Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 22, 1997, 237–244

FINE TOPOLOGY AND $\mathbf{A}_p$-HARMONIC MORPHISMS

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Abstract. As the main result we prove that each non-constant $\mathbf{A}_p$-harmonic morphism in a domain maps p-finely open sets to p-finely open sets. We also show that any $\mathbf{A}_p$-harmonic morphism is fine-to-fine continuous in the p-fine topology. These results apply to quasiregular mappings in the case $p = n$ and to mappings of bounded length distortion for all $1 < p \le n$.

1. Introduction

Harmonic morphisms are mappings preserving harmonic functions. Roughly speaking, a harmonic morphism is any continuous mapping $f\colon X \to Y$ between two spaces X and Y where harmonicity makes sense such that $h \circ f$ is harmonic in $f^{-1}(Y') \subset X$ whenever h is harmonic in $Y' \subset Y$. Classical examples of harmonic morphisms are given by analytic functions. In this paper, we consider $\mathbf{A}_p$-harmonic morphisms (Definition 1.1). This class of harmonic morphisms includes quasiregular mappings ($p = n$) and mappings of bounded length distortion ($1 < p \le n$).

One of the important topological properties of $\mathbf{A}_p$-harmonic morphisms is the fact that any non-constant $\mathbf{A}_p$-harmonic morphism $f\colon G \to \mathbf{R}^n$ in a domain $G \subset \mathbf{R}^n$ is an open mapping, i.e. $f(G')$ is open for any open subset G' of G ([5, 2.3]). The main result (Theorem 2.3) of this article completes this property by establishing that each non-constant $\mathbf{A}_p$-harmonic morphism $f\colon G \to \mathbf{R}^n$ in a domain $G \subset \mathbf{R}^n$ maps p-finely open subsets of G to p-finely open sets in $\mathbf{R}^n$ (the definition of the p-fine topology is given below). For quasiregular mappings and for mappings of bounded length distortion, Theorem 2.3 extends to the boundary in a natural way (Theorem 3.1 and Remark 3.2). We also prove that $\mathbf{A}_p$-harmonic morphisms are fine-to-fine continuous with respect to the p-fine topology and obtain some further results for quasiregular mappings.

We begin with recalling some notation and definitions. Let $n \ge 2$ be the dimension of $\mathbf{R}^n$ and let $p > 1$. In what follows, we use two different capacities. For an open set $G \subset \mathbf{R}^n$ and for any $E \subset G$ we denote by $\operatorname{cap}_p(E, G)$ the *variational p-capacity* of the condenser (E, G). The *Sobolev p-capacity* of an arbitrary set $A \subset \mathbf{R}^n$ is denoted by $C_p(A)$. We refer to [6] for the definitions and properties of these capacities.

1991 Mathematics Subject Classification: Primary 30C65; Secondary 31C45.

Fine topology. Any set $E \subset \mathbf{R}^n$ is called *p-thin* at $x \in \mathbf{R}^n$ if

$$\int_0^1 \left(\frac{\operatorname{cap}_p\bigl(E \cap B(x,t), B(x,2t)\bigr)}{\operatorname{cap}_p\bigl(B(x,t), B(x,2t)\bigr)} \right)^{1/p-1} \frac{dt}{t} < \infty.$$

The p-thinness is closely related to the p-fine topology. A point $x \in U \subset \mathbf{R}^n$ is a *p-fine interior point* of U if $\complement U$ is p-thin at x. It is not hard to see that p-finely open sets form a topology. This topology is called the *p-fine topology.*

Among the fine topology we need some quasi topological notions. A set $U \subset \mathbf{R}^n$ is said to be *p-quasi open* if for every $\varepsilon > 0$ there is an open set $G \subset \mathbf{R}^n$ such that $U \cup G$ is open and $C_p(G) < \varepsilon$. A set $E \subset \mathbf{R}^n$ is called *p-polar* if $C_p(E) = 0$.

$\mathbf{A}_p$-harmonic morphisms. We denote by $\mathbf{A}_p$ the set of all mappings $\mathscr{A}\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ satisfying the structural conditions (3.3)–(3.7) given in [6, p. 56]. Suppose that $\mathscr{A} \in \mathbf{A}_p$ and $G \subset \mathbf{R}^n$ is open. Then $h\colon G \to \mathbf{R}$ is called *$\mathscr{A}$-harmonic* in G if h is a continuous weak solution of the quasilinear elliptic equation

$$-\operatorname{div} \mathscr{A}(x, \nabla h) = 0$$

in G, i.e. h belongs to the Sobolev space $W^{1,p}_{\mathrm{loc}}(G)$ and

$$\int_G \mathscr{A}\bigl(x, \nabla h(x)\bigr) \cdot \nabla \varphi(x)\, dx = 0$$

for all test functions $\varphi \in \mathscr{C}_0^\infty(G)$.

$\mathbf{A}_p$-harmonic morphisms were introduced in [5, p. 116] as follows:

Definition 1.1. Let $G \subset \mathbf{R}^n$ be open. Then $f\colon G \to \mathbf{R}^n$ is called an *$(\mathscr{A}^*, \mathscr{A})$-harmonic morphism* if f is continuous and there are $\mathscr{A}$, $\mathscr{A}^* \in \mathbf{A}_p$ such that $h \circ f$ is $\mathscr{A}^*$-harmonic in $f^{-1}(G')$ whenever h is $\mathscr{A}$-harmonic in an open set $G' \subset \mathbf{R}^n$. Further, $f\colon G \to \mathbf{R}^n$ is called an *$\mathbf{A}_p$-harmonic morphism* if f is an $(\mathscr{A}^*, \mathscr{A})$-harmonic morphism for some $\mathscr{A}$, $\mathscr{A}^* \in \mathbf{A}_p$.

2. Fine topological properties of $\mathbf{A}_p$-harmonic morphisms

This section contains our results for $\mathbf{A}_p$-harmonic morphisms. Throughout this section, we consider an arbitrary $1 < p \le n$. There is no need to consider the case $p > n$, since then the p-fine topology coincides with the euclidean one. We begin with showing that $\mathbf{A}_p$-harmonic morphisms are fine-to-fine continuous with respect to the p-fine topology:

Theorem 2.1. *Let $f\colon G \to \mathbf{R}^n$ be an $\mathbf{A}_p$-harmonic morphism. Then f is continuous if both G and $\mathbf{R}^n$ are equipped with the p-fine topology.*

Proof. The idea of the proof goes back to [1, p. 118]. We are free to assume that G is a domain and f is non-constant. Let f be an $(\mathscr{A}^*, \mathscr{A})$-harmonic morphism with $\mathscr{A}$, $\mathscr{A}^* \in \mathbf{A}_p$. It is known (see [6, 12.17]) that the p-fine topology can be characterized as the coarsest topology of $\mathbf{R}^n$ making all $\mathscr{A}$-superharmonic functions in $\mathbf{R}^n$ continuous. Since $s \circ f$ is $\mathscr{A}^*$-superharmonic in G for each $\mathscr{A}$-superharmonic function $s\colon \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ ([5, 2.7]), it follows from [6, 12.3 and 12.17] that $s \circ f\colon G \to \mathbf{R}$ is p-finely continuous (with the euclidean topology in $\mathbf{R}$) for all $\mathscr{A}$-superharmonic functions $s\colon \mathbf{R}^n \to \mathbf{R}$. It is now a general fact of the induced topology that $f\colon G \to \mathbf{R}^n$ must be continuous if both G and $\mathbf{R}^n$ are equipped with the p-fine topology. □

Corollary 2.2. *Let $f\colon G \to \mathbf{R}^n$ be an $\mathbf{A}_p$-harmonic morphism. Then $f^{-1}(U)$ is p-quasi open for each p-quasi open set $U \subset \mathbf{R}^n$.*

Proof. Since G has a countable number of components, it suffices to prove the assertion for a domain G. We are also free to assume that f is not constant. Suppose that G is a domain and f is non-constant. Then by [8, 1.5], U can be expressed in a form $U = V \cup E$, where V is p-finely open and E is p-polar. By Theorem 2.1 and [5, 2.8], $f^{-1}(U)$ is the union of a p-finely open set $f^{-1}(V)$ and a p-polar set $f^{-1}(E)$. The claim follows from [8, 1.5]. □

Our main result is stated as follows:

Theorem 2.3. *Let $f\colon G \to \mathbf{R}^n$ be a non-constant $\mathbf{A}_p$-harmonic morphism in a domain G. Then $f(\Omega)$ is p-finely open for each p-finely open set $\Omega \subset G$.*

Proof. Let f be an $(\mathscr{A}^*, \mathscr{A})$-harmonic morphism with $\mathscr{A}$, $\mathscr{A}^* \in \mathbf{A}_p$. It is enough to prove the claim for any $(1,p)$-finely regular set (see [7, p. 110] for the definition). This is so because $(1,p)$-finely regular sets are p-finely open and they form a base for the p-fine topology ([7, 3.2 and 3.9]).

Suppose that Ω is $(1,p)$-finely regular and $f(\Omega)$ is not p-finely open. Then there exists $z \in \Omega$ such that $f(z)$ is not a p-fine interior point of $f(\Omega)$. We construct a bounded open set W,

$$W \supset \big(f(\Omega) \cap B\big(f(z), 1\big)\big) \setminus \{f(z)\},$$

such that $f(z)$ is a regular boundary point of W. First notice that $\complement f(\Omega)$ must be p-thick at $f(z)$ by [6, 12.8]. Since each $(1,p)$-finely regular set is a F_σ-set and f is continuous, we conclude that $\complement f(\Omega)$ is a Borel set. Hence there is ([6, 12.11]) a compact set $K \subset \complement f(\Omega) \cup \{f(z)\}$ such that K is p-thick at $f(z)$. Then $f(z) \in K$,

$$W = (\complement K) \cap B\big(f(z), 1\big)$$

is open, and

$$\big(f(\Omega) \cap B\big(f(z), 1\big)\big) \setminus \{f(z)\} \subset W.$$

Note that $f(z) \in \partial W$ by the continuity of f. Moreover, since $\complement W$ is p-thick at $f(z)$, it follows from [6, 6.27] that $f(z)$ is a regular boundary point of W.

Denote $y = f(z)$, $U = f^{-1}(W) \cup \{z\}$. Since $f^{-1}(\{y\})$ is p-polar ([5, 2.8]) and $\Omega \cap f^{-1}\bigl(B(y,1)\bigr)$ is a p-fine neighbourhood of z, we infer that

$$\bigl(\Omega \cap f^{-1}\bigl(B(y,1) \setminus \{y\}\bigr)\bigr) \cup \{z\}$$

is a p-fine neighbourhood of z (see [9, 2.6]). Accordingly, as

$$\Omega \cap f^{-1}\bigl(B(y,1) \setminus \{y\}\bigr) \subset f^{-1}(W),$$

we conclude that U is a p-finely open p-fine neighbourhood of z. Let $(g_i)_{i\in\mathbf{N}}$ be an increasing sequence of functions from $\mathscr{C}^\infty(\mathbf{R}^n)$ such that $g_i(y) = 0$, $0 \le g_i \le 1$ in $\overline{W}$ for all $i \in \mathbf{N}$ and $\lim_{i\to\infty} g_i = 1$ in $\overline{W} \setminus \{y\}$. For each $i \in \mathbf{N}$, let h_i be the solution of the Dirichlet problem

$$\begin{cases} -\operatorname{div} \mathscr{A}(x, \nabla h_i) = 0 & \text{in } W, \\ h_i = g_i & \text{on } \partial W \end{cases}$$

and let $u_i\colon U \setminus \{z\} \to \mathbf{R}$ be defined by $u_i = h_i \circ f$. Then the functions u_i are $\mathscr{A}^*$-harmonic in $f^{-1}(W)$ by Definition 1.1. Since y is a regular boundary point of W, we have $\lim_{t\to y,\, t\in W} h_i(t) = g_i(y)$. The continuity of f at z implies that

$$\lim_{x\to z,\, x\in f^{-1}(W)} (h_i \circ f)(x) = g_i(y) = 0.$$

Hence we can extend the functions u_i to U by setting

$$u_i(z) = 0 = \mathop{p\text{-fine-lim}}_{x\to z,\, x\in f^{-1}(W)} u_i(x).$$

According to Harnack's convergence theorem [6, 6.14], $(h_i)_{i\in\mathbf{N}}$ increases to an $\mathscr{A}$-harmonic function h in W. In fact $h \equiv 1$. To see this, let $y' \ne y$ be a regular boundary point of W. Then

$$\liminf_{x\to y'} h(x) \ge \lim_{i\to\infty} g_i(y') = 1.$$

Hence the inequality $h \ge 1$ in W follows from the comparison principle [6, 7.37] together with the Kellogg property [6, 9.11]. The inequality $h \le 1$ in W is trivial. Consequently, the sequence $(u_i)_{i\in\mathbf{N}}$ increases to 1 in $U \setminus \{z\}$. Let $u = \lim_{i\to\infty} u_i$, i.e. $u(z) = 0$ and $u = 1$ in $U \setminus \{z\}$. We will finish the proof by using some results of the fine potential theory of [9]. First, observe that functions $u_i\colon U \setminus \{z\} \to \mathbf{R}$ are finely $\mathscr{A}^*$-superharmonic ([9, 6.2]). By [9, 5.10], the extensions $u_i\colon U \to \mathbf{R}$ are finely $\mathscr{A}^*$-superharmonic as well. Hence also u must be finely $\mathscr{A}$-superharmonic in U by [9, 5.16]. This contradicts [9, 5.5], since

$$u(z) = 0 < 1 = \mathop{p\text{-fine-lim}\,\inf}_{x\to z} u(x). \ \square$$

3. Fine topological properties of quasiregular mappings

In this section we analyse more closely the fine behaviour of quasiregular mappings.

Let $G \subset \mathbf{R}^n$ be open. A continuous mapping $f\colon G \to \mathbf{R}^n$ is called *quasiregular*, abbreviated QR, if the coordinate functions of f belong to the Sobolev space $W^{1,n}_{\mathrm{loc}}(G)$ and

$$\max_{|h|=1} |f'(x)h|^n \le K J_f(x)$$

for a.e. $x \in G$ and for some $K \le 1$. Here $f'(x)$ is the Jacobi matrix at x and $J_f(x)$ is the determinant of $f'(x)$. A quasiregular mapping $f\colon G \to \mathbf{R}^n$ is called a *mapping of bounded length distortion*, abbreviated BLD, if there is $L \ge 1$ such that

$$\frac{1}{L} \le \min_{|h|=1} |f'(x)h| \le \max_{|h|=1} |f'(x)h| \le L$$

for a.e. $x \in G$ (see [11, pp. 424–425]).

QR and BLD maps are examples of $\mathbf{A}_p$-harmonic morphism. In fact, each QR map is an $\mathbf{A}_n$-harmonic morphism. Moreover, every BLD map is an $\mathbf{A}_p$-harmonic morphism for any $1 < p \le n$. For these results, see [2], [11] and [6]. Consequently, Theorems 2.1 and 2.3 have their counterparts for QR and BLD mappings. What is more, we obtain the following boundary versions of Theorem 2.3:

Theorem 3.1. *Let $f\colon G \to \mathbf{R}^n$ be a* BLD *map. Let $\complement G$ be p-thin at $z \in \partial G$. Then f has a p-fine limit $\alpha \in \mathbf{R}^n$ at z such that $\complement f(G)$ is p-thin at α.*

Proof. The existence of a finite p-fine limit α at z was proved in [4, 5.5 and 5.6]. Accordingly $\alpha \in \overline{f(G)}$. Moreover, since $f(G)$ is open, we are free to assume that $\alpha \in \partial f(G)$. Suppose that $\complement f(G)$ is not p-thin at α. Then α is a regular boundary point of $f(G) \cap B(\alpha, 1)$. Denote

$$W = f(G) \cap B(\alpha,1), \qquad U = f^{-1}(W) \cup \{z\}, \qquad y = \alpha.$$

Since f has a p-fine limit α at z, there is a p-fine neighbourhood V of z such that $V \setminus \{z\} \subset G$ and $f(V \setminus \{z\}) \subset B(\alpha, 1)$. Because $f^{-1}(W)$ is an open set containing $V \setminus \{z\}$, the set U is a p-finely open p-fine neighbourhood of z. We may now proceed as in the proof of Theorem 2.3. Although f does not necessarily have a euclidean limit at z (see [4, 5.6]), we have (using the same notation as in 2.3)

$$p\text{-fine-lim}_{x \to z,\, x \in f^{-1}(W)} u_i(x) = 0$$

since f has a p-fine limit α at z. This leads to a contradiction exactly the same way as in the proof of Theorem 2.3. □

Remark 3.2. Let $f\colon G \to \mathbf{R}^n$ be a bounded non-constant QR map in a domain $G \subset \mathbf{R}^n$. Let $\complement G$ be n-thin at $z \in \partial G$. Then f has an n-fine limit $\alpha \in \mathbf{R}^n$ at z such that $\complement f(G)$ is n-thin at α. This follows from [4, 5.13] by similar arguments as in Theorem 3.1.

Remark 3.3 (a) Let $f\colon G \to \mathbf{R}^n$ be a quasisimilarity (see [2] or [10] for the definition). Then, for each $x \in G$, there is $B(x, r_x)$ such that the restriction $f_{|B(x,r_x)}$ is a BLD map ([10, 2.1]). This implies that f is a p-finely open mapping for any $1 < p \le n$. Clearly f is also continuous if both G and $\mathbf{R}^n$ are equipped with the p-fine topology and $1 < p \le n$.

(b) Let $f\colon G \to \mathbf{C}$ be a conformal analytic mapping. Then f is a p-finely open mapping and fine-to-fine continuous with respect to the p-fine topology for any $1 < p \le 2$. This is so because conformal mappings are quasisimilarities (see [2, 3.7]).

Quasi topological properties of quasiregular mappings. We finish this article by considering the n-quasi topological counterpart of Theorem 2.3 for QR maps. Our arguments are based on the following capacity inequality (see [6, 14.77]): Let $f\colon G \to \mathbf{R}^n$ be a non-constant QR map in a domain $G \subset \mathbf{R}^n$. Then

$$\operatorname{cap}_n\bigl(f(E), f(D)\bigr) \le K_I(f)\operatorname{cap}_n(E, D) \tag{3.4}$$

for any open set $D \subset G$ and for any $E \subset D$. Here $K_I(f) \ge 1$ is the inner dilation of f.

Theorem 3.5. *Let $f\colon G \to \mathbf{R}^n$ be quasiregular. Then $f(U)$ is n-quasi open for each n-quasi open set $U \subset G$.*

Proof. Since G has a countable number of components, we are free to assume that f is a non-constant quasiregular mapping in a domain G. Let $x \in U$ and choose $R > 0$, $m \in \mathbf{N}$ such that $B(x, 2R) \subset G$ and

$$f\bigl(B(x, 2R)\bigr) \subset B\bigl(f(x), 2mR\bigr).$$

Take $r < R$ to be so small that

$$f\bigl(B(x, r)\bigr) \subset B\bigl(f(x), R\bigr).$$

We want to show that $f\bigl(U \cap B(x, \frac{1}{2}r)\bigr)$ is n-quasi open. To do this, we consider an open set $O \subset B(x, r)$ such that $O \cup \bigl(U \cap B(x, \frac{1}{2}r)\bigr)$ is open. Since $U \cap B(x, \frac{1}{2}r)$ is n-quasi open, we are free to assume $C_n(O)$ as small as we want. By [6, 2.38] and (3.4),

$$\begin{aligned}\operatorname{cap}_n\bigl(f(O), B\bigl(f(x), 2mR\bigr)\bigr) &\le \operatorname{cap}_n\bigl(f(O), f\bigl(B(x, 2R)\bigr)\bigr)\\ &\le K_I(f)\operatorname{cap}_n\bigl(O, B(x, 2R)\bigr) \le K_I(f)4^n(1 + R^{-n})C_n(O).\end{aligned}$$

Therefore, since $f(O) \subset f\big(B(x,r)\big) \subset B\big(f(x), mR\big)$, we may apply [6, 2.38] once more and write

$$\begin{aligned} C_n(f(O)) &\le (1 + cm^n R^n)\operatorname{cap}_n\big(f(O), B\big(f(x), 2mR\big)\big) \\ &\le K_I(f)(1 + cm^n R^n)4^n(1 + R^{-n})C_n(O). \end{aligned}$$

Let $\varepsilon > 0$. Since the constants in the previous inequality do not depend on O, we have $C_n\big(f(O)\big) < \varepsilon$ if $C_n(O)$ is small enough. This shows that $f\big(U \cap B(x, \frac{1}{2}r)\big)$ is n-quasi open, because the set

$$f(O) \cup f\big(U \cap B(x, \tfrac{1}{2}r)\big) = f\big(O \cup \big(U \cap B(x, \tfrac{1}{2}r)\big)\big)$$

is open in accordance with the fact that $f\colon G \to \mathbf{R}^n$ is an open mapping. To finish the proof, choose for each $x \in U$ a radius $r_x > 0$ such that $f\big(U \cap B(x, \frac{1}{2}r_x)\big)$ is n-quasi open. Since the covering $\{B(x, \frac{1}{2}r_x) : x \in U\}$ of U contains a countable subcovering $\{B(x_i, \frac{1}{2}r_{x_i}) : i \in \mathbf{N}\}$ of U, we have

$$f(U) = \bigcup_{i \in \mathbf{N}} f\big(U \cap B(x_i, \tfrac{1}{2}r_{x_i})\big).$$

Thus $f(U)$ is n-quasi open as a countable union of n-quasi open sets. □

Remark 3.6. Assume that $f\colon G \to \mathbf{R}^n$ is quasiregular and $U \subset G$ is n-polar. Then a slight modification of the proof of Theorem 3.5 reveals that $f(U)$ is n-polar. In fact, proceeding as in the proof of Theorem 3.5, it is sufficient to prove that $f\big(U \cap B(x, \frac{1}{2}r)\big)$ is n-polar. Let $\varepsilon > 0$ and let O be an open set with $U \cap B(x, \frac{1}{2}r) \subset O \subset B(x,r)$. By the estimates in the proof of Theorem 3.5, $C_n\big(f(O)\big) < \varepsilon$ if $C_n(O)$ is small enough. Hence $f\big(U \cap B(x, \frac{1}{2}r)\big)$ is n-polar and the claim follows.

Acknowledgements. The author would like to thank Jan Malý from the Charles University of Prague for inspiring ideas for the proof of Theorem 2.3 and Tero Kilpelinen from the University of Jyvskyl for reading the manuscript and making useful comments.

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Received 27 November 1995