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UNIVALENCE CRITERIA FOR CLASSES OF RECTANGLES AND EQUIANGULAR HEXAGONS

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Abstract. Let D be a simply connected plane domain and let B be the unit disk. The inner radius of D, $\sigma(D)$, is defined by

$$\sigma(D) = \sup\{a : a \geq 0,\ \|S_f\|_D \leq a \text{ implies } f \text{ is univalent in } D\}.$$

Here S_f is the Schwarzian derivative of f, ϱ_D the hyperbolic density on D and $\|S_f\|_D = \sup_{z\in D} |S_f(z)|\varrho_D^{-2}(z)$. Domains for which the value of $\sigma(D)$ is known include disks, angular sectors and regular polygons. All of the mentioned domains except non-convex angular sectors have an interesting property in common, namely that $\sigma(D) = 2 - \|S_h\|_B$, where h maps B conformally onto D. Because of the importance of this property, we say that D is a Nehari disk if $\sigma(D) = 2 - \|S_h\|_B$ is satisfied.

First we use the definition of a Nehari disk to give a new proof of the result on regular n-sided polygons P_n due to D. Calvis. Next we study rectangles and equiangular hexagons. We prove that if R is a rectangle whose ratio of longer over shorter side is bounded from above by a specific constant ($\cong 1.52346\ldots$), then R is a Nehari disk and $\sigma(R) = 1/2 = \sigma(P_4)$, and if H is an equiangular hexagon whose sides form the sequence *baabaa* with $b/a \leq 1.67117\ldots$, then H is a Nehari disk and $\sigma(H) = 8/9 = \sigma(P_6)$.

1. Introduction

The inner radius of a domain is a constant frequently used in the study of univalence criteria for analytic functions on a domain. This paper is devoted to studying some values and properties of the inner radius.

We use the symbol $\mathbf{C}$ to denote the complex plane and $\overline{\mathbf{C}}$ to denote the extended complex plane. Within $\overline{\mathbf{C}}$, we use the symbol B to refer to the unit disk ($B = \{z : |z| < 1\}$) and U for the upper half-plane ($U = \{z : \mathrm{Im}(z) > 0\}$). The symbol D will denote a domain in $\overline{\mathbf{C}}$ with at least two points on its boundary.

For $z \in B$, the *hyperbolic density* of B at z is the quantity $\varrho_B(z)$ given by $\varrho_B(z) = 1/(1 - |z|^2)$. For a general simply connected domain D, hyperbolic density ϱ_D is then defined in terms of ϱ_B and $h\colon B \to D$ where h maps B conformally onto D (see [10, p. 5]).

Next we recall an operator on locally univalent meromorphic functions, known as the Schwarzian derivative. If f is holomorphic in $D \subset \mathbf{C}$, with $f'(z) \neq 0$ for $z \in$

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D, the *Schwarzian derivative* S_f, of f, is defined in D by $S_f(z) = (f''/f')'(z) - \frac{1}{2}(f''/f')^2(z)$. This definition can easily be extended to include locally univalent meromorphic functions. A detailed explanation of the extended definition can be found in [10, p. 52]. To make our terminology more concise, locally univalent meromorphic functions will be referred to simply as locally univalent functions.

An important fact about the Schwarzian derivative is that, in a simply connected domain, it can be prescribed, i.e., given a holomorphic function φ in a simply connected domain D, there is a locally univalent function f in D such that $S_f = \varphi$ (see [10, p. 53]).

In order to discuss a univalence criterion for f we introduce a norm for S_f. Let D be a simply connected domain in $\overline{\mathbf{C}}$. For φ holomorphic in D, we define the *hyperbolic norm* of φ with respect to D by $\|\varphi\|_D = \sup_{z\in D} |\varphi(z)|\varrho_D^{-2}(z)$.

In particular, we will be concerned with $\|S_f\|_D = \sup_{z\in D} |S_f(z)|\varrho_D^{-2}(z)$, where f is locally univalent on D. We list some properties of $\|S_f\|_D$ (see [11, p. 4]):

(i) $\|S_f\|_D = 0$ if and only if f is a Möbius transformation;
(ii) $\|S_{f\circ g}\|_D = \|S_f - S_{g^{-1}}\|_{g(D)}$ when g is conformal on D;
(iii) $\|S_f\|_D = \|S_{f^{-1}}\|_{f(D)}$ when f is conformal on D;
(iv) $\|S_f\|_D = \|S_{\lambda\circ f\circ\mu}\|_{\mu^{-1}(D)}$ when λ and μ are Möbius transformations.

Suppose D is some fixed simply connected domain in $\overline{\mathbf{C}}$. If f is locally univalent on D and $\|S_f\|_D = 0$, then f is Möbius and hence univalent in D. Therefore it seems natural to consider constants $a \geq 0$ for which $\|S_f\|_D \leq a$ implies f is univalent on D. This brings us to the following definition of the inner radius of a domain.

Suppose D is a simply connected domain in $\overline{\mathbf{C}}$. We define the *inner radius* of D, $\sigma(D)$, by

$$\sigma(D) = \sup\{a : a \geq 0,\ \|S_f\|_D \leq a \text{ implies } f \text{ is univalent in } D\}.$$

The sup in the definition of $\sigma(D)$ can be replaced by max (see [10, p. 118]). Moreover, property (iv) implies that images of D under Möbius transformations have the same inner radius as D.

Nehari [12] and Hille [7] proved that $\sigma(B) = 2$. Later, Lehtinen showed in [8] that $\sigma(D) \leq 2$ for all simply connected domains in $\overline{\mathbf{C}}$ with equality occurring only when D is a disk in $\overline{\mathbf{C}}$ (i.e., an image of B under a Möbius transformation). In view of these results, the question that arises naturally is whether there are any domains D with $\sigma(D) = 0$. This question was answered by Ahlfors [1] and Gehring [5], who proved that when D is a simply connected domain, $\sigma(D) > 0$ if and only if D is quasidisk.

Finally, we list some known values of $\sigma(D)$. Let S denote the parallel strip defined as the image of U under $h(z) = \log z$. Since S is not a Jordan domain, it is not a quasidisk. Hence $\sigma(S)=0$. Lehto and Lehtinen have calculated the inner radii of angular sectors in [9] and [8]. If $A_k=\{z : z\in \mathbf{C},\ 0<\arg z<k\pi\}$ then,

$$\sigma(A_k)=\begin{cases} 2k^2 & \text{if } 0<k\le 1\ , \\ 4k-2k^2 & \text{if } 1<k<2 \end{cases}$$

for $0<k<2$. Another class of domains for which the inner radii have been calculated are regular polygons. Calvis [3], proved that $\sigma(P_n)=2(n-2)^2/n^2$ where P_n is an open regular n-sided polygon.

The section that follows will be devoted to demonstrating some elementary properties of $\sigma(D)$.

2. Elementary properties of $\sigma(D)$

We begin by stating a simple lemma which gives a natural lower bound for the inner radius of a domain.

Lemma 2.1. *If D is a simply connected domain and if h maps B conformally onto D, then $\sigma(D)\ge 2-\|S_h\|_B$.*

The proof of this lemma is elementary and it follows from the definitions and properties given in the previous section. It will be used to calculate a lower bound for $\sigma(D)$, with U replacing B whenever it is more convenient (the lower bound remains unaffected by this). Actually, it turns out that the lower bound $2-\|S_h\|_B$ is equal to $\sigma(D)$ in the case of many domains for which $\sigma(D)$ is already known—disks, parallel strips, convex angular sectors and regular polygons. In the case of disks this is trivial. For convex angular sectors, Lehto [9] showed that $\|S_h\|_U=2-2k^2$ when $0<k\le 1$ where $h(z)=z^k$, which maps U conformally onto A_k and thus $\sigma(A_k)=2-\|S_h\|_U$. Also, for the parallel strip S, $h(z)=\log z$ maps U onto S, so $S_h(z)=1/(2z^2)$ and consequently $\|S_h\|_U=2$, showing that $\sigma(S)=2-\|S_h\|_U$. The case of regular polygons will be treated in the next section, where we give a new proof of Calvis's result [3].

The notion of the lower bound in Lemma 2.1 being equal to $\sigma(D)$ leads us to the definition of a Nehari disk. A simply connected domain D in $\overline{\mathbf{C}}$ is called a *Nehari disk* if

$$\sigma(D)=2-\|S_h\|_B,$$

where h maps B conformally onto D. We remark that disks, parallel strips, convex angular sectors and regular polygons are all Nehari disks.

Of course, there exist many simply connected domains which are not Nehari disks. From the calculations in Lehto [9] and Lehtinen [8], one can easily see that the angular sectors A_k with $1<k<2$ are not Nehari disks. We provide more results on the nature of Nehari disks in [11]. In the next sections, we will show that rectangles and equiangular hexagons with certain restrictions on the ratio of the lengths of their sides furnish additional Nehari disks.

Next, we examine how $\sigma(D)$ is affected by convergence of domains. We derive a relation between the inner radii of D_n and D when $\{D_n\}$ converges to D in the sense of Carathéodory kernel convergence (for the definition of this convergence see [13, p. 13]). The following theorem describes this relation and its corollary will be used to prove our main results.

Theorem 2.2. *If D_n and D are simply connected domains and if $D_n \to D$ with respect to w_0, then*

$$\sigma(D) \geq \limsup_{n\to\infty} \sigma(D_n). \tag{2.1}$$

Proof. Without loss of generality, we may assume that w_0 is a finite point. For $n \in \mathbf{N}$, let g_n and g map B conformally onto D_n and D, normalized so that $g_n(0) = w_0$, $g_n'(0) > 0$ and $g(0) = w_0$, $g'(0) > 0$. By the Carathéodory kernel theorem, $g_n \to g$ locally uniformly in B (see [13, p. 14]).

Let F be a locally univalent function in D with $\|S_F\|_D < \limsup_{n\to\infty} \sigma(D_n)$. We will show that F is univalent in D. Let $f = F \circ g$ in B. Then f is locally univalent in B and $\|S_F\|_D = \|S_{f\circ g^{-1}}\|_D = \|S_f - S_g\|_B$. From a result mentioned in the introduction (see [10, p. 53]) it is clear that for every $n \in \mathbf{N}$, we can choose f_n locally univalent in B such that

$$S_{f_n} = S_f - S_g + S_{g_n} \tag{2.2}$$

in B. Then

$$\|S_{f_n\circ g_n^{-1}}\|_{D_n} = \|S_{f_n} - S_{g_n}\|_B = \|S_f - S_g\|_B < \limsup_{n\to\infty} \sigma(D_n)$$

for $n \in \mathbf{N}$.

Now by passing to a subsequence and relabeling, we can find $n_0 \in \mathbf{N}$ such that $\|S_{f_n\circ g_n^{-1}}\|_{D_n} < \sigma(D_n)$ for every $n \geq n_0$. Hence, $f_n \circ g_n^{-1}$ is univalent in D_n for $n \geq n_0$ and therefore f_n is univalent in B for $n \geq n_0$. Next, by replacing f_n with $\mu_n \circ f_n$ for suitable Möbius transformations μ_n and passing to a subsequence, we can assume that $f_n \to \varphi$ locally uniformly in B, where φ is a univalent mapping in B. As the Schwarzian derivative is unaffected by this type of substitution, $S_{f_n} \to S_\varphi$ in B and, from (2.2), $S_{f_n} \to S_f$ in B. Thus, $S_f = S_\varphi$ in B and $f = \mu \circ \varphi$ for some Möbius transformation μ. Since φ is univalent in B, f must be univalent in B. As $F = f \circ g^{-1}$, F is univalent in D, completing the proof. □

It is not difficult to provide examples which show that a strict inequality can occur in (2.1). If $D = U$ and $\{D_n\}$ is a sequence of open squares exhausting U, then $\sigma(D) = 2$ and $\sigma(D_n) = \frac{1}{2}$ for all $n \in \mathbf{N}$. Consequently, $\sigma(D) > \lim_{n\to\infty} \sigma(D_n)$.

There is a simple consequence of Theorem 2.2 that will prove to be very useful in the next sections.

Corollary 2.3. *Suppose P is a convex n-sided polygon. If P has $k\pi$ as an interior angle, then $\sigma(P) \le 2k^2$.*

Proof. As P is convex we know that $0 < k < 1$ and hence $\sigma(A_k) = 2k^2$. Now A_k can trivially be exhausted by an increasing sequence of domains—images of P under a succession of Möbius transformations (which consequently converge to A_k). Thus, from Theorem 2.2 we conclude that $\sigma(P) \le 2k^2$. □

We established a much broader generalization of Corollary 2.3 in [11, p. 69–72].

The general problem we are concerned with is as follows: Given a simply connected domain D in $\overline{\mathbf{C}}$, how does one calculate $\sigma(D)$?

A useful lower bound for $\sigma(D)$ is the one given in Lemma 2.1, $\sigma(D) \ge 2 - \|S_h\|_B$, where $h\colon B \to D$ is the Riemann mapping. To get an upper bound on $\sigma(D)$ one chooses a domain D' whose inner radius is known (such as an angular sector), and shows that a sequence of images of D under suitable Möbius transformations converges to D'. In this case, by Theorem 2.2, $\sigma(D')$ is an upper bound for $\sigma(D)$. If D is a convex polygon, Corollary 2.3 can be used directly to get the same upper bound. If the described upper and lower bounds are equal, we will have found the value of $\sigma(D)$ and will have demonstrated that D is a Nehari disk.

In the following sections we apply the above described method to regular polygons as well as rectangles and equiangular hexagons with restricted ratios of sides.

3. Regular polygons

Here, Lemma 2.1 and Corollary 2.3 are used to give a new proof and a refinement of a theorem due to Calvis. We restate the theorem, adding the notion of a Nehari disk to it.

Theorem 3.1. *Suppose P_n is an open regular n-sided polygon. Then*

$$\sigma(P_n) = 2\Big(\frac{n-2}{n}\Big)^2.$$

Moreover, P_n is a Nehari disk.

Proof. Since $\sigma(P_n)$ is invariant under Möbius transformations, we can assume that P_n is the image of B under the Schwarz–Christoffel transformation $g_n(w) = \int_0^w (1-z^n)^{-2/n}\,dz$. An elementary calculation shows that

$$S_{g_n}(w) = \frac{2(n-1)w^{n-2}}{(1-w^n)^2} \tag{3.1}$$

for $w \in B$. Now we will show that

$$\sup_{w\in B}\big|S_{g_n}(w)(1-|w|^2)^2\big| = \frac{8(n-1)}{n^2}.$$

By (3.1),

$$\lim_{r\to 1}\big|S_{g_n}(r)(1-r^2)^2\big| = \lim_{r\to 1}\Big(\frac{2(n-1)r^{n-2}}{(1-r^n)^2}\cdot(1-r^2)^2\Big) = \frac{8(n-1)}{n^2},$$

so

$$\sup_{w\in B}\big|S_{g_n}(w)(1-|w|^2)^2\big| \geq \frac{8(n-1)}{n^2}. \tag{3.2}$$

Next,

$$\frac{1}{|w|} - |w| = 2\sinh\Big(\log\frac{1}{|w|}\Big) \leq 2\cdot\frac{2}{n}\sinh\Big(\frac{n}{2}\log\frac{1}{|w|}\Big) \leq \frac{2}{n}\Big(\frac{1}{|w|^{n/2}} - |w|^{n/2}\Big),$$

which implies

$$\frac{|w|^{n-2}(1-|w|^2)^2}{(1-|w|^n)^2} \leq \frac{4}{n^2} \tag{3.3}$$

for all $w \in B$.

Then (3.1) and (3.3) yield

$$\big|S_{g_n}(w)(1-|w|^2)^2\big| = \Big|\frac{2(n-1)w^{n-2}}{(1-w^n)^2}(1-|w|^2)^2\Big| \leq \frac{8(n-1)}{n^2} \tag{3.4}$$

for all $w \in B$. Hence $\|S_{g_n}\|_B = 8(n-1)/n^2$ by (3.2) and (3.4).

From Lemma 2.1, we obtain $\sigma(P_n) \geq 2 - \|S_{g_n}\|_B = 2(n-2)^2/n^2$. From Corollary 2.3, we have $\sigma(P_n) \leq 2(n-2)^2/n^2$. Thus, $\sigma(P_n) = 2(n-2)^2/n^2$ and P_n is a Nehari disk. □

4. Rectangular domains

From Calvis [3] we know that $\sigma(P_4) = \frac{1}{2}$, where P_4 is an open square. We prove here that if we stretch two parallel sides of P_4 (thus obtaining a rectangle) up to a certain limit, the inner radius remains unchanged. It also turns out that all rectangles obtained in this way are Nehari disks.

For an open rectangle R, let $r(R)$ denote the ratio of the length of R's longer side to the length of its shorter side. We can state:

Theorem 4.1. *Suppose R is an open rectangle such that $1 \le r(R) \le c$, where $c = 1.52346\ldots$. Then $\sigma(R) = \frac{1}{2} = \sigma(P_4)$ and R is a Nehari disk.*

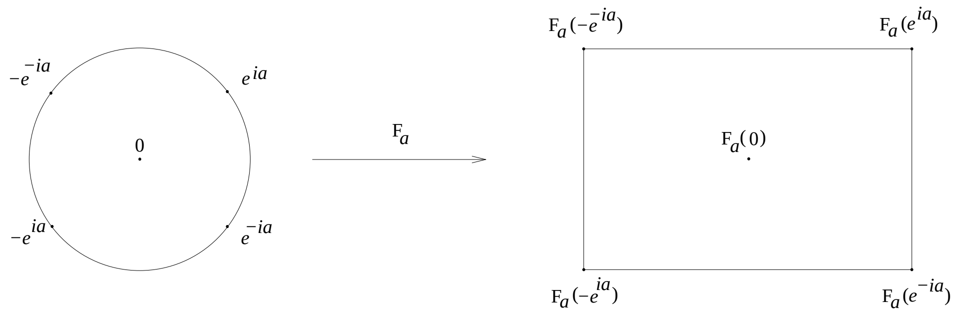

Figure 1. Schwarz–Christoffel mapping of B onto R_α

To obtain this result we will study Schwarz–Christoffel transformations mapping B onto open rectangles. See, for example, [2, pp. 228–233] for details. Let

$$F_\alpha(w) = \int_0^w \frac{1}{\sqrt{(z^2 - e^{2i\alpha})(z^2 - e^{-2i\alpha})}}\, dz, \tag{4.1}$$

where $0 < \alpha \le \frac{1}{4}\pi$. For $0 < \alpha \le \frac{1}{4}\pi$, let R_α be the image of B under F_α. (See Figure 1.) Then, R_α is an open rectangle centered at $F_\alpha(0)$ whose vertices coincide with the images of $e^{i\alpha}$, $-e^{i\alpha}$, $e^{-i\alpha}$ and $-e^{-i\alpha}$ under F_α. The ratio $r(R_\alpha)$ is a continuous monotone function of α which maps the interval $(0, \frac{1}{4}\pi]$ bijectively onto the interval $[1, \infty)$. When $\alpha = \frac{1}{4}\pi$, we have $r(R_\alpha) = 1$ and R_α is an open square; as α approaches 0, $r(R_\alpha)$ approaches ∞. Thus for any rectangle (up to a similarity transformation), there is a unique α, $0 < \alpha \le \frac{1}{4}\pi$ corresponding to it.

We can restate Theorem 4.1 in a form that will facilitate its proof.

Theorem 4.2. *If $\frac{1}{2}\arccos(\frac{3}{4}) \le \alpha \le \frac{1}{4}\pi$, then $\sigma(R_\alpha) = \frac{1}{2}$ and R_α is a Nehari disk.*

Our approach to proving Theorem 4.2 (and verifying that Theorem 4.1 is equivalent to it) is exactly the one outlined in general. Our first goal is to calculate $\|S_{F_\alpha}\|_B$—we show that $\|S_{F_\alpha}\|_B = \frac{3}{2}$ when $\frac{1}{2}\arccos(\frac{3}{4}) \le \alpha \le \frac{1}{4}\pi$. The calculation, as will be apparent later, reduces to showing that fourth degree polynomials of a certain type take only nonpositive values on the nonnegative reals. Our next goal is to verify this preliminary fact.

We begin by stating an elementary lemma.

Lemma 4.3. *Let $f(y;t,u) = C_4y^4 + C_3y^3 + C_2y^2 + C_1y + C_0$, where for real* *t and u*

$$\begin{aligned}
C_4 &= 16u^2 - 9,\\
C_3 &= 8(-9 - 3tu + 8u^2 + 4tu^3),\\
C_2 &= 8(u-t)(9t - 21u + 10tu^2 + 2u^3),\\
C_1 &= 288(u-t)^2(tu-1),\\
C_0 &= -144(u-t)^4.
\end{aligned}$$

Then, $f(y;t,u) \le 0$ *whenever* $-1 \le t \le 1$, $0 \le u \le \frac{3}{4}$ *and* $0 \le y$.

Proof. The proof rests on three observations. In order to simplify the presentation, we do not provide the computational details here, but the interested reader can find them in the appendix. We assert:

(i) $C_4, C_3, C_1, C_0 \le 0$ for $-1 \le t \le 1$ and $0 \le u \le \frac{3}{4}$;
(ii) $C_2 \le 0$ for $-1 \le t \le u$ and $0 \le u \le \frac{3}{4}$;
(iii) $D = C_2^2 - 4C_3C_1 \le 0$ for $u \le t \le 1$ and $0 \le u \le \frac{3}{4}$.

Now we are ready to complete the proof of the lemma. From (i) and (ii) we conclude that $C_0, C_1, \dots, C_4 \le 0$ when $-1 \le t \le u$ and $0 \le u \le \frac{3}{4}$. Thus $f(y;t,u) \le 0$ for $-1 \le t \le u$, $0 \le u \le \frac{3}{4}$ and $y \ge 0$.

From (i) and (iii) we get $C_3y^2 + C_2y + C_1 \le 0$. Consequently, $C_3y^3 + C_2y^2 + C_1y \le 0$ and $C_4, C_0 \le 0$, which implies that $f(y;t,u) \le 0$ for $u \le t \le 1$, $0 \le u \le \frac{3}{4}$ and $y \ge 0$. Thus $f(y;t,u) \le 0$ for $-1 \le t \le 1$, $0 \le u \le \frac{3}{4}$ and $y \ge 0$, so the lemma is proved. □

We proceed to the proofs of Theorems 4.2 and 4.1.

Proof of Theorem 4.2. Suppose α, $\frac{1}{2}\arccos(\frac{3}{4}) \le \alpha \le \frac{1}{4}\pi$, is arbitrarily fixed. We will verify that $\|S_{F_\alpha}\|_B \le \frac{3}{2}$. From (4.1), after a tedious but elementary calculation we get

$$\begin{aligned}
S_{F_\alpha}(w) &= \left(\frac{F''_\alpha(w)}{F'_\alpha(w)}\right)' - \frac{1}{2}\left(\frac{F''_\alpha(w)}{F'_\alpha(w)}\right)^2\\
&= 2\frac{\cos(2\alpha)w^4 - \big(\sin^2(2\alpha)+2\big)w^2 + \cos(2\alpha)}{\big(w^4 - 2\cos(2\alpha)w^2 + 1\big)^2}
\end{aligned} \tag{4.2}$$

for $w \in B$.

Now fix $w = re^{i\theta}$ where $0 \le r < 1$ and $0 \le \theta < 2\pi$. If $r = 0$, using (4.2) we get

$$|S_{F_\alpha}(w)|(1-|w|^2)^2 = 2|\cos(2\alpha)| \le \tfrac{3}{2}, \tag{4.3}$$

since $0 \le \cos(2\alpha) \le \frac{3}{4}$ whenever $\frac{1}{2}\arccos(\frac{3}{4}) \le \alpha \le \frac{1}{4}\pi$. Suppose $0 < r < 1$. Then (4.2) yields

$$\begin{aligned} S_{F_\alpha}(w) &= \frac{2}{w^2}\frac{\cos(2\alpha)(w^2+w^{-2}) - \big(\sin^2(2\alpha)+2\big)}{\big(w^2+w^{-2}-2\cos(2\alpha)\big)^2} \\ &= \frac{2\cos(2\alpha)}{w^2}\frac{w^2+w^{-2}-3\sec(2\alpha)+\cos(2\alpha)}{\big(w^2+w^{-2}-2\cos(2\alpha)\big)^2}, \end{aligned}$$

and hence

(4.4)
$$|S_{F_\alpha}(w)|(1-|w|^2)^2 = 2\cos(2\alpha)\frac{(1-|w|^2)^2}{|w|^2}\frac{|w^2+w^{-2}-3\sec(2\alpha)+\cos(2\alpha)|}{|w^2+w^{-2}-2\cos(2\alpha)|^2}.$$

Noting that

$$w^2+w^{-2} = \Big(r^2+\frac{1}{r^2}\Big)\cos(2\theta) + i\Big(r^2-\frac{1}{r^2}\Big)\sin(2\theta)$$

and setting $y = (r-r^{-1})^2$, $u = \cos(2\alpha)$ and $t = \cos(2\theta)$, one readily verifies that

$$\begin{aligned} |w^2+w^{-2}-3\sec(2\alpha)+\cos(2\alpha)|^2 &= \Big((y+2)t+u-\frac{3}{u}\Big)^2 + y(y+4)(1-t^2) \\ &= h(y;t,u) \end{aligned}$$

and

$$|w^2+w^{-2}-2\cos(2\alpha)|^2 = \big(ty+2(t-u)\big)^2 + y(y+4)(1-t^2) = g(y;t,u).$$

From (4.4) we see that

$$|S_{F_\alpha}(w)|(1-|w|^2)^2 = 2uy\frac{\sqrt{h(y;t,u)}}{g(y;t,u)}.$$

Now $|S_{F_\alpha}(w)|(1-|w|^2)^2 \le \frac{3}{2}$ whenever

$$f(y;t,u) = 16u^2y^2 \cdot h(y;t,u) - 9g^2(y;t,u) \le 0.$$

By expanding $f(y;t,u)$ as a polynomial in y we see that

$$f(y;t,u) = C_4y^4 + C_3y^3 + C_2y^2 + C_1y + C_0$$

where C_j, for $j = 0,1,\ldots,4$, are as in Lemma 4.3. Since $y = (r-r^{-1})^2 \ge 0$, $-1 \le t = \cos(2\theta) \le 1$ and $0 \le u = \cos(2\alpha) \le \frac{3}{4}$, we conclude from Lemma 4.3 that $f(y;t,u) \le 0$ for $w \in B \setminus \{0\}$. Consequently,

$$|S_{F_\alpha}(w)|(1-|w|^2)^2 \le \tfrac{3}{2}$$

for all $w \in B \setminus \{0\}$. From this fact and (4.3) it follows that

$$\|S_{F_\alpha}\|_B \le \tfrac{3}{2}. \tag{4.5}$$

Now, from (4.5), Lemma 2.1 and Corollary 2.3 it follows that

$$\tfrac{1}{2} \le 2 - \|S_{F_\alpha}\|_B \le \sigma(R_\alpha) \le \tfrac{1}{2}$$

and hence

$$\sigma(R_\alpha) = \tfrac{1}{2}, \qquad \|S_{F_\alpha}\|_B = \tfrac{3}{2};$$

and R_α is a Nehari disk. □

We remark that it is easy to demonstrate that $|S_{F_\alpha}(w)|(1-|w|^2)^2 \to \|S_{F_\alpha}\|_B$ as $w \to \partial B$ along the four rays corresponding to the angles α, $\pi-\alpha$, $\pi+\alpha$ and $2\pi-\alpha$, under the assumptions of Theorem 4.2.

At the beginning of this section, we mentioned the fact that the mapping $\alpha \to r(R_\alpha)$ is a monotone, continuous bijection between the intervals $(0, \frac{1}{4}\pi]$ and $[1,\infty)$. Since the inner radius is invariant with respect to Möbius transformations, Theorem 4.1 is proved except for establishing the estimate for c. The constant c in question can be written as a quotient of elliptic type integrals and, in this way, estimates for its value can be given. Now we complete the proof of Theorem 4.1.

Proof of Theorem 4.1. From the above observations, it is clear that all that is left to do is to calculate $r(R_\alpha)$ for $\alpha = \frac{1}{2}\arccos(\frac{3}{4})$. From the definition of F_α, for $0 < \alpha \le \frac{1}{4}\pi$, $r(R_\alpha) = L_\alpha / l_\alpha$ where

$$L_\alpha = \left| \int_{e^{i\alpha}}^{e^{i(\pi-\alpha)}} \frac{dw}{\sqrt{w^4 - 2\cos(2\alpha)w^2 + 1}} \right|$$

and

$$l_\alpha = \left| \int_{e^{-i\alpha}}^{e^{i\alpha}} \frac{dw}{\sqrt{w^4 - 2\cos(2\alpha)w^2 + 1}} \right|.$$

From the above we get

$$\begin{aligned} L_\alpha &= \left| \int_{\alpha}^{\pi-\alpha} \frac{ie^{i\theta}\,d\theta}{\sqrt{e^{4i\theta} - 2\cos(2\alpha)e^{2i\theta} + 1}} \right| \\ &= \left| \int_{\alpha}^{\pi-\alpha} \frac{i\,d\theta}{\sqrt{e^{2i\theta} + e^{-2i\theta} - 2\cos(2\alpha)}} \right| \\ &= \left| \int_{\alpha}^{\pi-\alpha} \frac{d\theta}{\sqrt{2\big(\cos(2\theta) - \cos(2\alpha)\big)}} \right| \end{aligned} \tag{4.6}$$

and, in the same way,

$$l_\alpha = \left| \int_{-\alpha}^{\alpha} \frac{d\theta}{\sqrt{2\big(\cos(2\theta) - \cos(2\alpha)\big)}} \right| \tag{4.7}$$

for $0 < \alpha \leq \frac{1}{4}\pi$.

The integrals in (4.6) and (4.7) are (improper) integrals of continuous real-valued functions, so consultation with a standard table of integrals or a symbolic mathematics software package can provide an estimate. We obtained

$$r(R_\alpha) = 1.52346\ldots$$

for $\alpha = \frac{1}{2}\arccos(\frac{3}{4})$. This completes the proof of the theorem. □

Finally, what remains unsettled is the behavior of the inner radius of a rectangle R with $r(R) > c$ (or analogously of $\sigma(R_\alpha)$ when $0 < \alpha < \frac{1}{2}\arccos(\frac{3}{4})$). However, the bound $\alpha = \frac{1}{2}\arccos(\frac{3}{4})$ is natural, since from (4.3) it is apparent that

$$|S_{F_\alpha}(0)|(1 - |0|^2)^2 \leq \tfrac{3}{2}$$

exactly when $\frac{1}{2}\arccos(\frac{3}{4}) \leq \alpha \leq \frac{1}{4}\pi$, with equality occurring for $\alpha = \frac{1}{2}\arccos(\frac{3}{4})$. This means that for $0 < \alpha < \frac{1}{2}\arccos(\frac{3}{4})$,

$$\|S_{F_\alpha}\|_B \geq 2\cos(2\alpha) > \tfrac{3}{2},$$

so the argument from the proof of Theorem 4.2 cannot be used. It is clear that a parallel strip S can be exhausted by an increasing sequence of rectangles with ratios of sides approaching infinity. In view of Theorem 2.2, this means that the inner radii of rectangles R approach 0 as $r(R)$ approach infinity (i.e., $\sigma(R_\alpha) \to 0$ as $\alpha \to 0$). Aside from this fact, the behavior of $\sigma(R_\alpha)$ for α in the interval $0 < \alpha < \frac{1}{2}\arccos(\frac{3}{4})$ remains unclear.

In the following section, we develop a completely analogous result for equiangular hexagons.

5. Hexagonal domains

If P_6 denotes an open, regular hexagon, from Calvis [3] we know that $\sigma(P_6) = \frac{8}{9}$. Here, we will study equiangular hexagons whose sides have lengths forming the sequence *baabaa* for some positive numbers a, b with $b \geq a$. We will establish a result about the inner radii of such hexagons, analogous to the result shown for rectangles. In fact, we show that, if we take a regular hexagon and stretch two of its parallel sides (thus obtaining a hexagon of the above description) up to a certain limit, the inner radius remains unchanged. It again turns out that such hexagons are Nehari disks.

Before stating our result, we introduce some notation to simplify matters. We will say that an equiangular hexagon H has the side sequence *baabaa*, if the lengths of its sides form the sequence *baabaa* for some a, b where $b \geq a$. In this case we set $r(H) = b/a$.

Theorem 5.1. *Suppose H is an open equiangular hexagon with side sequence baabaa, such that $1 \le r(H) \le c$, where $c = 1.67117\ldots$. Then $\sigma(H) = \frac{8}{9} = \sigma(P_6)$ and H is a Nehari disk.*

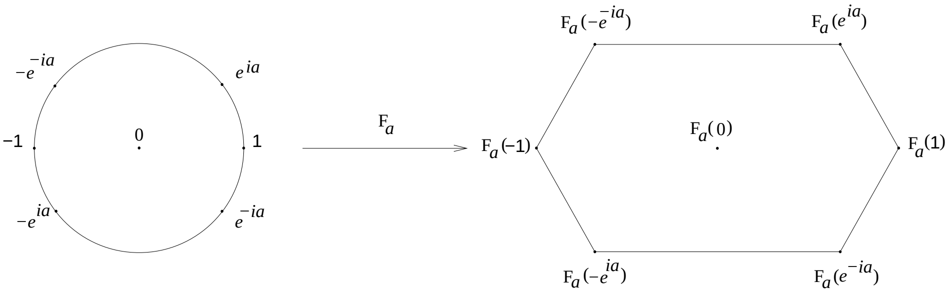

Figure 2. Schwarz–Christoffel mapping of B onto H_α

The hexagons under discussion here are images of B under Schwarz–Christoffel transformations of the form

$$F_\alpha(w) = \int_0^w \frac{1}{\left((z^2 - e^{2i\alpha})(z^2 - e^{-2i\alpha})(z^2 - 1)\right)^{1/3}}\, dz, \tag{5.1}$$

where $0 < \alpha \le \frac{1}{3}\pi$. For each α, $0 < \alpha \le \frac{1}{3}\pi$, $H_\alpha = F_\alpha(B)$ is an equiangular hexagon with side sequence *baabaa* for some $b \ge a > 0$, centered at $F_\alpha(0)$, whose vertices are the images of ± 1, $\pm e^{i\alpha}$ and $\pm e^{-i\alpha}$ under F_α. (See Figure 2.) It is easy to see that the mapping $\alpha \to r(H_\alpha)$ describes a continuous, monotone bijection from $(0, \frac{1}{3}\pi]$ onto $[1, \infty)$. When $\alpha = \frac{1}{3}\pi$, we have $r(H_\alpha) = 1$ and H_α is a regular hexagon, while as α approaches 0, $r(H_\alpha)$ approaches infinity. Hence for each equiangular hexagon with side sequence *baabaa* for some $b \ge a > 1$ (up to a similarity transformation) there is a unique α, $0 < \alpha \le \frac{1}{3}\pi$ corresponding to it.

As we did in the case of Theorem 4.1, we restate Theorem 5.1 in a way that enables us to use Schwarz–Christoffel transformations in the proof.

Theorem 5.2. *If $\frac{1}{2}\arccos(\frac{1}{3}) \le \alpha \le \frac{1}{3}\pi$, then $\sigma(H_\alpha) = \frac{8}{9}$ and H_α is a Nehari disk.*

The proof of Theorem 5.2 (and afterwards of Theorem 5.1) is analogous to the proof of the corresponding results on rectangles. We show that $\|S_{F_\alpha}\|_B = \frac{10}{9}$ when $\frac{1}{2}\arccos(\frac{1}{3}) \le \alpha \le \frac{1}{3}\pi$. The proof of this, again, reduces to showing that sixth degree polynomials of a certain type take only nonpositive values on the nonnegative reals.

We record the specific information required in a lemma, the counterpart of Lemma 4.3. Its elementary but rather lengthy proof is omitted. (The proof can be found in [11].)

Lemma 5.3. *Let* $f(y;t,u) = C_6y^6 + C_5y^5 + C_4y^4 + C_3y^3 + C_2y^2 + C_1y + C_0$, *where for real* t *and* u

$$
\begin{aligned}
C_6 &= (3u+4)(3u-1),\\
C_5 &= (-40u + 1 - 72u^2 + 24u^3)t + (72u + 72u^2 - 57),\\
C_4 &= (63 + 88u + 24u^3 - 64u^2)t^2 + (-40u - 432u^2 + 106 + 144u^3)t\\
&\quad + (16u^4 - 124u^3 + 260u + 244u^2 - 285),\\
C_3 &= (96u^3 + 104 + 112u^2 + 40u)t^3 + (-448u - 656u^2 - 48 + 96u^3)t^2\\
&\quad + (16u^4 + 584u - 136u^2 + 304u^3 + 288)t\\
&\quad + (-584 + 64u^4 + 464u + 200u^2 - 496u^3),\\
C_2 &= 8(t-1)(u-t)\big((32u^2 + 82u + 33)t^2 + (8u^3 - 114u^2 - 95u - 93)t\\
&\quad + (12u^3 + 2u^2 + 113u + 20)\big),\\
C_1 &= 400(t-1)(u-t)^2(u^2 + 2ut^2 - 4ut + t^2 - 2t + 2),\\
C_0 &= -400(t-1)^2(u-t)^4.
\end{aligned}
$$

Then, $f(y;t,u) \le 0$ *whenever* $-1 \le t \le 1$, $-\frac{1}{2} \le u \le \frac{1}{3}$ *and* $0 \le y$.

We now turn to the proof of Theorem 5.2.

Proof of Theorem 5.2. Suppose α, $\frac{1}{2}\arccos(\frac{1}{3}) \le \alpha \le \frac{1}{3}\pi$, is arbitrarily fixed. We will verify that $\|S_{F_\alpha}\|_B \le \frac{10}{9}$. From (5.1), we have that

$$
\frac{F''_\alpha(w)}{F'_\alpha(w)} = -\frac{2}{3}\cdot\frac{w\big(3w^4 - 2w^2 - 4w^2\cos(2\alpha) + 2\cos(2\alpha) + 1\big)}{(w^2 - e^{2i\alpha})(w^2 - e^{-2i\alpha})(w^2 - 1)}
$$

for $w \in B$. After some simplification, we obtain

$$
\begin{aligned}
S_{F_\alpha}(w) &= \left(\frac{F''_\alpha(w)}{F'_\alpha(w)}\right)' - \frac{1}{2}\left(\frac{F''_\alpha(w)}{F'_\alpha(w)}\right)^2\\
&= \frac{2}{9}\left[\frac{\big(3 + 6\cos(2\alpha)\big)(w^8 + 1)}{\big(w^4 - 2\cos(2\alpha)w^2 + 1\big)^2(w^2 - 1)^2}\right]\\
&\quad + \frac{2}{9}\left[\frac{\big(8\cos^2(2\alpha) - 28\cos(2\alpha) - 16\big)(w^6 + w^2)}{\big(w^4 - 2\cos(2\alpha)w^2 + 1\big)^2(w^2 - 1)^2}\right]\\
&\quad + \frac{2}{9}\left[\frac{\big(46 + 4\cos^2(2\alpha) + 4\cos(2\alpha)\big)w^4}{\big(w^4 - 2\cos(2\alpha)w^2 + 1\big)^2(w^2 - 1)^2}\right]
\end{aligned}
\tag{5.2}
$$

for $w \in B$.

Now fix $w = re^{i\theta}$ where $0 \le r < 1$ and $0 \le \theta < 2\pi$. If $r = 0$, using (5.2) we get

$$|S_{F_\alpha}(w)|(1-|w|^2)^2 = \tfrac{2}{3}|1+2\cos(2\alpha)| \le \tfrac{10}{9} \tag{5.3}$$

since $-\frac{1}{2} \le \cos(2\alpha) \le \frac{1}{3}$ whenever $\frac{1}{2}\arccos(\frac{1}{3}) \le \alpha \le \frac{1}{3}\pi$. Suppose $0 < r < 1$. Then, with the aid of (5.2), we obtain

(5.4)

$$|S_{F_\alpha}(w)|(1-|w|^2)^2 = \frac{2}{9} \cdot \frac{(1-|w|^2)^2}{|w|^2} \cdot \frac{|G_\alpha(w)|}{|w^2+w^{-2}-2\cos(2\alpha)|^2|w^2+w^{-2}-2|},$$

where

$$\begin{aligned} G_\alpha(w) = {} & \big(3+6\cos(2\alpha)\big)\big((w^2+w^{-2})^2-2\big) \\ & + \big(8\cos^2(2\alpha) - 28\cos(2\alpha) - 16\big)(w^2+w^{-2}) \\ & + \big(46 + 4\cos^2(2\alpha) + 4\cos(2\alpha)\big). \end{aligned}$$

Setting $y = (r - r^{-1})^2$, $u = \cos(2\alpha)$ and $t = \cos(2\theta)$, and noting that

$$w^2 + w^{-2} = \Big(r^2 + \frac{1}{r^2}\Big)\cos(2\theta) + i\Big(r^2 - \frac{1}{r^2}\Big)\sin(2\theta),$$

it is not difficult to see that there are polynomials h and g (in y, t, and u) such that $h(y;t,u) = |G_\alpha(w)|^2$ and

$$g(y;t,u) = |w^2 + w^{-2} - 2\cos(2\alpha)|^4 |w^2 + w^{-2} - 2|^2.$$

(The exact expressions are lengthy so we avoid writing them explicitly.) From (5.4), we get

$$|S_{F_\alpha}(w)|(1-|w|^2)^2 = \frac{2}{9} y \sqrt{\frac{h(y;t,u)}{g(y;t,u)}},$$

so $|S_{F_\alpha}(w)|(1-|w|^2)^2 \le \frac{10}{9}$ when $4 \cdot 81 y^2 h(y;t,u) - 100 \cdot 81 g(y;t,u) \le 0$. Setting $f(y;t,u) = \big(y^2 h(y;t,u) - 25 g(y;t,u)\big)/4$ we have that

$$|S_{F_\alpha}(w)|(1-|w|^2)^2 \le \tfrac{10}{9}$$

whenever $f(y;t,u) \le 0$. We expand $f(y;t,u)$ as a polynomial in y (we used the Maple software package), obtaining

$$f(y;t,u) = C_6 y^6 + C_5 y^5 + C_4 y^4 + C_3 y^3 + C_2 y^2 + C_1 y + C_0$$

with C_j, for $j = 0, 1, \dots, 6$, as in Lemma 5.3. Since $y = (r - r^{-1})^2 \ge 0$, $-1 \le t = \cos(2\theta) \le 1$ and $-\frac{1}{2} \le u = \cos(2\alpha) \le \frac{1}{2}$, we conclude from Lemma 5.3 that $f(y;t,u) \le 0$ when $w \in B \setminus \{0\}$. Consequently,

$$|S_{F_\alpha}(w)|(1-|w|^2)^2 \le \tfrac{10}{9}$$

for all $w \in B \setminus \{0\}$. From the above and (5.3) it follows that

$$\|S_{F_\alpha}\|_B \le \tfrac{10}{9}. \tag{5.5}$$

Now from (5.5), Lemma 2.1 and Corollary 2.3 it follows that

$$\tfrac{8}{9} \le 2 - \|S_{F_\alpha}\|B \le \sigma(H_\alpha) \le \tfrac{8}{9}$$

and hence

$$\sigma(H_\alpha) = \tfrac{8}{9}, \qquad \|S_{F_\alpha}\|_B = \tfrac{10}{9};$$

and H_α is a Nehari disk. □

We remark that it is simple to verify that, under the assumptions of Theorem 5.2, $|S_{F_\alpha}(w)|(1-|w|^2)^2$ approaches the value $\frac{10}{9}$ as w approaches ∂B along the rays corresponding to the angles 0, α, $\pi-\alpha$, π, $\pi+\alpha$ and $2\pi-\alpha$.

At the beginning of this section, we mentioned the fact that the mapping $\alpha \to r(H_\alpha)$ is a monotone, continuous bijection between the intervals $(0, \frac{1}{3}\pi]$ and $[1,\infty)$. Since the inner radius is invariant with respect to Möbius transformations, Theorem 5.1 is proved except for establishing the estimate for c. The last step of the proof mimics the last part of the proof of Theorem 4.1.

Proof of Theorem 5.1. From the above observations, it is clear that all we need to do is calculate $r(H_\alpha)$ for $\alpha = \frac{1}{2}\arccos(\frac{1}{3})$. From the definition of F_α, for $0 < \alpha \le \frac{1}{3}\pi$, $r(H_\alpha) = L_\alpha/l_\alpha$ where

$$L_\alpha = \left| \int_{e^{i\alpha}}^{e^{i(\pi-\alpha)}} \frac{dw}{\big((w^4 - 2\cos(2\alpha)w^2 + 1)(w^2-1)\big)^{1/3}} \right|$$

and

$$l_\alpha = \left| \int_{1}^{e^{i\alpha}} \frac{dw}{\big((w^4 - 2\cos(2\alpha)w^2 + 1)(w^2-1)\big)^{1/3}} \right|.$$

Now, from the above,

$$\begin{aligned} L_\alpha &= \left| \int_{\alpha}^{\pi-\alpha} \frac{ie^{i\theta}\,d\theta}{\big((e^{4i\theta} - 2\cos(2\alpha)e^{2i\theta} + 1)(e^{2i\theta}-1)\big)^{1/3}} \right| \\ &= \left| \int_{\alpha}^{\pi-\alpha} \frac{i\,d\theta}{\big((e^{2i\theta} + e^{-2i\theta} - 2\cos(2\alpha))(e^{i\theta}-e^{-i\theta})\big)^{1/3}} \right| \\ &= \left| \int_{\alpha}^{\pi-\alpha} \frac{i\,d\theta}{\big((2\cos(2\theta) - 2\cos(2\alpha))(2i\sin(\theta))\big)^{1/3}} \right| \\ &= \left| \int_{\alpha}^{\pi-\alpha} \frac{d\theta}{\big(4(\cos(2\theta) - \cos(2\alpha))\sin(\theta)\big)^{1/3}} \right| \end{aligned} \tag{5.6}$$

and, in the same way,

$$l_\alpha = \left| \int_0^\alpha \frac{d\theta}{\big(4\big(\cos(2\theta) - \cos(2\alpha)\big)\sin(\theta)\big)^{13}} \right| \tag{5.7}$$

for $0 < \alpha < \frac{1}{3}\pi$.

Referring to a standard table of integrals or using a symbolic mathematics software package, one arrives at

$$r(H_\alpha) = 1.67117\dots$$

for $\alpha = \frac{1}{2}\arccos(\frac{1}{3})$. □

There remains the question of what happens to the inner radius of an equiangular hexagon H with side sequence $baabaa$ ($b \geq a > 0$) for which $r(H) > c = 1.67117\dots$ (or analogously $\sigma(H_\alpha)$ when $0 < \alpha < \frac{1}{2}\arccos(\frac{1}{3})$). Proceeding along lines similar to those in our treatment of rectangles, one sees that

$$\|S_{F_\alpha}\|_B \geq \tfrac{2}{3}\big(1 + 2\cos(2\alpha)\big) > \tfrac{10}{9}$$

in this case, so the argument from the proof of Theorem 5.1 cannot be used. As with rectangles, it is clear that the parallel strip S can be exhausted by an increasing sequence of equiangular hexagons H with side sequence $baabaa$, with $r(H)$ approaching infinity. In view of Theorem 2.2, this means that the inner radii of these hexagons must approach 0 as $r(H)$ approaches infinity (i.e., $\sigma(H_\alpha) \to 0$ as $\alpha \to 0$). Aside from this fact, the behavior of $\sigma(H_\alpha)$ remains unclear for $\alpha \in \big(0, \frac{1}{2}\arccos(\frac{1}{3})\big)$.

One can also ask how the inner radius is affected if we shrink two parallel sides of a regular hexagon (instead of stretching them). In this case, we have that $\alpha \in \big[\frac{1}{3}\pi, \frac{1}{2}\pi\big]$ and consequently (5.3) still holds for all hexagons obtained this way. However, a rhombus with angles $\frac{2}{3}\pi$, $\frac{1}{3}\pi$ can be exhausted by an increasing sequence of these hexagons (with $\alpha \to \frac{1}{2}\pi$), which means that the inner radius becomes smaller than $\frac{8}{9}$ after a certain amount of shrinking is done. We conclude that, unlike the stretching case, nothing significant changes at the origin as α approaches $\frac{1}{2}\pi$, which makes this case more difficult to explore.

6. Appendix

Here we furnish the proofs of assertions (i), (ii) and (iii) from the proof of Lemma 4.3. Recall that

$$\begin{aligned}
C_4 &= C_4(t,u) = 16u^2 - 9, \\
C_3 &= C_3(t,u) = 8(-9 - 3tu + 8u^2 + 4tu^3), \\
C_2 &= C_2(t,u) = 8(u-t)(9t - 21u + 10tu^2 + 2u^3), \\
C_1 &= C_1(t,u) = 288(u-t)^2(tu-1), \\
C_0 &= C_0(t,u) = -144(u-t)^4.
\end{aligned}$$

Proof of (i). We see first that $C_4(t,u) \le 0$ since $0 \le u \le \frac{3}{4}$, $C_1(t,u) \le 0$ since $|tu| \le \frac{3}{4}$, and $C_0(t,u) \le 0$ for all u and t in the specified range.

Next, $C_3(t,u)$ is linear (and hence monotone) in t with

$$C_3(-1,u) = -32(u+1)(u-\tfrac{3}{2})^2 < 0$$

and

$$C_3(1,u) = 32(u-1)(u+\tfrac{3}{2})^2 < 0$$

since $0 \le u \le 1$. Hence $C_3(t,u) < 0$ for $-1 \le t \le 1$ and $0 \le u \le \frac{3}{4}$. □

Proof of (ii). If $-1 \le t \le u$ and $0 \le u \le \frac{3}{4}$, then

$$C_2(t,u) = -8(u-t)\big(9(u-t) + 10u(1-ut) + 2u(1-u^2)\big) \le 0$$

is obvious. □

Proof of (iii). The proof of this step is somewhat lengthy. The factorization that follows was generated by using the Maple software package.

$$\begin{aligned} D(t,u) &= \big(C_2(t,u)\big)^2 - 4C_3(t,u)C_1(t,u) \\ &= \big(8(u-t)(9t - 21u + 10tu^2 + 2u^3)\big)^2 \\ &\quad - 9216(u-t)^2(tu-1)(-9 - 3tu + 8u^2 + 4tu^3) \\ &= 64(u-t)^2\big(a_2(u)t^2 + a_1(u)t + a_0(u)\big) \\ &= 64(u-t)^2 F(t,u) \end{aligned}$$

where

$$\begin{aligned} a_2(u) &= -476u^4 + 612u^2 + 81, \\ a_1(u) &= 40u^5 - 960u^3 + 486u, \\ a_0(u) &= 4u^6 - 84u^4 + 1593u^2 - 1296, \end{aligned}$$

and

$$F(t,u) = a_2(u)t^2 + a_1(u)t + a_0(u).$$

Now, for $0 \le u \le \frac{3}{4}$,

$$a_2'(u) = -1904u^3 + 1224u^2 \ge 0$$

which implies that $a_2(u) \le 275$. Similarly,

$$a_1'(u) = 200u^4 - 2880u^2 + 486$$

has exactly one zero for u in $[0, \frac{3}{4}]$, which implies $a_1(u) \le 134$. Again, for $0 \le u \le \frac{3}{4}$,

$$a_0'(u) = 24u^5 - 336u^3 + 3186 \ge 0$$

which implies that $a_0(u) \le -424$. Therefore, since $0 \le u \le t \le 1$,

$$F(t,u) = a_2(u)t^2 + a_1(u)t + a_0(u) \le -15$$

and thus $D = C_2^2 - 4C_3C_1 \le 0$ for $u \le t \le 1$, $0 \le u \le \frac{3}{4}$. □

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