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PAIRS OF SYMMETRIES OF RIEMANN SURFACES

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Abstract. Let F_g be a compact Riemann surface of genus g. A symmetry S of F_g is an anticonformal involution acting on F_g. The fixed-point set of a symmetry is a collection of disjoint simple closed curves, called the mirrors of the symmetry. The number of mirrors $|S|$ of a symmetry of a surface of genus g can be any integer k with $0 \le k \le g+1$. However, if a Riemann surface F_g admits a symmetry S_1 with k mirrors then work of Bujalance and Costa [1] and Natanzon [9] on symmetries with $g+1$ mirrors suggest that there may possibly be restrictions on the number of mirrors of another symmetry S_2 of F_g. In the first three sections of this work we show that the number of such restrictions is few and only occur if one of the symmetries has $g+1$ or 0 mirrors. The main result of Sections 1–3 is Theorem 1.1 below. In Section 4 we study a finer classification than the number of mirrors, namely the species of a symmetry. The k mirrors of a symmetry S may or may not separate the surface F_g into two non-empty components. If the mirrors do separate, then we say that S has species $+k$, and if the mirrors do not separate then we say that the species is $-k$. (See [5].) The species of S determines S up to topological conjugacy. In Section 4 we investigate which pairs of species can occur for two symmetries S_1, S_2 of F_g. There are many more restrictions than when we just ask for the number of mirrors.

1. Introduction

Let F_g be a compact Riemann surface of genus $g \ge 2$. A symmetry S of F_g is an anticonformal involution acting on F_g. By Harnack's theorem the fixed points set of S consists of $k \le g+1$ simple closed curves, called *mirrors*. The number of mirrors of a symmetry S is denoted by $|S|$.

Using Hoare's theorem [6] Bujalance, Costa and Singerman [4] gave a method to calculate the total number of mirrors of two symmetries S_1, S_2 acting on a Riemann surface of genus g. The work there suggests that there may be some restrictions on the possible pairs of integers $(|S_1|, |S_2|)$ that can occur. In fact, we show that these restrictions are few and in Theorems 2.1, 2.2 and 3.2 we find all pairs $(|S_1|, |S_2|)$ that can occur. These results can be summarised as follows.

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Theorem 1.1. *Let k_1, k_2 be two integers with $0 \leq k_1 \leq k_2 \leq g+1$. Then we can find a Riemann surface F_g of genus g admitting a pair of symmetries S_1, S_2 with $|S_1| = k_1$, $|S_2| = k_2$ whenever $1 \leq k_1 \leq k_2 \leq g$ or when*

$k_2 = g+1$ with g even and $k_1 = 0, 1$ or $k_1 \equiv g+1 \bmod 2$, or
$k_2 = g+1$ with g odd and $k_1 = 0, 1, 2$ or $k_1 \equiv g+1 \bmod 2$, or
$k_1 = 0$ with g even and $k_2 = 0$, or k_2 is odd or
$k_1 = 0$ with g odd and $0 \leq k_2 \leq g+1$ arbitrary.

No pairs k_1, k_2 outside of this list can occur.

The results where $k_2 = g+1$ follow from work of Natanzon [9] and of Bujalance and Costa [1], (see Theorem 2.1). Further work of Natanzon [11] shows that the restrictions for a surface admitting more than two conjugacy classes of symmetries are likely to be more severe. For example it is shown there that if S_1, S_2, S_3 are three non-conjugate symmetries of a surface of genus g then $|S_1| + |S_2| + |S_3| \leq 2g+4$.

1.1. Real algebraic curves. One motivation for this study comes from real algebraic geometry. Whereas a compact Riemann surface corresponds to a complex algebraic curve, a compact symmetric surface corresponds to a real algebraic curve, each conjugacy class of symmetries in $\mathrm{Aut}(F_g)$, (the group of conformal and anticonformal automorphisms of F_g), corresponding to a different real model of the curve. The mirrors of the symmetry correspond to the components of the real curve. Thus, if there are two conjugacy classes of symmetries, S_1, S_2 with $|S_1| = k_1$ and $|S_2| = k_2$ then we have exactly two real models for the curve, one with k_1 components and one with k_2 components.

1.2. Preliminaries on NEC groups and Riemann surfaces. A Riemann surface of genus $g > 1$ is the quotient of the hyperbolic plane H by a *Fuchsian group*, a discrete subgroup of $\mathrm{Aut}^+(H)$ without elliptic elements. A discrete subgroup of $\mathrm{Aut}(H)$ with compact quotient is called an *NEC* (*non-Euclidean crystallographic*) *group*. Given an NEC group Γ the subgroup of Γ consisting of the orientation-preserving elements is called the *canonical Fuchsian group of* Γ. It is denoted by Γ^+. The algebraic structure of an NEC group Γ and the geometric structure of its quotient orbifold H/Γ are determined by the signature of Γ:

$$s(\Gamma) = \big(h; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}\big). \tag{1.1}$$

The quotient space H/Γ is an orbifold with underlying surface of genus h with r cone points and k mirror lines, each with $s_j \geq 0$ corner points. The signs $+$ or $-$ correspond to orientable or non-orientable orbifolds respectively. The integers m_i are called the proper periods of Γ, they are the orders of the cone points of H/Γ. The k brackets $(n_{j1}, \ldots, n_{js_j})$ are the period cycles of Γ and the integers n_{jh} are the link periods of Γ, the orders of the corner points of H/Γ.

Γ is called the *group* (or fundamental group) of the orbifold H/Γ.

Associated to the signature (1.1) there is a presentation for Γ with generators

$$
\begin{gathered}
x_1, \dots, x_r, \\
e_1, \dots, e_k, \\
c_{ij}, \quad 1 \leq i \leq k,\ 0 \leq j \leq s_i, \\
a_1, b_1, \dots, a_h, b_h \quad \text{if H}/\Gamma \text{ is orientable or} \\
a_1, \dots, a_h, \quad \text{if H}/\Gamma \text{ is non-orientable.}
\end{gathered}
$$

and relators

$$
\begin{gathered}
x_i^{m_i}, \quad i = 1, \dots, r, \\
c_{ij-1}^2, \quad c_{ij}^2, \quad (c_{ij-1}c_{ij})^{n_{ij}}, \quad i = 1, \dots, k,\ j = 0, \dots, s_i, \\
c_{i0}e_i^{-1}c_{is_i}e_i, \\
x_1x_2 \cdots x_r e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_h^{-1} b_h^{-1}, \quad \text{if H}/\Gamma \text{ is orientable or} \\
x_1 x_2 \cdots x_r e_1 \cdots e_k a_1^2 \cdots a_h^2, \quad \text{if H}/\Gamma \text{ is non-orientable.}
\end{gathered}
$$

These last two relators are sometimes called the *long relators*, which give rise to the *long relations* by putting them equal to 1. In these presentations, the only elements of finite order are the elliptic elements and the reflections. The elliptic elements are conjugates of powers of the x_i or $c_{ij-1}c_{ij}$ and the reflections are conjugates of the c_{ij}. The e_i generators are orientation preserving. They are called the *connecting generators*. An NEC group without elliptic elements is called a *surface group*. If Γ is an NEC group then H/Γ is a *Klein surface*, i.e., a surface with a dianalytic structure. A Klein surface whose complex double has genus greater than one can be expressed as H/Γ where Γ is an NEC surface group.

The hyperbolic area of the quotient orbifold is:

$$
\mu(\Gamma) = 2\pi\left(\varepsilon h - 2 + k + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right)\right), \tag{1.2}
$$

where $\varepsilon = 2$ if there is a $+$ sign and $\varepsilon = 1$ if there is a $-$ sign. If Γ^* is a subgroup of Γ of finite index then the Riemann–Hurwitz formula holds:

$$
|\Gamma : \Gamma^*| = \frac{\mu(\Gamma^*)}{\mu(\Gamma)}. \tag{1.3}
$$

Let S be a symmetry acting on a Riemann surface F with group Γ. Then $F/\langle S\rangle$ is a Klein surface that can be represented as H/Λ where Λ is a surface group with signature

$$
s(\Lambda) = \big(h_0; \pm; [\ ]; \{(\ )^k\}\big),
$$

and $|\Lambda:\Gamma|$ =2. The notation means that there are k empty period cycles in Λ. As $H/\Lambda = F/\langle S\rangle$, it follows that k is the number of mirrors of S.

We assume that a Riemann surface F with group Γ admits two symmetries S_1 and S_2, with k_1 and k_2 mirrors respectively, such that S_1S_2 has order n. Then S_1 and S_2 generate a dihedral group D_n of order $2n$. Let Δ be the NEC group generated by all the liftings to H of the elements of D_n. Then there is an epimorphism $\theta\colon \Delta \to D_n$ such that $\Gamma = \mathrm{Ker}(\theta)$. Note that Γ is a Fuchsian surface group. If $\Lambda_1 = \theta^{-1}(\langle S_1\rangle)$ and $\Lambda_2 = \theta^{-1}(\langle S_2\rangle)$, then

$$s(\Lambda_1) = \big(h_1;\pm;[\,];\{(\,)^{k_1}\}\big), \qquad \text{and} \qquad s(\Lambda_2) = \big(h_2;\pm;[\,];\{(\,)^{k_2}\}\big),$$

where

$$|\Delta:\Lambda_1| = |\Delta:\Lambda_2| = n \quad \text{and} \quad |\Lambda_1:\Gamma| = |\Lambda_2:\Gamma| = 2$$

Notation. From now on D_n is the group with presentation

$$\langle S, Q \mid S^2, Q^n, (SQ)^2\rangle, \tag{1.4}$$

where we identify S_1 with S and S_2 with SQ.

1.3. Strategy. Our aim is to compute the integers k_1, k_2, the numbers of mirrors of S_1, S_2 respectively. Now Λ_1 and Λ_2 are subgroups of the NEC group Δ and the permutation representation of Δ on the Λ_i-cosets is known, being the same as the permutation representation of D_n on the $\langle S_i\rangle$-cosets, $(i = 1,2)$. We can now use Hoare's theorem [6] which gives an algorithm to compute signatures of subgroups of NEC groups to calculate k_1, k_2. The use of Hoare's theorem in this context is explained in detail in [4]. As every reflection of Λ_i, $(i = 1,2)$ is conjugate to a reflection of Δ, the period cycles of Λ are 'induced' by the period cycles of Δ. (Topologically, we have an orbifold covering $H/\Lambda_i \to H/\Delta$ and we have to examine how the holes of H/Δ lift.) In [4] a graphical method was used to facilitate the application of Hoare's techniques in our context. We have modified this slightly in the description that we now give.

1.4. Hoare diagrams. If $\theta\colon \Delta \to D_n$ is the homomorphism above we associate a Hoare diagram to every period cycle of Δ, the purpose being to be able to read off from the diagram the number of period cycles of Λ_1 and Λ_2 and hence the number of mirrors of S_1 and S_2. The diagrams will have vertices and edges of two possible colours which we call blue and red and (as we shall see) the number of mirrors of S_1 will be the number of blue components and the number of mirrors of S_2 will be the number of red components. It turns out that the total number of mirrors is found just by adding the number of mirrors coming from each period cycle so we just need do the calculations for groups with one period cycle. Let us assume that Δ has signature

$$\big(0;+;[\,];\{(n_1,n_2,\dots,n_s)\}\big),$$

and presentation

$$\langle c_1, \dots, c_s \mid c_1^2, \dots, c_s^2, (c_1c_2)^{n_1}, \dots, (c_sc_1)^{n_s} \rangle.$$

i.e., Δ is the group generated by reflections in the sides of a hyperbolic polygon with angles $\pi/n_1, \pi/n_2, \dots, \pi/n_s$.

Case 1. Here we assume that at least one of the link periods is even. It then follows that n is even. The Hoare diagram of the pair (Δ, θ) is a coloured graph whose vertices are those of a regular s-sided polygon. We label the s vertices with the generating reflections $c_1, \dots, c_s$ and colour the vertex c_j blue or red if $\theta(c_j) = SQ^{u_j}$ with u_j even or odd respectively. The edges of the polygon are labelled with the link periods $n_1, \dots, n_s$ with the edge joining the vertices c_i and c_{i+1} labelled n_i. If we have two consecutive vertices c_i and c_{i+1} with the same colour then we colour the edge joining them by that colour if and only if n_i is odd. By [4], the number of period cycles of Λ_1 is the number of blue components and the number of period cycles of Λ_2 is the number of red components. These numbers are just the numbers of mirrors of S_1 and S_2 respectively.

If the above period cycle is just part of a signature then the situation is slightly different in that we have a connecting generator e and the relations

$$(c_0c_1)^{n_1} = (c_1c_2)^{n_2} = \cdots = (c_{s-1}c_s)^{n_s} = ec_0e^{-1}c_s = 1.$$

We now consider a polygon with $s+1$ vertices $c_0, c_1, \dots, c_s$. We consider conjugate reflections as representing the same vertex so that ec_0e^{-1} represents the same vertex as c_0. Then the relation $ec_0e^{-1}c_s = 1$ implies that c_0 and c_s are joined by an edge of the same colour as the vertices c_0 and c_s. Note that as c_0 and c_s are conjugate they must have the same colour. Thus the number of components of each colour is the same if the period cycle is on its own or just part of a signature.

Case 2. An empty period cycle. We split this up in two subcases.

2(i) n is even. The generators and relations associated to an empty period cycle are

$$\langle c, e \mid c^2 = ece^{-1}c = 1 \rangle.$$

Suppose that $\theta(c) = S$. Then as e and c commute, $\theta(e) = 1$ or $Q^{n/2}$. In the first case we find that there are two induced period cycles on Λ_1 and none on Λ_2 and in the second case we find that there is one induced period cycle on Λ_1 and none on Λ_2. As the empty period cycle case of Hoare's theorem requires careful interpretation and as there are some misprints in [4], we outline a proof of these results.

Writing $L_i = \langle S_i \rangle$ $(i = 1, 2)$, then

$$D_n = L_1 + L_1Q + \cdots + L_1Q^{n-1}.$$

Let $g \in \Delta$ obey $\theta(g) = Q$. Then

$$\Delta = \Lambda_1 + \Lambda_1 g + \cdots + \Lambda_1 g^{n-1}.$$

We are looking for the number of conjugacy classes of c in Λ_1. Every conjugate of c in Δ has the form $g^k c g^{-k}$ and $g^k c g^{-k} \in \Lambda_1$ if and only if $Q^k S Q^{-k} = S$, that is if and only if $k = 0$ or $n/2$. Thus Λ contains at most two conjugacy classes represented by c and $g^{n/2} c g^{-n/2}$. If these are conjugate in Λ_1 then for some $\lambda \in \Lambda$ we have

$$\lambda c \lambda^{-1} = g^{n/2} c g^{-n/2},$$

so that $\lambda^{-1} g^{n/2} \in$ centralizer (c). As centralizer (c) is just the group generated by e and c [12] we find that $g^{n/2} = \lambda e^k$ or $g^{n/2} = \lambda e^k c$. Applying θ we find that $Q^{n/2} = 1$ or S, which is false so that we have two conjugacy classes of reflections in Λ_1 and hence there are two induced period cycles in Λ_1. In the second case $\theta(e) = Q^{n/2}$ and a similar argument shows that there is now only one induced period cycle in Λ_1. In both cases it is easy to show that there are no induced period cycles in Λ_2.

2(ii) n is odd. A similar method shows that each empty period cycle induces one empty period cycle on Λ_1 and one on Λ_2.

Graphical representation. (i) n even. If $\theta(e) = 1$, and $\theta(c) = S$, or more generally, $\theta(c) = SQ^h$, h even, then our graph consists of two disjoint blue vertices; if $\theta(e) = 1$ and $\theta(c) = QS^k$, k odd, then our graph just consists of two disjoint red vertices. If $\theta(e) = Q^{n/2}$ and $\theta(c) = SQ^h$, h even, then our graph consists of one blue vertex; if $\theta(e) = Q^{n/2}$ and $\theta(c) = SQ^k$, k odd, then our graph consists of one red vertex. (ii) n odd. Here the graph consists of one red and one blue vertex.

Case 3. All link periods $n_1, \ldots, n_s$ are odd. These are called *odd period cycles* in [4].

3(i) n is even. Since $\theta(c_{j-1}c_j)$ has odd order, then $\theta(c_j) = SQ^{u_j}$, where u_j is either even for $1 \le j \le s$ or u_j odd for $1 \le j \le s$. Therefore an odd period cycle induces empty period cycles either in Λ_1 or in Λ_2, according to the parity of u_j in $\theta(c_j) = SQ^{u_j}$.

Similar arguments as in Case 2, shows that each odd period cycle induces either two empty period cycles in Λ_1 (respectively Λ_2), if $\theta(e_i) = 1$, or one empty period cycle if $\theta(e_i) \ne 1$.

3(ii) n is odd. It is shown that each odd period cycle induces one empty period cycle in Λ_1 and one in Λ_2.

Graphical representation. (i) n even. If $\theta(e) = 1$ and $\theta(c_j) = SQ^{u_j}$, u_j even, then our graph consists of two disjoint blue vertices; if $\theta(e) = 1$ and $\theta(c_j) = QS^{u_j}$, u_j odd, then our graph just consists of two disjoint red vertices. If $\theta(e) \ne 1$ and $\theta(c_j) = SQ^h$, h even, then our graph consists of one blue vertex; if $\theta(e) \ne 1$ and $\theta(c_j) = SQ^k$, k odd, then our graph consists of one red vertex. (ii) n odd. Here the graph consists of one red and one blue vertex.

As reflections from different period cycles cannot be conjugate the total number of induced period cycles on the Λ_i is just the sum of those induced from each period cycle. Thus the Hoare diagram associated to (Δ, θ) is just the disjoint union of the Hoare diagrams for each period cycle. The number of period cycles induced on Λ_1 which is equal to the number of mirrors of S_1 is the number of blue components, and the number of period cycles induced on Λ_2 which is equal to the number of mirrors of S_2 is the number of red components.

Note. Case 3(i) was missed in [4] which makes Theorem 2(i) of that paper incorrect. The equality there should be replaced by the following inequality.

$$\alpha + \beta + 2\gamma - \delta \leq t \leq \alpha + 2\beta + 2\gamma - \delta.$$

No other result of [4] is affected.

1.5. Examples. For a simple example, let us use the original example of Natanzon of a compact Riemann surface of genus g admitting two symmetries both having $g+1$ mirrors. To obtain this we use an NEC group with signature

$$\big(0; +; \{(2, 2, \dots, 2)\}\big)$$

where there are $2g+2$ link periods equal to 2 and the generating reflections are $c_1, \dots, c_{2g+2}$ with relations $(c_1c_2)^2 = \cdots = (c_{2g+1}c_{2g+2})^2 = (c_{2g+2}c_1)^2 = 1$ and the homomorphism $\theta\colon \Delta \to D_2$ is defined by the following action on the generators:

$$c_1 \to S, \quad c_2 \to SQ, \quad c_3 \to S, \quad \dots \quad c_{2g+2} \to SQ.$$

As there are no odd link periods the Hoare diagram has no edges and so just consists of $g+1$ blue vertices and $g+1$ red vertices. Thus we see that both S_1 and S_2 have $g+1$ mirrors as claimed.

For a more complicated example let Δ have signature

$$\big(0; +; \{(2, 2, 2, 2, 3, 3), (3), (\,)\}\big)$$

and a canonical presentation as in 1.2. We define a homomorphism $\theta\colon \Delta \to D_6$ by defining the action on the generators as follows:

$$\begin{array}{lll}
c_{10} \to S & c_{20} \to S & c_{30} \to S \\
c_{11} \to SQ^3 & c_{21} \to SQ^2 & e_3 \to 1 \\
c_{12} \to S & e_2 \to Q^2 & \\
c_{13} \to SQ^3 & & \\
c_{14} \to S & & \\
c_{15} \to SQ^4 & & \\
c_{16} \to SQ^2 & & \\
e_1 \to Q^4 & &
\end{array}$$

The Hoare diagram is as follows and from it we see that S_1 has 5 mirrors and S_2 has 2 mirrors. Theorem 2 of [4] just tells us that the total number of mirrors of S and SQ is equal to 7.

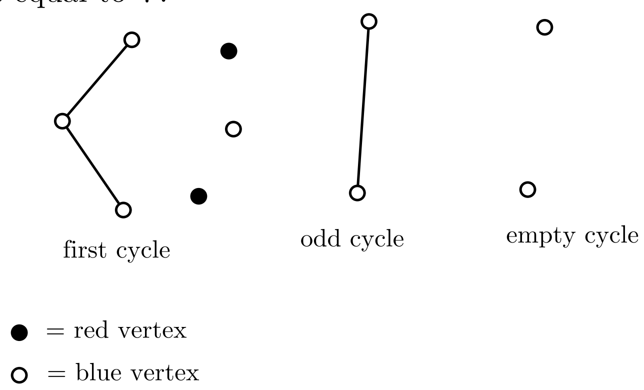

Figure 2.1

2. Calculation of possible pairs $|S_1|, |S_2|$

If F_g is a compact Riemann surface admitting a D_n action with $|S| = g+1$, then Bujalance and Costa [1], gave a complete list of possibilities for the number of mirrors of the paired symmetry SQ. Their result can be stated as follows:

Theorem 2.1. *Let D_n with presentation* (1.4) *act as a group of automorphisms of a compact Riemann surface F_g of genus g, with S and SQ acting as symmetries, and with $|S| = g+1$. If F_g is hyperelliptic then $|SQ| = g+1$, 0 or 1, if g is even and $|SQ| = g+1$, 0, 1 or 2 if g is odd. If F_g is non-hyperelliptic then $|SQ| = 0$ or $g+1-2t$, $0 \le t < \frac{1}{2}(g+1)$ if g is even and $|SQ| = g+1-2t$, $0 \le t \le \frac{1}{2}(g+1)$ if g is odd.*

Of course, the existence of surfaces with these pairs of symmetries can be obtained using Hoare diagrams. For example, to get a surface admitting symmetries with $g+1$ mirrors and $g+1-2t$ mirrors we use an NEC group Δ with signature

$$(0; +; [\,]; \{(2^{(r)}), (\,)^{(t)}\}) \tag{2.1}$$

with r even and $2t + \frac{1}{2}r = g+1$, (where the notation indicates t empty period cycles and r link periods equal to 2), and we map each of the connecting generators e_i to 1 and the reflection generators alternately to S and SQ in D_2. The Hoare diagram then consists of $2t + \frac{1}{2}r$ blue vertices and $\frac{1}{2}r$ red vertices.

Now that we have considered the case where one of the symmetries fixes the maximum number of mirrors, we investigate the other possibilities.

Theorem 2.2. *Let k_1 and k_2 be two integers with $1 \le k_1 \le g$, $1 \le k_2 \le g$. Then there exists a compact Riemann surface F_g of genus g admitting a pair of symmetries S_1, S_2, with $|S_1| = k_1$ and $|S_2| = k_2$.*

Proof. As above, we find an action of D_2 on a surface of genus g with $S_1 = S$ and $S_2 = SQ$. We first assume that $k_1 \equiv k_2 \bmod 2$ and that $k_1 \geq k_2$. Let Δ be a group with signature

$$\big(0; +; [2^{g-k_1+1}]; \{(2^{(2k_2)}), (\ )^{(k_1-k_2)/2}\}\big). \tag{2.2}$$

Consider the homomorphism $\theta\colon \Delta \to D_2$ that takes every elliptic generator x_i to Q, the reflection generators in the period cycle (2^{2k_2}) alternately to S and SQ, the connecting generators e_i of the empty period cycles to 1 and the reflection generators of the empty period cycles to S. We choose the image of the connecting generator e_1 of the non-empty period cycle to be Q or 1 so as to make the long relation

$$x_1 x_2 \cdots x_{g-k_1+1} e_1 \cdots e_{(k_1-k_2)/2+1} = 1$$

consistent with the homomorphism θ. In the Hoare diagram the period cycle (2^{2k_2}) contributes k_2 blue vertices and k_2 red vertices while the empty period cycles contribute $k_1 - k_2$ blue vertices, giving in total k_1 blue vertices and k_2 red vertices, as desired. Also, the genus of the kernel of θ is g, so the corresponding quotient surface has genus g and the symmetries S_1 and S_2 have k_1 and k_2 mirrors respectively.

If $k_1 \equiv k_2 + 1 \bmod 2$ then we use an NEC group with signature

$$\big(0; +; [2^{(g-k_1)}]; \{(2^{(2k_2)}), (\ )^{(k_1-k_2+1)/2}\}\big). \tag{2.3}$$

The homomorphism θ is defined as before except that θ maps exactly one of the connecting generators of an empty period cycle to Q and all the others to 1. Thus the empty period cycles contribute $2(k_1 - k_2 - 1)/2 + 1 = k_1 - k_2$ red vertices and now the calculation goes exactly as before.

3. Surfaces admitting a fixed-point free symmetry

We now investigate the case where the compact Riemann surface F_g admits a pair of symmetries S_1, S_2 with $|S_1| = 0$. For a group element A we let $o(A)$ denote the order of A.

Theorem 3.1. *If F_g admits a pair of symmetries S_1, S_2 with $|S_1| = 0$, and if $o(S_1S_2)$ is divisible by 4, then g is odd.*

Proof. Suppose that $o(S_1S_2) = 4a$, for some positive integer a. Then

$$T = (S_2S_1)^{a-1} S_2 S_1 S_2 (S_1S_2)^{a-1}$$

commutes with S_1, is conjugate to S_1 and is distinct from S_1. Let $A = \langle S_1, T\rangle$, a group generated by two commuting fixed-point free symmetries. Let Δ_1 be the lift of A to H. As S_1 and T act without fixed points, Δ_1 cannot have period cycles and so has signature $(h; -; [2^{(r)}]; \{\ \})$. In the homomorphism from Δ_1 to A the elliptic generators of Δ_1 must map to $S_1S_2 = Q$ so that r is even. The Riemann–Hurwitz formula gives

$$g - 1 = 2h - 4 + r$$

and so g is odd.

Theorem 3.2. *If g is even and if F_g admits a fixed-point free symmetry S_1 and a symmetry S_2 with non-empty fixed-point set, then $o(S_1S_2) \equiv 2 \bmod 4$ and $|S_2|$ is odd.*

Proof. By Theorem 3.1, $o(S_1S_2)$ is not divisible by 4. The order is not odd for then S_1 and S_2 would be conjugate which is impossible as S_2 has non-empty fixed-point set. Hence $o(S_1S_2) \equiv 2 \bmod 4$. If $o(S_1S_2) = 4a + 2$ then letting

$$T = (S_2S_1)^a S_2 (S_1S_2)^a$$

we see that T is conjugate to S_2 and $o(S_1T) = 2$. Let $B = \langle S_1, T\rangle$ and Δ_2 be the lift of B to H. We first note that Δ_2 cannot have non-empty period cycles. For otherwise these period cycles would consist of link periods equal to 2. In the homomorphism θ from Δ_2 to B the reflection generators would map alternately to S_1 and T. The Hoare diagram would then have both blue and red vertices contradicting the hypothesis that $|S_1| = 0$. Thus the signature of Δ_2 must have the form $\big(h; \pm; [2^r]; \{(\ )^s\}\big)$ and θ maps each reflection generator c_i $(i = 1, \dots, s)$ of Δ_2 to T, as S_1 acts freely. Let $e_1, \dots, e_s$ be the canonical generators of Δ_2 that commute with the c_i (the connecting generators). Suppose that θ maps $e_1, \dots, e_s$ to Q $(= S_1S_2)$ and $e_{u+1}, \dots, e_s$ to 1. Then the Hoare diagram consists of $2(s-u)+u$ red vertices and no blue vertices, so that $|T| = 2s-u$. As $\theta(x_j) = Q$ $(j = 1, \dots, r)$, then applying θ to the 'long relation' shows that $r + u$ is even. The Riemann–Hurwitz formula gives

$$4g - 1 = 4\varepsilon h + 2s - 4 + r$$

(where $\varepsilon = 2$ or 1 depending on whether Δ_2 is orientable or not) and as g is even, r is odd so that u is odd and thus $|S_2| = |T|$ is odd.

Theorem 3.2 puts some restrictions on the possible pairs $(0, |S_2|)$ describing the number of fixed curves of a pair of symmetries S_1, S_2. We now show by constructing examples that these are the only restrictions. In the following g denotes the genus of a surface admitting a fixed-point free symmetry S_1, and m denotes the number of mirrors of another symmetry S_2.

Case 1. g odd, m odd, $m \le g$. Let Δ be an NEC group with signature

$$\big(0; +; [4, 2^{(g-m)/2)}]; \{(2^{(m)})\}\big). \tag{3.1}$$

We can construct a homomorphism θ from Δ to D_4 by mapping the elliptic generator of order 4 to Q, the elliptic generators of order 2 to Q^2, the reflection generators alternately to SQ and SQ^3 and the connecting generator e_1 to Q or Q^{-1}. The kernel of θ is a Fuchsian surface group Γ. The Hoare diagram has m isolated red vertices and no blue vertices, so that S_1 and S_2 are symmetries of H/Γ with $|S_1| = 0$ and $|S_2| = m$ as claimed.

Case 2. g odd, m even, $1 < m < g$. We now let Δ have signature

$$(0; +; [2^{g+1-m}]; \{(\ )^{m/2}\}) \tag{3.2}$$

and construct a homomorphism $\phi\colon \Delta \to D_2$ that takes the elliptic generators to Q the reflection generators to SQ and the connecting generators e_i to 1. As above we see that the kernel of ϕ is a Fuchsian surface group of genus g and that the corresponding Riemann surface has a pair S_1, S_2 of symmetries with $|S_1| = 0$, $|S_2| = m$.

Case 3. g even, m odd. We now let Δ have signature

$$(0; +; [2^{g+2-m}]; \{(\ )^{(m+1)/2}\}) \tag{3.3}$$

and construct a homomorphism $\psi\colon \Delta \to D_2$ that takes the elliptic generators to Q, the reflection generators to SQ, the connecting generators $e_1, \dots, e_{(m-1)/2}$ to 1 and $e_{(m+1)/2}$ to Q. The kernel of ψ is a Fuchsian surface group of genus g and the corresponding Riemann surface has a pair of symmetries S_1, S_2 with $|S_1| = 0$, $|S_2| = m$.

Case 4. g odd, $m = 0$. This is achieved using an NEC group with signature $(1; -; [2^{g+1}]; \{\ \})$ and considering a homomorphism onto D_2, taking the elliptic generators to Q and the glide reflection generator to S.

Case 5. g even, $m = 0$. This is achieved using an NEC group with signature $(1; -; [3^{(g+2)/2}]; \{\ \})$ and considering a homomorphism onto D_3.

Theorems 2.1, 2.2 and 3.2 together with the above examples prove Theorem 1.1 announced in the introduction.

4. Computing pairs of species

In this section we consider a finer classification by taking into account whether the mirrors of S separate or do not separate the surface.

4.1. Separating symmetries. If S is a symmetry of F_g then either the fixed point set of S separates F_g into two homeomorphic components or the fixed point set of S does not separate. Thus we may have separating or non-separating symmetries. If $|S| = k$ and S is separating (respectively non-separating) then we say that S has *species* $+k$ (respectively $-k$), (see [5]). The species of a symmetry determines the symmetry up to topological conjugacy. In [7] an algorithmic method was described that can be used to determine whether a symmetry is separating. We briefly recall this method.

Let Γ be the Fuchsian surface group that uniformizes F_g and let G be a group of automorphisms of F_g that contains a symmetry S. Let Δ be the lift of G to the upper half-plane H and $\theta\colon \Delta \to G$ be the canonical epimorphism with kernel Γ. Let $\Lambda = \theta^{-1}(\langle S \rangle)$ and consider the Schreier coset graph $\mathscr{S}(\Delta, \Lambda)$. If c_i is a reflection in Δ that fixes a coset then this corresponds to a loop in $\mathscr{S}(\Delta, \Lambda)$. We let $\widehat{\mathscr{S}} = \widehat{\mathscr{S}}(\Delta, \Lambda)$ be the Schreier graph with all loops corresponding to reflection generators deleted. Each edge of $\widehat{\mathscr{S}}$ is labelled by a generator of Δ so every path corresponds to an element of Δ, namely the products of the labels of the edges. In [7] it is shown that S is a separating symmetry if and only if every closed path in $\widehat{\mathscr{S}}(\Delta, \Lambda)$ corresponds to an orientation preserving element. Thus we only need one closed path corresponding to an orientation-reversing element to imply that S is non-separating. We would thus expect S to be separating to impose strong restrictions on the signature of Δ and hence on the quotient orbifold F_g/G. For example it is well known that if S is separating then

$$|S| \equiv g+1 \bmod 2. \tag{4.1}$$

This follows by an easy Euler characteristic argument as the quotient $F_g/\langle S\rangle$ is orientable or see [5].

We now revert to our special case $G = D_n = \langle S, Q | S^2 = Q^n = (SQ)^2 = 1\rangle$.

The Schreier graph $\mathscr{S}$ is isomorphic to the Schreier graph of D_n with respect to the subgroup $\langle S\rangle$. As $D_n = \langle S\rangle \cup \langle S\rangle Q \cup \cdots \cup \langle S\rangle Q^{n-1}$ we can label the vertices of the graph by the elements of Z_n and the edges by the elements of D_n. Note that an edge labelled Q^i will join the vertex r to the vertex $r+i$ and SQ^j will join i to $j-i$.

Notation. We let Y denote the canonical set of generators of Δ and let Y^+, Y^- denote the subsets consisting of the orientation-preserving and orientation-reversing generators respectively. We may assume that $S \in \theta(Y^-)$.

Lemma 4.1. *Assume that S separates. Then if $n \equiv 2 \bmod 4$, $\theta(Y^+) \subseteq \{1, Q^{n/2}\}$; otherwise, $\theta(Y^+) = 1$.*

Proof. If $Q^i \in \theta(Y^+)$, $(i \neq n/2)$, then an orientation-reversing path lies in $\mathscr{S}$ as in Figure 4.1.

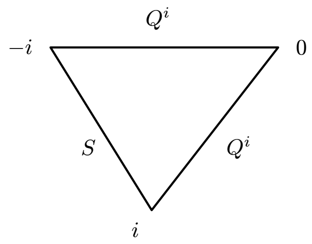

Figure 4.1

Thus if S separates and $Q^i \in \theta(Y^+)$ then $i = 0$ or $n/2$. If n is odd then $i = 0$; if $4|n$ then we can find the following orientation-reversing path with $k = n/4$.

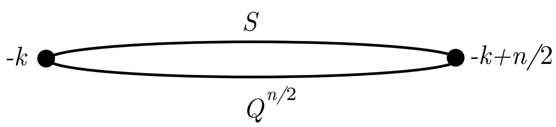

Figure 4.2

Lemma 4.2. *Assume that S separates. If n is odd then $\theta(Y^-) = \{S, SQ^i\}$ for some integer i with $(i, n) = 1$.*

Proof. If $SQ^i, SQ^j \in \theta(Y^-)$ then because n is odd, we can find k such that $i - j \equiv k \bmod n$ and we can now find an orientation-reversing path as in Figure 4.3.

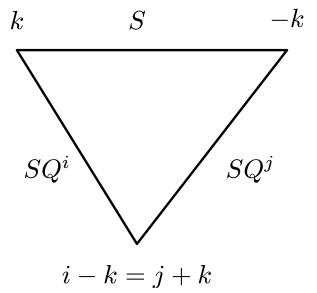

Figure 4.3

By Lemma 4.1, $\theta(Y^+) = \{1\}$, so $\theta(Y) = \{S, SQ^i\}$. As θ is an epimorphism $(i, n) = 1$.

As there is an automorphism of D_n that fixes S and maps Q to Q^r, for any r co-prime to n, we may assume that $i = 1$ in Lemma 4.2.

Lemma 4.3. *Assume that S separates. If n is even and if $S, SQ^i, SQ^j \in \theta(Y^-)$, with $i \not\equiv j \bmod n$ then i, j have different parities.*

Proof. If $i \equiv j \bmod 2$ then we can find k such that $i - j \equiv 2k \bmod n$ and then we can find an orientation-reversing path as in Lemma 4.2.

We may assume that i is odd and j is even.

Lemma 4.4. *Assume that S separates. If n is even and $S, SQ^i, SQ^j \in \theta(Y^-)$ with $i \not\equiv j \bmod n$, then $j \equiv 2i \bmod n$.*

Proof. We consider the following pentagon in the Schreier graph $\mathscr{S}$.

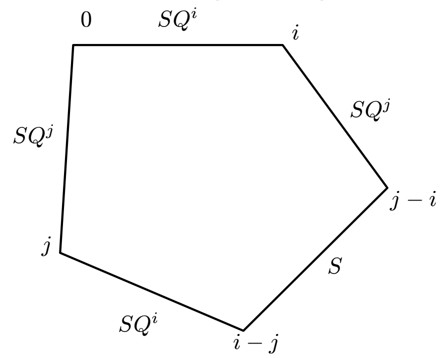

Figure 4.4

As S separates we cannot have this orientation-reversing path so that one of the edges must be a loop. We first consider the possibility that the edge labelled S is a loop. In this case, $i-j \equiv j-i \bmod n$ or $i-j \equiv \frac{1}{2}n \bmod n$. By Lemma 4.3, $n \equiv 2 \bmod 4$. Now $\theta(Y^+)$ contains at most 1 and $Q^{n/2}$ and $\theta(Y^-)$ contains at most $S, SQ^i, SQ^{i+(n/2)}$ with i odd. As $\theta(Y)$ generates D_n we must have $(i,n)=1$ and so we can apply an automorphism of D_n and assume that $i=1$ and so $j = 1+\frac{1}{2}n$. Now $SQ^j\colon 1 \to \frac{1}{2}n$ and so we can find the following path of odd length in $\mathscr{S}$:

$$\begin{array}{rcrl}
1 & \to & -1 & \text{(by } S)\\
-1 & \to & 2 & \text{(by } SQ)\\
2 & \to & -2 & \text{(by } S)\\
-2 & \to & 3 & \text{(by } SQ)\\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \\
\frac{1}{2}n-1 & \to & 1-\frac{1}{2}n & \text{(by } S)\\
1-\frac{1}{2}n & \to & \frac{1}{2}n & \text{(by } SQ)\\
\frac{1}{2}n & \to & 1 & \text{(by } SQ^j).
\end{array}$$

This is a contradiction so that the edge labelled S cannot be a loop. Hence the edge labelled SQ^j joining i to $j-i$ is a loop and then $i \equiv j-i \bmod n$ or $j \equiv 2i \bmod n$.

Now if n is odd and S separates then as all symmetries in D_n are conjugate SQ must necessarily separate so that both symmetries of the pair are separating symmetries. Now suppose that n is even and that both symmetries of the pair separate. By Lemma 4.4 we may suppose, after applying an automorphism of D_n that $\theta(Y^-) \subseteq \{S, SQ, SQ^2\}$.

Lemma 4.5. *If n is even and if both S and SQ separate then $\theta(Y^-) = \{S, SQ\}$.*

Proof. Suppose that $\theta(Y^-) = \{S, SQ, SQ^2\}$. Let $\Lambda_1 = \theta^{-1}(\langle SQ\rangle)$ and form the Schreier coset graph $\mathscr{S}_1 = \mathscr{S}(\Delta, \Lambda_1)$ and delete the reflection loops, as before, to form $\widehat{\mathscr{S}}_1$. Again, SQ is non-separating if there is an orientation-reversing path in $\mathscr{S}_1$. The cosets are now $\langle SQ\rangle, \langle SQ\rangle Q, \ldots, \langle SQ\rangle Q^{n-1}$, which we denote by $0, 1, \ldots, n-1$. We have $(0)SQ^2 = 1$, $1(SQ) = n-1$ and $(n-1)S = 0$, so we have an orientation reversing triangle.

These lemmas tell us about the possible link periods and proper periods when there are separating symmetries in the dihedral group.

Theorem 4.1. *If G is a group of automorphisms generated by a pair of symmetries and if Δ is the lift of G to the upper half-plane H, then if one of the symmetries separates,*

(a) (i) *if n is odd then Δ has no proper periods and all link periods are equal to n,*

(a) (ii) *if n is even then Δ has no proper periods if $4|n$ and all proper periods are equal to 2, if $n \equiv 2 \bmod 4$, and in both cases, all link periods are equal to n or $\frac{1}{2}n$.*

(a) (iii) *However, if Δ contains a proper period equal to 2 then Δ can have no link periods equal to n.*

If both of the symmetries separate then

(b) *Δ has no proper periods and all link periods are equal to n.*

Proof. a(i). Let $\theta\colon \Delta \to D_n$ be the canonical homomorphism. If Δ contains an elliptic element x then by Lemma 4.1, $\theta(x) = 1$ and then $x \in \ker\theta$ which contradicts $\ker\theta$ being a surface group. Now assume that c_k and c_{k+1} are reflections in Δ with $c_k c_{k+1}$ having finite order m. By Lemma 4.2 we may assume that $\theta(c_k) = S$, and $\theta(c_{k+1}) = SQ^i$ with $(i,n) = 1$. As $S(SQ^i)$ has order n and $\ker\theta$ is a surface group, $m = n$. The first part of a(ii) also follows from Lemma 4.1 and the second part from Lemma 4.4, or more particularly the remark at the end of the proof that $\theta(Y^-) \subseteq \{S, SQ, SQ^2\}$. For a(iii) we consider the cosets of $\langle S\rangle$ and label the coset $\langle S\rangle Q^r$ by r. We join 0 to $\frac{1}{2}n$ by a path of length $n-1$ alternately labelled S and SQ. This path goes from 0 to 1, 1 to $n-1$, $n-1$ to 2, 2 to $n-2$ etc. Once we have arrived at $\frac{1}{2}n$ we go back to 0, by a path labelled $Q^{n/2}$. The resulting closed path is an orientation-reversing path, which shows that if Δ contains a period equal to 2 and link periods equal to n then S is non-separating. To prove (b) we may assume, by Lemma 4.1, that $n \equiv 2 \bmod 4$. By Lemma 4.5 $S, SQ \in \theta(Y^-)$. We consider the cosets of $\langle SQ\rangle$ and label the coset $\langle SQ\rangle Q^r$ by r. We can find r such that $2r \equiv \frac{1}{2}n - 1 \bmod n$ and then $(r)S = -r-1$. Now if there is an elliptic period, then by Lemma 4.1, it must be equal to 2, and its image in G is $Q^{n/2}$. We then have

$$(r)Q^{n/2} = r + \tfrac{1}{2}n \equiv -r-1 \bmod n$$

and so we can find an orientation-reversing closed path in the Schreier graph. The fact that all link periods are equal to n follows directly from Lemma 4.5.

4.2. Signatures. We now have enough information to determine the possible signatures of Δ given that one or both symmetries of the pair separates.

Theorem 4.2. *If n is odd and one (and hence both) symmetries of the pair separates then the signature of Δ has the form*

$$\big(h; +; [\,]; \{(n,\ldots,n),\ldots,(n,\ldots,n),(\,),\ldots,(\,)\}\big). \tag{4.2}$$

where each non-empty period-cycle has even length.

Proof. By Theorem 4.1 above we only need prove that the period cycles have even length and that we have a positive sign in the signature. The first follows as the homomorphism θ must be of the form $c_{i0} \to S$, $c_{i1} \to SQ$, $c_{i2} \to S$, $\ldots$, $c_{is_i} \to S$, with c_{i0} and c_{is_i} conjugate in Δ and the second because a glide reflection generator must give an orientation-reversing loop in both the Schreier graphs.

Theorem 4.3. *If n is divisible by 4 and if one of the symmetries separates then Δ has a signature of the form*

$$(4.3) \quad \big(h;\pm;[\,];\{(\ldots,\tfrac{1}{2}n,n,n,\ldots,n,\tfrac{1}{2}n,\ldots)\ldots(\tfrac{1}{2}n,\ldots,\tfrac{1}{2}n)\ldots,(\,),\ldots,(\,)\}\big),$$

where there are always an even number of link periods equal to n between two link periods equal to $\frac{1}{2}n$, and the period cycles containing only link periods equal to $\frac{1}{2}n$ have even length.

Proof. The only statement that we have not proved concerns the even number of link periods equal to n between two link periods equal to $\frac{1}{2}n$. To see this we normalize the homomorphism so that the images of θ are S, SQ, SQ^2. We only get link periods equal to n when SQ is an image of θ. The images of the neighbouring reflections are then S and SQ^2 showing that the link periods equal to n occur in pairs.

Theorem 4.4. *If $n \equiv 2 \bmod 4$ and if one of the symmetries separates then Δ has signature either of the form*

(i) *as given in Theorem 4.3 or*
(ii) $\big(h;\pm;[2,\ldots,2];\{(\frac{1}{2}n,\ldots,\frac{1}{2}n),\ldots,(\frac{1}{2}n,\ldots,\frac{1}{2}n),(\,),\ldots,(\,)\}\big)$
where each period cycle has even length.

Proof. This follows from the above Lemmas and Theorem 4.1.

Theorem 4.5. *If both symmetries of the pair separate, then Δ has signature of the form*

$$\big(h;+;[\,];\{(n,\ldots,n),\ldots,(n,\ldots,n),(\,),\ldots,(\,)\}\big).$$

Proof. This follows from part (b) of Theorem 4.1.

Remarks. 1. Each of the above theorems tells us about the possible quotient orbifolds by the D_n action given that one of the symmetries separates. We have seen for example, that there can only be cone points of order 2 (and this occurs only in the case that $n \equiv 2 \bmod 4$) and the corner points can only have orders n and $\frac{1}{2}n$.

2. The converse of the above theorems will be true as long as the images of the hyperbolic and glide reflection generators obey the restrictions of Lemmas 4.1–4.4. In particular the converse always holds if the above NEC groups Δ have genus 0.

4.3. The existence of Riemann surfaces admitting pairs of symmetries with given species. We are now in a position to tackle the question: Let k_1, k_2, be two integers with $-g \le k_1 \le k_2 \le g+1$. Does there exist a Riemann surface X of genus g admitting a pair of symmetries S_1, S_2 with $\mathrm{sp}(S_1) = k_1$, $\mathrm{sp}(S_2) = k_2$? (Here we use the above convention that a positive species refers to a separating symmetry, and a non-positive species to a non-separating symmetry.) We shall see that most pairs of species do exist but that there are some interesting exceptions. We divide our investigation into a number of cases.

Case 1. $k_1 \leq -1$, $k_2 \leq -1$.

Theorem 4.7. *If* $-g \leq k_1 \leq k_2 \leq -1$ *then there exists a Riemann surface* F_g *of genus* g *admitting a pair of symmetries* S_1, S_2 *with* $\mathrm{sp}(S_1) = k_1$, $\mathrm{sp}(S_2) = k_2$.

Proof. In Theorem 2.2 we constructed symmetries with $|S_1| = |k_1|$, $|S_2| = |k_2|$. Signature (2.2) must correspond to a negative species by Theorems 4.4, 4.5 and (2.3) gives a negative species as the connecting generator maps to Q.

Case 2. $k_1 \leq 0$, $k_2 = 0$.

Theorem 4.8. *If* g *is odd and if* $-g \leq k_1 \leq 0$ *then there exists a Riemann surface* F_g *of genus* g *admitting a pair of symmetries* S_1, S_2 *with* $\mathrm{sp}(S_1) = k_1$, $\mathrm{sp}(S_2) = 0$. *If* g *is even and if* $k_1 = 0$ *or* $k_1 \leq g$ *is odd then there exists a Riemann surface* F_g *of genus* g *admitting a pair of symmetries* S_1, S_2 *with* $\mathrm{sp}(S_1) = k_1$, $\mathrm{sp}(S_2) = 0$.

Proof. If g is odd and k_1 is even then let Δ be an NEC group of signature

$$\big(1; -; [2^{(g+1-|k_1|)}]; \{(\ )^{|k_1|/2}\}\big). \tag{4.5}$$

We construct a homomorphism from Δ to D_2 by mapping the glide reflection and reflection generators to S_1, the elliptic generators to $S_1S_2 = Q$ and the connecting e-generators to the identity. This is a homomorphism as $g + 1 - k_1$ is even and so the long relation is preserved. As the glide reflection generator maps to S_1 the Schreier graph $\mathscr{S}(\Delta, \Lambda_1)$, where Λ_1 is inverse image of $\langle S_1 \rangle$, has orientation reversing loops not just coming from reflections so that S_1 is non-separating. As the connecting generators map to the identity each empty period cycle contributes two mirrors so the species of S_1 is k_1 as claimed. The Riemann–Hurwitz formula gives the genus of the Riemann surface as g. If g is odd and k_1 is odd we use the signature (3.1) to construct the symmetries. By forming the Schreier graph we see that the symmetry with non-empty fixed-point set does not separate. In this case the product of the symmetries has order 4, and by comparison with the above case where g is odd it is easy to see that we cannot achieve this when the product of the symmetries has order 2. If g is even and k_1 is odd then we let Δ be an NEC group of signature

$$\big(1; -; [2^{g-|k_1|}]; \{(\ )^{(|k_1|+1)}/2\}\big)$$

and construct a homomorphism from Δ to D_2, again sending the glide-reflection and reflection generators to S_1 and the elliptic generators to S_1S_2. This time, however we map one of the connecting generators to $S_1S_2 = Q$ and the others to the identity. As $g - k_1$ is odd this is a homomorphism as the long relation is preserved. The species is calculated as before. If $k_1 = 0$ we refer to Case 5 in Section 3 where we find a commuting pair of symmetries of a surface of odd genus both with species 0.

Case 3. $k_1 = 0\,,\ k_2 > 0\,.$

Theorem 4.9. *If g is odd and if $1 < k_2 \leq g+1$, with k_2 even then there exists a Riemann surface F_g of genus g admitting a pair of symmetries S_1, S_2 with* $\mathrm{sp}(S_1) = 0$, $\mathrm{sp}(S_2) = k_2$. *If g is even and k_2 is odd* $(1 \leq k_2 \leq g+1)$, *then there exists a Riemann surface F_g of genus g admitting a pair of symmetries S_1, S_2 with* $\mathrm{sp}(S_1) = 0$, $\mathrm{sp}(S_2) = k_2$.

Proof. We use Cases 2, 3 of Section 3 and verify that in each case that S_2 is a separating symmetry.

Note. By Theorem 3.2 there does not exist a Riemann surface of even genus admitting a symmetry of zero species and another one with a non-zero even species.

Case 4. $k_1 > 0\,,\ k_2 > 0\,.$ Note that by (4.1) we must have $k_1 \equiv k_2 \equiv g+1 \bmod 2$.

Theorem 4.10. (i) *If $1 < k_1 \leq k_2 \leq g+1$ then there exists a Riemann surface F_g of genus g admitting a pair of symmetries S_1, S_2 with* $\mathrm{sp}(S_1) = k_1$, $\mathrm{sp}(S_2) = k_2$. (ii) *If g is even, $1 \leq k_2 \leq g+1$, and $k_2 \equiv g+1 \bmod 4$ then there exists a Riemann surface F_g of genus g admitting a pair of symmetries of species S_1, S_2 with* $\mathrm{sp}(S_1) = +1$, $\mathrm{sp}(S_2) = k_2$.

Proof. Let Δ be an NEC group with signature

$$\big(h; +; [\,]; \{(2^{2t}), (\,)^{u_1+u_2}\}\big) \tag{4.7}$$

Let $\theta\colon \Delta \to D_2 = \langle S_1, S_2\rangle$ be defined by mapping the reflections of the non-empty period cycle alternately to S_1 and S_2, the reflections of u_1 of the period cycles to S_1 the periods of u_2 of the period cycles to S_2 and the connecting generators to the identity. Then letting $k_i = |S_i|$, $(i = 1, 2)$ we see that

$$k_i = t + 2u_i$$

and the Riemann–Hurwitz formula gives

$$g + 1 = 4h + 2u_1 + 2u_2 + t.$$

If $k_2 \equiv g+1 \bmod 4$ we let $u_1 = 0$, $u_2 = \frac{1}{2}(k_2 - k_1)$, $t = k_1$, $h = \frac{1}{4}(g+1-k_2)$ to give $\mathrm{sp}(S_1) = +k_1$, $\mathrm{sp}(S_2) = +k_2$. We get part (ii) by putting $t = 1$.

If $k_2 \equiv g-1 \bmod 4$ and $k_1 > 1$ we let $u_1 = 1$, $u_2 = \frac{1}{2}(k_2 - k_1)+1$, $t = k_1 - 2$, $h = \frac{1}{4}(g-1-k_2)$ to give $\mathrm{sp}(S_1) = +1$, $\mathrm{sp}(S_2) = +k_2$.

Theorem 4.9 does not give us information when $k_1 = 1$, $k_2 \equiv g-1 \bmod 4$. If $k_1 = 1$ we must have $t = 1$, $u_1 = 0$ so the Riemann–Hurwitz formula implies (when $n = 2$) that

$$g + 1 = 4h + 2u_2 + 1 = 4h + k_2$$

and so necessarily $k_2 \equiv g+1 \bmod 4$. Thus we cannot have a pair of commuting separating symmetries with one of the symmetries having just one mirror and the other having $k_2 \equiv g-1 \bmod 4$ mirrors. We now investigate the case where we have a pair S_1, S_2 of separating symmetries with S_1S_2 having even order $n > 2$. By Theorem 4.5 the only way we can do this is via a homomorphism from an NEC group of signature

$$\big(h; +; [\,]; \{(n,\ldots,n),\ldots,(n,\ldots,n),(\,)^{u_1+u_2}\}\big) \tag{4.8}$$

onto $D_n = \langle S_1, S_2\rangle$, where the reflections of the non-empty period cycles map alternately to S_1 and S_2 and the length of each period cycle is even, the reflections of u_1 of the empty period cycles map to S_1 and the reflections of u_2 of the empty period cycles map to S_2 and the connecting generators map to the identity. If there are q non-empty period cycles and $2t$ link-periods equal to n then the Riemann–Hurwitz formula gives

$$2g-2 = n(4h+2q+2u_1+2u_2-4+2t)-2t.$$

However, $k_1 = 1$ and so $t+2u_1 = 1$ giving $t=1$ and thus $q=1$ and $u_1 = 0$. We then obtain $g = n(2h+u_2)$ and $k_2 = 2u_2+1 = (2g/n)-4h+1$. Thus $k_2 \le (2g/n)+1$ and $k_2 \equiv (2g/n)+1 \bmod 4$. On the other hand we can choose u_2 and Δ with signature (4.8) to give this result and thus we have

Theorem 4.11. *If $k_2 \equiv g-1 \bmod 4$ then there exists a Riemann surface F_g of genus g admitting a pair of symmetries of species 1 and $k_2 > 0$ if and only if*

$$k_2 \equiv \left(\frac{2g}{n}+1\right) \bmod 4 \qquad \textit{and} \qquad k_2 \le \frac{2g}{n}+1,$$

where $n > 2$ is an integer dividing $2g$.

For example if $k_2 \equiv g-1 \bmod 4$ then there is no Riemann surface of genus g that admits a pair of symmetries of species 1 and k_2 where $k_2 > \frac{1}{2}g+1$.

Case 5. $k_1 < 0$, $k_2 > 0$. We first deal with the case when we have commuting symmetries. As $k_2 > 0$ we must have $k_2 \equiv g+1 \bmod 2$. What we now find is that $k_1 \equiv g+1 \bmod 2$ also.

Theorem 4.12. *If $-g \le k_1 < 0 < k_2 \le g$ then there exists a Riemann surface F_g of genus g admitting a commuting pair of symmetries S_1, S_2 with $\mathrm{sp}(S_1) = k_1$, $\mathrm{sp}(S_2) = k_2$ if and only if $k_1 \equiv k_2 \equiv g+1 \bmod 2$.*

Proof. To prove the existence of such a pair of symmetries we use an NEC group Δ of signature

$$\big(\tfrac{1}{2}(g+1-k_2); -; [\,]; \{(2^{2|k_1|}), (\,)^{(k_2-|k_1|)/2}\}\big)$$

and we define a homomorphism from Δ to D_2 which maps the glide reflection generators to S_1, the reflection generators of the non-empty period cycle alternately to S_1 and S_2, and the generators of the empty period cycles to S_2. On the other hand, it follows from Theorem 4.4 that the only possible signatures for Δ are $\big(h;\pm;[2^r];\{(\ )^u\}\big)$ or $\big(h;\pm;[\ ];\{(2^{s_1}),\dots,(2^{s_k})\}\big)$. In the first signature both symmetries do not separate, for the second for one of the symmetries to separate we must have all the connecting generators mapping to the identity. In this case every empty period cycle contributes two mirrors while the non-empty period cycles contribute the same number of mirrors to both symmetries. Thus $k_1 \equiv k_2 \bmod 2$.

Theorem 4.11 puts a restriction on the possible pairs k_1, k_2 when $k_1 < 0 < k_2$ for commuting symmetries S_1, S_2 that is, when S_1S_2 has order $n = 2$. We now see what other pairs k_1, k_2 are possible if S_1S_2 has order $n > 2$. We shall see that not only are restrictions needed but, unlike the other cases it is not that easy to find, in a uniform way, what pairs are possible for a given genus. The restrictions come about because of the inequality in the next theorem.

Theorem 4.13. *Let S_1, S_2 be a pair of symmetries of a Riemann surface F_g of genus $g > 1$ with S_1S_2 of even order $n > 2$. If $\mathrm{sp}(S_i) = k_i$ and if $k_1 < 0 < k_2$ then*

$$\tfrac{1}{2}n|k_1| + (\tfrac{1}{2}n - 1)k_2 \le g - 1 + n.$$

Proof. By Theorems 4.3 and 4.4 we find that the lifted NEC group Δ has a signature of the form

$$\big(p;\pm;[\ ];\{(\dots\tfrac{1}{2}n, n,\dots,n,\tfrac{1}{2}n,\dots),\dots,(\tfrac{1}{2}n,\dots,\tfrac{1}{2}n),\dots,(\ )^{u_1+u_2}\}\big)$$

with an even number of link periods equal to n between link periods equal to $\frac{1}{2}n$, and where for $i = 1,2$, u_i of the empty period cycles correspond to reflections that map to S_i. Suppose that there are $2s$ link periods equal to n and r link periods equal to $\frac{1}{2}n$. By Lemma 4.1 we need the connecting generators to map to the identity so by Section 1.4 we see that each empty period cycle contributes two mirrors and then by constructing the appropriate Hoare diagram we find that

$$k_2 = r + s + 2u_2,$$
$$k_1 = s + 2u_1.$$

If the total number of period cycles is $k \ge u_1 + u_2 + 1$ then the Riemann–Hurwitz formula now gives

$$\begin{aligned}
2g - 2 &= 2n\Big(\varepsilon p - 2 + k + s\Big(1 - \frac{1}{n}\Big) + \frac{r}{2}\Big(1 - \frac{2}{n}\Big)\Big)\\
&\ge n\Big(2u_1 + 2u_2 - 2 + 2s\Big(1 - \frac{1}{n}\Big) + r\Big(1 - \frac{2}{n}\Big)\Big)\\
&\ge n(2h_1 + 2h_2 + 2u + v) - 2u - 2v - 2h_1 - 2n\\
&\ge (n-2)(s + r + 2u_1) + n(s + 2u_2) - 2n\\
&\ge (n-2)k_2 + n|k_1| - 2n
\end{aligned}$$

and the result follows.

Corollary 4.14. *With the notation of Theorem* 4.12, *if* S_1S_2 *has order* $n \geq 4$ *then*

$$2|k_1| + k_2 \leq g + 3.$$

Proof. We can write the inequality of Theorem 4.12 in the form

$$2|k_1| + k_2 + (\tfrac{1}{2}n - 2)(|k_1| + k_2) \leq g + n - 1.$$

As $|k_1| \geq 1$, $|k_2| \geq 1$, the result follows.

Corollary 4.15. *If a Riemann surface* F_g *of genus* g *admits a non-separating symmetry* S_1 *with more than* $\frac{1}{2}g + 1$ *mirrors and if* S_2 *is another symmetry not commuting with* S_1 *then* S_2 *must be non-separating.*

5. Summary

The results 4.7–4.12 give necessary and sufficient conditions for the existence of a pair S_1, S_2 of symmetries of species k_1, k_2. However, 4.13 and 4.14 only give necessary conditions on the pair k_1, k_2. For example, there is no pair of symmetries of a Riemann surface of genus 2 of species $-2, +1$. Such a pair cannot exist for commuting symmetries by 4.12, and otherwise we have to solve the equations $r + s + 2u_2 = 1$, $s + 2u_1 = 2$. The only solution is $s = u_2 = 0$, $r = u_1 = 1$ and then the corresponding NEC group admits no surface-kernel homomorphism onto D_n for $n > 2$.

A way of summarising our results is to say that a symmetric Riemann surface F_g of genus g admits $(k_1, k_2)_n$ if it admits a pair of symmetries S_1, S_2 of species k_1, k_2 repectively with S_1S_2 having order n. Then we have shown:

If $k_1 \leq k_2 \leq -1$ then $(k_1, k_2)_2$ exists for all g (Theorem 4.7).

If $k_1 \leq -1$ is even then $(k_1, 0)_2$ exists if g is odd (Theorem 4.8).

If $k_1 \leq -1$ is odd then $(k_1, 0)_4$ exists if g is odd and $(0, 0)_2$ exists (Theorem 4.9).

If $1 < k_1 \leq k_2 \leq g + 1$ then $(k_1, k_2)_2$ exists. If $1 \leq k_2 \leq g + 1$, g even and $k_2 \equiv g + 1 \bmod 4$ then $(1, k_2)_2$ exists (Theorem 4.10).

If $k_2 \geq 1$, $k_2 \equiv g - 1 \bmod 4$ and $k_2 \equiv \big((2g/n) + 1\big) \bmod 4$, $k_2 \leq \big((2g/n) + 1\big)$, $n > 2$ then $(1, k_2)_n$ exists (Theorem 4.11).

If $k_1k_2 \geq 0$ and if there is a symmetric Riemann surface admitting $(k_1, k_2)_n$ for some n, then it appears in the above list with the least positive value of n.

If $-g < k_1 < 0 < k_2 < g$ then $(k_1, k_2)_2$ exists if and only if $k_1 \equiv k_2 \equiv g + 1 \bmod 2$ (Theorem 4.12).

If $-g \leq k_1 < 0 < k_2 < g$, $n > 2$ is even, and $(k_1, k_2)_n$ exists then it follows that $\frac{1}{2}n|k_1| + (\frac{1}{2}n - 1)k_2 \leq g - 1 + n$ (Theorem 4.13). This implies that $2|k_1| + k_2 \leq g + 3$ (Corollary 4.14).

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