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GEOMETRY OF SELF-SIMILAR MEASURES

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Abstract. Self-similar measures can be obtained by regarding the self-similar set generated by a system of similitudes $\Psi = \{\varphi_i\}_{i\in M}$ as the probability space associated with an infinite process of Bernoulli trials with state space Ψ. These measures are concentrated in normal Besicovitch sets, which are those sets composed of points with given asymptotic frequencies in their generating similitudes. In this paper we obtain some geometric-size properties of self-similar measures. We generalize the expression of the Hausdorff and packing dimensions of such measures to the case when M is countable. We give a precise answer to the problem of determining what packing measures are singular with respect to self-similar measures. Both problems are solved by means of a technique which allows us to obtain efficient coverings of balls by cylinder sets. We also show that normal Besicovitch sets have infinite packing measure in their dimension.

1. Introduction

The purpose of this paper is to investigate certain geometric properties of self-similar measures. Self-similar measures were introduced by Hutchinson in 1981 [Hut] (see also [Ban]). They provide the basic theoretical example of multifractal measures, a topic which generates active research amongst mathematicians and applied scientists [BMP], [CM], [O].

In [MR] it was shown how to get round the problem posed by the overlapping set when self-similar measures satisfy the open set condition (see below). The singularity of self-similar measures with respect to the Hausdorff measure was also analyzed there. In this paper we do an analogous analysis for the packing geometry, and generalize the formula for the Hausdorff and packing dimensions of self-similar measures to self-similar constructions with infinitely many similitudes. This requires new ideas, in particular a technique for the efficient covering of balls by cylinder sets. Here we develop this method, which we call the "travelling ball technique".

We now state the main results of the paper. We first introduce the self-similar objects this paper deals with. Let M denote either the set $\{1, 2, \dots, m\}$ or the set $\mathbf{N}$ of the positive integers. A collection $\Psi = \{\varphi_i : i \in M\}$ of similarity mappings in $\mathbf{R}^N$ is contractive if $\sup\{r_i : i \in M\} < 1$, where r_i stands for the contraction ratio of the mapping φ_i. We consider the space of similarities of $\mathbf{R}^N$ endowed with the topology of uniform convergence on bounded subsets of $\mathbf{R}^N$.

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Let Ψ be a contractive compact system of similarities, and let $S\Psi$ be the set mapping defined by

$$S\Psi(X) = \bigcup_{i\in M} \varphi_i(X) \tag{1}$$

for $X \subset \mathbf{R}^N$. It is well known that there is a unique compact $S\Psi$-invariant set E (that is $S\Psi(E) = E$), which is usually called the *self-similar set* generated by Ψ [Hut], [Wic]. The system Ψ satisfies the *open set condition* (denoted by OSC from now onwards) if there exists a nonempty bounded open set $V \subset \mathbf{R}^N$ such that $\varphi_i(V) \subseteq V$ for all $i \in M$, and $\varphi_i(V) \cap \varphi_j(V) = \emptyset$ for $i, j \in M$, $i \neq j$. An open set satisfying the OSC for the system Ψ will be denoted by V and its closure $\operatorname{cl} V$ by F. We will further assume throughout this paper that $V \cap E \neq \emptyset$. The set V is then said to satisfy the strong OSC for Ψ. When M is finite, Schief proved [Sch] that such V does always exist provided that the OSC holds. Without loss of generality we can assume that $|V| = 1$, where $|\cdot|$ stands for the diameter of a subset of $\mathbf{R}^N$. We denote by $\mathscr{S}(N, M)$ the set of compact contractive systems of similarities in $\mathbf{R}^N$, $\{\varphi_i : i \in M\}$, satisfying the strong OSC.

There exists a natural coding map $\pi \equiv \pi_\Psi$, from the product (or code) space $M^\infty := M \times M \times \cdots$ onto E, given by

$$\pi(\mathbf{i}) = \bigcap_{k\in \mathbf{N}} \bigl(\varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \circ \varphi_{i_k}(E)\bigr), \tag{2}$$

for $\mathbf{i} = (i_1 i_2 \ldots) \in M^\infty$. Let $\mathscr{P}^+ = \{(p_i)_{i\in M} : p_i > 0 \text{ for all } i,\ \sum_{i\in M} p_i = 1\}$. Given $\mathbf{p} = (p_i)_{i\in M} \in \mathscr{P}^+$, $\nu_{\mathbf{p}}$ denotes the infinite-fold product probability measure on M^∞, i.e.,

$$\nu_{\mathbf{p}} := \mathbf{p} \times \mathbf{p} \times \mathbf{p} \times \cdots \tag{3}$$

and $\mu_{\mathbf{p}}$ stands for the image measure of the measure $\nu_{\mathbf{p}}$ under the mapping π, i.e., $\mu_{\mathbf{p}} = \nu_{\mathbf{p}} \circ \pi^{-1}$. The measure $\mu_{\mathbf{p}}$ is a Borel measure, and it is called the *self-similar measure* associated with the pair $(\Psi, \mathbf{p})$. Let $\mathscr{M}^+ \equiv \mathscr{M}^+(\Psi) = \{\mu_{\mathbf{p}} : \mathbf{p} \in \mathscr{P}^+\}$ be the set of self-similar measures associated with the system Ψ. It can be seen that $\operatorname{supp} \mu = E$ for all $\mu \in \mathscr{M}^+$. Self-similar sets and measures, as considered here, were first introduced by Hutchinson [Hut]. In Section 3 we prove

Theorem A. *Let $\mathbf{p} = (p_i)_{i\in M} \in \mathscr{P}^+$, and let $\Psi = \{\varphi_i : i \in M\}$ be a finite or infinite countable compact contractive system of similarities of $\mathbf{R}^N$ fulfilling the strong open set condition. Then, the Hausdorff and packing dimensions of the self-similar measure $\mu_{\mathbf{p}}$ associated with the pair $(\Psi, \mathbf{p})$ are given by the formula*

$$\dim \mu_{\mathbf{p}} = \operatorname{Dim} \mu_{\mathbf{p}} = s(\mathbf{p}) := \frac{\sum_{i\in M} p_i \log p_i}{\sum_{i\in M} p_i \log r_i}, \tag{4}$$

provided that the series $\sum_{i\in M} p_i \log r_i$ converges.

We denote the Hausdorff and packing dimensions by dim and Dim respectively (see Section 2 for definitions). Formula (4) above generalizes the known formula in the finite case [DGSH].

We then prove Theorem B below, which embodies a non-trivial symmetry between the Hausdorff and packing measures in the context of deterministic self-similarity. The notations H^ψ and P^ψ below stand respectively for the Hausdorff and packing measures associated with an admissible dimension function ψ (see Section 2).

Theorem B. *Let $\Psi = \{\varphi_i : i \in M\}$ be a finite contractive system of similarities of $\mathbf{R}^N$ satisfying the open set condition, and let $\mathbf{p} = (p_i)_{i\in M} \in \mathscr{P}^+$. Let $\mathscr{G} = \{\psi_\alpha\}_{\alpha\in\mathbf{R}}$ be the family of real variable functions given by*

$$\psi_\alpha(\xi) = \xi^{s(\mathbf{p})} \exp\big(\alpha(2\log\xi^{c(\mathbf{p})} \log\log\log\xi^{c(\mathbf{p})})^{1/2}\big),$$

where $s(\mathbf{p})$ is defined in (4), and $c(\mathbf{p}) = \big(\sum_{i\in M} p_i \log r_i\big)^{-1}$. Let

$$d(\mathbf{p}) = \left(\sum_{i\in M} \big(\log p_i - s(\mathbf{p})\log r_i\big)^2 p_i\right)^{1/2}.$$

Then the self-similar measure $\mu_\mathbf{p}$ associated with the pair $(\Psi, \mathbf{p})$ satisfies

i) *For $\alpha < -d(\mathbf{p})$, $\mu_\mathbf{p}$ is singular with respect to (w.r.t) P^{ψ_α} (and thus it is singular w.r.t. H^{ψ_α}).*

ii) *For $|\alpha| < d(\mathbf{p})$, $\mu_\mathbf{p}$ is absolutely continuous w.r.t. P^{ψ_α} and it is singular w.r.t. H^{ψ_α}.*

iii) *For $\alpha > d(\mathbf{p})$, $\mu_\mathbf{p}$ is absolutely continuous w.r.t. H^{ψ_α} (and thus it is absolutely continuous w.r.t. P^{ψ_α}).*

iv) *$\mu_\mathbf{p}$ is not representable as an integral in terms of either the Hausdorff measure H^t or the packing measure P^t for $0 < t < s = \dim E$.*

v) *Let μ_s be the self-similar measure associated with the system Ψ and the probability vector $\mathbf{p}_s := (r_i^s)_{i\in M}$. Then μ_s admits an integral representation w.r.t. the measures H^s and P^s. In fact, the measures μ_s, H^s, and P^s coincide up to a constant factor.*

The value of $\dim E$ in parts iv) and v) is known to be the unique real number s such that $\sum_{i\in M} r_i^s = 1$ [Hut].

The statements made in the above theorem concerning Hausdorff measure properties were proven in [MR]. In Section 4 we prove their counterpart for the packing geometry of self-similar measures (see Theorem 4.5). In Theorem B we gather both results in order to illustrate the symmetric role of the Hausdorff and packing geometries in self-similar constructions. Part v) is well known, and it is included here for the sake of completeness.

Notice that $S\Psi$-invariant sets may be properly named 'self-similar' sets. From the $\mu_{\mathbf{p}}$-measure-theoretic point of view, only those sets with positive $\mu_{\mathbf{p}}$-measure are relevant. The normal Besicovitch sets $B_{\mathbf{p}}$ and $B_{\mathbf{p}}^{(\infty)}$ introduced in [MR] (see the definitions in (9) below) are thus (non-compact) 'self-similar' sets of full $\mu_{\mathbf{p}}$-measure. They can be regarded as the set-theoretic dual counterpart of self-similar measures and thus provide a natural alternative approach to self-similar geometry. In [MR] several Hausdorff measure and dimension properties of the normal Besicovitch sets were obtained. In particular, they were shown to have Hausdorff dimension given by $s(\mathbf{p})$, and an $s(\mathbf{p})$-dimensional Hausdorff measure that is either zero or infinity under the hypothesis that $\dim(B_{\mathbf{p}}\cap\Theta) < s(\mathbf{p})$, where Θ is the overlapping set associated with the geometric construction (see (6) for the definition). We prove in Section 5 that the Hausdorff and packing dimensions of Besicovitch sets associated with $(\Psi,\mathbf{p})$ are given by $s(\mathbf{p})$, and are thus 'fractal sets' in the sense of Taylor [Tay]. Moreover, using the results obtained in Section 4, we are able to prove that the normal Besicovitch sets have infinite packing measure. Our results are collected in Theorem 5.1. Some results concerning measure and dimension properties of other $\mu_{\mathbf{p}}$-full measure geometric point sets in self-similar constructions, characterized by the frequencies of their generating similarities, can be seen in [Rey].

The paper is organized in the following way. In Section 2 we introduce notation and prove some preliminary results. In Section 3 we prove Theorem A above. The statements of Theorem B concerning packing geometry are proved in Section 4. Finally, Section 5 gives an account of the packing geometric size of normal Besicovitch sets.

2. Notation and preliminary results

We first give some basic definitions and notation from geometric measure theory. Given $A\subset\mathbf{R}^N$ and $\delta>0$, a collection of balls $\{B_i : i\in\mathbf{N}\}$ is a δ-covering of the set A if $\bigcup_i B_i \supset A$ with $|B_i|\le\delta$ for all i. A δ-packing of A is a collection of closed balls $\{B(x_i,r_i) : x_i\in A\}_{i\in\mathbf{N}}$ satisfying $2r_i<\delta$ for all i, and $B(x_i,r_i)\cap B(x_j,r_j)=\emptyset$ for all $i\ne j$. Let $\mathscr{F}$ denote the set of dimension functions, that is, the set of those increasing continuous functions ϕ defined in some nonempty interval $(0,\varepsilon)$, satisfying $\lim_{\xi\to0^+}\phi(\xi)=0$ and $\limsup_{\xi\to0}\bigl(\phi(2\xi)/\phi(\xi)\bigr)=\phi^*<+\infty$. For $\phi\in\mathscr{F}$, the spherical ϕ-Hausdorff measure of $A\subset\mathbf{R}^N$ is given by

$$H^\phi(A)=\sup_{\delta>0}\inf\biggl\{\sum_{i\in\mathbf{N}}\phi(|B_i|) : \{B_i\}\text{ is a }\delta\text{-covering of }A\text{ by balls}\biggr\}.$$

It is a standard fact that H^ϕ is comparable to the standard Hausdorff measure [Fal]. In particular, if H^t denotes the Hausdorff measure associated with the dimension function $\phi(\xi) = \xi^t$, then the Hausdorff dimension of the set A is given by the threshold value

$$\dim A = \sup\{t : H^t(A) > 0\} = \inf\{t : H^t(A) < +\infty\}.$$

The ϕ-packing measure is defined in two steps. First the ϕ-packing premeasure of A is defined as

$$P_0^\phi(A) = \inf_{\delta>0} \sup\Big\{\sum_{i\in\mathbf{N}} \phi(|B_i|) : \{B_i\}_i \text{ is a } \delta\text{-packing of } A\Big\}.$$

Then the ϕ-packing measure of A is given by

$$P^\phi(A) = \inf\Big\{\sum_{i\in\mathbf{N}} P_0^\phi(A_i) : \bigcup_i A_i \supseteq A\Big\}.$$

The packing dimension of A is the value given by

$$\operatorname{Dim} A = \sup\{t : P^t(A) > 0\} = \inf\{t : P^t(A) < +\infty\},$$

P^t denoting, for each $t \geq 0$, the packing measure defined from the dimension function ξ^t.

Finally, the Hausdorff dimension of a Borel measure μ is defined by

$$\dim \mu = \inf\{\dim A : \mu(A) > 0\}, \tag{5}$$

where the infimum is taken over the class of Borel sets. The packing dimension of μ is defined in the same way, and will be denoted by $\operatorname{Dim}\mu$.

We now introduce notation and previous results which will be used in the remaining sections. These results guarantee in particular that the 'travelling ball' technique works for a set of full μ-measure on E.

The *overlapping set* Θ of the system Ψ is the set given by

$$\Theta = \big\{x \in E : \operatorname{card}\{\pi^{-1}(x)\} > 1\big\}, \tag{6}$$

i.e. the geometric set where π fails to be an injection. The notation M^* stands for the set $\bigcup_{k\in\mathbf{N}} M^k$, which is the set of finite sequences with terms in M. For $\mathbf{j} = (j_1 j_2 \cdots j_k) \in M^k$, $\mathbf{i} \in M^\infty$, and $n \in \mathbf{N}$, $n > k$, let

$$\delta_{\mathbf{j}}(\mathbf{i}, n) = \frac{1}{n}\operatorname{card}\{q : i_q = j_1,\ i_{q+1} = j_2, \ldots, i_{q+k-1} = j_k,\ 1 \leq q \leq n-k+1\}, \tag{7}$$

and write $\delta_{\mathbf{j}}(\mathbf{i}) = \lim_{n\to+\infty} \delta_{\mathbf{j}}(\mathbf{i}, n)$, whenever such a limit exists.

Given $\mathbf{p} \in \mathscr{P}^+$ and $k \in \mathbf{N}$, consider the sets of codes

$$\mathscr{B}_{\mathbf{p}}^{(k)} = \bigcap_{\mathbf{j} \in M^k} \{\mathbf{i} \in M^\infty : \delta_{\mathbf{j}}(\mathbf{i}) = p_{\mathbf{j}}\}, \tag{8}$$

where $p_{\mathbf{j}} = p_{j_1} p_{j_2} \cdots p_{j_k}$ for $\mathbf{j} = (j_1 j_2 \cdots j_k)$; and the set

$$\mathscr{B}_{\mathbf{p}}^{(\infty)} = \bigcap_{k \in \mathbf{N}} \mathscr{B}_{\mathbf{p}}^{(k)} = \bigcap_{\mathbf{j} \in M^*} \{\mathbf{i} \in M^\infty : \delta_{\mathbf{j}}(\mathbf{i}) = p_{\mathbf{j}}\}.$$

We define the geometric sets

$$B_{\mathbf{p}} := \pi(\mathscr{B}_{\mathbf{p}}^{(1)}) \qquad \text{and} \qquad B_{\mathbf{p}}^{(\infty)} := \pi(\mathscr{B}_{\mathbf{p}}^{(\infty)}), \tag{9}$$

which in [MR] were respectively called the *normal Besicovitch set* and the *super-normal Besicovitch set* associated with the pair $(\Psi, \mathbf{p})$.

Given $\mathbf{i} \in M^\infty$ and $k \in \mathbf{N}$, $\mathbf{i}(k)$ stands for the curtailed sequence $(i_1 i_2 \cdots i_k)$. For $\mathbf{j} \in M^k$, $[\mathbf{j}] = \{\mathbf{i} \in M^\infty : \mathbf{i}(k) = \mathbf{j}\}$ is called a cylinder set (the one whose heading sequence is $\mathbf{j}$). We write $\varphi_{\mathbf{j}}$ for the composite similitude $\varphi_{j_1} \circ \varphi_{j_2} \circ \cdots \circ \varphi_{j_k}$; and $E_{\mathbf{j}}$ (respectively $F_{\mathbf{j}}$, $V_{\mathbf{j}}$) for the image sets $\varphi_{\mathbf{j}}(E)$ (respectively $\varphi_{\mathbf{j}}(F)$, $\varphi_{\mathbf{j}}(V)$), which we call geometric cylinder sets. Recall that we abbreviated the product $p_{j_1} p_{j_2} \cdots p_{j_k}$ by $p_{\mathbf{j}}$ above. We also denote $r_{j_1} r_{j_2} \cdots r_{j_k}$ by $r_{\mathbf{j}}$. We call the projected sets $\pi([\mathbf{i}(k)]) = \varphi_{\mathbf{i}(k)}(E)$, geometric cylinders of the k-th generation.

We now state a result concerning the $\mu_{\mathbf{p}}$-sizes of the overlapping set Θ and of the Besicovitch sets.

Theorem 2.1. *Let $\Psi \in \mathscr{S}(N, M)$, $\mathbf{p} \in \mathscr{P}^+$, and let $\mu_{\mathbf{p}}$ be the self-similar measure associated with $(\Psi, \mathbf{p})$. Then*

i) $B_{\mathbf{p}}^{(\infty)} \cap \big(\Theta \cup (E \cap \partial V)\big) = \emptyset$;

ii) (Θ-lemma) $\mu_{\mathbf{p}}\big(\Theta \cup (E \cap \partial V)\big) = 0$;

iii) $\mu_{\mathbf{p}}(E_{\mathbf{j}}) = \mu_{\mathbf{p}}(F_{\mathbf{j}}) = \mu_{\mathbf{p}}(V_{\mathbf{j}}) = \nu_{\mathbf{p}}([\mathbf{j}]) = p_{\mathbf{j}}$ *for all* $\mathbf{j} \in M^*$.

The limits $\delta_{\mathbf{j}}(\mathbf{i})$ can be thought as the average time spent on the cylinder $\mathbf{j}$ by the forward shift orbit of $\mathbf{i}$ (see below for a definition of the shift mapping). From Birkhoff's ergodic theorem and the definition of $B_{\mathbf{p}}^{(\infty)}$ it follows that $\mu_{\mathbf{p}}(B_{\mathbf{p}}^{(\infty)}) = 1$. The Θ-lemma above then follows from part i). Part iii) follows from part ii). A proof for part i) can be seen in [MR].

Remark 2.2. Let $\mathscr{C}$ denote the σ-algebra generated by the class of cylinder sets and let $\mathscr{C}_\pi$ denote the σ-algebra induced on E by π. Notice that the Θ-lemma implies that $(E, \mathscr{C}_\pi, \mu_{\mathbf{p}})$ and $(M^\infty, \mathscr{C}, \nu_{\mathbf{p}})$ are isomorphic measure spaces. Remark 2.3 below strengthens the equivalence of those spaces from the dynamical viewpoint.

Let $\tau\colon M^\infty \mapsto M^\infty$ be the Bernoulli shift in the code space, i.e. $\tau(i_1 i_2 i_3 \cdots) = (i_2 i_3 \cdots)$. A Bernoulli shift, which will be denoted by T, is defined in the geometric space E by means of

$$T|_{E\setminus\Theta} = \pi \circ \tau \circ \pi^{-1}.$$

The geometric shift mapping T defined above will be fixed throughout the paper.

Remark 2.3. The Θ-lemma implies that T is $\mu_{\mathbf{p}}$-preserving for all $\mathbf{p} \in \mathscr{P}^+$, and furthermore that $(T, \mu_{\mathbf{p}})$ and $(\tau, \nu_{\mathbf{p}})$ are isomorphic measure-preserving mappings. Observe that T is also ergodic for any measure $\mu_{\mathbf{p}} \in \mathscr{M}^+$, since ergodicity is preserved under isomorphisms. The definitions from ergodic theory used above can be seen in [Wal].

We will need the following auxiliary result. Recall that the sets $E_i = \varphi_i(E)$ are geometric cylinders of the first generation for the system Ψ.

Lemma 2.4. *Let $\Psi \in \mathscr{S}(N, M)$. For $k \in \mathbf{N}$, let $\mathscr{A}_k$ be the class of sets*

$$\{\emptyset, \Theta, S\Psi^{k-1}(E_1), S\Psi^{k-1}(E_2), \ldots\}$$

($S\Psi^0$ is the identity map), and let $\sigma(\mathscr{A}_k)$ denote the σ-algebra generated by $\mathscr{A}_k$. Then, for each $p \in \mathbf{N}$ and $\mu \in \mathscr{M}^+$

$$\mu(A_1 \cap A_2 \cap \cdots \cap A_p) = \prod_{j=1}^{p} \mu(A_j), \tag{10}$$

whenever $A_j \in \sigma(\mathscr{A}_j)$ for each $j = 1, 2, \ldots, p$.

Proof. For each $k \in \mathbf{N}$, let $\tilde{\mathscr{A}_k}$ denote the class of sets in $\mathscr{A}_k$ plus all sets obtained as finite intersection of sets in $\mathscr{A}_k$. Let $p \in \mathbf{N}$. Observe that the Θ-lemma implies that equality

$$\mu\big(E_{i_1} \cap S\Psi(E_{i_2}) \cap \cdots \cap S\Psi^{p-1}(E_{i_p})\big) = \prod_{j=1}^{p} \mu(E_{i_j}),$$

holds for any $\mathbf{i} = (i_1, i_2, \ldots, i_p) \in M^p$. Thus (10) holds if $A_j \in \mathscr{A}_j$ for $1 \le j \le p$. Using the Θ-lemma again, it is easy to show that (10) actually holds when $A_j \in \tilde{\mathscr{A}_j}$, $j = 1, \ldots, p$. That is to say, the classes $\tilde{\mathscr{A}_1}, \tilde{\mathscr{A}_2}, \ldots, \tilde{\mathscr{A}_p}$ are independent [Bil]. Since the class $\tilde{\mathscr{A}_j}$ is a π-system for each j, a result by Billingsley [Bil, Theorem 4.2] implies that the σ-algebras $\sigma(\tilde{\mathscr{A}_1}), \sigma(\tilde{\mathscr{A}_2}), \ldots, \sigma(\tilde{\mathscr{A}_p})$ are also independent, and the lemma follows. ▫

For $i \in M$, let $\chi_i\colon E \mapsto \mathbf{N} \cup \{0, +\infty\}$ be the random variable defined by

$$\chi_i(x) = \sup\big\{p : x \in (E \setminus E_i) \cap T^{-1}(E \setminus E_i) \cap \cdots \cap T^{-p+1}(E \setminus E_i)\big\} \tag{11}$$

for $x \in E \setminus E_i$, and $\chi_i(x) = 0$ for $x \in E_i$. The following typical limit property for χ_i will be used in subsequent sections.

Proposition 2.5. *Let $\Psi \in \mathscr{S}(N, M)$. For $i \in M$ and $a > 0$, let*

$$G(i,a) = \Big\{ x \in E : \lim_{k \to +\infty} k^{-a} \chi_i\big(T^k(x)\big) = 0 \Big\}. \tag{12}$$

Then, $\mu\big(G(i,a)\big) = 1$ for any $\mu \in \mathscr{M}^+$.

Proof. Let $i \in M$, $a > 0$, and $\mu \in \mathscr{M}^+$. We will write A for the set $E \setminus E_i$. For positive integers k, j consider the set

$$A_{k,j} = T^{-k}\big(A \cap T^{-1}(A) \cap \dots \cap T^{-j+1}(A)\big). \tag{13}$$

Notice that, using the notation of Lemma 2.4, $T^{-q}(A) = S\Psi^q(A) \in \sigma(\mathscr{A}_{q+1})$ for all $q \in \mathbf{N}$. Thus, the T-invariance of μ (see Remark 2.3) together with Lemma 2.4 imply that

$$\mu(A_{k,j}) = \mu\big(A \cap T^{-1}(A) \cap \dots \cap T^{-j+1}(A)\big) = \mu(A)^j. \tag{14}$$

Let $n \in \mathbf{N}$. For each $k \in \mathbf{N}$, we define the set

$$G_n^*(i,a,k) = \Big\{ x : \chi_i\big(T^k(x)\big) > \frac{k^a}{n} \Big\}.$$

For each $k \in \mathbf{N}$, consider the integer $p(k) = \min\{p \in \mathbf{N} : pn > k^a\}$. From (11) and (13),

$$\mu\big(G_n^*(i,a,k)\big) \le \mu(A_{k,p(k)})$$

because the sequence $\{A_{k,j}\}_j$ is non-increasing. The choice of $p(k)$ and (14) then give

$$\mu\big(G_n^*(i,a,k)\big) \le \mu(A)^{p(k)} \le \big(\mu(A)^{1/n}\big)^{k^a}. \tag{15}$$

Writing $r = \mu(A)^{1/n}$, the estimate (15) and a change of variable give

$$\sum_{k=1}^{\infty} \mu\big(G_n^*(i,a,k)\big) \le \sum_{k=1}^{\infty} e^{-k^a |\log r|} \le a^{-1} |\log r|^{-1/a} \Gamma(a^{-1}), \tag{16}$$

where $\Gamma(\cdot)$ denotes the eulerian integral. Therefore the series in (16) converges, and the first Borel–Cantelli lemma implies that $\mu\big(\limsup_{k\to\infty} G_n^*(i,a,k)\big) = 0$. Since $n \in \mathbf{N}$ is arbitrary, $E \setminus G(i,a)$ is a μ-null set, which proves the result. □

3. Dimension of self-similar measures

In this section we assume that the set M is infinite and we find the Hausdorff and packing dimensions of a self-similar measure. Let $(\Psi, \mathbf{p}) \in \mathscr{S}(N, M) \times \mathscr{P}^+$. Assume that

(17) $$\sum_{i \in M} p_i \log r_i > -\infty$$

and let $s(\mathbf{p})$ be the real number defined in (4). The formula

(18) $$\dim \mu_{\mathbf{p}} = \operatorname{Dim} \mu_{\mathbf{p}} = s(\mathbf{p})$$

is known to hold when M is finite [DGSH]. The proof depends crucially on the existence of a positive minimum contraction ratio $u := \min\{r_i : i \in M\} > 0$, which does no longer hold for the infinite case. We prove in this section that formula (18) still holds for the infinite case. Using (18) for M finite it can be proved that the convergence hypothesis (17) implies that the series in the numerator of $s(\mathbf{p})$ also converges, namely

Lemma 3.1. *Let $\Psi = \{\varphi_i : i \in M\} \in \mathscr{S}(N, M)$, and $\mathbf{p} = (p_i)_{i \in M} \in \mathscr{P}^+$. If the series $\sum_{i \in M} p_i \log r_i$ converges, then the series $\sum_{i \in M} p_i \log p_i$ also converges.*

Proof. Suppose $\sum_{i \in M} p_i \log p_i = -\infty$. Let $l < 0$ be the sum of the series $\sum_{i \in M} p_i \log r_i$, and let $\{c_n\}_{n \in \mathbf{N}}$ be the sequence defined by $c_n = \sum_{i=1}^n p_i$, $n \in \mathbf{N}$. Choose a positive integer k satisfying

(19) $$\sum_{i=1}^k p_i \log p_i < Nl + c_k \log c_k.$$

Now consider the finite probability distribution $\widehat{\mathbf{p}} = (\widehat{p}_i)_{i \in K}$ on the set $K = \{1, 2, \ldots, k\}$ defined by $\widehat{p}_i = c_k^{-1} p_i$, $i \in K$. The system $\widehat{\Psi} = \{\varphi_i : i \in K\}$ is finite and satisfies the OSC. Thus the dimension formula given in (18) implies that the self-similar measure $\mu_{\widehat{\mathbf{p}}}$ associated with the pair $(\widehat{\Psi}, \widehat{\mathbf{p}})$ has Hausdorff and packing dimensions given by $s(\widehat{\mathbf{p}})$ (see (4)). From (19) it then follows that

$$s(\widehat{\mathbf{p}}) = \frac{\sum_{i=1}^k p_i \log p_i - c_k \log c_k}{\sum_{i=1}^k p_i \log r_i} > N,$$

which is a contradiction, since the measure $\mu_{\widehat{\mathbf{p}}}$ is defined in $\mathbf{R}^N$. □

Remark 3.2. In terms of ergodic theory, Lemma 3.1 asserts that the dynamical system $(E, T, \mu_{\mathbf{p}})$ has finite Kolmogorov–Sinai entropy provided the Liapunov exponent of the system is bounded above. An interpretation of (18) from this viewpoint can be seen in [MR].

We will often use the following notation. Let $\mathbf{p} \in \mathscr{P}^+$ and let $(M^\infty, \nu_\mathbf{p})$ be the corresponding product probability space (see (3)). For a random variable Z: $M \mapsto \mathbf{R}$, we call $\{Z_j\}_{j\in\mathbf{N}}$ the associated independent process in M^∞, i.e. for each $j \in \mathbf{N}$, Z_j: $M^\infty \mapsto \mathbf{R}$ is the r.v. defined by

$$Z_j = Z \circ \mathrm{pr}_1 \circ \tau^{j-1},$$

where pr_1: $M^\infty \mapsto M$ is the projection $\mathrm{pr}_1(i_1 i_2 \cdots) = i_1$, and τ is the Bernoulli shift in M^∞. For $\mathbf{i} \in M^\infty$ we write

$$S_k^Z(\mathbf{i}) = \sum_{j=1}^{k} Z_j(\mathbf{i}). \tag{20}$$

The expression $\mathscr{E}[Z]$ means the expectation of Z with respect to the probability $\mathbf{p}$ in M.

Let μ be a Borel measure in $\mathbf{R}^N$. The upper and lower spherical logarithmic densities of μ at $x \in \mathbf{R}^N$ are defined by

$$\overline{\alpha}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu\big(B(x,r)\big)}{\log r}, \qquad \underline{\alpha}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu\big(B(x,r)\big)}{\log r}.$$

We first prove that, for a given $\mathbf{p} \in \mathscr{P}^+$, the value $s(\mathbf{p})$ in (4) is an upper bound for the packing dimension of $\mu_\mathbf{p}$. This is a consequence of the following

Lemma 3.3. *Let $\Psi \in \mathscr{S}(N,M)$, $\mathbf{p} \in \mathscr{P}^+$, and let μ denote the self-similar measure associated with $(\Psi, \mathbf{p})$. Assume that* (17) *holds. Then $\overline{\alpha}_\mu(x) \le s(\mathbf{p})$ μ-a.e.*

Proof. Consider the random variables $W_1(i) = \log p_i$, $W_2(i) = \log r_i$, $i \in M$; and let

$$\mathscr{N} = \Big\{\mathbf{i} \in M^\infty : \lim_{k\to\infty} k^{-1} S_k^{W_j}(\mathbf{i}) = \mathscr{E}[W_j],\ j = 1,2\Big\}. \tag{21}$$

Let $\varepsilon > 0$. For $\mathbf{i} \in \mathscr{N}$, take k_0 such that

$$\max\big\{|k^{-1} S_k^{W_1}(\mathbf{i}) - \mathscr{E}[W_1]|,\ |(k-1)^{-1} S_{k-1}^{W_2}(\mathbf{i}) - \mathscr{E}[W_2]|,\ (k-1)^{-1}\big\} < \varepsilon$$

for all $k > k_0$. Let $0 < r < r_{\mathbf{i}(k_0)}$, and take $k_1 = \min\{k : r_{\mathbf{i}(k)} < r\}$. From the choice of k_1 and part iii) of Theorem 2.1, it follows that

$$\begin{aligned}\frac{\log \mu\big(B\big(\pi(\mathbf{i}), r\big)\big)}{\log r} &\le \frac{\log \mu(E_{\mathbf{i}(k_1)})}{\log r_{\mathbf{i}(k_1-1)}} \\ &= \Big(1 + \frac{1}{k_1 - 1}\Big) \frac{k_1^{-1} S_{k_1}^{W_1}(\mathbf{i})}{(k_1-1)^{-1} S_{k_1-1}^{W_2}(\mathbf{i})} \le (1+\varepsilon) \frac{\mathscr{E}[W_1] + \varepsilon}{\mathscr{E}[W_2] - \varepsilon}.\end{aligned}$$

Since $s(\mathbf{p}) = \mathscr{E}[W_1]/\mathscr{E}[W_2]$, we get $\overline{\alpha}_\mu\big(\pi(\mathbf{i})\big) \le s(\mathbf{p})$ for all $\mathbf{i} \in \mathscr{N}$. Hypothesis (17) and Lemma 3.1 together imply that the strong law of large numbers holds for both W_1 and W_2. Hence $\mu_\mathbf{p}\big(\pi(\mathscr{N})\big) = 1$ and the lemma follows. ◻

We introduce now the 'travelling ball' idea which allows us to obtain a geometric cylinder covering any small given ball, and then to show that $s(\mathbf{p})$ is a lower bound for the Hausdorff dimension of $\mu_{\mathbf{p}}$.

Lemma 3.4 ('Travelling ball' lemma, logarithmic version). *Let $\Psi \in \mathscr{S}(N,M)$, $\mathbf{p} \in \mathscr{P}^+$, and let μ be the self-similar measure associated with $(\Psi,\mathbf{p})$. Assume hypothesis* (17). *Then $s(\mathbf{p}) \le \underline{\alpha}_{\mu}(x)$ μ-a.e., where $s(\mathbf{p})$ is given in* (4).

Proof. Recall that V denotes an open set satisfying the strong OSC for the system Ψ. Thus there exist $\varepsilon > 0$ and $y = \pi(\mathbf{i}) \in E \cap V$ such that $B(y,\varepsilon) \subset V$. We choose a positive integer k_0 large enough so that

$$E_{\mathbf{i}(k_0)} \subset B(y,\varepsilon). \tag{22}$$

The set $E_{\mathbf{i}(k_0)}$ is a cylinder set of the first generation for the system

$$\Psi_0 := \Psi \circ \Psi \circ \overset{(k_0)}{\cdots} \circ \Psi = \{\varphi_{\mathbf{j}} : \mathbf{j} \in M^{k_0}\} \in \mathscr{S}(N, M^{k_0}),$$

which also has the set E as the unique compact $S\Psi_0$-invariant set. Furthermore, if the product probability $\mathbf{p}_0 := \mathbf{p}^{k_0} = \mathbf{p} \times \overset{(k_0)}{\cdots} \times \mathbf{p}$ is considered in M^{k_0}, ν_0 denotes the infinite-fold product measure $\times_1^{\infty}\mathbf{p}_0$, and π_0 is the natural coding map associated with the system Ψ_0 (see (2)), it can be seen that the induced measure $\nu_0 \circ \pi_0^{-1}$ coincides with the self-similar measure $\mu_{\mathbf{p}}$. This follows from a uniqueness argument, since both measures can be shown to be fixed points of the same contractive operator (see [Hut] for this theory).

Since there exists a $\varrho > 0$ such that $d\big(B(y,\varepsilon), \partial V\big) > \varrho$, we have from (22) that

$$d(B, \partial V) > \varrho, \tag{23}$$

where $B := E_{\mathbf{i}(k_0)}$ is a cylinder of the first generation for the system Ψ_0. Notice that the system Ψ_0 satisfies the OSC with open set V. Therefore, given an arbitrary system $\Psi \in \mathscr{S}(N,M)$, it can be assumed that there exists $l \in M$ and a cylinder $B := E_l$ of the first generation for the system Ψ satisfying (23). The geometric cylinder B and the real number $\varrho > 0$ will be fixed throughout the proof.

Recall the definitions of the sets $B_{\mathbf{p}}^{(\infty)}$, $G(i,a)$, and $\mathscr{N}$ given in (9), (12) and (21) respectively. Let $G = B_{\mathbf{p}}^{(\infty)} \cap G(l,1) \cap \pi(\mathscr{N})$ and take $x \in G$. Notice that Theorem 2.1 i) implies that $\pi^{-1}(x) = \mathbf{i} = (i_1, i_2, \ldots)$ is a singleton. Let $q = \min\{n : i_n = l\}$, and let $r > 0$ be such that $r < \varrho r_{\mathbf{i}(q)}$, where $\varrho > 0$ is the constant in (23). Consider the integer

$$k_r = \max\{k : r < \varrho r_{\mathbf{i}(k)}\}. \tag{24}$$

Hence the ('travelling') ball $B\big(T^{k_r}(x),\varrho\big)$ satisfies

$$\varphi_{\mathbf{i}(k_r)}\big(B(T^{k_r}(x),\varrho)\big)=B(x,\varrho r_{\mathbf{i}(k_r)})\supset B(x,r). \tag{25}$$

Since $d\big(T^k(x),\partial V\big)$ could take arbitrarily small values for $x\in B_{\mathbf{p}}^{(\infty)}$, we do not have that $B\big(T^{k_r}(x),\varrho\big)\subset V$ in general. That is why we consider the integer p_r defined by

$$p_r:=p(k_r)=\min\{j:T^{k_r-j}(x)\in B\}. \tag{26}$$

Notice that this choice guarantees that $B\big(T^{k_r-p_r}(x),\varrho\big)\subset V$ because of (23). Thus, from (24), we get

$$\begin{aligned}B(x,r)\subset B(x,\varrho r_{\mathbf{i}(k_r-p_r)})&=\varphi_{\mathbf{i}(k_r-p_r)}\big(B\big(T^{k_r-p_r}(x),\varrho\big)\big)\\ &\subset\varphi_{\mathbf{i}(k_r-p_r)}\big(\mathrm{cl}\,(V)\big)=F_{\mathbf{i}(k_r-p_r)},\end{aligned}$$

so that $\mu\big(B(x,r)\big)\le\mu(F_{\mathbf{i}(k_r-p_r)})$. Since $\varrho r_{\mathbf{i}(k_r+1)}<r$, we obtain

$$\frac{\log\mu\big(B(x,r)\big)}{\log r}\ge\frac{\log\mu(F_{\mathbf{i}(k_r-p_r)})}{\log(\varrho r_{\mathbf{i}(k_r+1)})}. \tag{27}$$

Using the random variables W_i $(i=1,2)$ and part iii) of Theorem 2.1, as we did in the proof of Lemma 3.3, inequality (27) can be written as

$$\frac{\log\mu\big(B(x,r)\big)}{\log r}\ge\left(1-\frac{p_r+1}{k_r+1}\right)\frac{(k_r-p_r)^{-1}S^{W_1}_{k_r-p_r}(\mathbf{i})}{(k_r+1)^{-1}\big(S^{W_2}_{k_r+1}(\mathbf{i})+\log\varrho\big)}, \tag{28}$$

where the notation is that in (20). From the choice of p_r (see (26)), and the definition (11) it follows that $\chi_l\big(T^{k_r-p_r+1}(x)\big)\ge p_r$ for all $x\in E$. Thus, if $p_r\ge1$, we have

$$\frac{\chi_l\big(T^{k_r-p_r+1}(x)\big)}{k_r-p_r+1}\ge\frac{p_r}{k_r}. \tag{29}$$

It can be seen that $k_r-p_r\to+\infty$ as $r\to0$ for $x\in B_{\mathbf{p}}^{(\infty)}$, because otherwise we would obtain that $\liminf_{k\to+\infty}\delta_l\big(\pi^{-1}(x),k\big)=0$ which is a contradiction (see the notation in (7)). Using (29), definition (12), and taking $\liminf$ as $r\to0$, we obtain from (28)

$$\liminf_{r\to0}\frac{\log\mu\big(B(x,r)\big)}{\log r}\ge\frac{\mathscr{E}[W_1]}{\mathscr{E}[W_2]}=s(\mathbf{p}).$$

This proves the theorem, since the Θ-lemma, Proposition 2.5, Lemma 3.1, and the strong law of large numbers imply that the set G has full μ-measure. ▫

Proof of Theorem A. It follows as a consequence of standard results [Tri], [Y] (see also [Cut1], [Cut2]) connecting the logarithmic densities $\underline{\alpha}_\mu$, $\overline{\alpha}_\mu$ of a Borel measure μ with the Hausdorff and packing dimensions of μ. More precisely, the equality $\underline{\alpha}_\mu(x)=\alpha_*$ μ-a.e. implies that $\dim\mu=\alpha_*$, whereas $\overline{\alpha}_\mu(x)=\alpha^*$ μ-a.e. implies that $\mathrm{Dim}\,\mu=\alpha^*$ (see e.g. [Tri, Theorem 1]). Lemma 3.3 and Lemma 3.4 together imply that $\alpha_*=\alpha^*=s(\mathbf{p})$ in our case. ▫

4. Absolute continuity of self-similar measures with respect to the packing measure

We assume that $M = \{1, 2, \ldots, m\}$ throughout this section. Consider $\Psi \in \mathscr{S}(N, M)$, and $\mathbf{p} \in \mathscr{P}^+$. We now address the problem of determining the behaviour of the self-similar measure $\mu_{\mathbf{p}}$ with respect to the packing measure in its dimension, i.e. deciding whether $\mu_{\mathbf{p}}$ is either singular or absolutely continuous w.r.t. $P^{s(\mathbf{p})}$.

We approach the problem by means of local techniques. Given a Borel measure μ and $\psi \in \mathscr{F}$, recall that the standard *lower spherical* ψ*-density* of μ at x is defined by

$$\underline{\theta}_\mu^\psi(x) = \liminf_{r\to 0} \frac{\mu\big(B(x,r)\big)}{\psi(2r)}.$$

Let $\mu \in \mathscr{M}^+$. We define the following *lower cylindrical* ψ*-density* of μ at x

$$\underline{d}_\mu^\psi(x) = \sup\left\{\liminf_{k\to+\infty} \frac{\mu(E_{\mathbf{i}(k)})}{\psi(r_{\mathbf{i}(k)})} : \mathbf{i} \in \pi^{-1}(x)\right\}.$$

Notice that part iii) in Theorem 2.1 implies that

$$\underline{d}_\mu^\psi(x) = \liminf_{k\to+\infty} \nu_{\mathbf{p}}\big([\mathbf{i}(k)]\big)/\psi(r_{\mathbf{i}(k)}) = \liminf_{k\to+\infty} p_{\mathbf{i}(k)}/\psi(r_{\mathbf{i}(k)})$$

for all $x = \pi(\mathbf{i})$ belonging to the set of μ-full measure $E \setminus \Theta$, where $\mathbf{p} \in \mathscr{P}^+$ is the probability μ is associated with. We write $\underline{\theta}_\mu^t(\cdot)$ and $\underline{d}_\mu^t(\cdot)$ when $\psi(\xi) = \xi^t$.

To obtain the main result of this section, the information supplied by the cylindrical densities, which are well-fitted to self-similar constructions, must be translated into geometrical information about spherical densities. We recall that $u := \min\{r_i : i \in M\}$. Let $\mathscr{F}^+ = \{\psi \in \mathscr{F} : \psi(x)\psi(y) \le \psi(xy) \text{ for all } x, y > 0 \text{ small enough}\}$.

Lemma 4.1. *Let* $\Psi \in \mathscr{S}(N, M)$, $\mathbf{p} \in \mathscr{P}^+$, *and let* μ *be the self-similar measure associated with* $(\Psi, \mathbf{p})$. *Let* $\psi \in \mathscr{F}^+$. *Then the inequality*

$$\psi(u/2)\underline{d}_\mu^\psi(x) \le \underline{\theta}_\mu^\psi(x) \tag{30}$$

holds for all $x \in E$.

Proof. The idea of the proof is similar to the one used in the proof of Lemma 3.3. For $x \in E$ and $r > 0$, we take $\mathbf{i} \in \pi^{-1}(x)$ and $k_0 = \min\{j : r_{\mathbf{i}(j)} < r\}$. Since $|V| = 1$, we get that $B(x, r) \supset E_{\mathbf{i}(k_0)}$, so that

$$\frac{\mu\big(B(x,r)\big)}{\psi(2r)} \ge \psi(u/2)\frac{\mu(E_{\mathbf{i}(k_0)})}{\psi(r_{\mathbf{i}(k_0)})},$$

because $r_{\mathbf{i}(k_0)} \geq ur$ and $\psi \in \mathscr{F}^+$. Letting $r \to 0$ and taking the infimum over the codes $\mathbf{i} \in \pi^{-1}(x)$ we obtain (30). □

Let $\mathscr{G} \equiv \mathscr{G}(\Psi, \mathbf{p}) = \{\psi_\alpha : \alpha \in \mathbf{R}\}$ be the one-parameter family of real functions defined by

$$\psi_\alpha(\xi) = \xi^{s(\mathbf{p})} \exp\bigl\{\alpha(2\log \xi^{c(\mathbf{p})} \log\log\log \xi^{c(\mathbf{p})})^{1/2}\bigr\}, \tag{31}$$

where $s(\mathbf{p})$ is given in (4), and $c(\mathbf{p})$ is the negative real number

$$c(\mathbf{p}) = \biggl(\sum_{i\in M} p_i \log r_i\biggr)^{-1}. \tag{32}$$

It can be seen that $\psi_\alpha \in \mathscr{F}^+$ for all $\alpha \leq 0$ (see Appendix). We recall that the definition of the normal Besicovitch set $B_{\mathbf{p}}$ was given in (9). We will need the following lemma, which was proven in [MR].

Lemma 4.2. *For $a < 0$, let f_a be the real variable function*

$$f_a(\xi) = (2\log \xi^a \log\log\log \xi^a)^{1/2}. \tag{33}$$

Then, for every $x \in B_{\mathbf{p}}$ there is an $\mathbf{i} \in \pi^{-1}(x)$ such that

$$\lim_{k\to+\infty} \frac{f_{c(\mathbf{p})}(r_{\mathbf{i}(k)})}{(2k\log\log k)^{1/2}} = 1, \tag{34}$$

where $c(\mathbf{p})$ is the constant defined in (32).

Lemma 4.3 ('Travelling ball' lemma, non-logarithmic version). *Let $\Psi \in \mathscr{S}(N, M)$, $\mathbf{p} \in \mathscr{P}^+ \setminus \{\mathbf{p}_s\}$, and let μ denote the self-similar measure associated with $(\Psi, \mathbf{p})$. Let $\mathscr{G}$ be the family defined in* (31). *Let*

$$d(\mathbf{p}) = \biggl(\sum_{i\in M} \bigl(\log p_i - s(\mathbf{p})\log r_i\bigr)^2 p_i\biggr)^{1/2}. \tag{35}$$

For $\psi_\alpha \in \mathscr{G}$, it holds

$$\text{For } \alpha < -d(\mathbf{p}), \qquad \underline{\theta}_\mu^{\psi_\alpha}(x) = \underline{d}_\mu^{\psi_\alpha}(x) = +\infty, \qquad \mu\text{-a.e.} \tag{36}$$

$$\text{For } \alpha > -d(\mathbf{p}), \qquad \underline{\theta}_\mu^{\psi_\alpha}(x) = \underline{d}_\mu^{\psi_\alpha}(x) = 0, \qquad \mu\text{-a.e.} \tag{37}$$

Proof. Let $X\colon M \mapsto \mathbf{R}$ be the random variable defined by

$$X(i) = \log p_i - s(\mathbf{p})\log r_i, \tag{38}$$

and define the set of codes

$$\mathscr{L} = \left\{ \mathbf{i} \in M^\infty : \liminf_{k\to+\infty} \frac{S_k^X(\mathbf{i})}{(2k \log\log k)^{1/2}} = -d(\mathbf{p}) \right\},$$

where the notation is that of (20). Since $\mathscr{E}[X] = 0$ and $d(\mathbf{p})$ is the standard deviation of X, it follows from the law of the iterated logarithm that

$$\mu\big(\pi(\mathscr{L})\big) = 1. \tag{39}$$

Observe that ψ_α can be written as $\psi_\alpha(\xi) = \xi^{s(\mathbf{p})} \exp\big(\alpha f_{c(\mathbf{p})}(\xi)\big)$, where $f_{c(\mathbf{p})}$ is the function defined in (33), and $c(\mathbf{p})$ is given in (32). In view of Theorem 2.1 part iii), the lower cylindrical ψ_α-density of μ at $x \in E$ satisfies, for any $\mathbf{i} \in \pi^{-1}(x)$,

$$\underline{d}_\mu^{\psi_\alpha}(x) \ge \exp\left(\liminf_{k\to+\infty} \left\{ f_{c(\mathbf{p})}(r_{\mathbf{i}(k)}) \left(\frac{S_k^X(\mathbf{i})}{f_{c(\mathbf{p})}(r_{\mathbf{i}(k)})} - \alpha \right) \right\} \right). \tag{40}$$

Let $\alpha < -d(\mathbf{p})$. It follows from (40), and the asymptotic identity (34) of Lemma 4.2 that $\underline{d}_\mu^{\psi_\alpha}(x) = +\infty$ for $x \in \pi(\mathscr{L})$. The identity in (36) is thus a consequence of Lemma 4.1 and (39).

The proof of (37) follows lines similar to those used in the proof of Theorem 3.4, even though the 'travelling ball' technique must be used here in a slightly different way.

Let $E = B \cup (E \setminus B)$ be the decomposition of the self-similar set E considered in the proof of Theorem 3.4, i.e. $B = E_l$ for some $l \in M$ with $d(B, \partial V) > \varrho$, where V is the open set satisfying the strong OSC for the system Ψ. The set B will be fixed throughout this proof.

Let $G = B_{\mathbf{p}}^{(\infty)} \cap G\big(l, \frac{1}{2}\big) \cap \pi(\mathscr{L})$, where $B_{\mathbf{p}}^{(\infty)}$ is the supernormal Besicovitch set defined in (9), and $G\big(l, \frac{1}{2}\big)$ is defined in (12). Take any $x = \pi(\mathbf{i}) \in G$. Let $q = \min\{j : i_j = l\}$, $k > q$, and choose some $\varepsilon_k > 0$ such that $\varepsilon_k < \varrho r_{\mathbf{i}(k)}$ but $\varepsilon_k \ge \varrho r_{\mathbf{i}(k+1)}$. Consider now, as in the proof of Theorem 3.4, the 'travelling ball' $B\big(T^k(x), \varrho\big)$ so that

$$B(x, \varepsilon_k) \subset \varphi_{\mathbf{i}(k)}\big(B(T^k(x), \varrho)\big).$$

The travelling ball will not in general be contained in the open set V. We thus define

$$p = p(k) = \min\{j : T^{k-j}(x) \in B\}.$$

Since $d(B, \partial V) > \varrho$, we have

$$F_{\mathbf{i}(k-p)} \supset \varphi_{\mathbf{i}(k-p)}\big(B(T^{k-p}(x), \varrho)\big) = B(x, \varrho r_{\mathbf{i}(k-p)}) \supset B(x, \varepsilon_k),$$

and thus part iii) in Theorem 2.1 gives

$$C(k)\mu(E_{\mathbf{i}(k)}) = \mu(E_{\mathbf{i}(k-p)}) \ge \mu\big(B(x, \varepsilon_k)\big), \tag{41}$$

where $C(k) = (p_{i_{k-p+1}} p_{i_{k-p+2}} \cdots p_{i_k})^{-1}$. Let $\alpha < 0$. Since $\varrho u r_{\mathbf{i}(k)} \le \varepsilon_k$, and $\psi_\alpha \in \mathscr{F}^+$, we obtain from (41) that

$$\psi_\alpha(2\varrho u)\frac{\mu\big(B(x,\varepsilon_k)\big)}{\psi_\alpha(2\varepsilon_k)} \le C(k)\frac{\mu(E_{\mathbf{i}(k)})}{\psi_\alpha(r_{\mathbf{i}(k)})}. \tag{42}$$

For $\psi \in \mathscr{F}$, let

$$\underline{\gamma}_\mu^\psi(x) := \liminf_{k\to\infty} C(k)\frac{\mu(E_{\mathbf{i}(k)})}{\psi(r_{\mathbf{i}(k)})}. \tag{43}$$

We will prove now that

$$\underline{\gamma}_\mu^{\psi_\alpha}(x) = \underline{d}_\mu^{\psi_\alpha}(x) = 0 \qquad \text{for } -d(\mathbf{p}) < \alpha < 0. \tag{44}$$

Notice that this also gives that $\underline{\gamma}_\mu^{\psi_\alpha}(x) = 0$ for $\alpha \ge 0$, since ψ_α is an increasing function of α.

Notice that $\underline{\gamma}_\mu^{\psi_\alpha}(x)$ can be written in this case as

$$\underline{\gamma}_\mu^{\psi_\alpha}(x) = \exp\left\{\liminf_{k\to+\infty}\left(f_{c(\mathbf{p})}(r_{\mathbf{i}(k)})\left(\frac{S_k^X(\mathbf{i})}{f_{c(\mathbf{p})}(r_{\mathbf{i}(k)})} - \alpha - \frac{\sum_{j=k-p+1}^k \log p_{i_j}}{f_{c(\mathbf{p})}(r_{\mathbf{i}(k)})}\right)\right)\right\}, \tag{45}$$

where the notation in (20) has been used. Since $x \in \pi(\mathscr{L})$, it follows from Lemma 4.2 that showing

$$\lim_{k\to+\infty} \frac{\sum_{j=k-p+1}^k \log p_{i_j}}{(2k\log\log k)^{1/2}} = 0 \tag{46}$$

proves (44). It can be assumed that $p \ge 1$. The reasoning used in Theorem 3.4 to obtain (29) also applies here, so that (46) follows from inequality

$$\frac{\chi_l\big(T^{k-p+1}(x)\big)}{(k-p+1)^{1/2}} \ge \frac{p}{k^{1/2}},$$

taking into account the estimate $\sum_{j=k-p+1}^k \log p_{i_j} \ge p(\min_{i\in M}\log p_i)$ and definition (12). Therefore the identity in (37) holds for all $x \in G$. The Θ-lemma, Proposition 2.5, and (39) together complete the proof. □

To prove the main result in this section we need Theorem 4.4 below, which provides a local characterization of both the singularity and the absolute continuity of a Borel measure w.r.t. ψ-packing measures. The theorem below stems from the work of C.A. Rogers and S.J. Taylor [RT] along with the density theorem for ψ-packing measures of S.J. Taylor and C. Tricot [TT, Theorem 5.4].

Theorem 4.4 (Rogers–Taylor theorem). *Let* μ *be a finite Borel measure in* $\mathbf{R}^N$*, and* $\psi \in \mathscr{F}$*. Then*

(a) μ *is singular w.r.t.* P^{ψ} *if and only if* $\underline{\theta}^{\psi}_{\mu}(x) = +\infty$ μ*-a.e.*

(b) μ *is absolutely continuous w.r.t.* P^{ψ} *if and only if* $\underline{\theta}^{\psi}_{\mu}(x) < +\infty$ μ*-a.e.*

(c) μ *has an integral representation w.r.t.* P^{ψ} *if and only if* $0 < \underline{\theta}^{\psi}_{\mu}(x) < +\infty$ μ*-a.e.*

Now we are ready to give a proof of

Theorem 4.5. *Let* $\Psi \in \mathscr{S}(N, M)$*,* $\mathbf{p} = (p_i)_{i \in M} \in \mathscr{P}^+$*, and let* $\mathscr{G} = \{\psi_\alpha\}_{\alpha \in \mathbf{R}}$ *be the family defined in* (31)*. Let* $d(\mathbf{p}) > 0$ *be the real number defined in* (35)*. Then, the self-similar measure* $\mu_{\mathbf{p}}$ *induced by the pair* $(\Psi, \mathbf{p})$ *satisfies*

i) $\mu_{\mathbf{p}}$ *is singular w.r.t.* P^{ψ_α} *if* $\alpha < -d(\mathbf{p})$.

ii) $\mu_{\mathbf{p}}$ *is absolutely continuous w.r.t.* P^{ψ_α} *if* $\alpha > -d(\mathbf{p})$.

iii) $\mu_{\mathbf{p}}$ *has an integral representation w.r.t.* P^t *for some* $t > 0$ *if and only if* $\mathbf{p} = \mathbf{p}_s$*, i.e.* $p_i = r_i^s$ *for* $i \in M$ *(and thus* $s = t$*).*

Proof. Part i) (respectively part ii)) above follows from combining part (a) (respectively part (b)) of Theorem 4.4 and identity in (36) (respectively identity in (37)) of Lemma 4.3.

We drop the subindex $\mathbf{p}$ from $\mu_{\mathbf{p}}$. The 'if' implication in part iii) is a consequence of Theorem 3 part ii) in [MR] together with the fact that the measures H^s and P^s coincide up to a constant factor [Haa], [Spe] (see also [MU]). To prove the 'only if' implication, we first prove that the statement of part iii) about integral representability of μ holds only if

$$0 < \underline{d}^t_{\mu}(x) < +\infty \qquad \mu\text{-a.e.} \tag{47}$$

The necessity of the second inequality above immediately follows from Theorem 4.4 part (c) and Lemma 4.1 considering $\psi(\mathrm{x}) = \mathrm{x}^t$ as dimension functions. The necessity of first inequality in (47) essentially follows from the work already done to prove Lemma 4.3. To see this, let $t > 0$ and notice that inequality (42) is also valid for $\psi(\mathrm{x}) = \mathrm{x}^t$. Let $\underline{\gamma}^t_{\mu}(x)$ denote the point function defined in (43) when $\psi(\mathrm{x}) = \mathrm{x}^t$. Using the notation of (20), $\underline{\gamma}^t_{\mu}(x)$ can be written in this case as

$$\underline{\gamma}^t_{\mu}(x) = \exp\left(\liminf_{k \to +\infty} k \left\{ \frac{1}{k} S_k^{X^{(t)}}(\mathbf{i}) - \frac{1}{k} \sum_{j=k+p-1}^{k} \log p_{i_j} \right\}\right),$$

where $X^{(t)}$ is the random variable $X^{(t)} = \log p_i - t \log r_i$. From the Θ-lemma and Lemma 2.5 it can be shown, following the lines of the proof of (44), that $\underline{\gamma}^t_{\mu}(x) > 0$ implies that $\underline{d}^t_{\mu}(x) > 0$ for x in a set of full μ-measure (e.g. taking $x \in B^{(\infty)}_{\mathbf{p}} \cap G(l, 1)$ works). Since $\underline{\theta}^t_{\mu}(x) \le \underline{\gamma}^t_{\mu}(x)$ μ-a.e. (see (42)), part (c) in the

Rogers–Taylor theorem completes the reasoning. Using the strong law of large numbers and the law of the iterated logarithm, as in the proof of Theorem 2 part iii) in [MR], the boundedness condition (47) it is shown to be equivalent to the choice $\mathbf{p} = \mathbf{p}_s$. This proves the theorem. □

As a corollary of Theorem 4.5 notice that the choice $\alpha = 0$ gives $\psi_0(\xi) = \xi^{s(\mathbf{p})}$, and thus ii) implies that the self-similar measure $\mu_{\mathbf{p}}$ is absolutely continuous w.r.t. the packing measure $P^{s(\mathbf{p})}$.

5. Packing geometry of Besicovitch sets

In this section we use results from Sections 3 and 4 to study the packing geometry of normal and supernormal Besicovitch sets. Let $\Psi \in \mathscr{S}(N, M)$, and $\mathbf{p} \in \mathscr{P}^+$. We assume in this section that $M = \{1, 2, \dots, m\}$, except when otherwise stated. Recall that $\mathbf{p}_s$ denotes the probability vector $(r_i^s)_{i \in M} \in \mathscr{P}^+$.

Theorem 5.1. *Let $\Psi \in \mathscr{S}(N, M)$ and $\mathbf{p} \in \mathscr{P}^+$. Let $B_{\mathbf{p}}$ and $B_{\mathbf{p}}^{(\infty)}$ be the normal and supernormal Besicovitch sets defined in* (9)*. Then*

i) $\operatorname{Dim} B_{\mathbf{p}}^{(\infty)} = \operatorname{Dim} B_{\mathbf{p}} = s(\mathbf{p})$.

ii) $0 < P^s(B_{\mathbf{p}_s}^{(\infty)}) = P^s(B_{\mathbf{p}_s}) = P^s(E) < +\infty$.

iii) *If $\mathbf{p} \neq \mathbf{p}_s$, every set with positive $\mu_{\mathbf{p}}$-measure has infinite* (*non-σ-finite*) *$s(\mathbf{p})$-packing measure. In particular*

$$P^{s(\mathbf{p})}(B_{\mathbf{p}}^{(\infty)}) = P^{s(\mathbf{p})}(B_{\mathbf{p}}) = +\infty \qquad (\textit{non-}\sigma\textit{-finite}).$$

Proof. The inequality $\operatorname{Dim} B_{\mathbf{p}}^{(\infty)} \geq s(\mathbf{p})$ follows from (5), (18) and the fact that $\mu_{\mathbf{p}}(B_{\mathbf{p}}^{(\infty)}) = 1$. Let $x \in B_{\mathbf{p}}$. From the definition of $B_{\mathbf{p}}$ there exists an $\mathbf{i}_x \in \pi^{-1}(x)$ such that $\delta_j(\mathbf{i}_x) = p_j$ for all $j \in M$. Taking $k_r := \min\{j : r_{\mathbf{i}_x(j)} < r\}$ and proceeding as in the proof of Lemma 3.3, we get that

$$\overline{\alpha}_{\mu_{\mathbf{p}}}(x) \leq \lim_{r \to 0} \frac{k_r}{k_r - 1} \frac{k_r^{-1} S_{k_r}^{W_1}(\mathbf{i}_x)}{(k_r - 1)^{-1} S_{k_r - 1}^{W_2}(\mathbf{i}_x)} = s(\mathbf{p}),$$

since $\lim_{k \to +\infty} k^{-1} S_k^{W_j}(\mathbf{i}_x) = \mathscr{E}[W_j]$ for $j = 1, 2$. The inequality $\operatorname{Dim} B_{\mathbf{p}} \leq s(\mathbf{p})$ then follows from the work of C. Tricot [Tri, Theorem 1]. This proves i).

Part ii) is a consequence of the uniqueness of the invariant measure associated with the pair $(\Psi, \mathbf{p})$ [Hut]. Since the measures H^s and P^s coincide up to a constant factor in the finite case [Haa], [Spe], [MU], ii) follows from Theorem 3 part ii) in [MR].

We omit $\mathbf{p}$ from $\mu_{\mathbf{p}}$. Let A be a borelian set of positive μ-measure. From Theorem 4.3 and the fact that $\underline{d}_{\mu}^{s(\mathbf{p})} = 0$ μ-a.e. (see (37)), it follows that there is a set $A^* \subset A$ such that $\mu(A) = \mu(A^*)$ and $\underline{\theta}_{\mu}^{s(\mathbf{p})}(x) = 0$ for all $x \in A^*$. Since $\mu(A^*) > 0$, the Taylor–Tricot density theorem [TT, Theorem 5.4] implies that $P^{s(\mathbf{p})}(A^*) = +\infty$. This proves part iii) because the Besicovitch sets are Borel sets of full μ-measure. □

The finiteness of M is essential to obtain the result in part iii) of Theorem 5.1. We are able, however, to obtain the same result for infinite M in some cases. This follows from the following proposition, which is a packing version of Proposition 3.7 in [MR].

Proposition 5.2. *Let* $\Psi \in \mathscr{S}(N, M)$*, with* M *infinite countable. Assume that* $\sum_{i\in M} r_i^s = 1$*. Let* $0 < t < s$*, and* $\psi(\xi) = \xi^t g(\xi) \in \mathscr{F}$ *with* g *non-increasing in some nonempty interval* $(0, \varepsilon)$*. Alternatively, let* $\psi \in \mathscr{F}^+$ *be such that* $\limsup_{\xi\to 0} \log\psi(\xi)/\log\xi < s$*. Then the* ψ*-packing measure of every* $S\Psi$*-invariant set with* $\operatorname{Dim} B > \operatorname{Dim}(B\cap\Theta)$ *is either zero or infinity.*

The proof of Proposition 5.2 is similar to that of Proposition 3.7 in [MR]. From the result above it follows that the normal Besicovitch sets $B_{\mathbf{p}}$ associated with systems in $\mathscr{S}(N, \mathbf{N})$ which intersect the overlapping set Θ in a set of packing dimension strictly less than $s(\mathbf{p})$, have either zero or infinite packing measure in their dimension. In particular, from part i) of Theorem 2.1 it follows that the $s(\mathbf{p})$-packing measure of supernormal Besicovitch sets is either null or infinite.

Remark 5.3. Notice that Theorem 5.1 applies to the classical case of the *Besicovitch–Eggleston sets* [Bes], [Egg]. These are subsets of the unit interval composed of points with given asymptotic frequencies in the figures of their m-base expansion. In particular, these sets (as well as the *supernormal Besicovitch–Eggleston sets*) have infinite packing measure in their dimension. A classical problem remains open, that is to decide whether the Hausdorff measure of the Besicovitch sets in their dimension is either zero or infinity, see [MR].

Appendix

Lemma A1. *Let* $t > 0$*, and* $c < 0$*. Consider the family of real variable functions*

$$\psi_\alpha(\mathrm{x}) = \mathrm{x}^t \exp\bigl(\alpha f_c(\mathrm{x})\bigr), \qquad \alpha \in \mathbf{R}, \tag{48}$$

where $f_c(\mathrm{x}) = (2\log \mathrm{x}^c \log\log\log \mathrm{x}^c)^{1/2}$*. Then* $\psi_\alpha \in \mathscr{F}^-$ *for* $\alpha \geq 0$*, and* $\psi_\alpha \in \mathscr{F}^+$ *for* $\alpha \leq 0$*. (*$\psi_\alpha \in \mathscr{F}^-$ *if* $\psi_\alpha(\mathrm{x})\psi_\alpha(\mathrm{y}) \geq \psi_\alpha(\mathrm{xy})$ *for all* $\mathrm{x}, \mathrm{y} > 0$ *small).*

Proof. The trivial case $\alpha = 0$ can be omitted. Let t, and c be fixed, and the subindex c be dropped from f_c. We write for convenience $f(\mathrm{x}) = h(\log \mathrm{x}^c)$, where

$$h(\xi) = (2\xi\log\log\xi)^{1/2}. \tag{49}$$

Notice that $\lim_{\xi\to+\infty} h(\xi) = +\infty$, $h'(\xi) > 0$ for $\xi > e$, and

$$\lim_{\xi\to+\infty} h'(\xi) = 0^+ \tag{50}$$

(this notation means that the limit is attained from positive values).

It is easy to show that ψ_α is positive and continuous in some interval $(0,\varepsilon)$. Since $t>0$, and $\lim_{\mathrm{x}\to 0} f(\mathrm{x})/\log\mathrm{x}=0$, writing

$$\psi_\alpha(\mathrm{x})=\exp\left(\log\mathrm{x}\left(t+\frac{\alpha f(\mathrm{x})}{\log\mathrm{x}}\right)\right), \tag{51}$$

we obtain $\lim_{\mathrm{x}\to 0^+}\psi_\alpha(\mathrm{x})=0$. Taking derivatives in (48)

$$\psi_\alpha'(\mathrm{x})=\mathrm{x}^{t-1}e^{\alpha f(\mathrm{x})}\big(t+\alpha\mathrm{x}f'(\mathrm{x})\big). \tag{52}$$

Since $f'(\mathrm{x})=c\mathrm{x}^{-1}h'(\log x^c)$ it follows from (50) and (52) that $\psi_\alpha'(\mathrm{x})>0$ for small x. This proves that ψ_α is increasing in a neighbourhood $(0,\varepsilon)$ for $\alpha\in\mathbf{R}$, and thus $\psi_\alpha\in\mathscr{F}$ for $\alpha\in\mathbf{R}$.

Notice that showing $\psi_\alpha\in\mathscr{F}^+$ for $\alpha<0$ concludes the proof. In order to check that for $\alpha<0$ and x, y small enough, inequality $\psi_\alpha(\mathrm{xy})\le\psi_\alpha(\mathrm{x})\psi_\alpha(\mathrm{y})$ holds, it is sufficient to prove that

$$f(\mathrm{xy})\le f(\mathrm{x})f(\mathrm{y}) \tag{53}$$

for x,y in a neighbourhood $(0,\varepsilon)$. It follows from (49) that proving inequality (53) holds is equivalent to showing that

$$h(\xi+\eta)\le h(\xi)+h(\eta) \tag{54}$$

holds for all ξ,η sufficiently large. To prove (54) we first notice that

$$\xi h'(\xi)=\frac{\xi}{h(\xi)}\big((\log\xi)^{-1}+\log\log\xi\big),$$

so that the inequality

$$\xi h'(\xi)<h(\xi) \tag{55}$$

holds for ξ large enough ($\xi>e^2$ suffices). On the other hand, after taking derivatives twice in (49) and some algebra, we obtain

$$h''(\xi)=-\big(h(\xi)^3\log^2(\xi)\big)^{-1}\big(1+2\log\log\xi+(\log\log\xi\log\xi)^2\big),$$

and thus $h''(\xi)<0$ if $\xi>e$. We now prove (54). Assume that $\xi<\eta$. From the intermediate value theorem it follows that

$$h(\xi+\eta)-h(\eta)=h'(z)\xi \tag{56}$$

for some $z\in[\eta,\eta+\xi]$. Since $h'(z)<h'(\xi)$, using inequality (55) we get from (56) that

$$h(\eta+\xi)-h(\eta)\le h'(\xi)\xi\le h(\xi)$$

for ξ,η large enough ($\xi,\eta>e^2$ will suffice). This proves (54), and therefore we are done. □

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