Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 22, 1997, 387–394

CONSERVATIVE ACTION AND THE HOROSPHERIC LIMIT SET

Pekka Tukia

University of Helsinki, Department of Mathematics
P.O. Box 4 (Yliopistonkatu 5), FIN-00014 Helsinki, Finland; pekka.tukia@helsinki.fi

Abstract. D. Sullivan [4] showed that a discrete Möbius group G of S^n acts conservatively on the horospheric limit set $\mathscr{H}(G)$ of G, that is if $A \subset \mathscr{H}(G)$ has positive measure, then so has $A \cap gA$ for infinitely many g in G. On the other hand, G has a measurable fundamental set outside $\mathscr{H}(G)$. We show that if the definition of the horospheric limit set slightly changed, these results are also valid for any conformal measure μ of G.

Let X be a topological space with a finite Borel measure μ and let G be a countable group of measurable, absolutely continuous bijections of X. If $A \subset X$ is setwise G-invariant, then the action of G is *conservative* on A (sometimes the word "recurrent" is also used) if, whenever $U \subset A$ is measurable and $\mu(U) > 0$, then $\mu(U \cap gU) > 0$ for infinitely many $g \in G$. In contrast, the action is *dissipative* on A if there is a measurable fundamental set for the action of G in A, that is there is a measurable set $F \subset A$ such that F contains exactly one point from each orbit Gz, $z \in A$. Conservative and dissipative action are not very compatible: if G acts conservatively and no point of A is fixed by some $g \neq \mathrm{id}$ in G, then there cannot be a measurable fundamental set for the action on A if $\mu(A) > 0$. Under fairly general conditions, it is possible to divide X into measurable disjoint pieces A and B, $X = A \cup B$, such that the action is conservative on A and dissipative on B. This result seems to be generally known but the only reference I know is [1] whose proof works if the set of points $x \in X$ fixed by some $g \neq \mathrm{id}$ in G is a nullset.

Let G be a discrete group of Möbius transformations of the n-sphere S^n and let m be the n-dimensional Hausdorff measure on S^n. In this case Sullivan [4] identified the conservative part as the horospheric limit set $\mathscr{H}(G)$. Recall that a Möbius group of S^n also acts on the open unit $(n+1)$-ball B^{n+1} and that a *horoball* at $x \in S^n$ is an open $(n+1)$-ball B properly contained in B^{n+1} such that ∂B is tangent to S^n at x. Now we can define that a point $x \in S^n$ is in the *horospheric limit set* $\mathscr{H}(G)$ if any horoball at x contains an infinite number of points from any orbit Gz, $z \in B^{n+1}$.

1991 Mathematics Subject Classification: Primary 30F40; Secondary 28D99.

We will use the following variation of the horospheric limit set: We define the *big horospheric limit set* $\mathscr{H}_b(G)$ as the set of points $x \in S^n$ such that, given $z \in B^{n+1}$, then there is some horoball B at x such that $B \cap Gz$ is infinite.

We will extend Sullivan's result for general conformal measures. A conformal (G-)measure of dimension δ on S^n is a finite Borel measure satisfying the transformation rule

$$\mu(gX) = \int_X |g'|^\delta \, d\mu$$

for $g \in G$ and Borel subsets X of S^n; here δ is the *dimension* of μ. Conformal measures are sometimes called Patterson–Sullivan measures and their general theory was developed in [5] (cf. also Nicholls [3]). Typically, they are supported by the limit set $L(G)$ of G but of course we can extend by zero to all of S^n. For instance, the Hausdorff n-measure of S^n is a conformal measure and, more generally, if $A \subset S^n$ is G-invariant and the δ-dimensional Hausdorff-measure m_δ of A is finite, then m_δ is a conformal measure.

Sullivan extended a method of Patterson in [5] and showed that there is a non-trivial conformal measure supported by $L(G)$ for any infinite discrete Möbius group; the dimension of this measure is the exponent of convergence δ_G of G. The measure is unique in many cases, and for the so-called convex cocompact groups the δ_G-dimensional Hausdorff measure of $L(G)$ is a conformal measure. Using conformal measures, it is possible to extend the measure theoretical aspects of the theory of Möbius groups of first kind (i.e. $L(G) = S^n$) to the groups of the second kind (i.e. $L(G) \neq S^n$).

We claim that the conservative part is the big horospheric limit set $\mathscr{H}_b(G)$. The ideas of the proof are much the same as in [4] but there are some complications, for instance we had to move to the big horospheric limit set.

The set $\mathscr{G} = \mathscr{H}_b(G) \setminus \mathscr{H}(G)$ is very similar to the "Garnett points" of [4] since, if $x \in \mathscr{G}$ and $z \in B^{n+1}$, there is a unique horoball B_x at x such that $Gz \cap B$ is finite if B is any smaller horoball at x and infinite if B is any larger horoball at x; in contrast, we recall that x is a Garnett point if $x \in \mathscr{G}$ and any smaller horoball $B \subset B_x$ does not contain any points of Gz. We denote the set of Garnett points by $\mathscr{G}_0$; unlike $\mathscr{H}(G)$ and $\mathscr{H}_b(G)$ it might depend on the orbit Gz but we assume a fixed reference point z and define $\mathscr{G}_0$ using it.

Part of Sullivan's proof (due to John Garnett) of [4] was to show that the n-dimensional Hausdorff measure m of $\mathscr{G}_0$ vanishes; the same proof would also show that $m(\mathscr{G}) = 0$. So Sullivan's original theorem could have been formulated as well using the big horospheric limit set. Unfortunately, the proof of the null-measure of Garnett points does not seem to generalize for general conformal measures (although it is possible to prove that $\mu(\mathscr{G} \setminus \mathscr{G}_0) = 0$, see the Appendix).

We recall that a Möbius transformation g of S^n can be classified as elliptic (including the identity map), loxodromic or parabolic: g is elliptic if g can be conjugated by a Möbius transformation of S^n onto $\bar{R}^n$ to an orthogonal linear map of R^n, g is loxodromic if g can be conjugated to a map of the form $x \mapsto \lambda\beta$ where $\lambda > 1$ and β is an orthogonal linear map, and g is parabolic if g can be conjugated to a map of the form $x \mapsto \beta(x) + a$ where $a \in R^n$ and β is an orthogonal linear map such that $\beta(a) = a$. We consider here only non-elementary Möbius groups defined as discrete Möbius groups such that the limit set contains more than two points.

Our aim is to prove

Theorem 1. *Let G be a non-elementary Möbius group on S^n and let μ be a conformal G-measure on S^n. Then the action of G is conservative on the big horospheric limit set $\mathscr{H}_b(G)$ and dissipative on $S^n \setminus \mathscr{H}_b(G)$. If $\mathscr{C}$ and $\mathscr{D}$ are G-invariant subsets of S^n and G acts conservatively on $\mathscr{C}$ and dissipatively on $\mathscr{D}$, then $\mathscr{D} \subset S^n \setminus \mathscr{H}_b(G)$ and $\mathscr{C} \subset \mathscr{H}_b(G)$ up to μ-nullsets and atoms at parabolic fixed points.*

Remark. So we have found the dissipative and conservative parts and they are unique up to nullsets and atoms at parabolic fixed points.

Note that a loxodromic fixed point cannot be an atom of μ since the dimension δ of μ is positive for non-elementary G ([5]) and since $|g'(v)| \neq 1$ at a fixed point v of a loxodromic g. Thus the stabilizer $G_v = \{g \in G : g(v) = v\}$ of an atom of μ can contain only parabolic and elliptic elements and if G_v is purely elliptic, it is finite by [6, Corollary C] and in this case v must be put into the dissipative part. Consequently, if G_v is infinite and v is an atom, then G_v must contain parabolic elements and obviously the orbit Gv can be put at will either to the conservative or the dissipative part.

We start the proof of Theorem 1 with the following characterization $\mathscr{H}_b(G)$. Here $|g'(z)|$ is the operator norm of the derivative. The same proof shows that $z \in \mathscr{H}(G)$ if and only if there are $g_i \in G$ such that $|g_i'(z)| \to \infty$, cf. (3).

Lemma 2. *A point $z \in S^n$ is in the big horospheric limit set $\mathscr{H}_b(G)$ if and only if there are $m > 0$ and a sequence of distinct $g_i \in G$ such that $|g_i'(z)| \geq m$.*

Proof. We will prove that if $g_i \in G$ are distinct, then $g_i^{-1}(0)$ are in some horoball at z if and only if there is $m > 0$ such that $|g_i'(z)| \geq m$.

Suppose that $h_i \in G$ are distinct. Then $|h_i(0)| \to 1$ by discreteness and a simple calculation shows that setting $d_i = 1 - |h_i(0)|$, then $h_i(0)$ are in some horoball at z if and only if

$$|z - h_i(0)| \leq a\sqrt{d_i} \tag{1}$$

for some suitable positive constant a.

Let $g_i = h_i^{-1}$. We claim that (1) is true for some $a > 0$ if and only if

$$|g_i'(z)| \geq m \tag{2}$$

for some $m > 0$.

The plan of the proof is simple. We first show that $|g_i'| = |\sigma_i'|$ when σ_i is the reflection on the isometric sphere S_i of g_i. Let τ_i be the center and r_i the radius of the isometric sphere S_i. We show that r_i is approximately $\sqrt{2d_i}$. Remembering that $|\sigma_i'(w)| = r_i^2|w-\tau_i|^{-2}$, to conclude that (1) and (2) are equivalent we will show that $|z-h_i(0)|$ is approximately $|z-\tau_i|$.

Let L_i be the hyperbolic line segment joining 0 and $h_i(0)$. The isometric sphere of g_i is the n-sphere S_i orthogonal to S^n such that S_i intersects L_i orthogonally at the hyperbolic midpoint c_i of L_i. We can see this as follows. Let σ_i be the reflection on S_i. Then $\sigma_i h_i(0) = 0$ and both σ_i and h_i preserve S^n. However, the only Möbius transformations preserving 0 and S^n are orthogonal linear maps of R^n. Thus $|(\sigma_i h_i)'| = 1$. Since also $|(g_i h_i)'| = 1$, it follows that $|g_i'| = |\sigma_i'|$, and so we can use σ_i in (2) instead of g_i. In particular, S_i is the isometric sphere of g_i, i.e. the n-sphere S such that $|g_i'| = 1$ on S.

Let the hyperbolic metric of B^{n+1} be given by the element of length $2|dx|/(1-|x|^2)$. Then the hyperbolic distance is $d(0,x) = \log\bigl(1+|x|)/(1-|x|)\bigr)$. Thus

$$\log\frac{1+|c_i|}{1-|c_i|} = \frac{1}{2}\log\frac{1+|h_i(0)|}{1-|h_i(0)|}$$

implying

$$1-|c_i| = a_i\sqrt{1-|h_i(0)|} = a_i\sqrt{d_i}$$

where $a_i \to \sqrt{2}$ as $i\to\infty$. We also obtain

$$|c_i - h_i(0)| = |h_i(0)| - |c_i| = (1-|c_i|) - (1-|h_i(0)| = a_i'\sqrt{d_i}$$

where $a_i' \to \sqrt{2}$ as $i\to\infty$.

The center τ_i of the isometric circle is mapped by σ_i onto ∞ and so $|\sigma_i'(\tau_i)| = |g_i'(\tau_i)| = \infty$ and so also $g_i(\tau_i) = \infty$. Let ϱ be the reflection on S^n. Then ϱ interchanges 0 and ∞ and commutes with Möbius transformations preserving S^n. Thus $0 = \varrho g_i(\tau_i) = g_i\varrho(\tau_i)$ and so $\varrho(\tau_i) = h_i(0)$. Since $|h_i(0)| \to 1$,

$$|h_i(0) - \tau_i| = b_i\bigl(1-|h_i(0)|\bigr) = b_i d_i$$

where $b_i \to 2$ as $i\to\infty$. It follows that the radius of the isometric circle

$$r_i = |c_i - \tau_i| = |c_i - h_i(0) + h_i(0) - \tau_i| = a_i''\sqrt{d_i}$$

where $a_i'' \to \sqrt{2}$ as $i\to\infty$. Similarly,

$$|z-\tau_i| = |z - h_i(0) + h_i(0) - \tau_i| = |z-h_i(0)| + \delta_i$$

where $|\delta_i| \le b_i d_i$.

We can now estimate the derivative

$$|g_i'(z)| = |\sigma_i'(z)| = \frac{r_i^2}{|z-\tau_i|^2} = \frac{(a_i'')^2 d_i}{\bigl(|z-h_i(0)|+\delta_i\bigr)^2}. \tag{3}$$

The conclusion is now immediate. If (1) is true, then (2) is true for big i with $m = a^{-2}$. If (2) is true, then (1) is true for big i with $a = 2/\sqrt{m}$.

We return to the proof of the theorem. If x is an atom of μ, then by the above Remark, x cannot be fixed by a loxodromic $g \in G$ and if x is fixed by a parabolic $g \in G$, then we can put the orbit of x at will either to the conservative or to the dissipative part. Consequently, to prove the theorem we can assume that no atom of μ is fixed by an element of G of infinite order. Under this assumption, we will prove:

A. $S^n \setminus \mathscr{H}_b(G)$ has a measurable fundamental set.

B. The action of G is conservative on $\mathscr{H}_b(G)$.

C. If $\mathscr{C} \subset S^n$ is G-invariant and G acts conservatively on $\mathscr{C}$, then $\mathscr{C} \subset \mathscr{H}_b(G)$ up to a nullset.

D. If $\mathscr{D} \subset S^n$ is G-invariant and G is dissipative on $\mathscr{D}$, then $\mathscr{D} \subset S^n \setminus \mathscr{H}_b(G)$ up to a nullset.

Proof of A. Denote $\mathscr{D} = S^n \setminus \mathscr{H}_b(G)$. If $z \in \mathscr{D}$, then $\Gamma_z = \{g \in G : |g'(z)| \geq |h'(z)| \text{ for all } h \in G\}$ is finite and non-empty by Lemma 2. Thus the set

$$F_0 = \{x \in \mathscr{D} : |f'(x)| \leq 1 \text{ for all } f \in G\} \tag{4}$$

which is also the union $\bigcup_{z \in \mathscr{D}} \Gamma_z z$, is a measurable set containing a finite and non-zero number of points from each orbit Gz, $z \in \mathscr{D}$. It is not difficult to extract a measurable fundamental set from F_0, for instance as follows. Let $\{e_i\}_{i>0}$, be a dense set of S^n and define measurable $F_1 \subset F_0$ by

$$F_1 = \{x \in F_0 : |x - e_1| \leq |z - e_1| \text{ for all } z \in Gx \cap F_0\}.$$

Even this may fail to give a fundamental set but we continue and set $F_2 = \{x \in F_1 : |x - e_2| \leq |z - e_2| \text{ for all } z \in Gx \cap F_1\}$, etc. The intersection $\bigcap_i F_i$ will be a measurable fundamental set for $\mathscr{D}$.

Proof of B. Unless the action of G on $\mathscr{H}_b(G)$ is conservative, there is a set $A \subset \mathscr{H}_b(G)$ of positive measure such that $\mu(A \cap gA) > 0$ for only finitely many $g \in G$. Applying Lemma 4 below, we can find measurable $B \subset A$ of positive measure and a finite subgroup $H \subset G$ such that the stabilizer $G_x = \{g \in G : g(x) = x\} = H$ for every $x \in B$ and that $gB \cap hB = \emptyset$ if $g, h \in G$ and $gH \neq hH$. In view of Lemma 2, and possibly by making B smaller but still of positive measure, we can assume that there is $m > 0$ such that, for each $z \in B$, $|g'(z)| \geq m$ for an infinite number of $g \in G$. It follows that we can find distinct $g_{11}, \dots, g_{1n_1} \in G$ such that if

$$E_1 = \{z \in B : |g'_{1i}(z)| \geq m \text{ for some } i,\ 1 \leq i \leq n_1\},$$

then $\mu(E_1) > \frac{1}{2}\mu(B)$. Since $|g'(z)| \geq m$ for an infinite number of g's, we can continue and find for each k elements $g_{k1}, \dots, g_{kn_k} \in G$ for $i \leq n_k$ such that $g_{ki} = g_{pr}$ only if $k = p$ and $i = r$ and such that if

$$E_k = \{z \in B : |g'_{ki}(z)| \geq m \text{ for some } i,\ 1 \leq i \leq n_k\}$$

then $\mu(E_k) > \frac{1}{2}\mu(B)$. Let $E_{ki} = \{z \in B : |g'_{ki}(z) \geq m\}$ so that $E_k = \bigcup_i E_{ki}$.

We can now estimate when $\#H$ is the number of elements of H

$$\infty > \mu\biggl(\bigcup_{g\in G} gB\biggr) = \frac{1}{\#H}\sum_{g\in G}\mu(gB) \geq \frac{1}{\#H}\sum_{k,i}\mu(g_{ki}B)$$
$$\geq \frac{1}{\#H}\sum_{k,i}\mu(g_{ki}E_{ki}) \geq \frac{1}{\#H}\sum_{k,i} m\mu(E_{ki}) \geq \frac{m}{\#H}\sum_{k}\mu(E_k) = \infty$$

since $\mu(E_k) \geq \frac{1}{2}\mu(B) > 0$. This contradiction implies B.

Proof of C. Suppose that G is conservative on $\mathscr{C}$ and that $\mu(\mathscr{C}\setminus\mathscr{H}_b(G)) > 0$. We derive a contradiction from this.

By A, there is a fundamental set F for $S^n \setminus \mathscr{H}_b(G)$. Replacing F by some gF, $g \in G$, we can assume that $\mu(\mathscr{C}\cap F) > 0$. Let $A = \mathscr{C}\cap F$. By Lemma 3 below, there are $B \subset A$ and a finite subgroup $H \subset G$ such that $\mu(B) > 0$ and that $g(x) \neq x$ if $x \in B$ and $g \in G \setminus H$. However, if $x \in B \cap gB$ for some $g \in G$, then x must be fixed by g since B is a subset of the fundamental set. Hence g is in the finite set H, contradicting the conservativity in $\mathscr{C}$.

Proof of D. Suppose that G is dissipative on $\mathscr{D}$ and $Z = \mathscr{H}_b(G)\cap\mathscr{D}$ has positive μ-measure. Since the action is dissipative on $\mathscr{D}$ and hence on Z, there is a fundamental set F_1 for Z. Then $\mu(F_1) > 0$ and hence by B, $\mu(F_1\cap gF_1) > 0$ for infinitely many $g \in G$. Pick one such g_1 and set $F_2 = F_1 \cap g_1F_1$. Find then $g_2 \neq g_1$ such that $F_3 = F_2 \cap g_2F_2$ has positive μ-measure. Continuing in this manner we find a sequence of distinct $g_i \in G$ and a decreasing sequence $F_1 \supset F_2 \supset \cdots$ of sets of positive μ-measure such that $\mu(F_i \cap g_iF_i) > 0$. Since F_i are subsets of the fundamental set F_1, $F_i \cap g_iF_i$ consists of fixed points of g_i. If g_i is of infinite order, then g_i fixes at most two points and neither of them is an atom by our assumptions. It follows that each g_i is elliptic and similarly, the group G_i generated by $g_1,\dots,g_i$, all of whose elements fix points of F_i, must be purely elliptic. Thus $\bigcup_i G_i$ would be a purely elliptic infinite group and this is impossible by [6].

We still need to prove two lemmas. As above μ is a conformal measure such that no parabolic or loxodromic fixed point is an atom of μ.

Lemma 3. *Let $A \subset S^n$ be measurable with positive μ-measure. Then there are a finite subgroup $H \subset G$ and measurable $B \subset A$ of positive measure such that $g(x) = x$ for every $x \in B$ and $g \in H$ but that $g(x) \neq x$ if $g \in G \setminus H$.*

Proof. Remove first all fixed points of loxodromic or parabolic elements of G. Since these points are not atoms of μ and their number is countable, still $\mu(A) > 0$. Thus if $x \in A$ and $g(x) = x$ for some $g \in G$, then g is elliptic.

In the following $X_F = \{x \in S^n : g(x) = x \text{ for all } g \in F\}$ when $F \subset G$ is a subgroup. We claim that there is a maximal finite subgroup $H \subset G$ such that $\mu(X_H \cap A) > 0$. If there is no such maximal group, we can find a sequence $H_1 \subset H_2 \subset \cdots$ with proper inclusions of finite subgroups of G such that $\mu(X_{H_i} \cap A) > 0$. Now $H = \bigcup_i H_i$ is a purely elliptic discrete subgroup and hence, by [6, Corollary C], H is finite. This is a contradiction and hence there is such a maximal H.

Let $B = X_H \cap A$. If $g \in G \setminus H$ and $x \in B$ and $g(x) = x$, then the group F generated by g and H is purely elliptic (since x is not a parabolic or loxodromic fixed point) and hence F is finite as we saw. So $\mu(X_F \cap B) = 0$ by maximality of H. Removing a countable number of nullsets of this form, we obtain the set B with the required properties.

Lemma 4. *Suppose that $A \subset S^n$ is of positive μ-measure and that $\mu(A \cap gA) > 0$ for only finitely many $g \in G$. Then there are a finite subgroup $H \subset G$ and a measurable subset $B \subset A$ of positive μ-measure such that $g(x) = x$ for all $x \in B$ and $g \in H$ but that $gB \cap B = \emptyset$ for all $g \in G \setminus H$.*

Proof. Let $B \subset A$ and $H \subset G$ be the set and subgroup given by Lemma 3. Since G is countable and each $g \in G$ is absolutely continuous with respect to μ, we can obtain by removing a countable number of nullsets that $\mu(B \cap gB) > 0$ if and only if $B \cap gB \neq \emptyset$ whenever $g \in G$.

Let $G_0 = \{g \in G : \mu(B \cap gB) > 0\} = \{g \in G : B \cap gB \neq \emptyset\}$. It is a finite union of cosets gH. Choose representatives $g_1, \dots, g_k$ from each coset. Pick $x \in B$ such that $\mu(U) > 0$ for every neighborhood U of x in B. Now $g_i(x)$, $i \leq k$, are all distinct and hence it is possible to choose a neighborhood U of x such that $g_i U$ are disjoint. If $U \cap gU \neq \emptyset$, then $g \in G_0$ and hence $gU = g_i U$ for some i. This is possible only if $g \in H$ and so $U \cap gU = \emptyset$ for all $g \in G \setminus H$. Thus the lemma is true if U is substituted for B.

Remark. It may be useful to see to what extent the proof depends on special properties of Möbius transformations. Proof of A works generally. Let X be a separable metric space, μ a finite Borel regular measure on X and G a countable group of absolutely continuous measurable bijections of X; we need the Radon–Nikodym derivative for $g \in G$, and hence we still need to assume, for instance, that that the family of balls of X satisfy the assumptions of the Vitali covering theorem [2, 2.8.16]. Interpret $|g'(x)|$ as the Radon–Nikodym derivative of g and the formula (4) gives a measurable set F_0 containing a finite and nonzero number from each orbit Gz when $z \in \mathscr{D}$ and $\mathscr{D}$ is the set of $x \in X$ such that the Radon–Nikodym derivative $|g'(x)|$ exists for all $g \in G$ and $E_m = \{g \in G : |g'(x)| \geq m\}$ is finite for every $m > 0$; this latter condition could even be relaxed to the form that E_m is finite and non-empty for some $m > 0$. The proof of A is valid and extracts a measurable fundamental set from F_0.

In contrast, to show that the action is conservative on $X \setminus \mathscr{D}$ and that the division to conservative and dissipative parts is unique up to nullsets, requires in addition two more specific assumptions: The set of points fixed by some $g \in G$ of infinite order is a nullset, and every subgroup of G consisting of elements of finite order is finite. With these assumptions our proof is valid.

Appendix. We now give the promised proof that $\mu(\mathscr{G} \setminus \mathscr{G}_0) = 0$.

Let $Z = \mathscr{G} \setminus \mathscr{G}_0$. We will show that the action is dissipative on Z. It follows by Theorem 1 that Z is up to a nullset a union of atoms at parabolic fixed points. However, a parabolic fixed point is not a point of Z as one easily sees.

We use here the method in Sullivan's paper [4]. Notice that if $x \in Z$, then there is a unique horoball B_x at x such that $Gz \cap \partial B_x \neq \emptyset$ but that $Gz \cap B_x$ is non-empty and finite; here z is the reference point used to define $\mathscr{G}_0$. Let $\Gamma_x = \partial B_x \cap Gz$ which is finite and non-empty. If $F \subset G$ is finite and non-empty, set

$$D_F = \{x \in Z : \Gamma_x = F\}.$$

The sets D_F, as F varies over finite and non-empty subsets of G, form a countable family of measurable, mutually disjoint sets covering Z. Obviously, $g(D_F) = D_{gF}$ and hence it is possible to find a countable family $\mathscr{F}$ of finite and non-empty subsets of G such that the countable union $\cup\{D_F : F \in \mathscr{F}\}$ will be a measurable fundamental set for Z.

References

[1] Agard, S.: Mostow rigidity on line: A survey. - In: Holomorphic Functions and Moduli II, edited by D. Drasin et al., MSRI Publications 11, Springer-Verlag 1988, 1–12.

[2] Federer, H: Geometric Measure Theory. - Springer-Verlag, 1969.

[3] Nicholls, P.J.: The Ergodic Theory of Discrete Groups. - London Math. Soc. Lecture Note Ser. 143, Cambridge University Press, 1989.

[4] Sullivan, D.: On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions. - In: Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference, edited by I. Kra and B. Maskit, Ann. of Math Studies 97, Princeton University Press, 1981, 465–496.

[5] Sullivan, D.: The density at infinity of a discrete group of hyperbolic motions. - Inst. Hautes Études Sci. Publ. Math. 50, 1979, 171–202.

[6] Waterman, P.L.: Purely elliptic Möbius groups. - In: Holomorphic Functions and Moduli II, edited by D. Drasin et al., MSRI Publications 11, Springer-Verlag, 1988, 173–178.

Received 18 January 1996