{z^2}\Big)u(z)=0. \tag{28}$$

With $w(z)=\sqrt{z}u(z)$ we can transform the equation to

$$w''(z)+\Big(1-\frac{\nu^2-\frac{1}{4}}{z^2}\Big)w(z)=0 \tag{29}$$

which has the form (1). The Bessel functions of first kind J_ν defined by

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1+\nu)} \left(\frac{z}{2}\right)^{2n}$$

are solutions of (28). Then

$$\tilde{J}_\nu(z) = \sqrt{z} J_\nu(z) \tag{30}$$

solves (29). Is is easy to see that $\tilde{J}_\nu / \tilde{J}'_\nu$ is meromorphic in $\mathbf{C}$. The only pole of $F(z) = (-\nu^2 + \frac{1}{4})/z^2$ is 0.

Corollary 3. *Let $w = \tilde{J}_\nu$ be as in* (30) *and f be as in* (2)*. If $f^n\left(\sqrt{\nu^2 - \frac{1}{4}}\right)$ converges to a finite limit, then $J_0(f)$ has zero measure. Moreover, if $f^n\left(\sqrt{\nu^2 - \frac{1}{4}}\right)$ converges to a finite limit, $f^n(z)$ converges to zeros of J_ν or to the origin almost everywhere.*

Remark 9. For our consideration we may choose an arbitrary branch of logarithm to define $\sqrt{z}$.

Proof. The function f is transcendental. It is easy to see that f is an odd function. Therefore $f^n\left(\sqrt{\nu^2 - \frac{1}{4}}\right)$ converges if and only if $f^n\left(-\sqrt{\nu^2 - \frac{1}{4}}\right)$ converges. The result follows from Theorem 1 and Corollary 1. □

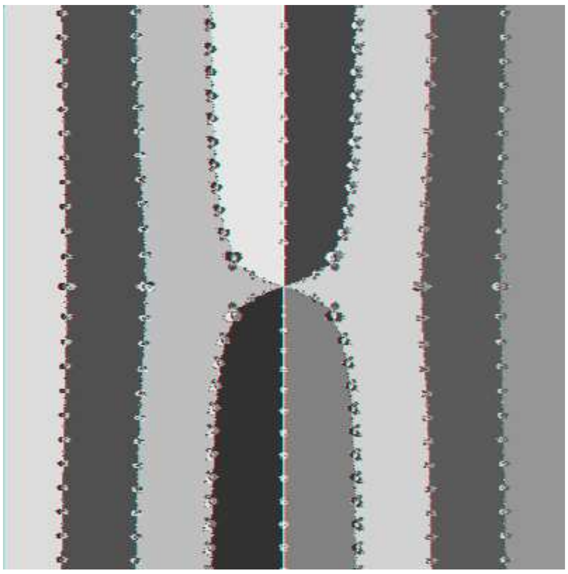

Figure 1: Newton's method for $\tilde{J}_{-5/2}$. We see the quasi-strips outside a small strip around the imaginary axis.

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