Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 22, 1997, 465–474

BILIPSCHITZ MAPS AND THE MODULUS OF RINGS

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Abstract. We show that a homeomorphism f of euclidean n-space is bilipschitz continuous if and only if there is a constant M such that

$$|M(f(A)) - M(A)| \le M$$

for all (spherical) annuli A, where $M(A)$ is the modulus of A. We also present a local version of this result and give an application concerning absolute continuity on lower dimensional sets.

1. Introduction and results

Let $n \ge 2$ be an integer and f a homeomeorphism of $\mathbf{R}^n$. There are three conceptually different but equivalent characterizations of quasiconformality, namely the geometric definition (using the modulus of curve families or rings), the metric definition (distortion of relative distances) and the analytic definition (absolute continuity on lines together with the relation $|Df|^n \le K|Jf|$ a.e. between the norm of the derivative and the Jacobian). The latter two definitions have obvious analogues for bilipschitz maps, and the purpose of this paper is to give a "geometric definition" of bilipschitz maps. Let us recall some definitions first.

It is usual to call a domain $A \subset \mathbf{R}^n$ a *ring* if the complement of A (with respect to the n-sphere $\overline{\mathbf{R}}^n$) has exactly two components. For a ring A, consider the family Γ of curves in A that join the components of the complement of A and let $M(\Gamma)$ denote the modulus of the curve family Γ, that is

$$M(\Gamma) = \inf_{\rho} \int_{\mathbf{R}^n} \rho^n \, dx,$$

where the infimum is over all non-negative Borel functions $\rho\colon \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ with the property that $\int_\gamma \rho \, ds \ge 1$ for all $\gamma \in \Gamma$. Here dx is Lebesgue n-measure and ds arclength. If A is a spherical ring $\{r < |x| < R\}$ then $M(\Gamma) = \omega_{n-1}\bigl(\log(R/r)\bigr)^{1-n}$

1991 Mathematics Subject Classification: Primary 30C65.

The author acknowledges support through the HCM network "Conformal geometry and geometric function theory".

where ω_k is the Lebesgue measure of the sphere S^k, see [Vä, 7.5]. The modulus of a ring A is defined by

$$M(A) = \left(\frac{M(\Gamma)}{\omega_{n-1}}\right)^{1/(1-n)}.$$

Thus $M(A) = \log(R/r)$ for spherical rings with radii $r < R$.

With the usual "geometric" definition of K-quasiconformal maps [Vä, 13.1] a homeomorphism is K-qc if and only if

$$\frac{M(A)}{K'} \le M\bigl(f(A)\bigr) \le K' M(A) \tag{1.1}$$

for all rings A, where $K' = K^{1/(n-1)}$ [Vä, 36.2]. For a qc-map f, we denote by $\|f\|_{\mathrm{qc}}$ the smallest number K such that K' satisfies (1.1) for all rings, that is the smallest K such that f is K-qc.

A homeomorphism f of $\mathbf{R}^n$ is called L*-bilipschitz* if

$$\frac{|x-y|}{L} \le |f(x) - f(y)| \le L|x-y| \tag{1.2}$$

for all $x, y \in \mathbf{R}^n$. We refer to the smallest L satisfying (1.2) as the bilipschitz norm $\|f\|_{\mathrm{bl}}$ of f. Bilipschitz maps are quasiconformal, whereas the converse is false. The counterpart of (1.2) for quasiconformal maps is the metric definition [Vä, 34.1]. Specialized to homeomorphisms of $\mathbf{R}^n$ it says that f is K-qc if and only if there is a number $H < \infty$ so that

$$\frac{\max_{|y-x|=r} |f(y) - f(x)|}{\min_{|y-x|=r} |f(y) - f(x)|} \le H \tag{1.3}$$

for all $x \in \mathbf{R}^n$ and all $r > 0$. The numbers H and K are bounded in terms of each other and n. It is well known that the right hand side of (1.3) may be replaced by the $\limsup_{r\to 0}$ of the same expression. A beautiful recent result of Heinonen and Koskela [HK] says that limsup may even be replaced by liminf.

The analytic definition of qc-maps [Vä, 34.6] has an analog in the bilipschitz world, too. Namely, a homeomorphism f is L-bilipschitz if and only if f is ACL and the derivative has bounded dilatation a.e., that is

$$\frac{1}{L}|y| \le \bigl|\bigl(Df(x)\bigr)y\bigr| \le L|y|$$

for a.e. $x \in \mathbf{R}^n$ and all $y \in \mathbf{R}^n$, $y \ne 0$.

In this paper, we will show that the geometric definition (1.1) has a counterpart for bilipschitz maps, and give an application.

Theorem 1.1. *Let f be a homeomorphism of $\mathbf{R}^n$. Then f is bilipschitz if and only if there is a constant M such that*

$$|M(f(A)) - M(A)| \leq M \tag{1.4}$$

for all rings $A \subset \mathbf{R}^n$.

If f is bilipschitz then M is bounded in terms of $\|f\|_{\mathrm{bl}}$ and n only. Conversely, if (1.4) *holds for all rings A, and if f fixes two points, then f is L-bilipschitz with L depending on M and n only.*

The proof shows that *rings* can be replaced by *spherical rings* in Theorem 1.1. In Section 2 (Theorem 2.2) we will also show that M is small if and only if $\|f\|_{\mathrm{bl}}$ is close to 1 (provided f fixes two points).

Note that conformal linear transformations of $\mathbf{R}^n$ satisfy (1.4) with $M = 0$, but can have arbitrarily large bilipschitz norm. Thus a normalization such as $f(0) = 0$, $f(1) = 1$ is necessary in order to obtain a bound for $\|f\|_{\mathrm{bl}}$.

Note further that it is essential to consider *rings* in Theorem 1.1. The corresponding statement with rings replaced by rectangles is false.

That bilipschitz maps satisfy (1.4) (for spherical rings in $n = 2$) has been observed and used already in [FH]. For the converse, note that there is no regularity of f assumed in Theorem 1.1. The first part of its proof consists in showing that f is quasiconformal if (1.4) holds. Taking this for granted, Theorem 1.1 follows from

Theorem 1.2. *Let f be a K-quasiconformal homeomorphism of $\mathbf{R}^n$ and $E \subset \mathbf{R}^n$ any set. Assume that there is a constant M such that*

$$|M(f(A)) - M(A)| \leq M$$

holds for all spherical rings A centered at points of E with the property that both boundary spheres meet E. Then the restriction of f to E is bilipshitz. If f fixes two points of E, then the bilipschitz norm (*on E*) *is bounded by K, M and n only.*

For many questions concerning bilipschitz maps, such as factorization questions, it would be desirable to have a characterization of bilipschitz maps in terms of analytic quantities such as the Beltrami coefficient $\mu_f(z) = \overline{\partial} f(z)/\partial f(z)$ in the plane. Beginning with the work of Carleson [Ca], several sufficient conditions for a quasiconformal homeomorphism of $\mathbf{R}^2$ to be absolutely continuous (with respect to one dimensional Hausdorff measure), when restricted to $\mathbf{R}$, have been given, see [Be], [Se], [AZ], [Dy] and the references therein, and [FKP] for generalizations. These results have in common that usually an a priori assumption on the regularity of $f(\mathbf{R})$ (for instance $f(\mathbf{R}) = \mathbf{R}$) is made, as well as a certain control on $|\mu_f(z)|$ as z approaches $\mathbf{R}$. Not much seems to be known (and can be said) for arbitrary sets E instead of $\mathbf{R}$. However, there is a classical Dini-type condition on the dilatation implying the modulus estimate (1.4). When combined with Theorem 1.2, it yields Corollary 1.3 below which is best possible.

For orientation preserving K-quasiconformal homeomorphisms f of $\mathbf{R}^2$, let $\mu_f = \overline{\partial} f/\partial f$ be the Beltrami coefficient and set

$$I_f(x) = \int_{\{|y-x|\le 1\}} \frac{|\mu_f(y)|}{|x-y|^2}\, dy.$$

Corollary 1.3. *Let $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ be quasiconformal , $E \subset \mathbf{R}^2$ bounded. If $I_f(x) \le M$ for some $M > 0$ and all $x \in E$, then f is bilipschitz on E.*

In particular, if $I_f(x) < \infty$ for all $x \in E$, then f is absolutely continuous for Hausdorff-measure H_t: For $t \ge 0$,

$$H_t(E) = 0 \quad \text{if and only if} \quad H_t\bigl(f(E)\bigr) = 0. \tag{1.5}$$

This is sharp in the following sense:

Theorem 1.4. *Let $h\colon [0,\infty) \to [0,\infty)$ be any decreasing function with $h(t)/t \to 0$ as $t \to 0$ and let $0 < d < 2$. Then there is $M > 0$, a compact set $E \subset \mathbf{R}^2$ and a quasiconformal homeomorphism f of $\mathbf{R}^2$ such that $0 < H_d(E) < \infty$, $H_d\bigl(f(E)\bigr) = 0$ and*

$$\tilde{I}_f(x) \equiv \int_{\{|y-x|\le 1\}} \frac{h\bigl(|\mu_f(y)|\bigr)}{|x-y|^2}\, dy \le M$$

for all $x \in E$.

In the next section, we discuss the relation between modulus and euclidean quantities, prove Theorems 1.2 and 1.3 and discuss the case that $\|f\|_{\mathrm{bl}}$ or M is small (Theorem 2.2 below). Section 3 is devoted to a discussion of Corollary 1.3 and the construction of an example as stated in Theorem 1.4.

I would like to thank the referee for his comments, leading to an improvement of the exposition.

2. Bilipschitz maps

Consider a ring $A \subset \mathbf{R}^n$. The idea behind Theorems 1.1 and 1.2 is the well-known fact that the modulus of A can be estimated by euclidean quantities. To make this precise, denote by A_1, respectively A_2, the bounded, respectively unbounded, component of $\mathbf{R}^n \setminus A$ and set $r(A) = \operatorname{diam} A_1$, $R(A) = \operatorname{dist}(A_1, A_2)$. The following lemma is just a combination of well-known estimates.

Lemma 2.1. *For each $n \ge 2$ there is a constant C such that*

$$\left| M(A) - \log\left(1 + \frac{R(A)}{r(A)}\right)\right| \le C \tag{2.1}$$

for all rings $A \subset \mathbf{R}^n$.

Notice that

$$M(A) \geq \log \frac{R(A)}{r(A)}$$

for $R(A) > r(A)$ by the monotonicity of the modulus (with respect to inclusion), see [Vä, 11.4]. The lower bound $M(A) - \log\bigl(1 + R(A)/r(A)\bigr) > -C$ of (2.1) (with $C = \log 2$) follows at once. The corresponding upper bound is deeper. For $n = 2$ it is not hard to prove using the Koebe distortion theorem. In general ($n \geq 2$) it follows from the monotonicity of the modulus under spherical symmetrization [Ge], together with an estimate of the modulus of the Teichmüller ring. See [Ge, Theorem 4] for $n = 3$ and for instance [Vu, Chapter 2] for estimates in all dimensions.

Proof of Theorem 1.2. Composing with a conformal transformation of $\mathbf{R}^n$, we may assume that f fixes two points of E. Of course, the bilipschitz norm could change.

Let $a, b \in E$ with $f(a) = a$, $f(b) = b$ and consider a conformal transformation T of R^n with $T(a) = 0$, $T(b) = 1$ (where we write 1 for the unit vector $(1, 0, \dots, 0)$ of $\mathbf{R}^n$). Then $T \circ f \circ T^{-1}$ fixes 0 and 1, and has the same bilipschitz norm (on $T(E)$) as f (on E). We thus may assume $f(0) = 0$, $f(1) = 1$ and $0, 1 \in E$.

First we show that f is bilipschitz at 0, that is

$$L^{-1}|x| \leq |f(x)| \leq L|x| \tag{2.2}$$

for $x \in E$. As f was supposed to be quasiconformal, it satisfies (1.3). A formally stronger but equivalent statement is that there is an increasing homeomorphism $\phi\colon (0, \infty) \to (0, \infty)$ (depending on K and n only) such that

$$|f(x) - f(z)| \leq \phi(t)|f(y) - f(z)| \tag{2.3}$$

whenever $|x - z| \leq t|y - z|$ (see [TV1]).

From this (2.2) (with L depending on K and n only) is immediate if $\frac{1}{2} < |x| < 2$ (even without the restriction $x \in E$).

We will write $a \sim b$ (respectively $a \lesssim b$) if $|a/b|$ is bounded above and below (respectively bounded above) by positive constants depending on K and n only.

For $|x| \leq \frac{1}{2}$, $x \in E$, let A be the spherical annulus with radii $|x|$ and 1, centered at 0. Then both boundary components of A meet E. By quasisymmetry (2.3) with $z = 0$, $r(f(A)) \sim |f(x)| \lesssim 1$ and $R\bigl(f(A)\bigr) \sim 1$ (since $f(1) = 1$). Together with (2.1) we get

$$\left|\log \frac{|f(x)|}{|x|}\right| = \left|\log \frac{1}{|x|} - \log \frac{1}{|f(x)|}\right| \lesssim \bigl|M(A) - M\bigl(f(A)\bigr)\bigr| + 1$$

and conclude (2.2). The case $|x| \geq 2$ is similar (take the annulus with radii 1 and $|x|$).

The same argument, switching the roles of 0 and 1, yields

$$L^{-1}|x-1| \le |f(x)-1| \le L|x-1|. \tag{2.4}$$

Now let $x, y \in E\setminus\{0\}$ be arbitrary. Choose conformal linear transformations T, S of $\mathbf{R}^n$ with $T(1)=x$, $S\bigl(f(x)\bigr)=1$ and set $F=S\circ f\circ T$. Then $F(0)=0$, $F(1)=1$, and by the invariance of the modulus under conformal mappings

$$\bigl|M\bigl(F(A)\bigr)-M(A)\bigr| = \bigl|M\bigl(f\bigl(T(A)\bigr)\bigr)-M\bigl(T(A)\bigr)\bigr| \le M$$

for all annuli A centered at points of $T^{-1}(E)$ with the property that both boundary components meet $T^{-1}(E)$. Hence (2.4) applies to F and $T^{-1}(E)$. In particular

$$L^{-1}|T^{-1}(y)-1| \le \bigl|F\bigl(T^{-1}(y)\bigr)-1\bigr| = \bigl|S\bigl(f(y)\bigr)-S\bigl(f(x)\bigr)\bigr| \le L|T^{-1}(y)-1|.$$

Using the linearity of S and T, and $|T(u)|=|x|\,|u|$, $|S(u)|=|u|/|f(x)|$ for all $u\in\mathbf{R}^n\setminus\{0\}$ we obtain

$$L^{-1}\frac{|y-x|}{|x|} \le \frac{|f(y)-f(x)|}{|f(x)|} \le L\frac{|y-x|}{|x|}.$$

Using (2.2) we conclude

$$\frac{|x-y|}{L^2} \le |f(x)-f(y)| \le L^2|x-y|$$

and are done.

Proof of Theorem 1.1. If f is L-bilipschitz and A a ring, then $L^{-1}r(A)\le r\bigl(f(A)\bigr)\le Lr(A)$ and $L^{-1}R(A)\le R\bigl(f(A)\bigr)\le LR(A)$, thus (2.1) yields

$$\bigl|M(A)-M\bigl(f(A)\bigr)\bigr| \le 2C+\left|\log\left(1+\frac{R(A)}{r(A)}\right)-\log\left(1+\frac{R\bigl(f(A)\bigr)}{r\bigl(f(A)\bigr)}\right)\right| \le 2(C+\log L).$$

The converse follows from Theorem 1.2, once we have shown that f is K-quasiconformal with K depending on M only. This will be achieved by verifying the metric definition (1.3).

Set $\varepsilon=\exp(-M-2C)$ where C is from (2.1) and M from (1.4). Given $x\in\mathbf{R}^n$ and $r>0$, consider the spherical rings $A_1=A(x,r\varepsilon,r)$, $A_2=A(x,r,r/\varepsilon)$ and $A_3=A(x,r\varepsilon,r/\varepsilon)$, where we used the notation $A(x,r,R)=\{y\in\mathbf{R}^n : r<|y-x|<R\}$. Write R_j and r_j for $R\bigl(f(A_j)\bigr)$ and $r\bigl(f(A_j)\bigr)$.

By Lemma 2.1 and (1.4) we have

$$C\le\log\left(1+\frac{R_j}{r_j}\right)\le 3M+5C$$

and conclude

$$c^{-1}\le\frac{R_j}{r_j}\le c$$

for $j=1,2,3$, with some $c>1$ depending on n and M only. Now

$$\frac{\max_{|y-x|=r}|f(y)-f(x)|}{\min_{|y-x|=r}|f(y)-f(x)|}\le\frac{r_2}{R_1}\le c\frac{r_2}{r_1}=c\frac{r_2}{r_3}\le c^2\frac{r_2}{R_3}\le c^2\frac{r_2}{R_2}\le c^3$$

and the proof is finished.

We now discuss the dependence of $\|f\|_{\mathrm{bl}}$ and M if one of the quantities is small. For a bilipschitz map f of $\mathbf{R}^n$, denote by $M(f)$ the smallest M such that (1.4) holds for all spherical rings.

Theorem 2.2. *For each $n \geq 2$ there are continuous functions ϕ: $[1,\infty) \to [0,\infty)$ and ψ: $[0,\infty) \to [1,\infty)$ with $\phi(1) = 0$, $\psi(0) = 1$, such that*

$$M(f) \leq \phi(\|f\|_{\mathrm{bl}}), \qquad \|f\|_{\mathrm{bl}} \leq \psi\bigl(M(f)\bigr)$$

for all bilipschitz maps f of $\mathbf{R}^n$ that fix two points.

In the special case $M = 0$, the conclusion is that 1-quasiconformal homeomorphisms of $\overline{\mathbf{R}}^n$ are Möbius transformations (since the isometries of $\mathbf{R}^n$ are conformal). But our proof relies on this well-known result of Gehring and Reshetnjak. See [TV2] for an elementary proof and references.

Proof. First consider a bilipschitz homeomorphism f of $\mathbf{R}^n$ with $L = \|f\|_{\mathrm{bl}}$ close to 1. Then f is quasiconformal with $\|f\|_{\mathrm{qc}} \leq L^{2n-2}$. If A is a spherical ring with $M(A) \leq 2$, we obtain (see (1.1)) $M\bigl(f(A)\bigr) - M(A) \leq (K'-1)M(A)$, $M(A) - M\bigl(f(A)\bigr) \leq (K'-1)M\bigl(f(A)\bigr) \leq K'(K'-1)M(A)$ and conclude

$$\bigl|M\bigl(f(A)\bigr) - M(A)\bigr| \leq 2L^2(L^2-1).$$

For $M(A) > 2$ we use the monotonicity of the modulus as in [FH]: If $A = A(x,r,R)$ is a spherical ring with $R/r > e^2$, then

$$A\bigl(f(x), r/L, RL\bigr) \subset f(A) \subset A\bigl(f(x), Lr, R/L\bigr)$$

(provided that $L < e$) and

$$\bigl|M\bigl(f(A)\bigr) - M(A)\bigr| \leq 2\log L$$

follows.

For the opposite direction, consider a homeomorphism f with $M(f)$ close to zero that fixes two points. We may assume $f(0) = 0$, $f(1) = 1$.

Observe first that $\|f\|_{\mathrm{qc}}$ is close to one: For (affine) linear maps this is easy, using a normal family argument. As the differential $y \mapsto Df(x)y$ satisfies $M\bigl(Df(x)\bigr) \leq M(f)$ in all points of differentiability of f, hence almost everywhere (we have already proven that f is bilipschitz), the claim follows.

The proof of Theorem 1.2 shows that it suffices to show

$$L^{-1}|x| \leq |f(x)| \leq L|x| \tag{2.5}$$

for all $x \in \mathbf{R}^n$ with L close to one. Again we distinguish three cases: If $\frac{1}{2} < |x| < 2$, (2.5) follows at once from the aforementioned fact that 1-quasiconformal maps of $\mathbf{R}^n$ are conformal, together with a normal family argument.

To deal with the other cases, we need an improved version of Lemma 2.1 applying to rings which are almost spherical: For each $n \geq 2$ there is a continuous function C: $[1, \varepsilon_0) \to [0, \infty)$ with $C(1) = 0$ such that

$$\left| M\big(f(A)\big) - \log\left(1 + \frac{R\big(f(A)\big)}{r\big(f(A)\big)}\right)\right| \leq C(\|f\|_{\mathrm{qc}})$$

for all spherical rings $A \subset \mathbf{R}^n$ with $M(A) > \log 2$ and all quasiconformal maps f of $\mathbf{R}^n$. This is easily proven using monotonicity of the modulus (under inclusion) as above, together with the conformality of 1-quasiconformal maps and normal families. Now (2.5) (and the theorem) follows as in the proof of Theorem 1.2.

3. Absolute continuity

Proof of Corollary 1.3. By [LV, Chapter V.6],

$$\big|M\big(f(A)\big) - M(A)\big| \leq C(K) \int_A \frac{|\mu_f(y)|}{|x-y|^2}\, dy$$

for rings $A = A(x, r, R) \subset \mathbf{R}^2$ and orientation preserving K-quasiconformal maps of A. Thus the assumptions of Theorem 1.2 are satisfied if $I_f(x) \leq M$ for all $x \in E$ and if E is bounded. Hence f is bilipschitz on E by Theorem 1.2.

The assumption $I_f(x) < \infty$ is very strong and implies (Teichmüller, Wittich and Belinskij) that f is conformal at x, see [LV, Chapter V.6]. Notice that (1.5) already follows from this (without using Corollary 1.3).

If f is a quasiconformal homeomorphism of the upper half plane $\mathbf{H}$ and if $\mu(t) = \operatorname{ess\,sup}_{\{z=x+iy:0<y<t\}}|\mu(z)|$, then the condition $I_f(x) < \infty$ is slightly weaker than the condition $\int_0^1 \mu(t)/t\, dt < \infty$. Carleson showed in [Ca] that the latter condition implies smoothness of the extension of f on $\mathbf{R}$, and that already $\int_0^1 \mu(t)^2/t\, dt < \infty$ implies absolute continuity (even $|f'| \in A_\infty$) on $\mathbf{R}$. The improvement in the exponent is possible because of the assumption that $f(\mathbf{R}) = \mathbf{R}$. This assumption, as well as the integral condition, has been somewhat weakened (see the references given in the introduction), but the exponent 2 is best possible. Roughly speaking, Theorem 1.4 says that the exponent 1 of $|\mu|$ in I_f is best possible in order to conclude (1.5).

Proof of Theorem 1.4. Choose $0 < \varepsilon < \frac{1}{4}$, an integer $2 \leq n \leq c/\varepsilon^2$ (where c is an absolute constant) and pick n disjoint closed disks $D_1, \ldots, D_n \subset \{z : |z| < \frac{1}{4}\}$ of radius ε. Let a_k denote the midpoint of D_k and set $\phi_k(z) = \varepsilon z + a_k$. Then the ϕ_k are contractions with $\phi_k(\overline{\mathbf{D}}) = D_k$, where $\mathbf{D}$ denotes the unit disk.

Let E be the (unique) compact set with $E = \bigcup_{1\leq k\leq n} \phi_k(E)$. Then the Hausdorff dimension d of E equals its similarity dimension: $n\varepsilon^d = 1$, that is

$$d = \frac{\log n}{|\log \varepsilon|}. \tag{3.1}$$

By choosing ε and n appropriately, every dimension d between 0 and 2 can be achieved this way. Furthermore, E has finite and nonzero d-dimensional Hausdorff measure. E can be obtained in the following way: Let F_m be the set of all compositions of m of the functions $\phi_1, \dots, \phi_n$ so that F_m has n^m elements. Then the sequence $\bigcup_{f\in F_m} f(\overline{\mathbf{D}})$ of compact sets decreases to E as $m \to \infty$. See [Fa, Chapter 8.3] for these facts.

Define a family g_s $(0 < s < 1)$ of $(1+s)$-quasiconformal homeomorphism of $\mathbf{D}$ by $g_s(z) = z|z|^s$ for $\frac{1}{2} < |z| < 1$ and $g_s(z) = \left(\frac{1}{2}\right)^s z$ for $|z| \le \frac{1}{2}$. Given a decreasing sequence $s_k < 1$ with $\lim_{k\to\infty} s_k = 0$, we define an infinite sequence f_m of $(1+s_1)$-quasiconformal homeomorphisms of $\mathbf{D}$ inductively as follows:

Let $f_1 = g_{s_1}$. Obtain f_2 from f_1 by modifying f_1 in $\cup D_k$ only: For each $k = 1, 2, \dots, n$, set $f_2 = f_1 \circ \phi_k \circ g_{s_2} \circ \phi_k^{-1}$ on D_k. As f_1 is linear in $|z| < \frac{1}{2}$, it maps each disk D_k to a disk D_k' (of radius $\left(\frac{1}{2}\right)^{s_1}\varepsilon$), and all we have done is to glue a scaled version of g_{s_2} into D_k. Notice that f_2 is still $(1+s_1)$-quasiconformal in $\mathbf{D}$: It is $(1+s_1)$-qc in $\left\{\frac{1}{2} < |z| < 1\right\}$, $(1+s_2)$-qc in $\cup D_k$ and linear in $\left\{|z| < \frac{1}{2}\right\} \setminus \cup D_k$.

Assume that f_m is already constructed, is $(1+s_1)$-qc in $\mathbf{D}$ and linear in each disk $f(\mathbf{D})$ for $f \in F_m$. Then define f_{m+1} to coincide with f_m on $\mathbf{D}\setminus\bigcup_{f\in F_m} f(\mathbf{D})$, and for each $f \in F_m$ define

$$f_{m+1} = f_m \circ f \circ g_{s_{m+1}} \circ f^{-1}$$

on $f(\mathbf{D})$. Now f_{m+1} is $(1+s_{m+1})$-quasiconformal in each disk $f(\mathbf{D})$ for $f \in F_m$ and linear in each disk $f(\mathbf{D})$ for $f \in F_{m+1}$, and the inductive step is complete.

Consider the limiting homeomorphism $f_\infty = \lim_{m\to\infty} f_m$. It is $(1+s_1)$-qc and maps the disks $f(\mathbf{D})$ $(f \in F_m)$ of radius ε^m to disks of radius

$$r_m = \varepsilon^m \left(\tfrac{1}{2}\right)^{(s_1+\cdots+s_m)}. \tag{3.2}$$

Extend f_∞ to $\mathbf{C}$ by the identity outside $\mathbf{D}$, call the resulting map f, and let us estimate the integral

$$\tilde{I}_f(x) = \int_{\{|y-x|\le 1\}} \frac{h\big(|\mu_f(y)|\big)}{|x-y|^2}\, dy$$

for $x \in E$. For each $m \ge 1$, f is $(1+s_m)$-quasiconformal in $A_m = A(x, \varepsilon^{m+1}, \varepsilon^m)$ and we get the estimate $|\mu_f| \le s_m$ in A_m. Thus

$$\tilde{I}_f(x) \le C + 2\pi\Big(\log\frac{1}{\varepsilon}\Big)\sum_{m\ge 1} h(s_m).$$

Choose the sequence s_m in such a way that $\sum h(s_m)$ converges but $\sum s_m$ diverges (assuming that $h(t)/t \to 0$ as $t \to 0$). Then (3.2) implies $H_d\big(f(E)\big) = 0$ for the d-dimensional Hausdorff measure, where d is the Hausdorff dimension of E given by (3.1).

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Received 7 March 1996