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THE EXPONENT OF CONVERGENCE OF A FINITE BLASCHKE PRODUCT

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Abstract. This paper describes the iteration theory of finite Blaschke products using the techniques of Fuchsian groups. In particular the Julia set is shown to decompose into points of conical approach and parabolic points. We also use the hyperbolic geometry of the unit disc to describe the construction and properties of conformal measures on the Julia set.

1. Introduction

It is a commonplace to remark the similarities between the theory of Kleinian groups and the iteration of rational maps. Here we restrict our attention to finite Blaschke products, those rational maps which fix a disc and examine their iteration theory from the point of view of Fuchsian groups. The techniques will be derived from the study of hyperbolic geometry and its influence on the geometry and ergodic theory of Fuchsian groups, both classically and from the work of Patterson and Sullivan [11], [13]. Though the results on conformal measures on S^1 we describe occur as special cases of the investigations of Denker and Urbanski [6], [5], [1] and Sullivan [12], the hyperbolic approach gives a different insight.

2. Blaschke products

Let B be a finite Blaschke product of degree $d > 1$. It is easy to show that J_B, the set on which the iterates B^n fail to be normal on any neighbourhood is either S^1, the unit circle, or a Cantor subset [2]. The Denjoy–Wolff theorem shows that the iterates of B converge locally uniformly to a unique point z_0 in the closure of $\triangle$, the unit disc.

If z_0 is contained in the open unit disc then it is an attracting fixed point for B and $|B'(z_0)| < 1$. Otherwise $z_0 \in S^1$ and either $|B'(z_0)| < 1$ or $B'(z_0) = 1$ and is a parabolic fixed point [2].

Each of the possible cases occur. If we write $f_r(z) = (z + r)/(1 + rz)$ and $B_r = f_r^2$. One checks easily that B_0 has $J_{B_0} = S^1$, 0 fixed. $B_{1/3}$ has $J_{B_{1/3}} = S^1$ parabolic point at 1, and $B_{2/3}$ has 1 as an attractive fixed point and Julia set

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a Cantor subset of S^1. If we consider $B(z) = z/(1+z-z^2)$, we obtain a map conjugate to a Blaschke product, with parabolic fixed point at the origin and J_B a Cantor subset of the extended real line [9].

We call those Blaschke products B with $J_B = S^1$ of the first kind by analogy with Fuchsian groups, and those with $J_B \neq S^1$ of the second kind. A Blaschke product B is parabolic if the fixed point z_0 of the Denjoy–Wolff theorem satisfies $B'(z_0) = 1$.

3. Exponent of convergence

Write P_c for the closure of the forward orbit of the critical points and $\triangle_c$ for $\triangle \setminus P_c$. If $z \in \triangle_c$ then there is a disc D, containing z, free of the images of critical points of B^n. By the monodromy theorem we have well defined inverses to B^n on D, and by shrinking D if necessary we may ensure that all pre-images of D are disjoint. Denote by B_n the family of univalent inverses of B^n defined on D and $O^-(z)$ for $\bigcup_n \bigcup_{f\in B_n} f(z)$, $O^-(z)$ accumulates only at J_B [2], for z not fixed. Given $z \in \triangle_c$ we form

$$S_{B,z,\alpha} = \sum_{n=1}^{\infty} \sum_{f\in B_n} |f'(z)|^{\alpha}.$$

Define the exponent of convergence as

$$\delta_{B,z} = \inf_{\alpha} \{S_{B,z,\alpha} < \infty\}.$$

Lemma 3.1. *Let B be a finite Blaschke product, then $\delta_{B,z}$ is constant, for $z \in \triangle_c$.*

Proof. For such z there is a disc $\overline{D(z,\varrho)} \subset \triangle_c$, so all inverse branches of all iterates of B are defined on D. Let f be the inverse branch of some iterate of B, and let w be some point in D. By the distortion theorem there is a constant M dependent on D, z, w but not on f for which $1/M|f'(w)| < |f'(z)| < M|f'(w)|$. Thus the series $S_{B,z,\alpha}$ converges independently of the choice of z in D. So $\delta_{B,z,\alpha}$ is constant off P_c.

We now use univalent function theory to replace our current definition of δ_B with one which emphasises the non-Euclidean geometry.

Lemma 3.2. *For $z \in \triangle_c$ let*

$$\delta'_B = \inf_{\alpha} \Big\{ \sum_n \sum_{f\in B_n} (1-|f(z)|)^{\alpha} < \infty \Big\},$$

$$\delta''_B = \inf_{\alpha} \Big\{ \sum_n \sum_{f\in B_n} \exp\bigl(-\alpha\varrho(x, f(z))\bigr) < \infty \Big\}$$

where $x \in \triangle$, and ϱ is the non-Euclidean metric on $\triangle$. Then $\delta_B = \delta'_B = \delta''_B$.

Proof. Consider $D(z,r)$ a Euclidean disc of radius r centred at z, which misses P_c. Let f be a choice of inverse to B^m some iterate of B on $\triangle$. From the Koebe $\frac{1}{4}$ theorem we have $1-|f(z)| > \frac{1}{4}|f'(z)|r$ and the Schwarz–Pick lemma asserts $|B^{m\prime}(x)| \le (1-|B^m(x)|^2)/(1-|x|^2)$. So for $x = f(z)$,

$$1-|f(z)|^2 \le |f'(z)|(1-|z|^2),$$

and we have

$$\frac{2}{1-|z|} \le \frac{|f'(z)|}{(1-|f(z)|)} \le \frac{4}{r}.$$

This shows $\delta'_B = \delta_B$. Now

$$2^{-s}\bigl(1-|f(x)|\bigr)^s < \exp -s\varrho\bigl(0,f(x)\bigr) < \bigl(1-|f(x)|\bigr)^s$$

and varying the basepoint from which the hyperbolic distance is calculated only contributes a constant term showing $\delta''_B = \delta'_B$.

The same argument shows

Theorem 3.1. *Let B be a finite Blaschke product acting on the unit disc, so that the closed disc $D = \overline{\triangle(z,r)} \subset \triangle_c$. Let D' be any pre-image of D, then the hyperbolic area of D' is bounded with bound only depending on B, D.*

Then by analogy with the classical results on Fuchsian groups one easily proves

Lemma 3.3. $0 < \delta_B \le 1$.

Proof. The upper estimate is a packing argument [10] in the hyperbolic metric. The lower bound [7] follows easily from the density of repelling points in J_B. If B is of degree d, with $K = \sup_{z\in J_B}|B'(z)|$ then

$$\delta_B \ge \frac{\log d}{\log K}.$$

4. A measure on the Julia set

Given a Blaschke product B a t-conformal measure μ for B is a Borel probability measure supported on the Julia set which satisfies the following equation

$$\mu\bigl(B(A)\bigr) = \int_A |B'(z)|^t\, d\mu(z) \tag{4.1.1}$$

on any Borel set A on which B is injective.

We follow Patterson's original construction [11] as described in [10] (pp. 45–55). Denoting the non-Euclidean distance $\varrho(x,y)$ by (x,y) we form the series

$$g_s(x,y) = \sum_{z\in O^-(y)} \exp\bigl(-s(x,z)\bigr)$$

which converges for $s > \delta_B$ and diverges for $s < \delta_B$. It is important that our series diverge at δ_B and to this end we introduce a weighting factor $h\big(\exp(x,z)\big)$ in $g_s(x,y)$ [10, Lemma 3.1.1] so that

$$g_s^*(x,y) = \sum_{z \in O^-(y)} \exp -s(x,z) h\big(\exp(x,z)\big)$$

diverges for $s \le \delta_B$ and converges for $s > \delta_B$. We now construct a set of measures, for $s > \delta$. Let

$$\mu_{x,s} = 1/g_s^*(y,y) \sum_{z \in O^-(y)} \exp -s(x,z) h\big(\exp(x,z)\big) D_z$$

where D_z denotes the Dirac point mass at z.

We can control the variation of $\mu_{x,s}$ with the basepoint x by the following simple estimate.

Lemma 4.1. *Given x, x' in $\triangle$ and $\xi \in \partial\triangle$ then*

$$\frac{\exp(x,w)}{\exp(x',w)} \longrightarrow \left[\frac{P(x,\xi)}{P(x',\xi)}\right]$$

as $w \longrightarrow \xi$. $P(x,\xi)$ is the Poisson kernel $(1-|x|^2)/(|x-\xi|^2)$.

Follow Nicholls [10, pp. 53–56] to estimate the behaviour of the measures under change of basepoint and the action of B, a series of straightforward estimates. To obtain a measure $\mu_{x,\delta}$ we note that $\mu_{x,s}(\bar{\triangle})$ is bounded as $s \to \delta$ and apply Helley's theorem to deduce that for $s \to \delta^+$ the measures $\mu_{x,s}$ converge weakly to $\mu_{x,\delta}$ on $\bar{\triangle}$. Since the series $g_s^*(\,\cdot\,,\cdot\,)$ diverges at $s = \delta$ the measure must be supported on the Julia set. Reading off the behaviour of $\mu_{x,\delta}$ under the action of B we get

Theorem 4.1. *Let B be a Blaschke product with exponent of convergence δ, and suppose $\mu_{x,\delta}$ is obtained as a limit of $\mu_{x,s}$ as $s \to \delta^+$. Then $\mu_{x,\delta}$ is a Borel measure supported on the Julia set J_B. Further if A is a Borel measurable subset of J_B on which B is injective then*

$$\mu_x\big(B(A)\big) = \int_{z \in A} \left[\frac{P(x,Bz)}{P(x,z)}\right]^{\delta} |B'(z)|^{\delta}\, d\mu_{x,\delta}(z).$$

In particular μ_0 satisfies

$$\mu_0\big(B(A)\big) = \int_{z \in A} |B'(z)|^{\delta}\, d\mu_0(z)$$

and so up to a constant multiple is a δ-conformal measure.

We can also control the influence of the change of base point on the measure we have constructed following the argument from Fuchsian groups. Denoting the collection of measures $\mu_{x,\delta}$ obtained as above by M_x we have

Theorem 4.2. *Let B be a Blaschke product with exponent of convergence δ. Choose $x, x' \in \triangle$ and for $\nu_x \in M_x$ define a new measure $\phi(\nu_x)$ by*

$$\phi(\nu_x)(E) = \int_E \left[\frac{P(x',\xi)}{P(x,\xi)}\right]^\delta d\nu_x(\xi).$$

Then ϕ is a homeomorphism of M_x onto $M_{x'}$. If ν_{x,s_j} converges weakly to ν_x as $s_j \to \delta^+$ then ν_{x',s_j} converges weakly to $\phi(\nu_x)$.

5. Conical and parabolic points

In the case of a Fuchsian group an important role is played by the way in which orbits accumulate at the limit set. The orbits control the way in which the Patterson measure accumulates on the limit set. We will make the same division of the Julia set into conical and parabolic points and show that the measure μ_0 respects this division.

A point $z \in J_B$ lies in the conical limit set of B if there is a sequence $\{x_n\} \subset O^-(x)$ such that $|x_n - z|/(1 - |x_n|)$ is bounded. That is the conical limit set consists of those points which are accumulated by pre-images in a Stolz angle. By Theorem 3.1 the conical limit set is independent of x the point used to define it.

We prove

Theorem 5.1. *Let B be a finite Blaschke product with Julia set J_B. Then for $z \in J_B$ either z is a conical limit point or z lies in $O^-(z_0)$, where z_0 is a parabolic fixed point of B.*

This is the analogue of the theorem of Beardon and Maskit [4] that the limit set of a finitely generated Fuchsian group splits into conical and parabolic points. In particular if B has no parabolic fixed point then every point in the Julia set is conical. We show first that every point in the Julia set which does not lie in $O^-(z_0)$ is conical, and then show that no member of $O^-(z_0)$ is conical.

First we observe if $x \in J_B \setminus O^-(z_0)$, then x has a forward limit point which is not z_0. If not there is an n_0 so that for $n > n_0$, $B^n(x) \in \triangle(z_0, \varepsilon)$ for some $\varepsilon > 0$. However the dynamics near z_0 shows this cannot be, since in some neighbourhood of a parabolic fixed point B maps points in J_B away from the parabolic point [2].

To show $x \in J_B \setminus O^-(z_0)$ is conical. Choose x_0 a forward limit point of x, not z_0, the parabolic fixed point. There is a disc $\triangle(x_0, r) \subset \triangle_c$, so all iterates of B have inverses defined and injective on $\triangle(x_0, r)$. Let f_n be a sequence of such inverses so that $f_n(x_0) \to x$. The functions f_n are univalent and map

$S^1 \cap \triangle(x_0, r)$ to a segment of S^1 so each function f_n is conformally conjugate to a function defined and univalent on the unit disc which maps the real axis to itself. Such a function is typically real. From [8] we have for f an analytic function univalent on the unit disc fixing the origin with $\operatorname{sign}\operatorname{Im} f(x) = \operatorname{sign}\operatorname{Im}(z)$ for non-real z then

$$\arg\frac{z}{(1+z)^2} \le \arg f(z) \le \arg\frac{z}{(1-z)^2}.$$

It is easy to show that there is a disc $\overline{D} \subset \mathbf{H} \cap \triangle$ and $D \cap I \neq 0$, $f(D) \cap I \neq 0$ where I is the imaginary axis. By expanding D in the upper-half plane we may arrange that $f(D) \cap I + \varepsilon \neq 0$ for all sufficiently small real ε.

Conjugating back to the unit disc and the Blaschke product, we consider the small disc $\triangle(x_0, r)$ on which inverses f_m of iterates of B are defined, so that $f_m(x_0)$ accumulate at x. Choose a disc D centred on $(0, x_0)$ whose closure is properly contained in $\triangle \cap \triangle(x_0, r)$. As the inverses f_m are conjugate to typically real functions, and the radius $\big(0, f_m(y)\big)$ intersects $f_m(D)$ for all y sufficiently close to x_0 in S^1. We may choose the $y = B^n(x)$ for n suitably large. So $f_m(D)$ intersects the radius $(0, x)$ for infinitely many m, by Theorem 3.1 $f_m(D)$ is of bounded hyperbolic diameter. From which the result follows.

5.1. Parabolic points. We now discuss parabolic points, giving a hyperbolic argument why pre-images of parabolic fixed points cannot be conical and then using some analytic estimates to describe the behaviour at a parabolic point.

Lemma 5.1. *Let B be a Blaschke product with z_0 a parabolic fixed point of B, then z_0 is not a conical limit point.*

Conjugate the Blaschke product so it acts on the upper-half plane with the origin a parabolic fixed point. Classical analysis of parabolic behaviour [2] shows the existence of Π the conformal image of a disc, such that $\Pi \cap \mathbf{H}$ subtends an angle π at the origin, and $B(\Pi) \subset \Pi$. Iterating we have $B^n(\Pi) \subset \Pi$ so if $x \notin \Pi$ then no pre-image $B^{-n}(x) \in \Pi$. If z_0 is conical then for any $x \in \triangle$ we must have $B^{-n}(x) \in \Pi$ for infinitely many n since Π contains any conical approach to z_0, a contradiction for $x \notin \Pi$.

Let $B^q(z) = z_0$, $z \neq z_0$. If z is conical there is a point y so that a sequence of images $B^{-m}(y)$ converge in a cone to z. B^q is univalent in some fixed neighbourhood of z and so the ratio

$$\frac{d\big(B^q(B^{-m}(y)), z_0\big)}{d\big(B^q(B^{-m}(y), S^1\big)} \approx \frac{d\big(B^{-m}(y), z\big)}{d\big(B^{-m}(y), S^1\big)}.$$

This is bounded above, just being the condition that $z = B^{-q}(z_0)$ is conical, so z_0 is conical and we have reached a contradiction, proving that the pre-images of a parabolic point are not conical.

Alternatively an induction shows

Lemma 5.2. *Let $R(z)$ be an analytic function with Taylor expansion at the origin given by $R(z) = z + z^{p+1} + O(z^{2p+1})$, and let f denote the inverse to R which fixes the origin, then for z small real and positive*

$$f^n(z) = an^{-1/p} + b_n(n^{-2/p})$$

where $a = 1/p^{1/p}$, and b_n is bounded over n and

$$\infty > C > |f^{n\prime}(z)|/n^{-(p+1)/p} > c > 0.$$

This allows us to repeat a classic Fuchsian groups argument [3] which established that groups with a parabolic generator have a definite bound below on their exponent of convergence.

Corollary 5.1. *Let B be a parabolic Blaschke product then*

$$\delta_B \geq p/(1+p) \geq \tfrac{1}{2}.$$

Proof. We certainly require

$$\sum_{n>0} |f^{n\prime}(z)|^s < \infty$$

for $s > \delta_B$. Each term is comparable to $n^{-s(p+1)/p}$ forcing $s(p+1) > p$ for all $s > \delta_B$, whence the result.

A stronger argument is required to counterpoint the Fuchsian group argument that $\delta > \frac{1}{2}$ for a Fuchsian group with parabolics.

Theorem 5.2. *If B is a Blaschke product, with a parabolic fixed point then $\delta_B > \frac{1}{2}$.*

If B is of the first kind then the previous corollary gives $\delta_B \geq \frac{2}{3}$ so we are only concerned about Blaschke products of the second kind. Let f, g be branches of B^{-1} defined in a neighbourhood N of z with $f(z) = z$ and $g(N) \cap N = 0$. We consider images of a point y of the form

$$F_{n_1 n_2 \ldots n_j}(y) = f^{n_1} g f^{n_2} g \ldots f^{n_j} g(y).$$

We can now compute a lower bound for δ_B since

$$\sum_n \sum_{f \in B_n} |f'(y)|^s \geq \sum_j \sum_{n_1,\ldots,n_j}^{\infty} |F'_{n_1 \ldots n_j}(y)|^s.$$

Now we estimate $|F'_{n_1\dots n_j}(y)|$ by the chain rule. On some neighbourhood N of z, $|g'(x)| > M$ and for n sufficiently large Lemma 5.2 tells us $f^{n\prime}\big(g(z)\big) > cn^{-2}$. So

$$\sum_j \sum_{n_1,\dots,n_j}^{\infty} |F'_{n_1\dots n_j}(y)|^s \geq \sum_j \sum_{n_1\dots n_j} |M^j c^j n_1^{-2}\dots n_j^{-2}|^s.$$

This is bounded below by

$$\sum_j \left[(Mc)^s \sum_{n=m}^{\infty} n^{-2s}\right]^j$$

which diverges if $M^s c^s < \sum_{n=1}^{\infty} n^{-2s}$, which certainly occurs for some $s > \frac{1}{2}$.

6. No atoms

Finally we show that the δ-conformal measure we constructed in Section 4 has no atoms. The difficulty is with parabolic points of the second kind, and we treat this following Patterson's nifty argument in the Fuchsian case [10, §6.5], a different argument to that of [1]. It is clear no conical limit point z can be charged by a conformal measure otherwise $|B^{n\prime}(z)|$ is uniformly bounded, which is impossible.

Lemma 6.1. *Let B be a Blaschke product with $J_B = S^1$, then*

$$g_1(x,y) = \sum_{n=1}^{\infty} \sum_{f\in B_n} \exp -\varrho\big(x, f(y)\big)$$

diverges.

Whence the conformal measure charges no single point.

Proof. Select any point y in the unit disc not fixed by the Blaschke product. If the series in the lemma converges we may form the non-trivial infinite Blaschke product B_∞ with zeroes at $O^-(y)$. This has non-tangential boundary values one almost everywhere, but at all points save the countable pre-images of a parabolic point there are zeroes of the function converging conically, a contradiction.

It is harder to show that for B a Blaschke product of the second kind the critical series diverges. We sketch a proof exhibiting the non-Euclidean behaviour. Conjugate B, a Blaschke product with a parabolic fixed point of the second kind so it acts on the upper-half plane with the parabolic fixed point at infinity. We assume that the series $g_\delta(x,y)$ converges, and consider how the measure $\mu_{x,\delta}$ varies as x moves on the imaginary axis. We assume B has a power series expansion at ∞ of the form $z\big(1 - 1/z + O(1/z^2)\big)$. The inverse to this function which fixes ∞ is denoted by f, we may analytically continue it along the positive real axis where it still satisfies the functional relation $f(B) =$ identity. Then we have f^n is the inverse to B^n fixing ∞ and estimates on the behaviour near parabolic points are provided by [2], or inversion of Lemma 5.2 to give $\lim_{n\to\infty} |f^n(z) - z|/n = 1$ for z real. So the long term behaviour of f^n is roughly translation.

To mimic the Fuchsian argument we consider Y the set of points in $O^-(y)$ so that $\{\operatorname{Re} z < N : z \in O^-(y)\}$, corresponding to a fundamental region for a subgroup of a Fuchsian group generated by a translation. We claim that $g_s^*(x,y)$ is boundedly comparable to

$$\Sigma_{s,N} = \sum_{z\in Y}\sum_{n=1}^{\infty} \exp -s\varrho\bigl(x, f^n(z)\bigr) h\bigl(\exp \varrho(x, f^n(z))\bigr)$$

and so we may estimate the series $g_s^*(x,y)$ by the behaviour of $\Sigma_{s,N}$. This we will do by controlling the 'iterates' of f acting on a bounded part of the plane. Then as we allow $x = \lambda i$, λ real, to vary we will control $g_s(x,y)$ by comparing $\mu_{x,s}(\infty)$ as λ varies.

We estimate

$$\text{(6.1.1)} \qquad \frac{\mu_{\lambda i,s}(\infty)}{\mu_{i,s}(\infty)} = \left[\frac{P(\lambda i, \infty)}{P(i,\infty)}\right]^s = \lambda^s .$$

Now we show that if N is chosen appropriately then $g_s^*(\lambda i, y) \approx \Sigma_{s,N}$. We have $g_s^* > c_{s,N}\Sigma_{s,N}$ as the latter sum is over a subset of the former up to finite multiplicity. The opposite inequality follows by comparing the mass left at ∞ by the two series, which varies with λ as (6.1.1). As λ increases both series deposit an increasing proportion of their total mass at ∞. So an estimate at $\lambda = 1$ gives an estimate of the ratio $g_s^*/\sum_{s,N}$ independent of λ, s, and is non-zero if we choose N sufficiently large.

The estimate we need is the following.

Theorem 6.1. *For all ε sufficiently small there are constants A, A' independent of s, $\delta < s < \delta + 1$, so that for $\lambda > 1$*

$$\begin{aligned} g_s^*(\lambda i, y) &= \sum_{\zeta\in O^-(y)} h\bigl(\exp \varrho(\lambda i, \zeta)\bigr) \exp -s\varrho(\lambda i, \zeta) \\ &\le A\lambda^{\varepsilon+1-s} \sum_{\{\zeta\in O^-(y):\operatorname{Re}\zeta<N\}} \operatorname{Im}(\zeta)^s h\bigl(\exp\varrho(i,\zeta)\bigr) \\ &\le A'\lambda^{\varepsilon+1-s} g_s^*(i,y). \end{aligned}$$

The proof follows that given by Nicholls [10, Theorem 3.5.5] of Patterson's result, being careful to check that we only introduce bounded distortion on preimages, and the result is as in the Fuchsian case. We finish the proof of the absence of parabolic point masses by computing

$$g_s^*(\lambda i, y) < A'\lambda^{\varepsilon+1-s} g_s^*(i,y)$$

for ε sufficiently small. Taking a limit as $s \to \delta$

$$\mu_{\lambda i,\delta}(\overline{\mathbf{H}}) < A'\lambda^{\varepsilon+1-\delta}\mu_i(\overline{\mathbf{H}}).$$

So for some constant C we have $\mu_{\lambda i}(\infty) < C\lambda^{\varepsilon+1-\delta}$. But (6.1.1) forces

$$\lambda^{\delta} < C\lambda^{\varepsilon+1-\delta}$$

for all ε sufficiently small, and $\lambda > 1$. But $\delta > \frac{1}{2}$ from Theorem 5.2 and we have reached a contradiction, $\mu(\infty) = 0$, and hence by δ-conformality the support of μ has empty intersection with the parabolic points.

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