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ON ISOMORPHISMS OF BERS FIBER SPACES

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Abstract. The main purpose of this paper is to describe all fiber-preserving isomorphisms among Bers fiber spaces in most general cases. Some extension problems in the theory of Bers fiber spaces are also discussed in this paper.

1. Introduction and statement of results

In this paper we study relationships among Bers fiber spaces. All groups under consideration are finitely generated Fuchsian groups of the first kind which act on the upper half plane U. It is well known that the Bers fiber spaces depend on the signatures of the groups; that is, two Bers fiber spaces are fiber-preservingly isomorphic to each other whenever their groups have the same signature. The purpose of this paper is to determine all fiber-preserving isomorphisms among Bers fiber spaces in general cases.

Let Γ be of signature $\sigma = (g,n;\nu_1,\dots,\nu_n)$, where g is the genus of U/Γ, n is the number of distinguished points on U/Γ, and ν_i is an integer or ∞ with $2 \le \nu_1 \le \dots \le \nu_n \le \infty$. Let $M(\Gamma)$ denote the space of measurable functions μ on U satisfying the following conditions:

(i) $\| \mu \|_\infty = \operatorname{ess\,sup}\ \{|\mu(z)|;\ z \in U\} < 1$, and

(ii) $(\mu \circ \gamma) \cdot \bar{\gamma}'/\gamma' = \mu$, for all $\gamma \in \Gamma$.

Two elements $\mu,\ \mu' \in M(\Gamma)$ are *equivalent* if $w^\mu = w^{\mu'}$ on $\widehat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$, where w^μ is the unique quasiconformal mapping on $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ which fixes $0,\ 1,\ \infty$, is conformal on $L = \{z\,; \operatorname{Im} z < 0\}$, and satisfies the Beltrami equation $w_{\bar z} = \mu w_z$ on U. The equivalence class of μ is denoted by $[\mu]$. The *Teichmüller space* $T(\Gamma)$ of Γ is the space of equivalence classes $[\mu]$ with $\mu \in M(\Gamma)$. The *Bers fiber space* $F(\Gamma)$ over $T(\Gamma)$ is defined as

$$\big\{([\mu], z) \in T(\Gamma) \times \mathbf{C}\,;\ z \in w^\mu(U)\big\}.$$

Let $T(g,n)$ denote the Teichmüller space $T(\Gamma)$ for Γ of type (g,n) (see Bers–Greenberg [2]) and $F(g,n;\nu_1,\dots,\nu_n)$ denote the Bers fiber space $F(\Gamma)$ for Γ of signature $(g,n,\nu_1,\dots,\nu_n)$. For more details about Bers fiber spaces, see Bers [1], Earle–Kra [3], [4].

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Throughout this paper, we only consider those finitely generated Fuchsian groups of the first kind whose types are not $(0,3)$, $(1,1)$ or $(0,4)$.

Let Γ and Γ' be two such groups whose signatures are distinct, and let $\sigma = (g,n;\nu_1,\ldots,\nu_n)$, $\sigma' = (g',n';\nu'_1,\ldots,\nu'_{n'})$ be the signatures of Γ and Γ', respectively. A natural question arises as to whether or not there are fiber-preserving isomorphisms of $F(\Gamma)$ onto $F(\Gamma')$ in the case that $\sigma \neq \sigma'$. It is known that

$$F(2,0;\_\_) \cong F(0,6;2,\ldots,2)$$
$$F(1,2;\nu,\nu) \cong F(0,5;2,2,2,2,\nu)$$

for ν an integer ≥ 2 or ∞. In these two exceptional cases, the isomorphisms can be defined by carrying

$$([\mu],z) \in F(\Gamma) \text{ to } \gamma'\big(\Phi([\mu]),z\big) \in F(\Gamma'),$$

where $\gamma' \in \Gamma'$, and Φ stands for the canonical isomorphism $T(2,0) \cong T(0,6)$ (or $T(1,2) \cong T(0,5)$).

We are interested in finding all fiber-preserving, orientation-preserving isomorphisms among Bers fiber spaces. Let $Q(\Gamma)$ be the group of quasiconformal self-homeomorphisms w of U such that $w \circ \gamma \circ w^{-1} \in \mathrm{PSL}(2,\mathbf{R})$, for all $\gamma \in \Gamma$. We consider the case when $\sigma = \sigma'$. Then, as we see, there is $w \in Q(\Gamma)$ such that $w\Gamma w^{-1} = \Gamma'$. By a theorem of Bers [2], w induces an isomorphism $[w]_*$ of $F(\Gamma)$ onto $F(\Gamma')$. More precisely, the isomorphism $[w]_*$ can be described by sending every point $([\mu],z) \in F(\Gamma)$ to the point $\big([\nu], w^\nu \circ w \circ (w^\mu)^{-1}(z)\big) \in F(\Gamma')$, where $\nu \in M(\Gamma')$ is the Beltrami coefficient of $w^\mu \circ w^{-1}$. It is easy to check that $[w]_*$ is a fiber-preserving isomorphism. An isomorphism defined in this way is called a *Bers allowable mapping.*

The main result of this paper is the following:

Main Theorem. *Let Γ be a finitely generated Fuchsian group of the first kind whose signature is σ. Assume that $\dim T(\Gamma) \geq 2$, and that σ is not $(2,0;\_\_)$, $(0,6,2,\ldots,2)$, $(1,2;\infty,\infty)$, or $(0,5;2,\ldots,2,\infty)$. Assume also that Γ contains at least one parabolic element if $g \leq 1$. Let Γ' be a Fuchsian group of signature σ'. Then there is a fiber-preserving isomorphism $\varphi\colon F(\Gamma) \to F(\Gamma')$ if and only if $\sigma = \sigma'$ and φ is a Bers allowable mapping.*

This paper is organized as follows. In Section 2, we review some basic definitions and results which play a crucial role in proving our main result. In Section 3, we enumerate, in general cases, all pairs $\big(F(\Gamma),F(\Gamma')\big)$ of Bers fiber spaces for which $F(\Gamma)$ is fiber-preservingly isomorphic to $F(\Gamma')$. To establish this result we must study those holomorphic automorphisms of the Bers fiber space $F(\Gamma)$ which leave each fiber invariant. We show that the automorphisms with this property are elements of Γ if Γ is also viewed as a subgroup of holomorphic automorphisms of $F(\Gamma)$. In Section 4, we prove our main theorem, and also, we prove that the group of fiber-preserving automorphisms of $F(\Gamma)$ coincides with the modular group of $F(\Gamma)$. In Section 5, we study some extension problems in the theory of Bers fiber space with the aid of the main theorem.

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2. Preliminaries

An automorphism θ of Γ is called *geometric* if there is an element $w \in Q(\Gamma)$ such that $\theta(\gamma) = w \circ \gamma \circ w^{-1}$ for all $\gamma \in \Gamma$. The modular group $\operatorname{mod}\Gamma$ of Γ is defined as the group of geometric automorphisms of Γ. The Teichmüller modular group $\operatorname{Mod}\Gamma$ is the quotient group $\operatorname{mod}\Gamma/\Gamma$, where Γ acts by conjugation as an automorphism of Γ. We thus have a quotient homomorphism ζ of $\operatorname{mod}\Gamma$ onto $\operatorname{Mod}\Gamma$, and Ker ζ is the group of inner automorphisms of Γ which is identified with Γ (since Γ is centerless).

An element $\theta \in \operatorname{mod}\Gamma$ acts biholomorphically on $F(\Gamma)$ by the formula

$$\theta([\mu], z) = ([\nu], \hat{z}), \tag{2.1}$$

where ν is the Beltrami coefficient of the map $w^\mu \circ w^{-1}$ and $\hat{z} = w^\nu \circ w \circ (w^\mu)^{-1}(z)$. Throughout this paper, we identify the group $\operatorname{mod}\Gamma$ with its action on $F(\Gamma)$. It is well known that the action defined in (2.1) is effective, properly discontinuous, and fiber-preserving. The element $\zeta(\theta) \in \operatorname{Mod}\Gamma$ acts biholomorphically on $T(\Gamma)$ by sending $[\mu]$ to $[\nu]$. Let $\theta \in \operatorname{mod}\Gamma$, and let $\chi = \zeta(\theta) \in \operatorname{Mod}\Gamma$. It is easy to see that $\pi \circ \theta(x) = \chi \circ \pi(x)$ for all $x \in F(\Gamma)$; that is, the following diagram is commutative:

$$\begin{array}{ccc} F(\Gamma) & \xrightarrow{\ \theta\ } & F(\Gamma) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ T(\Gamma) & \xrightarrow{\ \chi\ } & T(\Gamma) \end{array}$$

where $\pi\colon F(\Gamma) \to T(\Gamma)$ is the natural holomorphic projection onto the first factor.

As we see, Γ can be considered as the group of inner automorphisms of Γ. With this point of view, Γ becomes a normal subgroup of $\operatorname{mod}\Gamma$. In particular, Γ acts on $F(\Gamma)$ by the formula (as a special case of (2.1)):

$$\gamma([\mu], z) = \big([\mu], w^\mu \circ \gamma \circ (w^\mu)^{-1}(z)\big), \tag{2.2}$$

for $\gamma \in \Gamma$, $[\mu] \in T(\Gamma)$, and $z \in w^\mu(U)$. The quotient space

$$V(\Gamma) = F(\Gamma)/\Gamma \tag{2.3}$$

is a complex manifold. Since the action of Γ keeps all fibers invariant, the natural projection π induces a holomorphic projection $\pi_0\colon V(\Gamma)\to T(\Gamma)$ with $\pi_0^{-1}([\mu])$ a Riemann surface conformally equivalent to $w^\mu(U)/\Gamma^\mu$, where the quasifuchsian group Γ^μ is defined by $\Gamma^\mu = w^\mu\Gamma(w^\mu)^{-1}$. $V(\Gamma)$ is called the *Teichmüller curve.*

For a moment, we assume that Γ contains no parabolic elements. In this case, $V(\Gamma)$ is called *n-pointed Teichmüller curve* and is denoted by $V(g,n)$. In particular, we see that $V(g,n)$ is a complex manifold with a holomorphic projection $\pi_n\colon V(g,n)\to T(g,n)$ onto $T(g,n)$ such that for each point $x\in T(g,n)$, $\pi_n^{-1}(x)$ is the closed orbifold of genus g determined by the surface of type (g,n) represented by x.

In what follows, we assume that Γ is of type (g,n) and may or may not contain parabolic elements. Let

$$U_\Gamma = \{z\in U\,;\ z \text{ is not a fixed point of any elliptic element of } \Gamma\}.$$

We define the *punctured Bers fiber space* $F_0(\Gamma)$ as the space

$$\bigl\{([\mu],z)\in T(\Gamma)\times \mathbf{C}\,;\ \mu\in M(\Gamma) \text{ and } z\in w^\mu(U_\Gamma)\bigr\}.$$

Clearly, the group Γ acts on $F_0(\Gamma)$ freely and discontinuously as a group of holomorphic automorphisms which keeps each fiber invariant. The quotient space $V(g,n)' = F_0(\Gamma)/\Gamma$ is called a *punctured Teichmüller curve.* Let $\pi_n'\colon V(g,n)'\to T(g,n)$ denote the natural projection.

To each elliptic element e of Γ, we can associate a canonical section of $\pi\colon F(\Gamma)\to T(\Gamma)$ determined by the fixed point of e. This section projects (via (2.3)) to a global holomorphic section, which is called a *canonical section* of $\pi_0\colon V(\Gamma)\to T(\Gamma)$. Conjugate elliptic elements of Γ determine a single holomorphic section of π_0. The above discussion leads to the following relation:

$$V(g,n)' = V(g,n) - \{\text{the images of all canonical sections of } \pi_n\}.$$

If Γ contains k conjugacy classes of parabolic elements, then we have

$$V(\Gamma) = V(g,n) - \bigcup_{j=1}^{k} s_j\bigl(T(g,n)\bigr),$$

where s_j, $j=1,\dots,k$, are the sections of $\pi_n\colon V(g,n)\to T(g,n)$ determined by k punctures.

Several important results, due to Hubbard [8], Earle–Kra [3], [4], give us almost full information on global holomorphic sections of $\pi_0\colon V(\Gamma)\to T(\Gamma)$. The following result, which is a weak version of their results (the main theorem of [8], Theorem 4.6 of [3] and Theorem 2.2, Theorem 10.3 of [4]), is good enough for our application in this paper.

Theorem 2.1. *Let Γ be a finitely generated Fuchsian group of the first kind of type (g,n), and let k (may be zero) be the number of conjugacy classes of parabolic elements of Γ. Then the number of global holomorphic sections of $\pi_0\colon V(\Gamma)\to T(\Gamma)$ is finite (it is zero in most torsion free cases) provided that Γ satisfies one of the following conditions:*

(1) $g \geq 2$;
(2) $g = 1$, $n \geq 2$, *and* $k > 0$;
(3) $g = 0$, $n \geq 5$, *and* $k > 0$.

Remark. The condition that $k > 0$ guarantees that there is at least one puncture on U/Γ. When $g \leq 1$ and there are no punctures on U/Γ, we then have uncountably many conformal involutions of $V(\Gamma)$. Then the above theorem is not true.

The number of the holomorphic sections in the above theorem can be counted. We omit the calculations since they are not needed in this paper. Nevertheless, a particularly interesting case is still worth mentioning. When $g = 2$ and $k = 0$, there are 6 Weierstrass sections (see [4]) which are defined by the fixed point locus of the holomorphic involution $J\colon V(2,n)\to V(2,n)$ (the restriction of J to each fiber is the usual hyperelliptic involution on the corresponding compact Riemann surface of genus 2). There are also n canonical sections $s_1,\dots,s_n$. In this case, there are altogether $2n+6$ holomorphic sections:

$$\{s_1,\dots,s_n\,;\ J\circ s_1,\dots,J\circ s_n\,;\ \text{six Weierstrass sections}\}.$$

However, when $g = 2$ and $k > 0$, the number of sections is just $n - k$.

The following table is excluded in our discussion throughout this paper.

$(g,n) = (0,3)$, $(0,4)$, or $(1,1)$		
$g = 1$	$n \geq 2$	Γ contains no parabolic elements
$g = 0$	$n \geq 5$	Γ contains no parabolic elements

Table 1.

3. A classification of Bers fiber spaces

The objective of this section is to prove a weak version of the main theorem which classifies (in a fiber-preserving way) all Bers fiber spaces in general cases.

Theorem 3.1. *Let Γ be a finitely generated Fuchsian group of the first kind whose type (or signature) is not in Table 1. Assume that $(g,n;\nu_1,\dots,\nu_n)$ is not $(2,0;\_)$, $(0,6;2,\dots,2)$, $(1,2;\infty,\infty)$, or $(0,5;2,\dots,2,\infty)$. Let Γ' be a group with signature $\sigma' = (g',n';\nu'_1,\dots,\nu'_{n'})$. Then the following conditions are equivalent:*

(i) $F(\Gamma)$ *is fiber-preservingly isomorphic to* $F(\Gamma')$;
(ii) $\sigma = \sigma'$.

We begin with a result whose proof is a direct consequence of Theorem 2.1. Let K denote the set of all images of holomorphic sections of $\pi_0: V(\Gamma) \to T(\Gamma)$. By Theorem 2.1, if the type (or signature) of Γ is not in Table 1, then the cardinality of K is always finite. Let

$$p_0: F(\Gamma) \to V(\Gamma)$$

be the natural projection determined by (2.3); that is, the image $p_0(x)$ of $x \in \big([\mu], w^\mu(U)\big) \subset F(\Gamma)$ is its image under the natural projection

$$w^\mu(U) \to w^\mu(U)/\Gamma^\mu \subset V(\Gamma).$$

Proposition 3.2. *Suppose that K is not empty. Then for each $[\mu] \in T(\Gamma)$, $K^\mu = \pi^{-1}([\mu]) \cap p_0^{-1}(K)$ is discrete and invariant under the action of $\Gamma^\mu = w^\mu \Gamma (w^\mu)^{-1}$.* □

To prove Theorem 3.1, we first need to study those holomorphic automorphisms which keep each fiber invariant. In what follows, we use the same symbol Γ to denote the Fuchsian group as well as the automorphism group of $F(\Gamma)$ it induces. The symbol Γ^μ, $\mu \in M(\Gamma)$, stands for the quasifuchsian group $w^\mu \Gamma (w^\mu)^{-1}$, which can be identified with $\Gamma|_{\pi^{-1}([\mu])}$ in the action of Γ on $F(\Gamma)$. We need

Lemma 3.3. *Assume that the type (g,n) (or signature) of Γ is not in Table 1, and assume that τ is a holomorphic automorphism of $F(\Gamma)$ which keeps each fiber invariant. Then $\Gamma_0 = \langle \Gamma, \tau|_{\pi^{-1}([0])}\rangle$ is again a finitely generated Fuchsian group of the first kind.*

Proof. If Γ is torsion free, then the Bers isomorphism theorem [1] asserts that

$$F(\Gamma) \cong T(g, n+1).$$

Thus, the group $\mathrm{Aut}\big(F(\Gamma)\big)$ of holomorphic automorphisms of $F(\Gamma)$ is isomorphic to the group of holomorphic automorphisms of $T(g,n+1)$ which is, by Royden's theorem [11] (and its extension proved by Earle–Kra [3]), the Teichmüller modular group $\mathrm{Mod}(g,n+1)$. Since $\mathrm{Mod}(g,n+1)$ acts discontinuously on $T(g,n+1)$, $\mathrm{Aut}\big(F(\Gamma)\big)$ acts discontinuously on $F(\Gamma)$ as well. It follows that Γ_0 is discrete.

Now we assume that Γ contains elliptic elements. Let U be the central fiber of $F(\Gamma)$. (By the central fiber we mean $U = \pi^{-1}([0])$, where $[0] \in T(\Gamma)$ is the origin.) Observe that $\tau|_{\pi^{-1}([0])}$ is a real Möbius transformation.

The set $K^0 = p_0^{-1}(K) \cap U$ $(\neq \emptyset)$ constructed above is not only Γ-invariant, but also Γ_0-invariant. For otherwise there is a point $x_0 \in K^0$ with the property that $\tau|_{\pi^{-1}([0])}(x_0) \notin K^0$. If we denote by $s_0: T(\Gamma) \to F(\Gamma)$ the global canonical section defined by sending $[\mu] \in T(\Gamma)$ to $\big([\mu], w^\mu(x_0)\big)$, the above argument shows that $\tau \circ s_0$ is a global holomorphic section of π whose image is not in $p_0^{-1}(K)$, contradicting Theorem 2.1. To prove that Γ_0 is discontinuous, it is equivalent to showing that Γ_0 is discrete (see, for example, Farkas–Kra [5]). Suppose for the contrary that there is a sequence $\{\gamma_n\} \in \Gamma_0$ such that $\gamma_n \to \mathrm{id}$. This implies that for any point, in particular, for $x \in K^0$, $\gamma_n(x) \to x$. Since K^0 is Γ_0-invariant and discrete in U by Proposition 3.2, we find a contradiction. Lemma 3.3 is proved. ▫

Lemma 3.4. *Let Γ be a finitely generated Fuchsian group of the first kind. Then any fiber-preserving holomorphic automorphism φ: $F(\Gamma) \to F(\Gamma)$ projects to a holomorphic automorphism χ of $T(\Gamma)$ in the sense that*

$$\pi \circ \varphi(x) = \chi \circ \pi(x), \qquad \text{for all } x \in F(\Gamma). \tag{3.1}$$

Proof. Given the map φ, we define χ by (3.1). Clearly, χ is well defined. The only issue is to show that χ is holomorphic. Choose an arbitrary point $x \in \big([\mu], w^\mu(U)\big) \subset F(\Gamma)$. There is a local holomorphic section s: $T(\Gamma) \to F(\Gamma)$ with $s([\mu]) = x$. For $[\nu]$ close to $[\mu]$, we can write

$$\chi([\nu]) = \chi \circ \big(\pi(x')\big) = \pi \circ \varphi(x') = \pi \circ \varphi \circ s([\nu]),$$

where $s([\nu]) = x'$. Since s, φ, and π are holomorphic, χ is holomorphic as well. The lemma is proved. ▫

Lemma 3.5. *Let D be a simply connected domain in $\widehat{\mathbf{C}}$ which misses at least three points of $\widehat{\mathbf{C}}$, let f is a conformal self-map of D. Then f has at most one fixed point in D.*

Proof. Let α: $U \to D$ be a Riemann mapping, where U is the upper half plane. Then $\alpha^{-1} \circ f \circ \alpha$ is conformal, and hence belongs to $\mathrm{PSL}(2, \mathbf{R})$. Therefore, $\alpha^{-1} \circ f \circ \alpha$ has at most one fixed point in U. ▫

It is well known that for any $\mu \in M(\Gamma)$, there is a unique quasiconformal self-map w_μ of U which fixes 0, 1, ∞, and satisfies the Beltrami equation $w_{\bar z} = \mu w_z$. Hence, to each $\mu \in M(\Gamma)$, there corresponds a Fuchsian group $\alpha_\mu(\Gamma) = \Gamma_\mu$ which depends only on the equivalence class $[\mu]$ of μ, where α_μ: $\Gamma \to \Gamma_\mu$ is an isomorphism defined by taking $\gamma \in \Gamma$ to $w_\mu \circ \gamma \circ (w_\mu)^{-1} \in \Gamma_\mu$. We see that $T(\Gamma)$ is identified with the set $\big\{\alpha_\mu\colon \Gamma \to \Gamma_\mu \subset \mathrm{PSL}(2, \mathbf{R})\,;\ [\mu] \in T(\Gamma)\big\}$. Let us denote by $\mathrm{Max}(\Gamma)$ the set of points $[\mu]$ in $T(\Gamma)$ which corresponds to a finite maximal Fuchsian group; that is, the group Γ_μ for which there does not exist any other Fuchsian group G such that $\Gamma_\mu \subset G$ and the index $[G : \Gamma_\mu]$ is finite.

Lemma 3.6. *Under the condition of Lemma* 3.3*, let τ be a holomorphic automorphism of $F(\Gamma)$ which keeps each fiber invariant. Suppose that for all $[\mu] \in T(\Gamma)$, $\tau|_{\pi^{-1}([\mu])}$ is not in Γ^μ. Then the set* $\mathrm{Max}(\Gamma)$ *is empty.*

Proof. Let h_μ: $w^\mu(U) \to U$ be the Riemann mapping with $h_\mu(0) = 0$, $h_\mu(1) = 1$, and $h_\mu(\infty) = \infty$. It is easy to see that $h_\mu = w_\mu \circ (w^\mu)^{-1}$ and that $h_\mu \Gamma^\mu (h_\mu)^{-1}$ is properly contained in $h_\mu \Gamma_0^\mu (h_\mu)^{-1}$ for all $[\mu]$, where $\Gamma_0^\mu =$

$\langle\Gamma^\mu, \tau|_{\pi^{-1}([\mu])}\rangle$. Also, a simple computation shows that $h_\mu\Gamma^\mu(h_\mu)^{-1} = w_\mu\Gamma(w_\mu)^{-1}$. Since $h_\mu \circ \tau|_{\pi^{-1}([\mu])} \circ (h_\mu)^{-1}$ is a real Möbius transformation which is not in Γ_μ, Γ_μ is properly contained in $h_\mu\Gamma_0^\mu(h_\mu)^{-1}$. The discreteness of $h_\mu\Gamma_0^\mu(h_\mu)^{-1}$ for any μ follows from the proof of Lemma 3.3. Since $\alpha_\mu\colon \Gamma \to \Gamma_\mu$ runs over all points in $T(\Gamma)$, the lemma is established. □

Lemma 3.7. *Let* Γ *be the group which contains elliptic elements. Under the condition of Lemma* 3.3, *suppose that for some* $[\mu] \in T(\Gamma)$, Γ^μ *is properly contained in* $\Gamma_0^\mu = \langle\Gamma^\mu, \tau|_{\pi^{-1}([\mu])}\rangle$. *Then the set* $\mathrm{Max}(\Gamma)$ *is empty.*

Proof. Let P denote the set

$$\{[\mu] \in T(\Gamma)\,;\ \Gamma^\mu \text{ is properly contained in } \langle\Gamma^\mu, \tau|_{\pi^{-1}([\mu])}\rangle\}.$$

We claim that P is open. Indeed, we may assume, without loss of generality, that the origin $[0]$ of $T(\Gamma)$ belongs to P. (Otherwise a Bers allowable mapping will be constructed to carry a fiber over a point in P to a fiber over $[0]$ of another Teichmüller space.) Choose a point x in U which is not fixed by any non-trivial element of Γ. We see that

$$\delta = \varrho_U\big(\tau(x), \Gamma(x)\big) = \inf\big\{\varrho_U\big(\tau(x), \gamma(x)\big)\,;\ \gamma \in \Gamma,\ \text{and } \gamma \neq \mathrm{id}\big\}$$

is positive, where ϱ_E is the Poincaré metric on a domain E. For any sequence $\{\mu_n\} \in M(\Gamma)$ with $\mu_n \to 0$ almost everywhere, the sequence $\{w^{\mu_n}\}$ converges to the identity uniformly on compact sets (see [6]). This implies that $w^{\mu_n}(U) \to U$ in the sense that for any compact set $E \subset U$, there exists a large n_0 such that $E \subset w^{\mu_n}(U)$ whenever $n \geq n_0$. Therefore, $\varrho_{w^{\mu_n}(U)}(x,y)$ must converge to $\varrho_U(x,y)$ for any pair $x, y \in U$. We conclude that if $[\mu] \in T(\Gamma)$ is in a sufficiently small neighborhood of $[0]$, the point x stays in $w^\mu(U)$ and satisfies the condition that

$$\varrho_{w^\mu(U)}\big(\tau(x), \Gamma^\mu(x)\big) > \tfrac{1}{2}\delta.$$

This implies that Γ^μ is properly contained in Γ_0^μ as well. Hence, P is open.

To show that $P = T(\Gamma)$, it remains to verify that P is also closed. By assumption, Γ contains elliptic elements and its type (or signature) does not belong to Table 1. Choose a fixed point z_0 of some elliptic element, and choose an arbitrary $[\mu] \in P^c$. By definition of P^c, there is a $\gamma \in \Gamma$ so that $\tau|_{\pi^{-1}([\mu])} = \gamma^\mu = w^\mu \circ \gamma \circ (w^\mu)^{-1}$. (Note that γ^μ depends only on the equivalence class $[\mu]$ of μ.) Let Q be a finite subset of $p_0^{-1}(K)$ whose cardinality is ≥ 2. Then in particular, we have

$$\tau|_{\pi^{-1}([\mu])}\big(Q \cap \pi^{-1}([\mu])\big) = \gamma^\mu\big(Q \cap \pi^{-1}([\mu])\big).$$

Consider now an arbitrary sequence $\{\mu_i\} \in M(\Gamma)$ with $[\mu_i] \to [\mu]$ as $i \to \infty$. By Proposition 3.2, $p_0^{-1}(K) \cap \pi^{-1}([\mu'])$ is discrete for any $[\mu'] \in T(\Gamma)$. For sufficiently large i, we must have

$$\tau|_{\pi^{-1}([\mu_i])}\big(Q \cap \pi^{-1}([\mu_i])\big) = \gamma^\mu\big(Q \cap \pi^{-1}([\mu_i])\big).$$

(Otherwise there would be a new holomorphic section whose image is not an element of $p_0^{-1}(K)$, contradicting Theorem 2.1). This implies that there is a neighborhood N_μ of $[\mu]$ such that for any $[\nu]\in N_\mu$, the restriction of the conformal self-map $\tau|_{\pi^{-1}([\nu])}$ to $Q\cap\pi^{-1}([\nu])$ coincides with $\gamma^\nu|_{Q\cap\pi^{-1}([\nu])}$. Now both $\tau|_{\pi^{-1}([\nu])}$ and γ^ν are conformal self-maps of $w^\nu(U)$, $(\tau|_{\pi^{-1}([\nu])})^{-1}\circ\gamma^\nu$ is thus a conformal self-map of $w^\nu(U)$ which fixes all points in $Q\cap\pi^{-1}([\nu])$. Hence, by Lemma 3.5, we see that $\tau|_{\pi^{-1}([\nu])}=\gamma^\nu$. (Since they are self-maps in the quasidisk and have the same values in at least two points.) It follows that $N_\mu\subset P^c$, and P^c is open. This implies that for each $[\mu]\in T(\Gamma)$, Γ^μ is properly contained in $\langle\Gamma^\mu,\tau|_{\pi^{-1}([\mu])}\rangle$. Lemma 3.6 then implies that $\mathrm{Max}(\Gamma)$ is empty. □

For torsion free Fuchsian group Γ, we have

Lemma 3.8. *Let Γ be a torsion free group whose type is not $(0,3)$, $(0,4)$, or $(1,1)$. $\tau\in\mathrm{Aut}\,F(\Gamma)$ keeps each fiber invariant. Further assume that for some $[\mu]\in T(\Gamma)$, Γ^μ is properly contained in $\Gamma_0^\mu=\langle\Gamma^\mu,\tau|_{\pi^{-1}([\mu])}\rangle$. Then the signature of Γ must be either $(2,0;\_\_)$ or $(1,2;\infty,\infty)$.*

Proof. Suppose that the signature of Γ is neither $(2,0;\_\_)$ nor $(1,2;\infty,\infty)$. Since there are no holomorphic sections of π_0, the argument applied in Lemma 3.7 does not work at this time, we must use another method.

By the Bers isomorphism theorem [1], there is an isomorphism

$$\psi\colon F(\Gamma)\to T(g,n+1).$$

By Lemma 5.4 of Bers [1], given an arbitrary $a\in U$, any point of $F(\Gamma)$ can be represented as a pair $\big([\mu],w^\mu(a)\big)$ for some $\mu\in M(\Gamma)$. Our first claim is that $\tau\in\mathrm{mod}\,\Gamma$.

Suppose τ is not in $\mathrm{mod}\,\Gamma$. By Royden's theorem [11] (and its generalization [3]), $\psi\circ\tau\circ\psi^{-1}$ is induced by a self-map f_0 of a surface S of type $(g,n+1)$, by Theorem 10 of Bers [1], f_0 must send the special puncture $\hat a$, where $\hat a$ is the image of a under the projection $U\to U/\Gamma$, to another puncture. This means that f_0 does not define a self-map of $\widehat S$, where $\widehat S=S\cup\{\hat a\}$. On the other hand, by Lemma 3.4, $\tau\in\mathrm{Aut}\big(F(\Gamma)\big)$ projects to a trivial action on $T(\Gamma)$, which says that $\pi\circ\tau=\pi$. Since the Bers isomorphism identifies the projection π onto the first factor with the forgetful map ϑ, we have the following commutative diagram:

$$\begin{array}{ccc} F(g,n;\infty,\dots,\infty) & \xrightarrow{\ \psi\ } & T(g,n+1) \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\vartheta} \\ T(g,n) & \xrightarrow{\ \mathrm{id}\ } & T(g,n) \end{array}$$

which gives

$$\vartheta\circ\psi\circ\tau\circ\psi^{-1}=\pi\circ\tau\circ\psi^{-1}=\pi\circ\psi^{-1}=\vartheta.$$

This implies that $\psi \circ \tau \circ \psi^{-1}$ projects to the identity via the forgetful map ϑ. But $\psi \circ \tau \circ \psi^{-1} \in \mathrm{Mod}(g, n+1)$ is induced by a self-map f_0 of a Riemann surface of type $(g, n+1)$. We see that f_0 fixes $\hat{a}$, which leads to a contradiction. Therefore, $\tau \in \mathrm{mod}\,\Gamma$, and we conclude that τ is induced by a self-map f of U with $f \circ \gamma \circ f^{-1} \in \Gamma$ for all $\gamma \in \Gamma$. Since Γ is torsion free and the signature of Γ is neither $(2,0;\_\_)$ nor $(1,2;\infty,\infty)$, the Teichmüller modular group $\mathrm{Mod}\,\Gamma$ acts effectively on $T(\Gamma)$. Observe also that the kernel of the quotient homomorphism $\zeta\colon \mathrm{mod}\,\Gamma \to \mathrm{Mod}\,\Gamma$ is Γ. We must have $\tau \in \Gamma$. This implies that for all $[\mu] \in T(\Gamma)$, $\tau|_{\pi^{-1}([\mu])}$ is an element of Γ^{μ}, contradicting our hypothesis. ▫

Lemma 3.9. *Under the condition of Lemma* 3.3, *assume that the signature of* Γ *is neither* $(2,0;\_\_)$ *nor* $(1,2;\infty,\infty)$. *Then any holomorphic automorphism on* $F(\Gamma)$ *which leaves each fiber invariant is an element of* Γ.

Proof. By Lemma 3.8, we know that the lemma holds when Γ is torsion free. Now we assume that Γ has torsion. Let τ be a holomorphic automorphism which satisfies the condition of the lemma. By Lemma 3.3, Γ and $\Gamma_0 = \langle \Gamma, \tau|_{\pi^{-1}([0])} \rangle$ are two finitely generated Fuchsian groups of the first kind with $\Gamma \subset \Gamma_0$. Suppose that Γ is properly contained in Γ_0. The index

$$[\Gamma_0 : \Gamma] = \mathrm{Area}(U/\Gamma)/\mathrm{Area}(U/\Gamma_0) < \infty.$$

It follows from Lemma 3.7 that the set $\mathrm{Max}(\Gamma)$ is empty. Hence, by Theorem 3A of Greenberg [7] or Theorem 1 of Singerman [12], we see that there is a unique group G such that Γ is a subgroup of finite index in G (the index can be proved to be equal to 2) and $T(G) \cong T(\Gamma)$. Furthermore, Γ must be of the signature $(1,2;\nu,\nu)$, where $\nu \geq 2$ is an integer or ∞. (Γ cannot be of signature $(2,0;\_\_)$ since our assumption says that Γ contains elliptic elements.) This is a contradiction. We conclude that $\Gamma = \Gamma_0$, and thus $\tau|_{\pi^{-1}([0])}$ is an element of Γ. By the argument of Lemma 3.7, we see that $\tau \in \Gamma$, as a group of automorphisms of $F(\Gamma)$. ▫

Proof of Theorem 3.1. Suppose that there is a fiber-preserving isomorphism $\varphi\colon F(\Gamma') \to F(\Gamma)$. The upper half plane U can be viewed as the central fiber of both $F(\Gamma)$ and $F(\Gamma')$. By composing with a Bers allowable mapping (which is fiber-preserving, see Bers [1]), we may assume, without loss of generality, that $\varphi(U) = U$. Consider the homomorphism α_φ of Γ' to $\mathrm{Aut}\big(F(\Gamma)\big)$ defined by $\alpha_\varphi(\gamma') = \varphi \circ \gamma' \circ \varphi^{-1}$ for all $\gamma' \in \Gamma'$. Since $\gamma' \in \mathrm{mod}\,\Gamma'$ leaves each fiber invariant, $\alpha_\varphi(\gamma')$ is an automorphism of $F(\Gamma)$ which keeps each fiber invariant. Since σ is neither $(2,0;\_\_)$ nor $(1,2;\infty,\infty)$, by Lemma 3.9, $\alpha_\varphi(\gamma') \in \Gamma$ for all $\gamma' \in \Gamma'$. It follows that α_φ is a monomorphism of Γ' to Γ. Since σ is neither $(0,6;2,\ldots,2)$ nor $(0,5;2,\ldots,2,\infty)$, by Theorem 3A of Greenberg [7] or Theorem 1 of Singerman [12], we conclude that α_φ is an isomorphism of Γ' onto Γ. Since $\varphi|_U\colon U \to U$ is a real Möbius transformation, $\alpha_\varphi\colon \Gamma \to \Gamma'$ is type-preserving. We conclude that

Γ and Γ' have the same signature. The reverse direction is trivial. This completes the proof of Theorem 3.1. □

4. Proof of the main theorem

First, let us recall a lemma which is proved by Royden [11] (see also Earle–Kra [4]) in the case when Γ is torsion free. However, it remains true even if Γ has torsion. See Gardiner ([6, Section 9.6, p. 184–185]) for a proof. We formulate it as

Lemma 4.1. *Assume that Γ is a finitely generated Fuchsian group of the first kind, and let $\chi\colon T(\Gamma) \to T(\Gamma)$ be a biholomorphic map. If for each $[\mu] \in T(\Gamma)$, there exists a $\chi_{[\mu]} \in \operatorname{Mod}\Gamma$ such that*

$$\chi([\mu]) = \chi_{[\mu]}([\mu]), \tag{4.1}$$

then $\chi \in \operatorname{Mod}\Gamma$. □

The following result is a consequence of Lemma 4.1.

Proposition 4.2. *Under the condition of Lemma* 3.3*, assume that the signature of Γ is neither $(2,0;\_)$ nor $(1,2;\infty,\infty)$. Then the following conditions are equivalent*:

(i) *$\theta \in \operatorname{Aut}\bigl(F(\Gamma)\bigr)$ is fiber-preserving*;
(ii) *θ projects to an element $\chi \in \operatorname{Mod}\Gamma$ (induced by a self-map of U/Γ)*;
(iii) *θ is an element of $\operatorname{mod}\Gamma$.*

Proof. The proof that (iii) $\Rightarrow$ (i) is trivial. The implication (i) $\Rightarrow$ (ii) is not so obvious since we do not know θ can be projected to a modular transformation of $T(\Gamma)$. To prove (i) $\Rightarrow$ (ii), we use Lemma 3.4, and see that θ projects to a holomorphic automorphism χ of $T(\Gamma)$ under the projection $\pi\colon F(\Gamma) \to T(\Gamma)$. Now Royden's theorem [11] (and its generalization [3]) asserts that $\chi \in \operatorname{Mod}(g,n)$; that is, χ is induced by a self-map f of the punctured Riemann surface U_Γ/Γ. We claim that f defines a self-map of U/Γ in the sense of orbifolds. For this purpose, let $\chi([0]) = [\mu]$. If we think of Γ as a group of automorphisms of $F(\Gamma)$, then for each $\gamma \in \Gamma$, $\theta \circ \gamma \circ \theta^{-1}$ is again an automorphism which leaves each fiber of $F(\Gamma)$ invariant. It follows from Lemma 3.9 that $\theta \circ \gamma \circ \theta^{-1} = \gamma_1$ for some $\gamma_1 \in \Gamma$. This implies that θ conjugates Γ to itself; in other words, θ can be projected to a biholomorphic self-map ξ on the Teichmüller curve $V(\Gamma)$. We thus obtain the following commutative diagram:

$$\begin{array}{ccc} V(\Gamma) & \xrightarrow{\ \xi\ } & V(\Gamma) \\ \Big\downarrow{\scriptstyle \pi_0} & & \Big\downarrow{\scriptstyle \pi_0} \\ T(\Gamma) & \xrightarrow{\ \chi\ } & T(\Gamma) \end{array} \tag{4.2}$$

Note that the Riemann surface $\pi_0^{-1}([0]) = U/\Gamma$ is represented by $[0] \in T(\Gamma)$, and $\pi_0^{-1}([\mu]) = w^\mu(U)/\Gamma^\mu$ is represented by $[\mu] \in T(\Gamma)$. Since ξ is biholomorphic, the restriction of ξ to each fiber of $V(\Gamma)$ is clearly conformal. By construction, ξ carries a branch point with ramification number ν to a branch point with the same ramification number. In particular, ξ realizes a conformal equivalence between $\pi_0^{-1}([0]) = U/\Gamma$ and $\pi_0^{-1}([\mu]) = w^\mu(U)/\Gamma^\mu$. This implies that the two points $[0]$ and $[\mu] \in T(\Gamma)$ are modular equivalent. Let us denote by χ_0 the corresponding modular transformation of $T(\Gamma)$ induced by $\xi|_{\pi_0^{-1}([0])}$ and by f_ξ the self-map of U/Γ which induces χ_0. Note also that $\chi_0 \in \operatorname{Mod}\Gamma$.

Since the diagram (4.2) commutes, we have

$$\chi_0([0]) = [\mu] = \chi([0]).$$

Now choose an arbitrary point $[\nu] \in T(\Gamma)$. By using the same argument as above, we see that there exists a modular transformation $\chi_\nu \in \operatorname{Mod}\Gamma$ such that $\chi_\nu([\nu]) = \chi([\nu])$. We arrive at the situation of Lemma 4.1, by which we conclude that $\chi \in \operatorname{Mod}\Gamma$; that is, χ is induced by a self-map f of U/Γ which is isotopic to f_ξ keeping all distinguished points fixed. This finishes the argument of (i) $\Rightarrow$ (ii).

To verify (ii) $\Rightarrow$ (iii), let $\chi \in \operatorname{Mod}\Gamma$ denote the projection of θ. By assumption of (ii), χ is induced by a self-map f of U/Γ. f can be lifted to a self-map $\tilde{f}$ of U. Then the geometric isomorphism $\widetilde{\chi}$ induced by $\tilde{f}$ is an element of $\operatorname{mod}\Gamma$. We thus have the following commutative diagram:

$$\begin{array}{ccc} F(\Gamma) & \xrightarrow{\widetilde{\chi}} & F(\Gamma) \\ \downarrow \pi & & \downarrow \pi \\ T(\Gamma) & \xrightarrow{\chi} & T(\Gamma) \end{array}$$

Then $\theta \circ \widetilde{\chi}^{-1} \in \operatorname{Aut}\big(F(\Gamma)\big)$ and $\pi \circ \theta \circ \widetilde{\chi}^{-1} = \chi \circ \pi \circ \widetilde{\chi}^{-1} = \pi$. Hence, $\theta \circ \widetilde{\chi}^{-1}$ leaves each fiber invariant. Lemma 3.9 then asserts that $\theta \circ \widetilde{\chi}^{-1} = \gamma \in \Gamma$. It follows that $\theta = \gamma \circ \widetilde{\chi} \in \operatorname{mod}\Gamma$. This completes the proof of Theorem 4.2. □

Proof of the main theorem. Suppose that $\varphi\colon F(\Gamma') \to F(\Gamma)$ is a fiber-preserving isomorphism. By Lemma 3.4, φ can be projected to a biholomorphic map $\chi\colon T(\Gamma') \to T(\Gamma)$. By Theorem 3.1, the signatures of Γ and Γ' are the same. Note that Γ and Γ' have the same signature if and only if there is $w \in Q(\Gamma')$ so that

$$w\Gamma' w^{-1} = \Gamma. \tag{4.3}$$

Hence, by Theorem 2 of Bers [8], w induces a Bers allowable mapping $[w]_*$ of $F(\Gamma')$ onto $F(\Gamma)$.

Consider the automorphism $\varphi_0 = [w]_*^{-1} \circ \varphi \colon F(\Gamma') \to F(\Gamma')$. It is easy to see that φ_0 is fiber-preserving and holomorphic. By Proposition 4.2, we assert that φ_0 is an element of $\operatorname{mod}\Gamma'$. This implies that there is a quasiconformal self-map $\hat{f}$ of U with

$$\hat{f}\,\Gamma'\hat{f}^{-1} = \Gamma' \tag{4.4}$$

such that $\varphi_0 = [\hat{f}\,]_*$. It follows that $\varphi = [w]_* \circ [\hat{f}\,]_* = [w \circ \hat{f}\,]_*$, which says that φ is the Bers allowable mapping induced by the quasiconformal self-map $w \circ \hat{f}$ of U.

To be more precise, we see from (4.3) and (4.4) that $(w \circ \hat{f}\,)\Gamma'(w \circ \hat{f}\,)^{-1} = \Gamma$. Thus $w \circ \hat{f} \in Q(\Gamma')$. Hence, by a construction of Bers [1], the mapping φ is given by

$$\varphi([\nu], z) = [w \circ \hat{f}\,]_*([\nu], z) = \big(\chi([\nu]), w^{\mu} \circ w \circ \hat{f} \circ (w^{\nu})^{-1}(z)\big),$$

for all $([\nu], z) \in F(\Gamma')$, where μ is the Beltrami coefficient of $w^{\nu} \circ (w \circ \hat{f})^{-1}$. By (4.3) and (4.4) again, we know that $\mu \in M(\Gamma)$. It is also not hard to see that χ is defined by sending the conformal structure $[\nu] \in T(\Gamma')$ to the conformal structure $[\mu] \in T(\Gamma)$. The reverse direction is completely trivial. This completes the proof of our main theorem. □

5. Holomorphic extensions of motions on sections

In this section we study some extension problems which are related to our discussion on isomorphisms of Bers fiber spaces. Let Γ be a Fuchsian group with torsion, and let f be a quasiconformal self-map of U/Γ in the sense of orbifolds; that is, f maps regular points to regular points, punctures to punctures, and branch points to branch points of the same order. More specifically, we assume that the map f fixes a branch point $\hat{z}_0$ which is determined by an elliptic element e of Γ. Let s be the canonical section of $\pi \colon F(\Gamma) \to T(\Gamma)$ which is defined by the element e.

The map f induces a modular transformation χ_f on $T(\Gamma)$. Now the automorphism $s_*(\chi_f)$ of $s\big(T(\Gamma)\big)$ is defined by

$$s_*(\chi_f)(x) = s \circ \chi_f \circ \pi(x) \qquad \text{for } x \in s\big(T(\Gamma)\big).$$

Lemma 5.1 of [13] shows that $s_*(\chi_f)$ is the restriction of a global holomorphic automorphism of $F(\Gamma)$. Natural questions to be asked are:

(1) Except for the obvious ones constructed in Lemma 5.1 of [13], is there any other holomorphic extension of $s_*(\chi_f)$?

(2) For any $\theta \in \operatorname{Aut} T(\Gamma)$ and any holomorphic section s of π, is there a holomorphic extension of $s_*(\chi)$?

The following result, which is an interesting application of the main theorem, answers these questions in general cases.

Proposition 5.1. *Under the same notations as above, assume that the type (or signature) of* Γ *is not in Table* 1. *Then*

(1) χ *and* $\chi \circ e$ $(= e \circ \chi)$ *are the only two fiber-preserving holomorphic extensions of* $s_*(\chi_f)$.

(2) *A fiber-preserving holomorphic extension of* $s_*(\theta)$ *exists only when* $\theta = \chi_f$ *for an* f *which fixes a branch point and* s *is determined by that branch point.*

Proof. We first assume that the signature of Γ is neither $(2, 0; \_\_)$ nor $(1, 2; \infty, \infty)$.

(1) Proposition 4.2 asserts that every fiber-preserving automorphism of $F(\Gamma)$ is an element of $\operatorname{mod}\Gamma$. Suppose that χ_0 is a fiber-preserving extension of $s_*(\chi_f)$ distinct from χ and $\chi \circ e$ $(= e \circ \chi)$, then $\chi_0 \in \operatorname{mod}\Gamma$. Now $\chi_0 \circ \chi^{-1}$ is an element of $\operatorname{mod}\Gamma$ whose restriction to $s(T(\Gamma))$ is the identity. This implies that $\chi_0 \circ \chi^{-1}$ lies in the kernel of the quotient map $\zeta\colon \operatorname{mod}\Gamma \to \operatorname{Mod}\Gamma$. Therefore, $\chi_0 \circ \chi^{-1} \in \Gamma$. On the other hand, since the restriction of $\chi_0 \circ \chi^{-1}$ to $s(T(\Gamma))$ is the identity, in particular, $\chi_0 \circ \chi^{-1}$ restricts to the central fiber U of $F(\Gamma)$ and fixes the fixed point of $e \in \Gamma$. It follows that $\chi_0 \circ \chi^{-1} = \mathrm{id}$ or e; that is, either $\chi_0 = \chi$ or $\chi_0 = e \circ \chi = \chi \circ e$. This completes the proof of (1).

(2) Suppose that $s_*(\theta)$ admits a fiber-preserving holomorphic extension $\chi \in \operatorname{Aut} T(\Gamma)$ with $\chi|_{s(T(\Gamma))} = \theta$. By Proposition 4.2, $\chi \in \operatorname{mod}\Gamma$. This means that χ is induced by an $\hat{f}\colon U \to U$ with $\hat{f}\Gamma\hat{f}^{-1} = \Gamma$, which in turn implies that $\hat{f}$ can be projected to a self-map $f\colon U/\Gamma \to U/\Gamma$ in the sense of orbifolds. Since χ keeps $s(T(\Gamma))$ invariant, f must fix a branch point (determined by e, say), and s must be a section defined by e. We are done.

Now we assume that Γ is of signature $(2, 0; \_\_)$. Let $s\colon T(\Gamma) \to F(\Gamma)$ be a Weierstrass section. Suppose that $s_*(\chi_f)$ can be extended holomorphically to a fiber-preserving $\chi \in \operatorname{mod}\Gamma$. We can conclude that χ and $\chi \circ e$ (which equals $e \circ \chi$) are the only two fiber-preserving extensions of $s_*(\chi_f)$, where $e|_U$ is a lift of the hyperelliptic involution, it is an elliptic Möbius transformation with order 2. Clearly, $e|_U$ is not in the group Γ but in the group $\langle \Gamma, \tilde{J} \rangle$ (which is generated by 5 elliptic elements of order 2), where $\tilde{J}$ is one of lifts of the hyperelliptic involution. Our assertion can be verified by applying Theorem 2.1, Lemma 3.3, and Theorem 3A of Greenberg [7] (or Theorem 1 of Singerman [12]). As a matter of fact, those results imply that any fiber preserving automorphism of $F(2, 0; \_\_)$ which acts trivially on the image $s(T(2, 0))$ of a canonical section s is actually a lift of the holomorphic involution on $V(2, 0; \_\_)$.

If Γ is of signature $(1, 2; \infty, \infty)$, the conclusion is also true; the proof is basically the same as above. However, the assertion is no longer true if Γ is of signature $(1, 2; \nu, \nu)$ for $2 \le \nu < \infty$ since there are uncountably many (global) holomorphic sections of $\pi_0\colon V(1, 2; \nu, \nu) \to T(1, 2)$. ◻

Remark. (i) When Γ is of type $(0, 3)$, $T(\Gamma)$ is a point, so $F(\Gamma)$ is U. For each point $a \in U$, there are uncountably many Möbius transformations in

$\mathrm{PSL}(2,\mathbf{R})$ fixing a. Proposition 5.1 fails in this case.

(ii) When Γ is of signature $(0,4;2,2,\infty,\infty)$ or $(0,4;2,2,2,\infty)$, Proposition 5.1 remains true, but the proof involves different methods (see Theorem 5.2).

The rest of this paper is also devoted to the study the same extension problems, but we do not require the extension to be fiber-preserving. A partial result towards the problems is as follows:

Theorem 5.2. *Let Γ be of signature $(0,4;2,2,\infty,\infty)$, $(0,4;2,2,2,\infty)$, $(0,5;2,2,2,2,\infty)$, or $(0,6;2,2,2,2,2,2)$. Then the only holomorphic extensions of $s_*(\chi_f)$ in $\mathrm{Aut}\,F(\Gamma)$ are χ and $\chi\circ e$ $(=e\circ\chi)$.*

A Riemann surface S of type (g,n) with $2g-2+n>0$, $n\geq 1$, is called *hyperelliptic* if S admits a hyperelliptic involution. Here by a hyperelliptic involution on S we mean a conformal involution on S (hence on the compactification $\overline{S}$) which has $2g+2$ fixed points on $\overline{S}$, interchanges pairwise the n punctures if n is even, and fixes one puncture and interchanges the other $n-1$ punctures pairwise if n is odd. The subset of a Teichmüller space consisting of hyperelliptic Riemann surfaces is called *hyperelliptic locus.* In general, the hyperelliptic locus is not connected.

Lemma 5.3. *Let Γ be a torsion free finitely generated Fuchsian group of type (g,n) with $2g-2+n>0$, and let $\chi\in\mathrm{Mod}\,\Gamma$ be an elliptic modular transformation of prime order, with the property that the restriction of χ to a component l of the hyperelliptic locus is the identity. Then χ is either the identity or a hyperelliptic involution.*

Proof. See [13]. □

More precise information can be obtained for some special group Γ.

Lemma 5.4. *Let Γ be of type $(2,1)$, $(1,3)$ or $(0,n)$, for $n\geq 5$. Assume that $\chi\in\mathrm{Mod}\,\Gamma$ is a modular transformation whose restriction to l is the identity. Then χ must be of prime order, and hence χ is either the identity, or equal to a hyperelliptic involution.*

Proof. Suppose for the contrary that $n=mp$ is the order of χ, where $m\geq 2$, and p is a prime. Then $\chi_*=\chi^m$ is of order p. By using Lemma 5.3, we deduce that $p=2$. Therefore, n must be of form 2^r for $r\geq 1$ an integer. Observe that

$$l\subset T(\Gamma)^{\chi}\subset\cdots\subset T(\Gamma)^{\chi^{2^{r-1}}}, \tag{5.1}$$

where $\chi^{2^{r-1}}$ is induced by a hyperelliptic involution. We obtain

$$\dim T(\Gamma)^{\chi^{2^{r-1}}}=\dim l.$$

It follows from (5.1) that

$$\dim l = \dim T(\Gamma)^{\chi} = \cdots = \dim T(\Gamma)^{\chi^{2^{r-1}}}. \tag{5.2}$$

In the proof given below, we denote by S a Riemann surface of type $(2,1)$, $(1,3)$, or $(0,n)$ for $n \geq 5$, by h the conformal automorphism of $\overline{S}$ which induces χ (its order is 2^r by the above argument). Let $\mathrm{Fix}(h)$ be the set of the fixed points of h on $\overline{S}$, k the number of fixed points of h on $\overline{S}$, and g_* the genus of $\overline{S}/\langle h\rangle$.

Case 1. Γ is of type $(2,1)$. It is obvious that

$$\mathrm{Fix}(h) \subset \mathrm{Fix}(h^2) \subset \mathrm{Fix}(h^4) \subset \cdots.$$

Since $h^{2^{r-1}}$ is a hyperelliptic involution, it fixes all Weierstrass points $x_1, \ldots, x_6$ of $\overline{S}$. Let x_6 be the puncture. Observe that h fixes x_6 and at least one another Weierstrass point, say x_1. Thus, h determines a permutation of the remaining 4 Weierstrass points $x_2, \ldots, x_5$. If $\{x_2, \ldots, x_5\}$ is divided into two orbits under the iteration of h, then h^2 is hyperelliptic and

$$\mathrm{Fix}(h^3) \subset \mathrm{Fix}(h^6) = \mathrm{Fix}(h^2).$$

It follows that all fixed points of h^j are the Weierstrass points, which implies that the surface $S/\langle h\rangle$ has 4 distinguished points. As an immediate consequence, $\dim T(\Gamma)^{\chi} = 3g_* + 1$, contradicting that $\dim T(\Gamma)^{\chi} = \dim l = 2g_* - 1 + [\frac{1}{2}n] = 3$.

If $\{x_2, \ldots, x_5\}$ is a cycle under the iteration of h, then by the same argument as above, the surface $S/\langle h^2\rangle$ has 4 distinguished points. This implies that $\dim T(\Gamma)^{\chi^2} = 3g_* + 1$, contradicting that $\dim T(\Gamma)^{\chi^2} = \dim T(\Gamma)^{\chi} = 3$ (where g_* is the genus of $\overline{S}/\langle h^2\rangle$).

If h fixes all 6 Weierstrass points, then h is a hyperelliptic involution.

Case 2. Γ is of type $(1,3)$. In this case, $h^{2^{r-1}}$ is a hyperelliptic involution. By definition, it fixes only one puncture and interchanges other two punctures. Since the fixed points of h are contained in the set of fixed points of $h^{2^{r-1}}$, h must fix one puncture and interchanges the other two punctures as well. But $h^{2^{r-1}}$ is of even order (unless $r=1$), it must fix all three punctures. This is a contradiction. We conclude that $r=1$ and h is a hyperelliptic involution.

Case 3. Γ is of type $(0,n)$, $n \geq 5$. In this case, the number of the fixed points of all conformal automorphisms (Möbius transformations) h^j, $j = 1, \ldots, 2m-1$, is two. Moreover, the fixed points of h^j coincide with those of h^i for all $i, j = 1, \ldots, 2m-1$. Note that $n \geq 5$. By a simple calculation, we obtain

$$\begin{aligned} \dim T(0,n)^{\chi_*} &= \dim T(S/\langle h^m\rangle) \\ &= \begin{cases} -3 + (n-2)/2 + 2 & \text{if two fixed points of } h \text{ are punctures;} \\ -3 + (n-1)/2 + 2 & \text{if one fixed point of } h \text{ is a puncture;} \\ -3 + n/2 + 2 & \text{if no fixed points of } h \text{ are punctures.} \end{cases} \end{aligned}$$

We thus have

$$\dim T(0,n)^{\chi_*} \geq -3 + \frac{n-2}{2} + 2 = \frac{n-2}{2} - 1. \tag{5.3}$$

On the other hand, by the same argument as above, we see that

$$\dim T(0,n)^{\chi} \leq -3 + \frac{n}{2m} + 2 = \frac{n}{2m} - 1. \tag{5.4}$$

Since $m \geq 2$, and $n \geq 5$, a simple computation shows that

$$\frac{n}{2m} < \frac{n-2}{2}.$$

Together with (5.3) and (5.4), we thus have

$$\dim T(0,n)^{\chi_*} > \dim T(0,n)^{\chi},$$

contradicting (5.2). □

Lemma 5.5. *Under the same condition as in Lemma* 5.4*, there is a unique non-trivial* $\chi \in \operatorname{Mod}\Gamma$ *whose restriction to* l *is the identity.*

Proof. We first consider the case when Γ is of type $(2,1)$. Let $S \in l$ be a hyperelliptic Riemann surface of type (2,1). By Lemma 5.4, χ is induced by a hyperelliptic involution on S. By Corollary 2 to Proposition III(7.9) of Farkas–Kra [21], there is only one hyperelliptic involution on $\overline{S}$. It follows that there is only one hyperelliptic involution on S. The assertion then follows.

Next, we consider the case when Γ is of type $(1,3)$. Again, let $S \in l$ be a (marked) hyperelliptic Riemann surface of type $(1,3)$, and let x_1, x_2 and x_3 denote the three punctures on S. Since every hyperelliptic involution on S must fix one and only one puncture, if there are two hyperelliptic involutions J and J_* on S which fix the same puncture, then $J \circ {J_*}^{-1}$ is either a hyperelliptic involution or the identity by Lemma 5.4. Since $J \circ {J_*}^{-1}$ fixes all three punctures, we see that $J \circ {J_*}^{-1}$ is the identity; that is, $J = J_*$. Now we assume that there are two distinct hyperelliptic involutions J_1 and J_2, and that J_i fixes x_i, $i = 1, 2$. By a simple calculation, $J_1 \circ J_2$ permutes all three punctures. On the other hand, by the same argument as above, $J_1 \circ J_2$ is either a hyperelliptic involution or the identity, it must fix at least one puncture. This is a contradiction.

Finally, if Γ is of type $(0,n)$, $n \geq 5$, then we choose $S \in l$ and assume that there are two distinct hyperelliptic involutions J_1 and J_2 on S. Let $h = J_1 \circ J_2$. The modular transformation χ induced by h is elliptic, and its restriction to l is the identity. By Lemma 5.4, χ is either the identity or a hyperelliptic involution. If h is the identity, there is nothing to prove. We thus assume that h is hyperelliptic. Similarly, $h_* = J_2 \circ J_1$ is also a hyperelliptic involution. But $h \circ h_* = J_1 \circ J_2 \circ J_2 \circ J_1 = \mathrm{id}$. It follows that $h_* = h$. We conclude that $J_1 \circ J_2 = J_2 \circ J_1$. Since h, J_1, and J_2 are half-turns on $\overline{S} = \mathbf{S}^2$, by Proposition B.5 of Maskit [10], the axes of h, J_1 and J_2 constitute an orthogonal basis as shown in Figure 1.

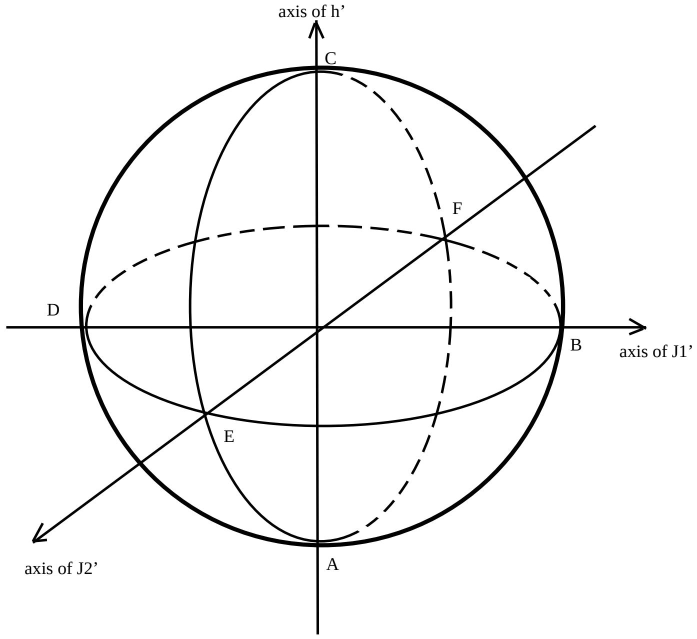

Figure 1.

If n is odd, then since h is a hyperelliptic involution, by definition, either C or A (but not both) is a puncture. Without loss of generality, we assume that C is a puncture, and A is a regular point. Since J_2 is also hyperelliptic, it sends C to A, which is impossible. We see that n must be even, and all the points A, B, $\ldots$, F, are not punctures.

Now let BCE denote the triangle in the sphere bounded by the geodesics BE, CF, and BC, and so forth. (In the spherical metric.) Observe that J_1, J_2, and h send the triangle BCE to triangles ABF, ADE, and CDF, respectively. The punctures which are contained in BCE are mapped to punctures contained in ABF, ADE, and CDF, respectively. The same situation occurs for any of other triangles. This implies that $n = 4k$ for some $k \in \mathbf{Z}^+$. Let χ_1 and χ_2 be the modular transformations induced by J_1 and J_2, respectively, and let Λ denote the subgroup of $\operatorname{Mod}\Gamma$ generated by χ_1 and χ_2. Since the quotient surface $S/\langle J_1, J_2\rangle$ has k punctures and three branched points of order 2, the dimension of $T(0,n)^\Lambda$ is k. On the other hand, by assumption, $\dim l = \dim T(0,n)^\Lambda$. We thus obtain

$$k = \dim T(0,n)^\Lambda = \dim l = \dim T(0,n)^{\chi_1} = -1 + [\tfrac{1}{2}n] = 2k - 1.$$

But this occurs only if $k = 1$ and $n = 4$. □

Remark. If $(g,n) = (0,4)$, Earle–Kra [3] proved that any Riemann surface of type $(0,4)$ has three (hyperelliptic) involutions, all of which induce the identity on $T(0,4)$. Lemma 5.5 fails in this special case.

Proof of Theorem 5.2. It is known that

$$F(0,4;2,2,\infty,\infty)\cong F(0,4;2,2,2,\infty)\cong T(0,5)$$
$$F(0,5;2,2,2,2,\infty)\cong T(1,3)$$

and

$$F(0,6;2,2,2,2,2,2)\cong T(2,1).$$

Let Γ be of the signature which is one of those mentioned above, and let φ denote the corresponding isomorphism. Also, we denote by $s_*(\chi_f)$ a motion of the image $s\big(T(\Gamma)\big)$ of a canonical section s which extends to a holomorphic automorphism χ of $F(\Gamma)$ (see Proposition 5.1). Suppose that there is another holomorphic extension χ_0 of $s_*(\chi_f)$. Then $\chi\circ\chi_0^{-1}\in\operatorname{Aut}F(\Gamma)$ is non-trivial but restricts to the identity map on $s\big(T(\Gamma)\big)$. From Lemma 3.5 of [13], we see that $l'=\varphi\circ s\big(T(\Gamma)\big)$ is a component of hyperelliptic locus, and $\varphi\circ\chi\circ\chi_0^{-1}\circ\varphi^{-1}\in\operatorname{Mod}\Gamma'$ is non-trivial but restricts to an identity map on l' as well. By Lemma 5.4, $\varphi\circ\chi\circ\chi_0^{-1}\circ\varphi^{-1}\in\operatorname{Mod}\Gamma'$ is a hyperelliptic involution J' (since it is not the identity). On the other hand, we denote by $e\in\Gamma$ the elliptic element corresponding to the canonical section s. Then $\varphi\circ e\circ\varphi^{-1}\in\operatorname{Mod}\Gamma'$ is a hyperelliptic involution by Lemma 3.5 of [13] again. Hence, from Lemma 5.5, we conclude that $J'=\varphi\circ e\circ\varphi^{-1}$; that is, $e=\chi\circ\chi_0^{-1}$. This completes the proof of Theorem 5.2. ▫

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