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ENTIRE FUNCTIONS OF ORDER ≤ 1, WITH BOUNDS ON BOTH AXES

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Abstract. Results are proved on the existence of a not identically vanishing entire function f, of order ≤ 1, and satisfying $\log|f(x)| \leq k(x)$, $\log|f(ix)| \leq a|x|$, $x \in \mathbf{R}$, where k is a given odd function and a a positive constant. Existence is proved if $k \in C^2(\mathbf{R})$ with $|xk''(x)| \leq 2a\pi^{-1}$, $x \in \mathbf{R}$, or, for every a, if k is convex on $\mathbf{R}^+$, with k' bounded. The second result is used to produce non-trivial translation invariant subspaces of the corresponding weighted $l^p(\mathbf{Z})$.

0. Introduction

Throughout this paper a denotes a positive number and k a continuous real-valued function on $\mathbf{R}$.

Question. Is there a not identically vanishing entire function f, of order ≤ 1, such that

$$\log|f(x)| \leq k(x), \qquad x \in \mathbf{R}, \tag{0.1}$$

$$\log|f(iy)| \leq a|y|, \qquad y \in \mathbf{R}. \tag{0.2}$$

A famous result of A. Beurling and P. Malliavin [1, Theorem I] gives the following:

Theorem (Beurling–Malliavin). *The answer is* yes *for every a if k, outside some compact interval, is absolutely continuous with bounded derivative, and*

$$\int_{-\infty}^{\infty} \frac{\min\big(0, k(x)\big)}{1+x^2}\, dx > -\infty.$$

This paper contains some simple complementary results in the case when k is an *odd function*. We will prove the following theorems.

Theorem 1. *Let k be odd on $\mathbf{R}$, differentiable outside some compact interval I, with absolutely continuous derivative. Then the answer is* yes *if*

$$|xk''(x)| \leq \frac{2a}{\pi}, \qquad x \notin I. \tag{0.3}$$

Theorem 2. *Let k be odd on $\mathbf{R}$, and convex (concave) with bounded derivative for sufficiently large positive values of the variable. Then the answer is* yes, *for every a.*

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These three theorems can be combined to more general results, since positive answers for the pairs (a_1, k_1) and (a_2, k_2) give positive answers for the pairs $(a_1 + a_2, k_1 + k_2)$ and $(a_1 + a_2, k_1 - k_2)$.

We will give some comments to Theorem 1. Take $a = \frac{1}{2}\pi$ and

$$k(x) = x \log|x|, \qquad x \neq 0,\ k(0) = 0.$$

Then

$$xk''(x) = 1, \qquad x \neq 0. \tag{0.4}$$

By Theorem 1 the answer in this case is *yes*. This can also be seen directly by choosing

$$f(z)^{-1} = z \exp(z) \Gamma(z).$$

But if $b > 1$ and

$$k(x) \geq bx \log|x|, \qquad x > 0,$$

then the answer is *no*, for $a = \frac{1}{2}\pi$. For if f satisfies the conditions in the question, then

$$g(z) = f(z) \exp\bigl(-bz \log(-z)\bigr),$$

with the principal branch of log, is analytic and of order at most 1 in the left half-plane. It is bounded on the negative real axis and vanishes of exponential order at ∞ and $-\infty$ on the imaginary axis. A standard Pragmén–Lindelöf argument shows that g is bounded in the left half-plane. Hence a classical corollary of Jensen's formula gives

$$\int_{\mathbf{R}} \log|g(iy)|(1 + y^2)^{-1}\, dy > -\infty, \tag{0.5}$$

a contradiction.

Theorems 1 and 2 are proved in Sections 1 and 2, respectively. Section 3 contains essentially a discussion on the sharpness of Theorem 2 and of Theorem 2′, a slightly more general variant of it. In Section 4, Theorem 2 is applied to produce non-trivial invariant subspaces with respect to translation in certain weighted $l^p(\mathbf{Z})$.

1. Proof of Theorem 1

Let u be a convex function on $[0,\infty)$ with $u(0)=0$ and

$$\int_1^\infty u(x)x^{-3}\,dx < \infty.$$

Observe that the integrand has a summable negative part. The Poisson formula for quadrants can be applied to extend u continuously to a function U in the closed right half-plane with boundary value 0 on the imaginary axis and harmonic inside the first and fourth quadrants.

Lemma. *U is subharmonic in the open right half-plane.*

Proof of the lemma. It suffices to show that U satisfies the mean value inequality locally at each point in the open half-plane. Hence it is enough to find, for every $x_o > 0$, a harmonic function U_o, such that $U_o \leq U$ in the open half-plane, and $U_o(x_o) = U(x_o)$.

By the convexity of u there is a function l of the form

$$l(x) = cx + d, \qquad x \in \mathbf{R},$$

such that

$$l(x_o) = u(x_o), \qquad l(x) \leq u(x),\ x \geq 0.$$

Take

$$U_o(x+iy) = l(x), \qquad x \geq 0.$$

In each of the two quadrants, the harmonic function U_o is given by the corresponding Poisson integral, and since $U_o \leq U$ on the boundaries, $U_o \leq U$ in the half-plane. Since $U_o(x_o) = U(x_o)$, the lemma is proved.

To prove the theorem, let us first compare two functions k_1 and k_2, each satisfying the conditions of the theorem, and such that $k_1'' = k_2''$ outside some compact interval. Then

$$k_1(x) - k_2(x) - bx, \qquad x \in \mathbf{R},$$

is bounded, for some real constant b. It follows from this that if f is an entire function, giving a positive answer of the question for (a, k_1), then, for some constant C, $Cf(z)\exp(bz)$, $z \in \mathbf{C}$, gives a positive answer for (a, k_2). Hence we can without loss of generality assume that (0.3) holds if $x \neq 0$. We can obviously choose $a = 1$, hence

$$|xk''(x)| \leq \frac{2}{\pi}, \qquad x \neq 0. \tag{1.1}$$

As a consequence of the growth restriction on k, we can use Poisson's formula in each quadrant to produce a continuous function U in $\mathbf{C}$, harmonic in each quadrant, 0 on the imaginary axis and with values $k(x)$, $x \in \mathbf{R}$. By the symmetry, U is harmonic in the upper and the lower half-plane. Let V be the function obtained, if we perform the corresponding extension of $2\pi^{-1}x \log|x|$, $x \in \mathbf{R}$. It is easy to see that for $z = x + iy \in \mathbf{C}$

$$V(z) = \mathrm{Re}\Big(\frac{2}{\pi} z \log z\Big) + |y|.$$

(0.4) and (1.1) show that

$$\frac{2}{\pi} x \log|x| - k(x) \qquad \text{and} \qquad \mathrm{k(x)} + \frac{2}{\pi}\mathrm{x} \log|\mathrm{x}|$$

are convex, for $x > 0$. By the lemma, both $V - U$ and $U + V$ are subharmonic in the open right half-plane. This means that in distribution sense

$$-\Delta V \le \Delta U \le \Delta V. \tag{1.2}$$

in the open right half-plane. Let m denote Lebesgue measure on $\mathbf{R}$, considered as a Borel measure in the plane. Since $\Delta|y| = 2m$, the fact that V is odd shows that

$$\Delta V = 2(\mathrm{sign})m + v,$$

where the right hand member stands for the measure m, multiplied with the function with values $2 \operatorname{sign} x$, $x \in \mathbf{R}$, plus a distribution v with support in 0. If $v \neq 0$, we can find a sequence $\{\phi_n\}$ of test functions, such that $\langle \Delta V, \phi_n \rangle \to 1$, while $\langle V, \Delta\phi_n \rangle \to 0$, as $n \to \infty$, a contradiction. Hence $v = 0$, and we obtain

$$\Delta V = 2(\mathrm{sign})m.$$

Since ΔU is odd, (1.2) shows that

$$\Delta U = gm + w,$$

where g is an odd Lebesgue measurable function, satisfying

$$-2 \le g(x) \le 2, \qquad x \in \mathbf{R},$$

and w is a distribution with support in 0. Exactly as above we see that $w = 0$, hence

$$\Delta U = gm.$$

Let us now form the function W, defined by

$$W(x+iy) = U(x+iy) + |y|, \qquad x + iy \in \mathbf{C}.$$

This is a subharmonic function in $\mathbf{C}$, since, with $h = g + 2$,

$$\Delta W = \Delta U + 2m = hm \ge 0.$$

As a subharmonic function, W is of order ≤ 1, and it satisfies the conditions

$$W(x) = k(x), \quad x \in \mathbf{R}, \quad W(iy) = |y|, \quad y \in \mathbf{R}.$$

Our theorem is proved if we can modify W to a function $W_o = \log|f|$, where f is entire of order ≤ 1, and satisfies (0.1) and (0.2).

To construct W_o we use a procedure of atomizing the measure ΔW. The method was introduced by B. Kjellberg [5, pp. 39–41]. A generalization of it, the KKK method, is described in W. Hayman [3, Ch. 10.5].

Let us first observe that

$$0 \leq h \leq 4. \tag{1.3}$$

Let H be the absolutely continuous function on $\mathbf{R}$ with $H' = h$, and $H(0) = 0$. Let

$$A(x) = 2\pi\big[(2\pi)^{-1}H(x)\big] - H(x), \qquad x \in \mathbf{R},$$

where [ ] denotes the integral part. The distributional derivative A' of A is then a Borel measure on $\mathbf{R}$. We think of it as a measure on $\mathbf{C}$, supported by the real axis. Since $\Delta \log$ equals 2π times the Dirac distribution at 0, we know that if T is upper semicontinuous on $\mathbf{C}$, and $\Delta T = A'$, then $W + T$ is a function of the form $\log|f_o|$, where f_o is entire.

It suffices to choose T so that it can be estimated from above by an expression of the type $C\log(|z|+2)$, $z \in \mathbf{C}$. For since H is unbounded, f_o has infinitely many zeros, and hence we can divide away finitely many of them in order to obtain an entire function of order ≤ 1 and satisfying (0.1) and (0.2).

A convenient choice is

$$T(z) = (2\pi)^{-1}\int_{|t|<1} \log|z-t|\,dA(t) + (2\pi)^{-1}\int_{|t|\geq 1} \log\Big|1 - \frac{z}{t}\Big|\,dA(t). \tag{1.4}$$

It follows from (1.3) that there is a constant C_1, such that the contribution from the set $[-1,1]\cup[x-1,x+1]$ to (1.4) is bounded above by $C_1\log(|z|+2)$. In the remaining intervals, the integrand has, for fixed z, just one interior extremal point, and its value varies in some interval

$$\big[-C_2\log(|z|+2), C_2\log(|z|+2)\big].$$

A partial integration shows that the integral over these intervals converges and is bounded by an expression of the form $C_3\log(|z|+2)$ since A is a bounded function. It is easy to see that T is upper semicontinous and that $\Delta T = A'$. Hence the theorem is proved.

2. Proof of Theorem 2

We can restrict ourselves to the case when k is convex on $[0,\infty)$. We represent k as a difference $k = k_1 - k_2$, where

$$k_1(x) = \int_{1/2}^{3/2} k(xy)\,dy, \qquad x \in \mathbf{R}. \tag{2.1}$$

Then k_1 and k_2 are odd, and $k_2 \geq 0$ on $\mathbf{R}^+$, by the convexity. k_1 fulfils the differentiability assumption of Theorem 1, and k_2 the differentiability assumption of the Beurling–Malliavin theorem. Furthermore k_2' is bounded. We will prove

$$\lim_{x\to\infty} x\, k_1''(x) = 0 \tag{2.2}$$

and

$$\int_2^\infty k_2(x) x^{-2}\, dx < \infty. \tag{2.3}$$

Then Theorem 1, applied to $(\frac{1}{2}a, k_1)$, and the Beurling–Malliavin theorem, applied to $(\frac{1}{2}a, k_2)$, prove our theorem.

For $x \geq 0$, we obtain from (2.1)

$$k_1''(x) = x^{-3} \int_{x/2}^{3x/2} t^2\, dk'(t),$$

and since k' increases on $\mathbf{R}^+$,

$$0 \leq k_1''(x) \leq 3x^{-1}\bigl(k'(\infty) - k'(\tfrac{1}{2}x)\bigr), \qquad x \in \mathbf{R}^+.$$

Since $k'(\infty)$ is finite, (2.2) is proved.

Partial integrations give, for $x > 0$,

$$8xk_2(x) = \int_{x/2}^{x} (x-2t)^2\, dk'(t) + \int_x^{3x/2} (3x-2t)^2\, dk'(t) \leq x^2 \int_{x/2}^{3x/2} dk'(t).$$

Hence

$$\int_2^\infty x^{-2} k_2(x)\, dx \leq \int_1^\infty \int_{2t/3}^{2t} (8x)^{-1}\, dx\, dk'(t) \leq \int_1^\infty dk'(t) = k'(\infty) - k'(1) < \infty,$$

and (2.3) is proved.

3. Comments on Theorem 2

Theorem 2 can obviously be stated in the following slightly more general form.

Theorem 2′. *Let* k *be odd on* R, *absolutely continuous in some interval* $[b,\infty)$, *and with its derivative equivalent to a function of bounded variation. Then the answer is* yes, for every a.

Take any a and any positive function ϱ on $\mathbf{R}^+$, such that $\varrho(x)\to\infty$, as $x\to\infty$. We will prove that Theorem $2'$ is no longer true if the condition of bounded variation is changed to the condition that k' is equivalent to a bounded function, satisfying

$$\int_b^x |dk'(t)| = O\big(\varrho(x)\big), \qquad x\to\infty, \tag{3.1}$$

and

$$\int_{-x}^x \frac{\max\big(0,k(t)\big)}{1+t^2}\,dt = O\big(\varrho(x)\big), \qquad x\to\infty. \tag{3.2}$$

A consequence of this is that, in Theorem 2, the condition of boundedness of k' can not be weakened to the condition

$$k'(x) = O\big(\varrho(x)\big), \qquad x\to\infty.$$

Let us first form the harmonic function

$$u(z) = \operatorname{Re}\big(-iz\log(-iz)\big) = -y\log\frac{1}{|z|} - ix\operatorname{Arg}(-iz).$$

in the closed upper half-plane. As always we choose the principal branch of log. Let l be the odd continuous function on $\mathbf{R}$, vanishing on $(0,\frac12)$ and $(\frac32,\infty)$, and such that $l(1)=1$, and l is linear in the complementary intervals. Put

$$k_o(x) = \sum_{n=-\infty}^{\infty} 3^n l(3^{-n}x), \qquad x\in\mathbf{R}.$$

Extend k_o by Poisson's formula for the first quadrant to a function U_o, harmonic inside, and with $U_o(iy)=ay$, $y\in\mathbf{R}^+$. A standard comparison between the Poisson integrals $U_o(z)$ and $u(z-1)$ in the first quadrant shows that

$$U_o(z) = 1 + Au(z-1) + O(|z-1|) = 1 - By\log\frac{1}{|z-1|} + O(|z-1|), \qquad z\to 1,$$

where A and B are certain positive constants. Extend U_o to a function in $\mathbf{C}$, even in y, odd in x. Then there is a r, $0<r<\frac12$, such that, for every $x\in[1-r,1+r]$, the mean value of U_o, taken over a circular disc with center x and radius r, is $< U_o(x)-r$. Since

$$U_o(3z) = 3U_o(z), \qquad z\in\mathbf{C},$$

we have, for every integer n and every $x\in[3^n(1-r),3^n(1+r)]$, that the mean value of U_o, taken over a circular disc with center x and radius $3^n r$, is $< U_o(x)-3^n r$.

Now we are ready for the final construction, which does not depend on a. It is possible to choose a strictly increasing infinite sequence $\{n_\nu\}$, of non-negative integers such that k, defined by

$$k(x) = \sum_\nu 3^{n_\nu} l(3^{-n_\nu} x), \qquad x \in \mathbf{R},$$

satisfies (3.1) and (3.2). Obviously k is absolutely continuous and k' is bounded. Extend k by Poisson's formula to a continuous function U, harmonic in each quadrant, and satisfying $U(iy) = a|y|$, $y \in \mathbf{R}$. Since $U \le U_o$ in the right half-plane we we can conclude, for every ν and every $x \in [3^{n_\nu}(1-r), 3^{n_\nu}(1+r)]$, that the mean value of U taken over a circular disc with center x and radius $3^{n_\nu} r$, is

$$< U_o(x) - 3^{n_\nu} r = k(x) - 3^{n_\nu} r.$$

If f is an entire function, giving a positive answer for (a, k), then $\log|f| \le U$ in $\mathbf{C}$, and hence, by the mean value inequality for $\log|f|$, we have for every ν that

$$\log|f(x)| \le k(x) - 3^{n_\nu} r, \qquad x \in [3^{n_\nu}(1-r), 3^{n_\nu}(1+r)].$$

Define g by

$$g(z) = f(z) f(-z), \qquad z \in \mathbf{C}.$$

g is an entire function of exponential type, and $\log|g| \le 0$ on $\mathbf{R}$. On every interval of the form $[3^{n_\nu}(1-r), 3^{n_\nu}(1+r)]$ we have

$$\log|g| \le -3^{n_\nu} r,$$

and hence

$$\int_1^\infty x^{-2} \log|g(x)|\, dx = -\infty.$$

As in the introduction this gives a contradiction.

Observe that the results here and in the introduction on non-existence of functions $\log|f|$, satisfying (0.1) and (0.2), where f is entire of order ≤ 1, are valid as well with $\log|f|$ exchanged to an arbitrary subharmonic function of order ≤ 1, not $\equiv -\infty$.

4. Translation-invariant subspaces of $l^p(w, \mathbf{Z})$

Let $\{k(n)\}$, $n \in \mathbf{Z}$, be a real odd sequence, such that $\{k(n+1) - k(n)\}$, $z \in \mathbf{Z}$, is bounded and

$$\sum_{-\infty}^{\infty} |k(n+1) - 2k(n) + k(n-1)| < \infty.$$

Put $\exp k(n) = w_n$, $w = \{w_n\}$, $n \in \mathbf{Z}$. For a given p, $1 \leq p < \infty$, $l^p(w, \mathbf{Z})$ denotes the Banach space of complex sequences $c = \{c_n\}$, $n \in \mathbf{Z}$, with

$$\|c\|^p = \sum_{-\infty}^{\infty} |c_n|^p {w_n}^p < \infty.$$

In $l^p(w, \mathbf{Z})$ translation, defined by $\{c_n\} \mapsto \{c_{n-1}\}$, is an bounded operator with a bounded inverse. We will use Theorem 2 to construct a non-trivial subspace of $l^p(w, \mathbf{Z})$, invariant under translation.

Extend k to an odd continuous function on $\mathbf{R}$, linear in each interval in $\mathbf{R} \setminus \mathbf{Z}$. Since k can be written $k_1 - k_2$, where k_1 and k_2 satisfy the conditions of Theorem 2, there is for every given a an entire function f giving a positive answer to the question for the pair (a, k). Take any a in the interval $(0, \frac{1}{2}\pi)$. It follows from the proofs of Theorems 1 and 2, that we can assume that f has infinitely many zeros. After dividing away two of these zeros, if necessary, we can assume that

$$|f(x)| \leq (1 + x^2)^{-1} \exp\bigl(k(x)\bigr), \qquad x \in \mathbf{R}.$$

Let us, for any given integer m, form the entire function g_m, defined by

$$g_m(z) = f(z) f(m - z), \qquad z \in \mathbf{C}.$$

A comparison between $\log |f|$ and convenient linear functions in each quadrant shows that $\log |g_m(iy)| - 2a|y|$, $y \in \mathbf{R}$, is bounded above. Since g_m is bounded on the real axis and of order ≤ 1, it is thus of exponential typ $\leq 2a$. g_m is summable on the real axis, and a theorem of Paley and Wiener [6, Theorem V] shows that the Fourier transform of the restriction of g_m to $\mathbf{R}$ vanishes outside $[-2a, 2a]$. Since $g_m(x)(1 + x^2)$, $x \in \mathbf{R}$ is summable and g_m is continuous, we can apply the Poisson summation formula (see for instance Y. Katznelson [4]) to the function

$$x \mapsto g_m(x) \exp(\pi i x),$$

and since the Fourier transform of this function vanishes at all integer multiples of $2\pi i$ we obtain

$$\sum_{n \in \mathbf{Z}} f(n)(-1)^n f(m - n) = 0.$$

For every p, $\{f(m - n)\}$, $n \in \mathbf{Z}$, belongs to $l^p(w, \mathbf{Z})$, and $\{f(n)(-1)^n\}$, $n \in \mathbf{Z}$, belongs to the dual Banach space. Obviously we can choose f such that it does not vanish identically at the integer points. Then the closed linear span of the translates of $\{f(-n)\}$ is a non-trivial closed translation-invariant subspace.

Theorems 1 and $2'$ can be used in a similar way. For a particular case, see Example 2 in §2 of [2].

References

[1] BEURLING, A., and P. MALLIAVIN: On Fourier transforms of measures with compact support. - Acta Math. 107, 1962, 291–309.

[2] BORICHEV, A.A.: The generalized Fourier transformation, Titchmarsh's theorem and asymptotically holomorphic functions. - Leningrad Math. J. 1, 1990, 825–857.

[3] HAYMAN, W.: Subharmonic Functions, II. - Academic Press, London, 1989.

[4] KATZNELSON, Y.: An Introduction to Harmonic Analysis. - Dover, New York, 1968.

[5] KJELLBERG, B.: On certain integral and harmonic functions. - Appelbergs, Uppsala, 1948.

[6] PALEY, R.E.A.C., and N. WIENER: Fourier transforms in the complex domain. - Amer. Math. Soc. Colloq. Publ. 19, New York, 1934.

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