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A NOTE ON ISOLATED POINTS IN THE BRANCH LOCUS OF THE MODULI SPACE OF COMPACT RIEMANN SURFACES

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Abstract. Let g be an integer ≥ 3 and let $\mathscr{B}_g = \{X \in \mathscr{M}_g \mid \operatorname{Aut} X \neq e\}$, where $\mathscr{M}_g$ denotes the moduli space of a compact Riemann surface.

The geometric structure of $\mathscr{B}_g$ is of substantial interest because $\mathscr{B}_g$ corresponds to the singularities of the action of the modular group on the Teichmüller space of surfaces of genus g (see [H]).

Surprisingly R.S. Kulkarni [K] has found isolated points in $\mathscr{B}_g$. He showed that they appear if and only if $2g+1$ is an odd prime distinct from 7. The aim of this paper is to find a geometrical explanation of this phenomenon using the fact that the isolated points are given by surfaces admitting anticonformal involutions (symmetries). The points in the Teichmüller space, corresponding to groups uniformizing surfaces with a symmetry, is a (non disjoint) union of submanifolds. We shall obtain that the isolated intersections of such submanifolds give us the isolated points in the branch loci.

Also we prove that there are no isolated points in the moduli space of Klein surfaces which are not Riemann surfaces.

1. Preliminaries

A *non-euclidean crystallographic group*, (NEC) group, is a discrete group Γ of the group $\mathscr{G}$ of isometries of the hyperbolic plane H with compact quotient space H/Γ. If the group Γ is a subgroup of the group $\mathscr{G}^+$ of orientation-preserving isometries of H, then it is called a *Fuchsian group*. Otherwise $\Gamma^+ = \Gamma \cap \mathscr{G}^+$ is a subgroup of index 2 in Γ, called its *canonical Fuchsian subgroup*.

Let Γ be an NEC group. Then there is a fundamental region P for Γ which is a polygon in H whose perimeter, described counterclockwise, is one of the following:

(1) $\varepsilon_1\varepsilon_1'\cdots\varepsilon_r\varepsilon_r'\delta_1\gamma_{10}\cdots\gamma_{1s_1}\delta_1'\cdots\delta_k\gamma_{k0}\cdots\gamma_{ks_k}\delta_k'\alpha_1\beta_1\alpha_1'\beta_1'\cdots\alpha_h\beta_h\alpha_h'\beta_h'$,
(2) $\varepsilon_1\varepsilon_1'\cdots\varepsilon_r\varepsilon_r'\delta_1\gamma_{10}\cdots\gamma_{1s_1}\delta_1'\cdots\delta_k\gamma_{k0}\cdots\gamma_{ks_k}\delta_k'\alpha_1\alpha_1^*\cdots\alpha_h\alpha_h^*$.

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In these symbols each letter denotes an oriented side of the polygon P. The apostrophe means that the corresponding sides of P are identified by generators of the group which preserve the orientation, and the asterisk means that the corresponding sides are identified by generators of the group which reverse the orientation. As a consequence, if we identify corresponding points on the related edges of the polygon, we obtain from P a surface with boundary. In the case (1), the surface will be a sphere with k disks removed and g handles added. In the case (2), the surface will be a sphere with k disks removed and h cross-caps added. The angle determined by the sides $\varepsilon_i\varepsilon_i'$ is $2\pi/m_i$, $i=1,\dots,r$, and the angle given by $\gamma_{ij}\gamma_{ij+1}$ is π/n_{ij}, $i=1,\dots,k$, $j=0,\dots,s_{i-1}$. In the group Γ there are elliptic elements that identify ε_i with ε_i' and reflections whose axe contain the sides γ_{ij}. The fundamental region P determines a presentation for Γ. The side pairing and the reflections give rise to a set of generators and the relations are given by the way as the images of P by elements of Γ fit together around the vertices of P.

Then the algebraic structure of an NEC group is determined by its signature

$$(1.1) \qquad s(\Gamma) = \big(h; \pm; [m_1,\dots,m_r]; \{(n_{11},\dots,n_{1s_1}),\dots,(n_{k1},\dots,n_{ks_k})\}\big).$$

The presentation for Γ associated to the signature (1.1) have generators

$$\begin{gathered} x_1,\dots,x_r, \\ e_1,\dots,e_k, \\ c_{ij}, \quad 1\le i\le k,\ 0\le j\le s_i, \\ a_1,b_1,\dots,a_h,b_h \text{ if } H/\Gamma \text{ is orientable or} \\ a_1,\dots,a_h, \text{ if } H/\Gamma \text{ is non-orientable} \end{gathered}$$

and relators

$$\begin{gathered} x_i^{m_i}, \qquad i=1,\dots,r, \\ c_{ij-1}^2,\ c_{ij}^2,\ (c_{ij-1}c_{ij})^{n_{ij}}, \qquad i=1,\dots,k,\ j=1,\dots,s_i, \\ c_{i0}e_i^{-1}c_{is_i}e_i, \\ x_1x_2\cdots x_re_1\cdots e_ka_1b_1a_1^{-1}b_1^{-1}\cdots a_h^{-1}b_h^{-1}, \text{ if } H/\Gamma \text{ is orientable or} \\ x_1x_2\cdots x_re_1\cdots e_ka_1^2\cdots a_h^2, \text{ if } H/\Gamma \text{ is non-orientable.} \end{gathered}$$

These last two relations are sometimes called the *long relation.* In these presentations, the only elements of finite order are the elliptic elements and the reflections. The elliptic elements are conjugate of powers of the x_i or $c_{ij-1}c_{ij}$ and the reflections are conjugate of the c_{ij}. The e_i generators are orientation preserving. They are called the *connecting generators.*

The hyperbolic area of H/Γ, which we shall call $\mu(\Gamma)$, is equal to the hyperbolic area of a fundamental polygon P, then

$$\mu(\Gamma) = 2\pi\left(\varepsilon h - 2 + k + \sum_{i=1}^{r}\Big(1 - \frac{1}{m_i}\Big) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\Big(1 - \frac{1}{n_{ij}}\Big)\right), \tag{1.2}$$

where $\varepsilon = 2$ if there is a $+$ sign and $\varepsilon = 1$ if there is a $-$ sign. If Γ^* is a subgroup of Γ of finite index then the Riemann–Hurwitz formula holds:

$$|\Gamma : \Gamma^*| = \frac{\mu(\Gamma^*)}{\mu(\Gamma)}. \tag{1.3}$$

Let Γ^+ be the canonical Fuchsian of an NEC group Γ. Then H/Γ^+ is a 2-sheeted covering of H/Γ, called its *complex double.* The genus $h^+ = \varepsilon h + k - 1$ of H/Γ^+ is the *algebraic genus* of H/Γ. An NEC group Γ without elliptic elements is called a *surface group.* Its signature is $\big(g; \pm; [\,]; \{(\,)^k\}\big)$. A Klein surface whose complex double has genus greater than one can be expressed as H/Γ where Γ is an NEC surface group. An orientable Klein surface without boundary can be thought as a *Riemann surface.* If G is a finite group, then G is a group of automorphisms of a Klein surface H/Γ if and only if there exists an NEC group Γ' and a homomorphism from Γ' on to G having Γ as the kernel (see [BEGG]). Then for each symmetry there is a group Γ' containing Γ as index two subgroup.

Given an NEC group Γ, we denote by $\mathbf{R}(\Gamma)$ the set of monomorphisms $r\colon \Gamma \to \mathscr{G}$ such that $r(\Gamma)$ is discrete and $H/r(\Gamma)$ is compact. Two elements $r_1, r_2 \in \mathbf{R}(\Gamma)$ are said to be equivalent if there exists $g \in \mathscr{G}$ such that for each $\gamma \in \Gamma$, $r_1(\gamma) = g r_2(\gamma) g^{-1}$. The orbit space $\mathbf{T}(\Gamma)$ is called the Teichmüller space of Γ and it is homeomorphic to a real ball considering in $\mathbf{T}(\Gamma)$ the Teichmüller metric (see [MS]).

Let $A(\Gamma)$ denote the automorphism group of Γ, $A(\Gamma)^+$ be the orientation preserving automorphism group if Γ is a Fuchsian group, and $I(\Gamma)$ the subgroup of inner automorphisms. The modular group $M(\Gamma) = A(\Gamma)/I(\Gamma)$ or $M(\Gamma)^+ = A(\Gamma)^+/I(\Gamma)$ if Γ is a Fuchsian group, acts on $\mathbf{T}(\Gamma)$ as follows. If $[r] \in \mathbf{T}(\Gamma)$ and $[\alpha] \in M(\Gamma)$, then $[\alpha][r] = [r \circ \alpha]$. The *moduli space of* Γ is the quotient space

$$\mathscr{M}_g = \mathbf{T}(\Gamma)/M(\Gamma)$$

and $\mathscr{M}_g = \mathbf{T}(\Gamma)/M(\Gamma)^+$ if Γ is a Fuchsian group (see [MS]). Let

$$\pi\colon \mathbf{T}(\Gamma) \to \mathscr{M}_g$$

be the natural projection.

As an application of the extended Nielsen theorem we have that any automorphism of a surface group can be geometrically realized as a homeomorphism of the surface. Hence we can identify the branch locus of the action of $M(\Gamma)$ on $\mathbf{T}(\Gamma)$ with the set

$$\mathscr{B}_g = \{X \in \mathscr{M}_g \mid \operatorname{Aut} X \neq e\}.$$

This is called the *branch locus* of $\mathcal{M}_g$.

For surfaces X with algebraic genus 2, $\mathcal{B}_2 = \{X \in \mathcal{M}_g \mid \operatorname{Aut} X \neq Z_2\}$.

Let now $\Gamma \leq \Gamma'$ be NEC groups and let $i\colon \Gamma \to \Gamma'$ be the inclusion mapping. Then i induces $m\colon \mathbf{T}(\Gamma') \to \mathbf{T}(\Gamma)$ defined by $m[r] = [r \circ i]$, where m is an isometric embedding (see [MS]). Then $m\big(\mathbf{T}(\Gamma')\big)$ is a submanifold of $\mathbf{T}(\Gamma)$. So each Γ^* containing Γ as subgroup of index two and each $i\colon \Gamma \to \Gamma^*$, give rise to a submanifold in $\mathbf{T}(\Gamma)$.

2. Isolated points in the branch locus of Riemann surfaces

We consider the set N of classes $[r,s,t]$ of unordered triples $\{r,s,t\}$ of numbers counted $\bmod q$ such that none of them is $\equiv 0 \mod q$ and $r+s+t \equiv 0 \mod q$ under the action of the multiplicative group $Z_q - \{0\}$. There is a bijection from N onto the set of Riemann surfaces X admitting Z_q-action with quotient orbifold a sphere with three cone points of order q. Thus we can associate a symbol $[r,s,t]$ in an one-to-one manner to such Riemann surfaces. The symbol $[r,s,t]$ is called the *characteristic symbol* of the Riemann surface X.

In [K] it is shown that

Theorem 2.1. *The number of isolated points in $\mathcal{B}_g$ is 1 if $g = 2$, $[(g-2)/3]$ if $q = 2g+1$ is a prime > 7, and 0 otherwise. The isolated point in $\mathcal{B}_2$ is the hyperelliptic surface $w^2 = z^5 - 1$. For $g \geq 5$ the isolated Riemann surfaces are precisely those X' for which $\operatorname{Aut}^+ X'$ is isomorphic to Z_q and such that in their characteristic symbols $[r,s,t]$ no two of r,s,t are equal, and if $q \equiv 1 \mod 3$ and λ is a root of unity then $[r,s,t] \neq [1,\lambda,\lambda^2]$.*

Here for a real number x, $[x]$ denotes the greatest integer $\leq x$.

The following algorithm, with $q = 2g+1$ prime, yields geometric models for Riemann surfaces which correspond to isolated points in $\mathcal{B}_g$; see [K]. Its characteristic symbol is chosen to be $[1,s,t]$ without loss of generality.

Geometrical description. Let T be an hyperbolic triangle with angles π/q, π/q, π/q. We label the vertices with the capital letters N, U, V and the opposite sides with n, u, v. We paste $2q$ copies of T around the vertices labelled N and identifying u_{2i-1} with u_{2i} and v_{2i} with v_{2i+1}, and then U_{2i-1} with U_{2i} and V_{2i} with V_{2i+1}. In this way we obtain an hyperbolic polygon P. The surface X with characteristic symbol $[1,s,t]$ is obtained from P by the identification of the side n_{2i-1} with n_{2i+2s}.

The polygon P admits as symmetry group a dihedral group D_q containing the reflections on the diagonals of P and rotations of order q. The symmetries of P are compatible with the identifications of the sides that produce X. One of the rotations, R, of D_q transforms the side n_x in the side n_{x+2}. The rotation R is compatible with the sides identification because $n_{2i-1+2} = n_{2(i+1)-1}$ is identified with $n_{2i+2s+2} = n_{2(i+1)+2s}$. One of the reflections, F, of D_q transforms the side n_x in the side n_{q-x}. To prove that F is compatible with the side identifications it is enough to remark that, if $0 < j \leq q$ is such that $j \equiv \frac{1}{2}(-2i-2s+1) \mod q$

then $n_{q-(2i-1)} = n_{2j+2s}$ is identified with $n_{q-2i-2s} = n_{2j-1}$. Then the surface X admits q symmetries (anticonformal involutions). In particular there are two symmetries s_1 and s_2 that are conjugate in $\mathrm{Aut}(X)$ and that generate D_q as the automorphism group of X.

The above proves the following

Lemma 2.2. *An isolated point X in $\mathscr{B}_g$ admits two symmetries which generate D_q as the automorphism group of X.*

Note. There are two possible signatures of NEC groups, namely $s(\Lambda_1) = \big(0;+;[\,];\{(q,q,q)\}\big)$, $s(\Lambda_2) = \big(0;+;[q];\{(q)\}\big)$ corresponding to groups that admit the Fuchsian group Λ with signature $(0;+;[q,q,q];\{\,\})$ as their canonical Fuchsian subgroup. Only for the first signature we have that the epimorphisms $\theta\colon \Lambda \to Z_q$ with kernel a surface group extend to an epimorphism $\theta_1\colon \Lambda_1 \to D_q$. Hence if X represents an isolated point in $\mathscr{B}_g$, then $\mathrm{Aut}\, X = D_q$ and the orbifold $X/\mathrm{Aut}\, X$ is a disc with three corner points of order q.

Let X be a Riemann surface representing an isolated point of $\mathscr{B}_g$. By Lemma 2.2 X has an automorphism group D_q then there is an NEC group Δ and an epimorphism $\theta\colon \Delta \to D_q$ such that $X = H/\mathrm{Ker}\,\theta$. Let $s \in D_q$ be a symmetry of X and $\theta^{-1}\{1,s\} = \Gamma_s$ then $\mathrm{Ker}\,\theta \subset \Gamma_s$. Let Γ be the abstract Fuchsian group such that Γ is isomorphic to $\mathrm{Ker}\,\theta$ and let Γ^* be the abstract group isomorphic to Γ_s. Let $t\colon \Gamma^* \to \Gamma_s$ be an isomorphism and $i\colon \Gamma \to \Gamma^*$ be a monomorphism such that $t \circ i(\Gamma) = \mathrm{Ker}\,\theta$. Then i induces $m\colon \mathbf{T}(\Gamma^*) \to \mathbf{T}(\Gamma)$. Hence $m\mathbf{T}(\Gamma^*)$ give us a submanifold in $\mathbf{T}(\Gamma)$ containing X.

Let v be a symmetry of D_q different from s. Then there is $w \in D_q$ such that $v = wsw^{-1}$. Let $g \in \Delta$ such that $\theta(g) = w$. Conjugation by g induces an automorphism in $\mathrm{Ker}\,\theta$, c_g, and

$$\alpha = (t \circ i)^{-1} \circ c_g \circ t \circ i \tag{2.1}$$

is an automorphism of Γ. Let $[\alpha]$ be the element of the modular group represented by α.

Lemma 2.3. $[t \circ i] \in m\big(\mathbf{T}(\Gamma^*)\big) \cap [\alpha] m\big(\mathbf{T}(\Gamma^*)\big)$.

Proof. By the definition of m we have $[t \circ i] = m[t]$ and by the definition of the action of $[\alpha]$ we have: $[\alpha]m[t] = [\alpha][t\circ i] = [t\circ i\circ\alpha] = [t\circ i\circ(t \circ i)^{-1} \circ c_g \circ t \circ i] = [c_g \circ t \circ i] = [t \circ i]$.

Lemma 2.4. *There exists $\varepsilon > 0$ such that, if $B_\varepsilon([t \circ i])$ is the ball of radius ε around $[t \circ i]$ in $\mathbf{T}(\Gamma)$, then*

$$\{[t \circ i]\} = m\big(\mathbf{T}(\Gamma^*)\big) \cap [\alpha] m\big(\mathbf{T}(\Gamma^*)\big) \cap B_\varepsilon([t \circ i]).$$

Proof. Since X is an isolated point in $\mathscr{B}_g$ there is an $\varepsilon > 0$ such that $\mathrm{Aut}^+\big(H/r*(\Gamma)\big) = \{\text{identity}\}$ for all $r*$ such that $r* \in B_\varepsilon([t \circ i]) - \{[t \circ i]\}$.

Assume that there exists

$$[r*] \in m\big(\mathbf{T}(\Gamma^*)\big) \cap [\alpha] m\big(\mathbf{T}(\Gamma^*)\big) \cap \big(B_\varepsilon([t\circ i]) - \{[t\circ i]\}\big).$$

Then $[r*] = m[r_1] = [r_1 \circ i] = [\alpha] m[r_2] = [r_2 \circ i \circ \alpha]$, where $[r_1]$ and $[r_2] \in m\big(\mathbf{T}(\Gamma^*)\big)$. Hence there is $h \in \mathscr{G}^+$ such that $r_1 \circ i = c_h \circ r_2 \circ i \circ \alpha$, where c_h is defined by $c_h(x) = hxh^{-1}$. Let $r_3 = c_h \circ r_2$. Then

$$r_1 \circ i = r_3 \circ i \circ \alpha, \qquad r_1 \circ i(\Gamma) \subset r_1(\Gamma^*) \tag{2.2}$$

and

$$r_3 \circ i(\Gamma) = r_3 \circ i \circ \alpha(\Gamma) \subset r_3(\Gamma^*).$$

If $r_1(\Gamma^*) \neq r_3(\Gamma^*)$ then there are two symmetries a and b of $Y = H/r_1 \circ i(\Gamma)$ and then there is a conformal automorphism of Y, ab, different from the identity in contradiction with the choice of ε. Thus $r_1(\Gamma^*) = r_3(\Gamma^*)$. Hence

$$\beta = {r_3}^{-1} \circ r_1 \tag{2.3}$$

is an automorphism of Γ^*. Then by (2.2) and (2.3)

$$\beta \circ i = {r_3}^{-1} \circ r_1 \circ i = {r_3}^{-1} \circ r_3 \circ i \circ \alpha = i \circ \alpha. \tag{2.4}$$

Given an NEC group, Λ, we define

$$E(\Lambda) = \{g e_i g^{-1} : g \varepsilon \Lambda \text{ and } e_i \text{ is a connecting generator}\}.$$

We have $E(\Gamma_s) \subset \operatorname{Ker}\theta$ and $E\big(c_g(\Gamma_s)\big) \subset \operatorname{Ker}\theta$. Since Γ_s and $c_g(\Gamma_s)$ produce different symmetries on X then

$$E(\Gamma_s) \cap E\big(c_g(\Gamma_s)\big) = \Phi. \tag{2.5}$$

On the other hand

$$c_g\big(E(\Gamma_s)\big) = E\big(c_g(\Gamma_s)\big). \tag{2.6}$$

We shall translate the situation to Γ using $t \circ i$ from (2.5) we have

$$(t\circ i)^{-1}\big(E(\Gamma_s)\big) \cap (t\circ i)^{-1}\big(E\big(c_g(\Gamma_s)\big)\big) = \Phi,$$

and from (2.6)

$$(t\circ i)^{-1}\big(c_g\big(E(\Gamma_s)\big)\big) = (t\circ i)^{-1}\big(E\big(c_g(\Gamma_s)\big)\big),$$

so from the definition of α, (2.1), we obtain

$$\alpha \circ (t \circ i)^{-1}\big(E(\Gamma_s)\big) = (t \circ i)^{-1}\big(E\big(c_g(\Gamma_s)\big)\big).$$

Also $E(\Gamma^*) = t^{-1}\big(E(\Gamma_s)\big)$. Then

$$\alpha \circ i^{-1}\big(E(\Gamma^*)\big) = (t \circ i)^{-1}\big(E\big(c_g(\Gamma_s)\big)\big) = \alpha \circ i^{-1}\big(t^{-1}E(\Gamma_s)\big).$$

So

$$\alpha \circ i^{-1}\big(E(\Gamma^*)\big) \cap i^{-1}\big(E(\Gamma^*)\big) = \Phi. \tag{2.7}$$

Since β is an automorphism of Γ^* the following equality holds: $\beta\big(E(\Gamma^*)\big) = E(\Gamma^*)$. We have $E(\Gamma^*) \subset i(\Gamma)$, and by (2.4)

$$E(\Gamma^*) = \beta \circ i\big(i^{-1}E(\Gamma^*)\big) = i \circ \alpha \circ i^{-1}\big(E(\Gamma^*)\big).$$

Then $\alpha \circ i^{-1}\big(E(\Gamma^*)\big) = i^{-1}\big(E(\Gamma^*)\big)$ that is a contradiction with (2.7).

In the above lemma choosing a symmetry different from s and v we can replace the submanifold $[\alpha]m\big(\mathbf{T}(\Gamma^*)\big)$ by $[\alpha^j]m\big(\mathbf{T}(\Gamma^*)\big)$ with $j \in \{2, \dots, q-1\}$. Applying Lemma 2.4 to every pair of symmetries of D_q we have

Theorem 2.5. *Let Γ be a Fuchsian surface group and $[r] \in \mathbf{T}(\Gamma)$ such that $H/r(\Gamma)$ represents an isolated point in the branch loci with automorphism group D_q. Let $[\alpha^j]m\big(\mathbf{T}(\Gamma^*)\big)$, $j \in \{1, \dots, q\}$, be the submanifolds of $\mathbf{T}(\Gamma)$ given by the groups uniformizing surfaces with a symmetry and such that $[r] \in [\alpha^j]m\big(\mathbf{T}(\Gamma^*)\big)$. Then there exists a neigborhood V of $[r]$ such that*

$$\{[r]\} = V \cap \big(\cap[\alpha^j]m\big(\mathbf{T}(\Gamma^*)\big)\big).$$

Remark. Let S_1 and S_2 be two submanifolds of $\mathbf{T}(\Gamma)$ such that S_1 and S_2 correspond to groups uniformizing Riemann surfaces with a symmetry. Since $\dim S_1 = \dim S_2 = \frac{1}{2}\dim \mathbf{T}(\Gamma)$, if S_1 and S_2 cut transversally in a point p then such a point projects by $\pi\colon \mathbf{T}(\Gamma) \to \mathscr{M}_g$ in an isolated point of the branch loci.

3. On the non-existence of isolated points for non-orientable surfaces and surfaces with boundary

We shall see in this section that, on the contrary of what happens with Riemann surfaces, there are not isolated points in the branch locus $\mathscr{B}$ of Klein surfaces which either are non-orientable or have nonempty boundary.

We first consider non-orientable surfaces Y without boundary. It is well known that if $G = \operatorname{Aut} Y$, then $G = \operatorname{Aut}^+ Y_c$, where Y_c denotes the complex double of Y.

Theorem 3.1. *There are not isolated points in the branch locus* $\mathscr{B}_g$ *of non-orientable Klein surfaces without boundary.*

Proof. Assume that Y is a non-orientable surface without boundary of genus g representing an isolated point in $\mathscr{B}_g$. Using Theorem 2.1, we obtain that $\operatorname{Aut} Y = \operatorname{Aut}^+ Y_c = Z_q$, with $q = 2g^+ + 1$ a prime, then $q = 2g - 1$. On the other hand the quotient orbifold $Y/\operatorname{Aut} Y$ must be either a disc with three corner points of order q, or a disc with a corner point of order q and a cone point of order q. Now, as Y is non-orientable and q is an odd integer, then the orbifold Y/Z_q is non-orientable and we are in a contradiction.

Let Γ be an NEC surface group with signature $\bigl(h;\pm;[\,];\{(\,)^k\}\bigr)$, the notation means that there are k empty period cycles in Γ.

If G' is a subgroup of $\operatorname{Aut}(H/\Gamma)$, then the G'-action on H/Γ induces a stratum of the branch locus of the surfaces with group Γ. The stratum has dimension 0 if the quotient orbifold of H/Γ by G' is the disc with three corner points, or the disc with a corner point of order q and a cone point of order q. But the quotient orbifold of a bordered Klein surface contains a boundary component without corner points or a boundary component with at least two corner points of order 2 (see [BM]). This proves the following

Theorem 3.2. *There are no isolated points in the branch locus of Klein surfaces with nonempty boundary.*

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