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POLARIZATION, CONFORMAL INVARIANTS, AND BROWNIAN MOTION

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Abstract. The polarization $\mathbf{P}(A)$ of a closed or open set $A \subset \mathbf{C}$ is defined as follows: If $z, \bar{z} \in A$ then $z, \bar{z} \in \mathbf{P}(A)$. If neither $z, \bar{z} \in A$ then neither $z, \bar{z} \in \mathbf{P}(A)$. If exactly one of $z = x + iy$, $\bar{z}$ belongs to A then $x + i|y| \in \mathbf{P}(A)$ and $x - i|y| \notin \mathbf{P}(A)$. We prove theorems that describe the behaviour of harmonic measure, Green function, Robin function, Brownian motion and extremal length under polarization. These theorems, combined with an approximation technique due to Dubinin, lead to a new proof of symmetrization results of Baernstein.

1. Introduction

Polarization with respect to the real axis is a geometric transformation that preserves the symmetric part of a set and moves the nonsymmetric part to the upper half plane. To give the precise definition we need the following notation:

$\mathbf{C}_+$ is the upper half plane and $\mathbf{C}_-$ is the lower half plane. If F is a closed set in $\mathbf{C}$, then $F_+ = \operatorname{clos} \mathbf{C}_+ \cap F$ and $F_- = \operatorname{clos} \mathbf{C}_- \cap F$. If O is an open set in $\mathbf{C}$, then $O_+ = \mathbf{C}_+ \cap O$ and $O_- = \mathbf{C}_- \cap O$. Here clos means closure in the topology of the plane. The reflection of a set A in the real axis is denoted by $\bar{A}$, i.e., $\bar{A} = \{\bar{z} : z \in A\}$.

The polarization $\mathbf{P}(A)$ of a closed or open set $A \subset \mathbf{C}$ is defined as follows: If $z, \bar{z} \in A$ then $z, \bar{z} \in \mathbf{P}(A)$. If neither $z, \bar{z} \in A$ then neither $z, \bar{z} \in \mathbf{P}(A)$. If exactly one of $z = x + iy$, $\bar{z}$ belongs to A then $x + i|y| \in \mathbf{P}(A)$ and $x - i|y| \notin \mathbf{P}(A)$. Below there is an equivalent definition.

Definition 1.1. Let A be a closed or open set. The *polarization* $\mathbf{P}(A)$ *of* A (with respect to $\mathbf{R}$) is $\mathbf{P}(A) = (A \cup \bar{A})_+ \cup (A \cap \bar{A})_-$.

We define also the polarization $\mathbf{P}_l(A)$ of A with respect to any oriented line l.

Definition 1.2. Let l be an oriented line and let $T_l \colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be the Möbius transformation that maps ∞ to ∞ and l to $\mathbf{R}$ preserving orientation. The *polarization* $\mathbf{P}_l(A)$ *of* A *with respect to* l is $\mathbf{P}(A) = T_l^{-1}\mathbf{P}(T_l A)$.

Thus in the notation of Definitions 1.1 and 1.2, we have $\mathbf{P} = \mathbf{P}_{\mathbf{R}}$. Another piece of notation: If $F = \{z\}$ is a singleton, $\mathbf{P}(z)$ is the element of the singleton $\mathbf{P}(F)$, i.e., if $\operatorname{Im} z \geq 0$ then $\mathbf{P}(z) = z$ and if $\operatorname{Im} z < 0$ then $\mathbf{P}(z) = \bar{z}$.

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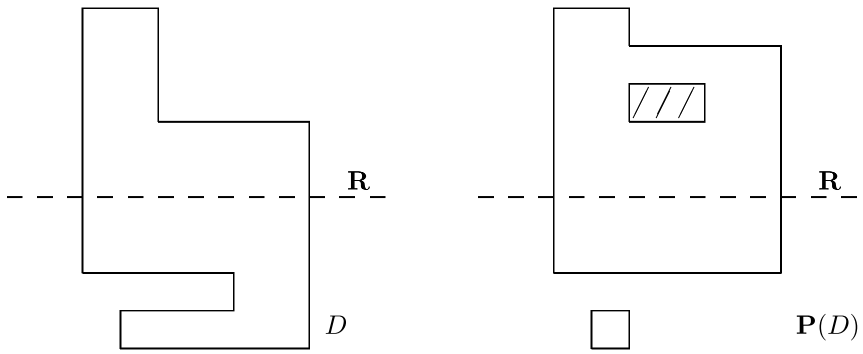

Figure 1. A domain D and its polarization $\mathbf{P}(D)$.

Remarks. (1) If F is a closed set then $\mathbf{P}(F)$ is a closed set. If O is an open set then $\mathbf{P}(O)$ is open set.

(2) $\mathbf{P}(D)$ need not be connected even if D is. It is not necessarily simply connected even if D is.

(3) The area of $\mathbf{P}(O)$ is equal to the area of O.

(4) $\mathbf{P}(D)$ has always the property: $\overline{\mathbf{P}(D)_-} \subset \mathbf{P}(D)_+$.

Polarization appeared in a 1952 paper by V. Wolontis [23], who proved results on the behavior of certain extremal lengths under polarization. We will use the following notation for extremal distances: Let $D \subset \mathbf{C}$ be a domain and let K_1, K_2 be two compact subsets of $\operatorname{clos} D$. Then $\lambda(K_1, K_2, D)$ is the extremal distance between K_1 and K_2 with respect to D. More precisely $\lambda(K_1, K_2, D)$ is the extremal length of the family of all piecewise rectifiable curves that lie in $D \backslash K_1 \backslash K_2$ and run from K_1 to K_2. See [16] for more information on extremal length.

If F_1 and F_2 are two closed disjoint sets, Wolontis proved that

$$\lambda(F_1, F_2, \mathbf{C}) \leq \lambda\big(\mathbf{P}(F_1), \overline{\mathbf{P}(F_2)}, \mathbf{C}\big). \tag{1.3}$$

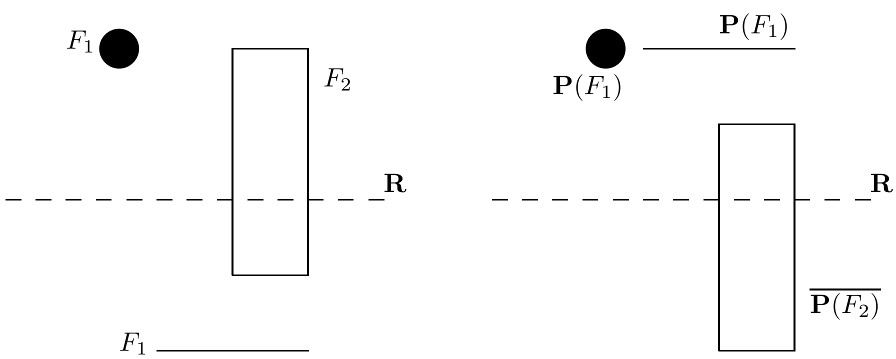

Figure 2. An illustration for inequality (1.3).

Wolontis used the important idea that symmetrization results can be proven by successive applications of polarization results. Polarization was also used by Baernstein and Taylor (see [5] and [4]) for the proof of rearrangement inequalities. Dubinin used successive polarizations to prove theorems for the capacity of condensers in n-space (see [9], [10] and references therein).

We will use the following notation for harmonic measure: Let $\Omega \subset \widehat{\mathbf{C}}$ be an open set and K be any set in $\widehat{\mathbf{C}}$. $\omega(z, K, \Omega)$ will denote the harmonic measure at z of the set $\operatorname{clos} K \cap \operatorname{clos} \Omega$ with respect to the component of $\Omega \setminus \operatorname{clos} K$ that contains z.

Øksendal [17] used Brownian motion to prove a primitive polarization theorem for harmonic measure.

Theorem 1.4 (Øksendal). *Let Ω be a domain in $\mathbf{C}$, symmetric with respect to $\mathbf{R}$. Let $K \subset \Omega$ be compact and put $\widetilde{K} = \big(\mathbf{P}(K)\big)_+$. Then*

$$\omega(x, K, \Omega) \geq \omega(x, \widetilde{K}, \Omega), \qquad x \in \mathbf{R} \cap \Omega \setminus K. \tag{1.5}$$

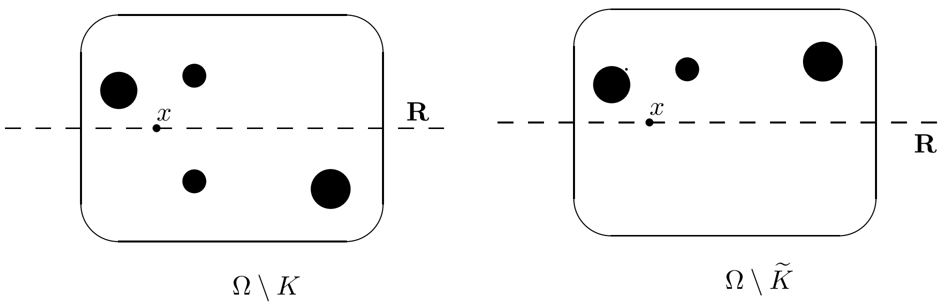

Figure 3. An illustration for inequality (1.5).

Øksendal used this theorem to prove projection estimates for harmonic measure. He proved, for example the Beurling–Nevanlinna projection theorem and Hall's lemma. Theorem 1.4 was rediscovered by Baernstein (see Lemmas 1 and 2 in [14]) whose proof is similar to the original proof of the Beurling–Nevanlinna theorem. Baernstein showed that Theorem 1.4 implies the following result for the Green function.

Theorem 1.6 (Baernstein). *Let E be a compact set with positive capacity. Let $\widetilde{E} = \{x - i|y| : x + iy \in E\}$. If $x \in \mathbf{R} \cup \{\infty\}$ and $u + iv \in \mathbf{C}$, then $g(x, u + iv, \widehat{\mathbf{C}} \setminus E) \leq g(x, u + i|v|, \widehat{\mathbf{C}} \setminus \widetilde{E})$.*

In Sections 2 and 3 we state and prove a polarization theorem for harmonic measure which generalizes Øksendal's theorem. The subsequent sections contain polarization results for the Green and Robin functions, and for Brownian motion. Some of these results are closely related to the results obtained independently by A.Yu. Solynin in [21]. The polarization theorem for harmonic measure (Theorem 2.2) is stated in [21] without proof and the theorem on the Green function (Theorem 4.1) is proved in [21] with a different method. Now we present some of our main polarization inequalities.

Let D be a domain regular for the Dirichlet problem. Let $E \subset \partial D$ be a closed set such that $\overline{E} \cap D = \emptyset$. Then we have:

(i) For $x \in \mathbf{R} \cap D$,

$$\omega(x, E, D) \le \omega\big(x, \mathbf{P}(E), \mathbf{P}(D)\big). \tag{1.7}$$

(ii) For $z \in D \cap \overline{D}$,

$$\omega(z, E, D) + \omega(\bar{z}, E, D) \le \omega\big(z, \mathbf{P}(E), \mathbf{P}(D)\big) + \omega\big(\bar{z}, \mathbf{P}(E), \mathbf{P}(D)\big). \tag{1.8}$$

(iii) Let l be the vertical line passing through $x \in \mathbf{R} \cap D$, and let $\Phi\colon \mathbf{R} \to \mathbf{R}$ be an increasing convex function. Then

$$\int_l \Phi\big(\omega(\zeta, E, D)\big)\,|d\zeta| \le \int_l \Phi\big(\omega\big(\zeta, \mathbf{P}(E), \mathbf{P}(D)\big)\big)\,|d\zeta|. \tag{1.9}$$

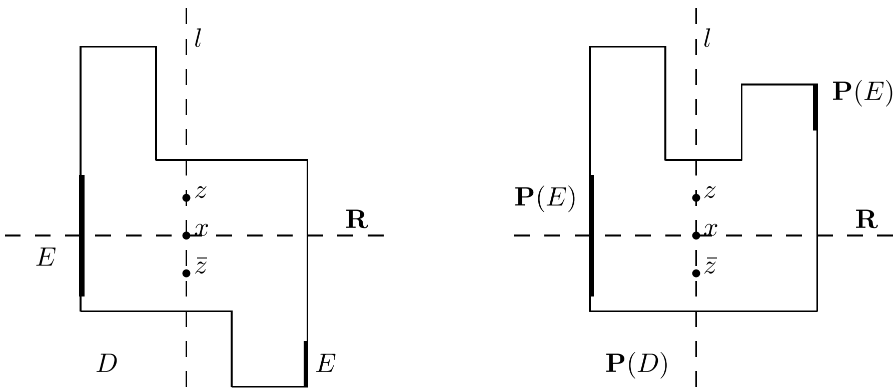

Figure 4.

(iv) For the Green functions of D and $\mathbf{P}(D)$, and for $x, y \in \mathbf{R} \cap D$, we have:

$$g(x, y, D) \le g\big(x, y, \mathbf{P}(D)\big), \qquad x, y \in \mathbf{R} \cap D \tag{1.10}$$

(v) Let B_t denote Brownian motion in the plane and τ_D be the exit time from D. Let P^x be the probability measure corresponding to Brownian motion starting at $x \in \mathbf{R} \cap D$. Then, for $0 \le \tau_1 < \tau_2 \le \infty$,

$$P^x(B_{\tau_D} \in E, \tau_1 < \tau_D < \tau_2) \le P^x\big(B_{\tau_{\mathbf{P}(D)}} \in \mathbf{P}(E), \tau_1 < \tau_{\mathbf{P}(D)} < \tau_2\big). \tag{1.11}$$

2. Polarization and harmonic measure

In this section we state a theorem that describes the behaviour of harmonic measure under polarization. Solynin [21] has discovered it independently.

In the sequel we will use the following notational convention for harmonic measure: $\omega(z, E, \Omega)$ is set to be 0 outside the component of $\Omega \setminus \operatorname{clos} E$ that contains z.

Let D be a domain in $\mathbf{C}$, regular for the Dirichlet problem. Let $E \subset \partial D$ be a closed set and assume that E satisfies the condition:

$$\overline{E} \cap D = \emptyset. \tag{2.1}$$

Elementary set-theoretic considerations show that E satisfies (2.1) if and only if it is the union of two sets E_1 and E_2 such that

(i) $\overline{E}_1 \subset \partial D$ and

(ii) $\mathbf{P}(E_2) \subset \partial \mathbf{P}(D)_+$.

For the equality statements of the theorem it is assumed, in addition, that D is bounded by a finite number of curves or arcs and E consists of a finite number of curves or arcs.

The theorem compares the harmonic measures $\omega\big(\mathbf{P}(z), \mathbf{P}(E), \mathbf{P}(D)\big)$ and $\omega(z, E, D)$. Before stating it, we need some more notation. The polarization $\mathbf{P}(D)$ of D will be denoted (sometimes) by G, i.e. $G = \mathbf{P}(D)$. We give the names (a), (b), (c) to the following conditions:

(a) $E = \underline{\mathbf{P}(E)}$, $D = \underline{\mathbf{P}(D)}$.

(b) $E = \overline{\mathbf{P}(E)}$, $D = \overline{\mathbf{P}(D)}$.

(c) $D = \overline{D}$.

Theorem 2.2. *With the above notation we have*

$$\omega(x, E, D) \le \omega\big(x, \mathbf{P}(E), \mathbf{P}(D)\big), \qquad x \in \mathbf{R} \cap D. \tag{2.3}$$

Equality holds in (2.3) *for some* $x \in \mathbf{R} \cap D$ *if and only if* (a) *or* (b) *or* (c) *holds.*

$$\omega(\bar{z}, E, D) \le \omega\big(z, \mathbf{P}(E), \mathbf{P}(D)\big), \qquad z \in \overline{D_-}. \tag{2.4}$$

Equality holds in (2.4) *for some* $z \in \overline{D_-}$ *if and only if* (b) *holds.*

$$\omega(z, E, D) \le \omega\big(z, \mathbf{P}(E), \mathbf{P}(D)\big), \qquad z \in D_+. \tag{2.5}$$

Equality holds in (2.5) *for some* $z \in D_+$ *if and only if* (a) *holds.*

$$\omega(z, E, D) + \omega(\bar{z}, E, D) \le \omega\big(z, \mathbf{P}(E), \mathbf{P}(D)\big) + \omega\big(\bar{z}, \mathbf{P}(E), \mathbf{P}(D)\big), \quad z \in D. \tag{2.6}$$

Equality holds in (2.6) *for some* $z \in D$ *if and only if* (a) *or* (b) *or* (c) *holds.*

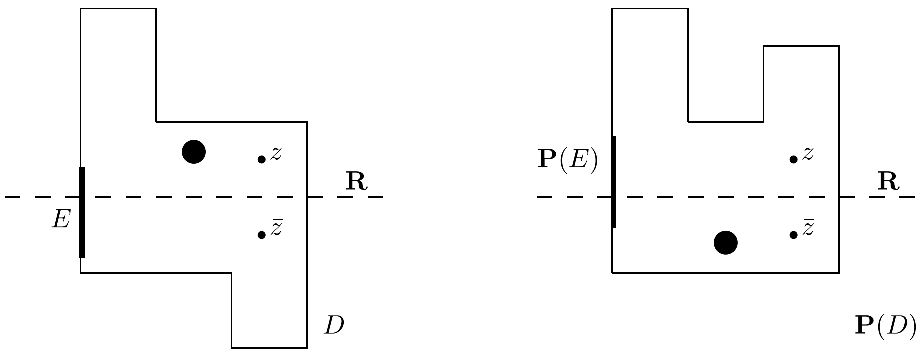

Figure 5. An illustration for Theorem 2.2.

The proof of the theorem is deferred until the next section. Here we will present some of its consequenses. Øksendal's Theorem 1.4 follows from Theorem 2.2 by setting $D=\Omega\setminus K$ and $E=\partial\Omega$. In the corollaries that follow, we use the notation set before the statement of Theorem 2.2.

Corollary 2.7. *For all* $x\in\mathbf{R}$ *and all increasing convex functions* $\Phi\colon\mathbf{R}\to\mathbf{R}$

$$\int_{D^x}\Phi\bigl(\omega(x+iy,E,D)\bigr)\,dy\le\int_{G^x}\Phi\bigl(\omega\bigl(x+iy,\mathbf{P}(E),\mathbf{P}(D)\bigr)\bigr)\,dy, \tag{2.8}$$

where $D^x=\{y:x+iy\in D\}$ *and* $G^x=\{y:x+iy\in\mathbf{P}(D)\}$.

Equality holds in (2.8) *for some* $x\in\mathbf{R}\cap D$ *and some nonconstant, increasing, convex function* Φ *if and only if at least one of the conditions* (a), (b), (c) *holds.*

The inequality (2.8) follows from (2.6) and the following fact about convexity: Suppose $a\le b$, $c\le d$, $a\le d$ and $a+b\le c+d$. Let Φ be a convex increasing function. Then $\Phi(a)+\Phi(b)\le\Phi(c)+\Phi(d)$.

The equality statement, which is due to Solynin [21], follows from the equality statement for the above convexity inequality: If $\Phi(a)+\Phi(b)=\Phi(c)+\Phi(d)$ then $a+b=c+d$.

By taking $\Phi(t)=t^p$ in the above corollary and letting $p\to\infty$ we obtain:

Corollary 2.9. *With the above notation we have*:

$$\max_{z\in D_x}\omega(z,E,D)\le\max_{z\in G_x}\omega\bigl(z,\mathbf{P}(E),\mathbf{P}(D)\bigr),\qquad x>0, \tag{2.10}$$

where $D_x=\{z\in D:\operatorname{Re}z=x\}$ *and* $G_x=\{z\in\mathbf{P}(D):\operatorname{Re}z=x\}$.

By applying a sequence of polarizations (a technique due to Dubinin [10], see also [5], [4]), we see that Corollary 2.7 implies:

Corollary 2.11. *Let Ω be a domain with $\Omega \subset \mathbf{D}$ and let $\alpha = \partial\Omega \cap \partial\mathbf{D}$. Denote by Ω^* and α^* the circular symmetrizations of Ω and α respectively with respect to the positive semiaxis. Set*

$$u(z) = \begin{cases} \omega(z, \alpha, \Omega), & \text{if } z \in \Omega, \\ 0, & \text{if } z \in \mathbf{D} \setminus \Omega \end{cases} \tag{2.12}$$

and

$$v(z) = \begin{cases} \omega(z, \alpha^*, \Omega^*), & \text{if } z \in \Omega^*, \\ 0, & \text{if } z \in \mathbf{D} \setminus \Omega^*. \end{cases} \tag{2.13}$$

Then for all $r \in (0,1)$ and all increasing convex functions $\Phi\colon \mathbf{R} \to \mathbf{R}$

$$\int_0^{2\pi} \Phi\big(u(re^{i\theta})\big)\, d\theta \le \int_0^{2\pi} \Phi\big(v(re^{i\theta})\big)\, d\theta. \tag{2.14}$$

Equality holds in (2.14) for some $r \in \Omega^*$ and some nonconstant, increasing and convex function Φ if and only if $\Omega = e^{i\phi}\Omega^*$ and $\alpha = e^{i\phi}\alpha^*$ for some $\phi \in \mathbf{R}$.

This is the main symmetrization result for harmonic measure. It was proven by Baernstein [2] who used the star function method. The equality statement improves a result due to Essén and Shea [13].

3. Proof of the polarization theorem for harmonic measure

The proof uses a method of Øksendal [17] and involves successive applications of the strong Markov property of harmonic measure: *Let Ω_1 and Ω_2 be two domains in $\mathbf{C}$. Assume that $\Omega_1 \subset \Omega_2$ and let $F \subset \partial\Omega_2$ be a closed set. Let $\sigma = \partial\Omega_1 \setminus \partial\Omega_2$. Then for $z \in \Omega_1$,*

$$\omega(z, F, \Omega_2) = \omega(z, F, \Omega_1) + \int_\sigma \omega(z, ds, \Omega_1)\omega(s, F, \Omega_2). \tag{3.1}$$

The equation (3.1) is a consequence of the probabilistic interpretation of harmonic measure as hitting probability of Brownian motion (see [18]). One can also prove it directly using the potential-theoretic definition of harmonic measure (see [15, p. 114]).

We explain the notation $\omega(z, ds, \Omega_1)$ that appears in (3.1): $\omega(z, \cdot\,, \Omega_1)$ is a measure for fixed $z \in \Omega_1$. Call this measure $\mu_z^{\Omega_1}$. In integrals the usual notation is $d\mu_z^{\Omega_1}(s)$ where s is the variable of integration. Instead of this notation we will use the notation $\omega(z, ds, \Omega_1)$, i.e. $d\mu_z^{\Omega_1}(s) = \omega(z, ds, \Omega_1)$.

We start with the proof of (2.3). Let $\Omega = D \cap \overline{D} = \mathbf{P}(D) \cap \overline{\mathbf{P}(D)}$. We observe that

$D = \Omega \cup (D_- \setminus \Omega) \cup (D_+ \setminus \Omega)$ (disjoint union) and
$G = \Omega \cup \overline{(D_- \setminus \Omega)} \cup (D_+ \setminus \Omega)$ (disjoint union).

Let $\lambda = \partial\Omega \cap D_+$, $\varrho = \mathbf{R} \cap \Omega$ $(= \mathbf{R} \cap D = \mathbf{R} \cap G)$ and $\sigma = \partial\Omega \cap D_-$. Note that if $\lambda = \emptyset$, then $\overline{D_+} \subset D_-$ and hence $D = \overline{G}$. Also, if $\sigma = \emptyset$, then $\overline{D_-} \subset D_+$ and hence $D = G$. If both $\lambda = \emptyset$ and $\sigma = \emptyset$, then $D = \overline{D}$. In this case (2.3) holds with equality. It is also easy to see that if at least one of the conditions (a), (b) holds, then (2.3) holds with equality. From now on we assume that either $\lambda \neq \emptyset$ or $\sigma \neq \emptyset$.

The set λ lies in D_+ and σ lies in D_-. Note also that $\lambda \subset G_+$ and $\bar{\sigma} \subset G_+$. We will denote the points of λ, ϱ, σ by l, r, s, respectively.

We first prove (2.3) under the following additional assumptions:

$$\operatorname{clos}\lambda \cap \mathbf{R} = \emptyset \qquad \text{and} \qquad \operatorname{clos}\sigma \cap \mathbf{R} = \emptyset. \tag{3.2}$$

and

$$D \text{ is bounded} \tag{3.3}$$

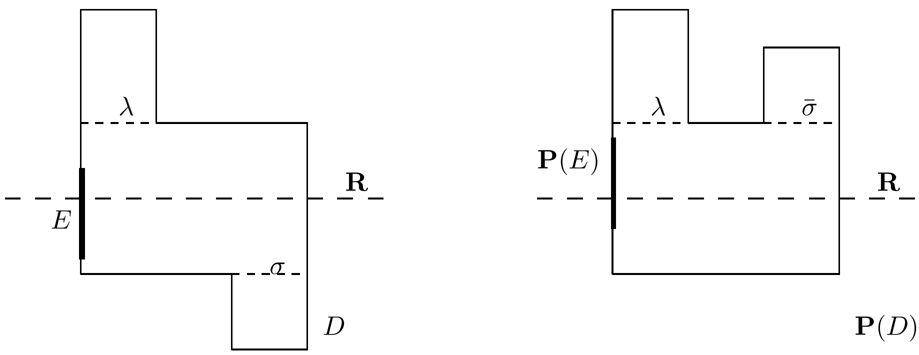

Figure 6. The sets λ, σ, and $\bar{\sigma}$ in D and $\mathbf{P}(D)$.

Fix $x \in \mathbf{R} \cap D$. An application of the strong Markov property shows that

$$\omega(x,E,D) = \omega(x,E,\Omega) + \int_\lambda \omega(x,dl,\Omega)\omega(l,E,D) + \int_\sigma \omega(x,ds,\Omega)\omega(s,E,D). \tag{3.4}$$

Again the strong Markov property yields

$$\omega(l,E,D) = \omega(l,E,D_+) + \int_\varrho \omega(l,dr,D_+)\omega(r,E,D) \quad \text{and} \tag{3.5}$$

$$\omega(s,E,D) = \omega(s,E,D_-) + \int_\varrho \omega(s,dr,D_-)\omega(r,E,D). \tag{3.6}$$

We substitute (3.5) and (3.6) in (3.4) and obtain

$$\begin{aligned} \omega(x,E,D) &= \omega(x,E,\Omega) + \int_\lambda \omega(x,dl,\Omega)\omega(l,E,D_+) \\ &+ \int_\lambda \int_\varrho \omega(x,dl,\Omega)\omega(l,dr,D_+)\omega(r,E,D) \\ &+ \int_\sigma \omega(x,ds,\Omega)\omega(s,E,D_-) \\ &+ \int_\sigma \int_\varrho \omega(x,ds,\Omega)\omega(s,dr,D_-)\omega(r,E,D). \end{aligned} \tag{3.7}$$

A similar triple application of the strong Markov property applied to the harmonic measure $\omega(x,\mathbf{P}(E),G)$ gives

$$\begin{aligned} \omega\big(x,\mathbf{P}(E),G\big) &= \omega\big(x,\mathbf{P}(E),\Omega\big) + \int_\lambda \omega(x,dl,\Omega)\omega\big(l,\mathbf{P}(E),G_+\big) \\ &+ \int_\lambda \int_\varrho \omega(x,dl,\Omega)\omega(l,dr,G_+)\omega\big(r,\mathbf{P}(E),G\big) \\ &+ \int_{\bar\sigma} \omega(x,ds,\Omega)\omega\big(s,\mathbf{P}(E),G_+\big) \\ &+ \int_{\bar\sigma} \int_\varrho \omega(x,ds,\Omega)\omega(s,dr,G_+)\omega\big(r,\mathbf{P}(E),G\big). \end{aligned} \tag{3.8}$$

Next we write (3.7) with x replaced by r and r,s,l replaced by r_1,s_1,l_1. We thus obtain a formula for $\omega(r,E,D)$. We substitute this formula into (3.7).

Performing this argument n times we see that $\omega(x,E,D)$ can be written as a sum whose first term is $\omega(x,E,\Omega)$ and the other terms are certain integrals. The integrands are products of harmonic measures. These harmonic measures have one of the following eight forms:

(1) $\omega(r,E,\Omega)$	(2) $\omega(l,E,D_+)$	(3) $\omega(s,E,D_-)$	(4) $\omega(l,dr,D_+)$
(5) $\omega(s,dr,D_-)$	(6) $\omega(r,dl,D_s)$	(7) $\omega(r,ds,\Omega)$	(8) $\omega(r,E,D)$.

It is important to note that a term of the form (8) appears only in 2^n integrals $I_1,I_2,\dots,I_{2^n}$. The integrand of I_j is a product of $2n+1$ factors: n factors of the forms (4) or (5), n factors of the forms (6) or (7) and one factor of the form (8).

Because of the assumptions (3.2) and (3.3), there exists a positive constant $\delta=\delta(D)<1$ such that

$$\omega(l,\varrho,G_+) \le \delta, \tag{3.9}$$

$$\omega(l,\varrho,D_+) \le \delta, \tag{3.10}$$

$$\omega(\bar s,\varrho,G_+) \le \delta, \tag{3.11}$$

$$\omega(s,\varrho,D_-) \le \delta. \tag{3.12}$$

We also have, by symmetry,

$$\omega(r,\lambda,\Omega)\le \tfrac{1}{2}, \tag{3.13}$$

$$\omega(r,\sigma,\Omega)\le \tfrac{1}{2}. \tag{3.14}$$

The inequalities (3.9)–(3.14) imply that $I_j \le \delta^n/2^n$, $j=1,2,\dots,2^n$ and therefore

$$\sum_{j=1}^{2^n} I_j \le \delta^n \to 0, \qquad \text{as } n\to\infty. \tag{3.15}$$

Similarly we can obtain a formula for $\omega\bigl(x,\mathbf{P}(E),G\bigr)$: It can be written as a sum whose first term is $\omega\bigl(x,\mathbf{P}(E),\Omega\bigr)$ and the other terms are certain integrals. The integrands now are products of harmonic measures of the following eight forms:
$(1)^*$ $\omega(r,\mathbf{P}(E),\Omega)$ $(2)^*$ $\omega(l,\mathbf{P}(E),G_+)$ $(3)^*$ $\omega(\bar s,\mathbf{P}(E),G_+)$
$(4)^*$ $\omega(l,dr,G_+)$ $(5)^*$ $\omega(\bar s,dr,G_+)$ $(6)^*$ $\omega(r,dl,\Omega)$
$(7)^*$ $\omega(r,d\bar s,\Omega)$ $(8)^*$ $\omega(r,\mathbf{P}(E),G)$.

Again we remark that a term of the form $(8)^*$ appears only in 2^n integrals $I_1^*,I_2^*,\dots,I_{2^n}^*$ such that

$$\sum_{j=1}^{2^n} I_j^* \to 0, \qquad \text{as } n\to\infty. \tag{3.16}$$

It is easy to see that by symmetry and the domain monotonicity of harmonic measure each term of the form $(\mathrm{i})^*$ is at least as large as the corresponding term of the form (i), $i=1,2,3,4,5,6,7$. For the measures of the form (4) and $(4)^*$, $\omega(l,dr,D_+)\le\omega(l,dr,G_+)$ means that $\omega(l,I,D_+)\le\omega(l,I,G_+)$ for all $l\in\lambda$ and all Borel subsets $I\subset\varrho$. Similarly for the measures of the forms (5)–(7) and $(5)^*$–$(7)^*$. For example, consider harmonic measures of the forms (3) and $(3)^*$. We have for all:

$$s\in\sigma, \qquad \omega(s,E,D_-)\le\omega\bigl(\bar s,\mathbf{P}(E),G_+\bigr). \tag{3.17}$$

This holds because $\overline{D_-}\subset G_+$ and $\overline{E_-}\subset\mathbf{P}(E)_+$.

It remains to observe that

$$\omega(x,E,\Omega)=\omega\bigl(x,\mathbf{P}(E),\Omega\bigr) \tag{3.18}$$

This finishes the proof of (2.3) with the assumptions (3.2) and (3.3). We show now how we can remove assumption (3.2).

We continue to assume that D is bounded. Let $a = \inf \varrho$ and $b = \sup \varrho$. For $n \in \mathbf{Z}^+$, let

$$O_n = \left\{ z : a < \operatorname{Re} z < b,\ -\frac{1}{n} < \operatorname{Im} z < \frac{1}{n} \right\} \setminus E \setminus \mathbf{R}$$

and

$$D_n = D \cup O_n.$$

Then D_n satisfies (3.2) and hence, by the first part of the proof,

$$\omega(x, E, D_n) \le \omega\big(x, \mathbf{P}(E), \mathbf{P}(D_n)\big), \qquad x \in \varrho. \tag{3.19}$$

Since D_n is a decreasing sequence converging to D, the sequence of harmonic functions $h_n(z) = \omega(z, E, D_n)$ is a decreasing sequence and converges to a function h harmonic in D.

Also $\omega(z, E, D_n) \ge \omega(z, E, D)$, for all $n \in \mathbf{Z}^+$. Hence

$$\omega(z, E, D) \le h(z), \qquad z \in D. \tag{3.20}$$

Claim 3.12. *The sequence* $h(z) = \omega(z, E, D)$, $z \in D$.

Proof. Let $\zeta \in \partial D \setminus E$. We must prove that

$$h(z) \to 0, \qquad \text{as } D \ni z \to \zeta. \tag{3.22}$$

Choose $k \in \mathbf{Z}^+$ large enough so that $\zeta \in \partial D_k$. Then

$$0 \le \limsup_{D \ni z \to \zeta} h(z) \le \lim_{z \to \zeta} h_k(z) = 0 \tag{3.23}$$

and so (3.22) is proven. The claim follows from (3.20) and (3.22).

Similarly we prove that $\omega\big(z, \mathbf{P}(E), \mathbf{P}(D_n)\big)$ converges to $\omega\big(z, \mathbf{P}(E), \mathbf{P}(D)\big)$ as n tends to ∞. Thus (3.19) implies (2.3).

To remove assumption (3.3), we consider the increasing sequence of bounded open sets $D_m = D \cap \{|z| < m\}$, $m \in \mathbf{Z}^+$ which converges to D and we proceed as above using simple convergence arguments.

So (2.3) has been proved.

The inequality (2.4) follows immediately after applying the maximum principle to the function $\omega(\bar{z}, E, D) - \omega\big(z, \mathbf{P}(E), \mathbf{P}(D)\big)$, $z \in \overline{D_-}$. (2.5) and (2.6) are proven similarly.

We proceed with the proof of the equality statement for (2.3). Assume that there exists $x \in \varrho$ such that

$$\omega(x, E, D) = \omega\big(x, \mathbf{P}(E), \mathbf{P}(D)\big). \tag{3.24}$$

Claim 3.25. (i) *If* $E \ne \mathbf{P}(E)$, *then* $D = \overline{G}$.

(ii) *If* $E \ne \overline{\mathbf{P}(E)}$, *then* $D = G$.

Proof. Assume that $D\neq\overline{G}$. So $\lambda\neq\emptyset$. (3.24) and the strong maximum principle applied to (2.6) imply that $\omega(z,E,D)=\omega\big(z,\mathbf{P}(E),\mathbf{P}(D)\big)$ for all $z\in\Omega$. Hence

$$\omega(l,E,D)=\omega\big(l,\mathbf{P}(E),\mathbf{P}(D)\big),\qquad l\in\lambda. \tag{3.26}$$

The assumption $E\neq\mathbf{P}(E)$ implies that $A:=E\setminus\mathbf{P}(E)\neq\emptyset$. Note that $A\subset\partial D_-$ and $\bar{A}\subset\partial\mathbf{P}(D)_+$. By the strong Markov property,

$$\omega(l,A,D)=\int_\varrho\omega(l,dr,D_+)\,\omega(r,A,D) \tag{3.27}$$

$$\omega\big(l,\bar{A},\mathbf{P}(D)\big)=\omega(l,\bar{A},G_+)+\int_\varrho\omega(l,dr,G_+)\,\omega(r,\bar{A},G). \tag{3.28}$$

(3.27) and (3.28) imply

$$\omega(l,A,D)<\omega\big(l,\bar{A},\mathbf{P}(D)\big). \tag{3.29}$$

Therefore

$$\omega(l,E,D)<\omega\big(l,\mathbf{P}(E),\mathbf{P}(D)\big), \tag{3.30}$$

which contradicts (3.26). So $D=\overline{G}$ and (i) is proven. A similar argument proves (ii).

Claim 3.31. (i) *If $E\neq\mathbf{P}(E)$, and $D\neq\overline{D}$, then $E=\overline{\mathbf{P}(E)}$.*
(ii) *If $E\neq\overline{\mathbf{P}(E)}$, and $D\neq\overline{D}$, then $E=\mathbf{P}(E)$.*

Proof. Assume $E\neq\mathbf{P}(E)$. By Claim 3.25 $D=\overline{G}$. Since $D\neq\overline{D}$, $\sigma\neq\emptyset$. If $E\neq\overline{\mathbf{P}(E)}$ we obtain a contradiction as in Claim 3.25. This proves (i). The proof of (ii) is similar.

Claim 3.32. *If $E=\overline{E}$, then either $D=G$ or $D=\overline{G}$.*

Proof. Assume that $E=\overline{E}$ and $D\neq\overline{G}$ and $D\neq G$. Then $\lambda\neq\emptyset$ and $\sigma\neq\emptyset$. The maximum principle and (2.6) give

$$\omega(r,E,D)=\omega(r,E,G), \tag{3.33}$$

$$\omega(l,E,D)=\omega(l,E,G). \tag{3.34}$$

Now by the strong Markov property,

$$\omega(l,E,D)=\omega(l,E_+,D)+\int_\varrho\omega(l,dr,D_+)\,\omega(r,E,D), \tag{3.35}$$

$$\omega(l,E,G)=\omega(l,E_+,G)+\int_\varrho\omega(l,dr,G_+)\,\omega(r,E,G). \tag{3.36}$$

Since $\sigma \neq \emptyset$, D_+ is strictly contained in G_+. So by (3.35), (3.36), and (3.33) we have $\omega(l,E,D) < \omega(l,E,G)$, which contradicts (3.34) and the claim is proven.

Now Claims 3.25, 3.31 and 3.32 imply the equality statement for (2.3).

If equality holds in (2.4) for some $z \in \overline{D_-}$, then equality holds for all $z \in \overline{D_-}$. Therefore equality holds in (2.3) for all $x \in \varrho$. Hence at least one of the conditions (a), (b), (c) holds. The cases (a) and (c) can be easily discarded (unless $D = \overline{D}$ and $E = \overline{E}$, in which case (a) and (c) coincide with (b)). The remaining equality statements are proven similarly.

4. Polarization and Green function

In this section we describe the behavior of the Green function under polarization and present some consequences of this behavior. The main theorem on this subject (Theorem 4.1) has been proved independently by Solynin [21]. All the domains we consider in this section are assumed to possess a Green function. We note also that for the equality statements it is assumed (but not stated explicitly) that the domain is bounded by a finite number of Jordan curves. With this assumption we avoid some trivial cases of equality that involve sets of capacity 0. Set $\mathbf{P}(\infty) = \infty$.

Theorem 4.1. *Let D be a domain in $\widehat{\mathbf{C}}$.*

(i) *For all z, w in D,*

$$g(z,w,D) \leq g\big(\mathbf{P}(z),\mathbf{P}(w),\mathbf{P}(D)\big). \tag{4.2}$$

Equality holds in (4.2) for some $z \in D$ and some $w \neq z$ if and only if either $z = \mathbf{P}(z)$, $w = \mathbf{P}(w)$ and $D = \mathbf{P}(D)$ or $z = \overline{\mathbf{P}(z)}$, $w = \overline{\mathbf{P}(w)}$ and $D = \overline{\mathbf{P}(D)}$.

(ii) *For all $w \in D$ and for all $z \in \mathbf{C} \setminus \{w\}$,*

$$g(z,w,D) + g(\bar{z},w,D) \leq g\big(z,w,\mathbf{P}(D)\big) + g\big(\bar{z},w,\mathbf{P}(D)\big). \tag{4.3}$$

Equality holds in (4.3) for some $w \in D$ and some $z \in \mathbf{C} \setminus \{w\}$ if and only if either $z = \mathbf{P}(z)$, $w = \mathbf{P}(w)$ and $D = \mathbf{P}(D)$ or $z = \overline{\mathbf{P}(z)}$, $w = \overline{\mathbf{P}(w)}$ and $D = \overline{\mathbf{P}(D)}$.

Proof. Step 1. Assume first that $z = x \in \mathbf{R}$ and $w = y \in \mathbf{R}$. By an inversion we may assume that $y = \infty$. So we have to prove that

$$g(x,\infty,D) \leq g\big(x,\infty,\mathbf{P}(D)\big). \tag{4.4}$$

But (see [22, p. 14–16])

$$g(x,\infty,D) = \lim_{n\to\infty} u_n(x)\log n, \tag{4.5}$$

$$g\big(x,\infty,\mathbf{P}(D)\big) = \lim_{n\to\infty} v_n(x)\log n, \tag{4.6}$$

where

$$u_n(x) = \omega\big(x, \partial D(0,n), D \cap D(0,n)\big), \tag{4.7}$$

$$v_n(x) = \omega\big(x, \partial D(0,n), \mathbf{P}(D) \cap D(0,n)\big); \tag{4.8}$$

n is assumed to be a large enough positive integer so that $\mathbf{C} \setminus D \subset D(0,n)$ and $\mathbf{C} \setminus \mathbf{P}(D) \subset D(0,n)$. Now Theorem 2.2 implies

$$u_n(x) \le v_n(x). \tag{4.9}$$

Thus (4.5), (4.6), (4.9) give (4.4).

Step 2. Assume that $z = x \in \mathbf{R}$ and $w \in D_-$. So we have to prove that

$$g(x,w,D) \le g\big(x, \bar{w}, \mathbf{P}(D)\big). \tag{4.10}$$

Let $u(w) = g(x,w,D) - g\big(x,\bar{w},\mathbf{P}(D)\big)$, $w \in D_-$, x is fixed. u is harmonic in D_- and $u \le 0$ on ∂D_-. So (4.10) holds. Note that we have used the result of step 1.

Step 3. Now assume that $z, w \in D_-$. So we have to prove

$$g(z,w,D) \le g\big(\bar{z},\bar{w},\mathbf{P}(D)\big). \tag{4.11}$$

For fixed z we define

$$v(w) = g(z,w,D) - g\big(\bar{z},\bar{w},\mathbf{P}(D)\big). \tag{4.12}$$

Step 2 implies $v \le 0$ on ∂D_- and thus (4.11) follows from the maximum principle.

The remaining cases $z \in D_-$, $w \in D_+$ and $z \in D_+$, $w \in D_+$ are treated similarly.

Step 4. We prove here a special case of the equality statement. We use the notation Ω, λ, σ taken from the proof of Theorem 2.2.

Assume that $g(x,\infty,D) = g\big(x,\infty,\mathbf{P}(D)\big)$ for some $x \in \mathbf{R} \cap D$. We must prove that $D = \mathbf{P}(D)$ or $D = \overline{\mathbf{P}(D)}$. Assume that $D \ne \overline{\mathbf{P}(D)}$. So $\lambda \ne \emptyset$.

The function

$$h_1(z) = g(z,\infty,D) + g(\bar{z},\infty,D) - g\big(z,\infty,\mathbf{P}(D)\big) - g\big(\bar{z},\infty,\mathbf{P}(D)\big)$$

is harmonic and nonpositive in Ω and $h_1(x) = 0$. So the maximum principle implies that $h(z) = 0$, for all $z \in \Omega$. Hence

$$g(l,\infty,D) = g\big(l,\infty,\mathbf{P}(D)\big), \qquad \text{for all } l \in \lambda. \tag{4.13}$$

The maximum principle implies that

$$g(z,\infty,D) = g\big(z,\infty,\mathbf{P}(D)\big), \qquad \text{for all } z \in D_+. \tag{4.14}$$

If $\sigma \ne \emptyset$, (4.14) leads to a contradiction. So $\sigma = \emptyset$ and hence $D = \mathbf{P}(D)$. The other equality statements are proven by similar arguments.

Corollary 4.15. *Let D be a domain in $\mathbf{C}$. For $w \in D$ define*

$$u_w(z) = \begin{cases} g(w, z, D), & \text{if } z \in D, \\ 0, & \text{if } z \in \mathbf{C} \setminus D, \end{cases} \tag{4.16}$$

and

$$v_w(z) = \begin{cases} g\big(\mathbf{P}(w), z, \mathbf{P}(D)\big), & \text{if } z \in \mathbf{P}(D), \\ 0, & \text{if } z \in \mathbf{C} \setminus D. \end{cases} \tag{4.17}$$

If $w \in D$, $x \in \mathbf{R}$ and $\Phi\colon \mathbf{R} \to \mathbf{R}$ is a convex, increasing function then

$$\int_{\mathbf{R}} \Phi(u_w(x+it))\, dt \le \int_{\mathbf{R}} \Phi\big(v_w(x+it)\big)\, dt. \tag{4.18}$$

Equality holds in (4.18) for some $w \in D$, some $x \in \mathbf{R} \cap D \setminus \{w\}$ and some nonconstant, convex, increasing function Φ if and only if either $w = \mathbf{P}(w)$ and $D = \mathbf{P}(D)$ or $w = \overline{\mathbf{P}(w)}$ and $D = \overline{\mathbf{P}(D)}$.

This follows from (4.3) and its equality statement.

Corollary 4.19. *Let D be a domain in $\widehat{\mathbf{C}}$. If $z \in D \cap \mathbf{C}$, let $R(z, D)$ be the conformal radius of D at z. Then $R(z, D) \le R\big(\mathbf{P}(z), \mathbf{P}(D)\big)$.*

Corollary 4.19 is obtained at once from Theorem 4.1 and the definition of conformal radius (see [10]). An essentially equivalent formulation is the following:

Corollary 4.20. *Let E be a compact set in the plane. Then* $\operatorname{cap} E \ge \operatorname{cap} \mathbf{P}(E)$.

Corollaries 4.19 and 4.20 have been proven by different methods (see [23], [10]).

The following corollary is the basic symmetrization result for a Green function. It follows from Corollary 4.15.

Corollary 4.21. *Let $D \subset \mathbf{C}$ be a domain and D^* be its circular symmetrization with respect to the positive semiaxis. Define the functions*

$$u(z) = \begin{cases} g(z, z_o, D), & \text{if } z \in D \setminus \{z_o\}, \\ 0, & \text{if } z \in \mathbf{C} \setminus D \end{cases} \tag{4.22}$$

and

$$v(z) = \begin{cases} g(z, |z_o|, D^*), & \text{if } z \in D^* \setminus \{|z_o|\}, \\ 0, & \text{if } z \in \mathbf{C} \setminus D^* \end{cases} \tag{4.23}$$

where z_o is a point in D. If $r > 0$ and $\Phi\colon \mathbf{R} \to \mathbf{R}$ is a convex increasing function, we have

$$\int_0^{2\pi} \Phi\big(u(re^{i\theta})\big)\, d\theta \le \int_0^{2\pi} \Phi\big(v(re^{i\theta})\big)\, d\theta. \tag{4.24}$$

Equality holds in (4.24) for some $r \in D^*$ and some nonconstant, convex, increasing function $\Phi\colon \mathbf{R} \to \mathbf{R}$ if and only if $D = e^{i\phi} D^*$ for some $\phi \in \mathbf{R}$.

This result was proven by Baernstein [2] who used the star function method. The equality statement, which is due to Solynin [21], improves a result due to Essén and Shea [13].

5. Polarization and Robin function

We start with some definitions.

Definition 5.1. Let D be a domain in the plane bounded by a finite number of analytic Jordan curves, and let F be a closed set consisting of a finite number of arcs or curves on ∂D. The *Robin function* of F at $z \in D$ with pole at $z_o \in D$ is defined by the properties:

(i) N is harmonic in D and continuous in $\operatorname{clos} D$.

(ii) $N(z) \to 0$, as $D \ni z \to \zeta \in F$.

(iii) $(\partial N/\partial n)(\zeta) = 0$, for $\zeta \in \partial D \setminus F$. ($\partial/\partial n$ denotes the inner normal derivative).

(iv) $N(z) + \log|z - z_o|$ is a harmonic function of z in a neighborhood of z_o.

For the Robin function we will use the notation $N(z, z_o, F, D)$. The existence of the Robin function can be shown by the Perron method (according to Ohtsuka [16, p. 241]). Duren and Schiffer [11] gave a proof based on the variational method. The uniqueness of N is a consequence of Hopf's maximum principle (see [19, p. 70]).

Definition 5.2. Let $\gamma(z_o, F, D) = \lim_{z \to z_o} \bigl(N(z, z_o, F, D) + \log|z - z_o|\bigr)$ and $R(z_o, F, D) = e^{-\gamma(z_o, F, D)}$. $R(z_o, F, D)$ is the *Robin capacity* of F at z_o with respect to D.

Robin capacity is invariant under certain normalized conformal maps. Using this conformal invariance and suitable conformal maps we can drop the assumption about analyticity of the boundary of D. However, for simplicity, in the sequel we will retain this assumption.

We will need the following proposition. Its proof is very similar to the proof of the corresponding result for the Green function (see p. 14–16 of [22]).

Proposition 5.3. *Let D be a domain that contains ∞ and is bounded by a finite number of analytic Jordan curves. Let F be a set on ∂D consisting of a finite number of arcs or curves. If $R > 0$ is such that $\partial D \subset D(0, R)$, let u_R be the function with the properties $\Delta u_R = 0$ in $D \cap D(0, R)$, $u_R = 0$ on F, $(\partial u_R/\partial n) = 0$ on $\partial D \setminus F$ and $u_R = 1$ on $\partial D(0, R)$. Then*

$$N(z, \infty, F, D) = \lim_{R \to \infty} (\log R) u_R(z), \qquad z \in D \setminus \{\infty\}. \tag{5.4}$$

The convergence is locally uniform.

Now we can prove a polarization result. The interior of a Jordan curve γ is denoted by $\operatorname{IN} \gamma$.

Proposition 5.5. *Let* $\gamma_1, \gamma_2, \ldots, \gamma_N$ *;* $\sigma_1, \sigma_2, \ldots, \sigma_M$ *,* $(N, M \in \mathbf{Z}^+)$ *be disjoint analytic curves in* $\mathbf{D}$ *with disjoint interiors. Let* $K = \bigcup_{i=1}^N \operatorname{clos}(\mathrm{IN}\,\gamma_i)$ *,* $E = \bigcup_{j=1}^M \operatorname{clos}(\mathrm{IN}\,\sigma_j)$ *. Assume that* K *is symmetric, i.e.* $K = \overline{K}$ *. Let* $D = \mathbf{D}\setminus(K \cup E)$ *,* $D^* = \mathbf{P}(D) = \mathbf{D} \setminus \big(K \cup \overline{\mathbf{P}(E)}\big)$ *. Let* u *be such that* $\Delta u = 0$ *in* D *,* $u = 1$ *on* $\partial\mathbf{D}$ *,* $u = 0$ *on* σ_j *,* $j = 1, 2, \ldots, M$ *, and* $\partial u/\partial n = 0$ *on* γ_i *,* $i = 1, 2, \ldots, N$ *. Let* u^* *be such that* $\Delta u^* = 0$ *in* D^* *,* $u^* = 1$ *on* $\partial\mathbf{D}$ *,* $u^* = 0$ *on* $\overline{\mathbf{P}(\sigma_j)}$ *,* $j = 1, 2, \ldots, M$ *, and* $\partial u^*/\partial n = 0$ *on* γ_i *,* $i = 1, 2, \ldots, N$ *. Then*

$$u(x) \le u^*(x), \qquad x \in (-1, 1) \cap D. \tag{5.6}$$

Proof. We consider the Schottky double R of D with respect to $\bigcup_{i=1}^N \gamma_i$ (see [22, p. 32]). R is a Riemann surface "above" D. Its boundary consists of two copies of $\partial\mathbf{D} \cup \bigcup_{j=1}^M \sigma_j$. Similarly we consider the Schottky double R^* of $\mathbf{P}(D)$. We extend u and u^* on R and R^* respectively by defining them as follows: Let $z_1, z_2 \in R$ be the two points above $z \in D$. Then $u(z_1) = u(z_2) = u(z)$ and similarly for u^*. Then (see [22, p. 32]) $u(z) = \omega(z, \partial\mathbf{D}, R)$ and $u^*(z) = \omega(z, \partial\mathbf{D}, R^*)$.

Now we apply the method of proof of Theorem 2.2 (successive applications of the Markov property) and conclude that $u(x) \le u^*(x)$ for all $x \in (-1, 1) \cap D$. Note that the strong Markov property holds for harmonic measures on Riemann surfaces by the same proof as in the planar case.

Proposition 5.7. *Let* $D \subset \mathbf{C}$ *be a domain bounded by a finite number of analytic Jordan curves* $\gamma_1, \gamma_2, \ldots, \gamma_N$ *;* $\sigma_1, \sigma_2, \ldots, \sigma_M$ $(N, M \in \mathbf{Z}^+)$*. The exterior boundary of* D (*which coincides with one of the above curves*) *is assumed to be symmetric with respect to* $\mathbf{R}$*. Let* $K = \bigcup_{i=1}^N \gamma_i$ *and* $F = \bigcup_{j=1}^M \sigma_j$ *. Assume that* $K = \overline{K}$ *. Then, for* $z, w \in D$ *, the following inequalities hold*:

$$N(z, w, F, D) \le N\big(\mathbf{P}(z), \mathbf{P}(w), \overline{\mathbf{P}(F)}, \mathbf{P}(D)\big),$$

$$N(z, w, F, D) + N(\bar z, \bar w, F, D) \le N\big(z, w, \overline{\mathbf{P}(F)}, \mathbf{P}(D)\big) + N\big(\bar z, \bar w, \overline{\mathbf{P}(F)}, \mathbf{P}(D)\big).$$

(*We define the Robin function to be* 0 *outside* D*.*)

Proof. As in the corresponding result for the Green function, the main inequality to be proved is

$$N(x, y, F, D) \le N\big(x, y, \overline{\mathbf{P}(F)}, \mathbf{P}(D)\big), \qquad x, y \in \mathbf{R} \cap D. \tag{5.8}$$

By applying an inversion we may assume that $\infty \in D$ and $y = \infty$. Then (5.8) follows immediately from Propositions 5.3 and 5.5. The proof proceeds as the proof of Theorem 4. Instead of the usual maximum principle we use Hopf's maximum principle (see [19, p. 70]).

Corollary 5.9. *Let D, γ_i, σ_j, F be as in Proposition* 5.7*. Then for the Robin capacity we have*

(i) *For all $z \in \mathbf{P}(D)_+$,*

$$R(z,F,D) \geq R\big(\mathbf{P}(z), \overline{\mathbf{P}(F)}, \mathbf{P}(D)\big). \tag{5.10}$$

(ii) *For all $z \in D_- \cup \mathbf{P}(D)_+$,*

$$R(z,F,D) + R(\bar{z},F,D) \geq R\big(z, \overline{\mathbf{P}(F)}, \mathbf{P}(D)\big) + R\big(\bar{z}, \overline{\mathbf{P}(F)}, \mathbf{P}(D)\big). \tag{5.11}$$

This corollary comes at once from Proposition 5.7 and Definition 5.2.

6. Polarization and Brownian motion

The method of proof of Theorem 2.2 can be applied in the study of the behaviour under polarization of some functions related to Brownian motion. For the definition and main properties of Brownian motion we refer to [18], [12].

Let W be the Wiener space, the collection of all continuous functions (paths) w from $[0,\infty)$ to $\mathbf{C}$. By B_t, $t > 0$, we denote a Brownian motion in the plane, i.e. $B_t(w) = w(t)$, $w \in W$. The corresponding probability measures are denoted by P^z, where $z = B_0$. For a domain $D \subset \mathbf{C}$, τ_D is the exit time from D, i.e. $\tau_D(w) = \inf\{t > 0,\ B_t(w) \notin D\}$, $w \in W$.

Let now D be a domain regular for the Dirichlet problem. For $t > 0$, $z \in D$, and for an open set $A \subset D$, let

$$Q_D^t(z,A) = P^z(B_t \in A, t < \tau_D). \tag{6.1}$$

Thus $Q_D^t(z,A)$ is the probability that a Brownian motion B_s starting at z does not exit D for $s \leq t$ and $B_t \in A$.

We will also consider the corresponding density function (transition probability) $Q_D^t(z,w)$, $z,w \in D$, $t > 0$, for which we have

$$Q_D^t(z,A) = \int_A Q_D^t(z,x+iy)\,dx\,dy. \tag{6.2}$$

$Q_D^t(z,w)$ is related to the Green function of D via the formula

$$g(z,w,D) = \int_0^\infty Q_D^t(z,w)\,dt. \tag{6.3}$$

For fixed A, $Q_D^t(z,A)$ as a function of $t > 0$ and $z \in D$, satisfies the heat equation $2u_t = \Delta u$ with the initial-boundary conditions $u(t,\zeta) = 0$, $\zeta \in \partial D$ and $u(0,z) = \chi_A(z)$, $z \in D$. $Q_D^t(z,w)$ is the *heat kernel* of D.

Theorem 6.4. *With the above notation we have*

$$Q^t_D(x,A) \le Q^t_{\mathbf{P}(D)}\big(x,\mathbf{P}(A)\big), \qquad x\in \mathbf{R}\cap D,\ t>0, \tag{6.5}$$

$$Q^t_D(x,z) \le Q^t_{\mathbf{P}(D)}\big(x,\mathbf{P}(z)\big), \qquad x\in \mathbf{R}\cap D,\ t>0,\ z\in D. \tag{6.6}$$

Proof. The inequality (6.6) for $z=y\in\mathbf{R}\cap D$ follows from (6.5) by setting $A=D(y,\varepsilon)$ and letting $\varepsilon\to 0$. For general $z\in D$, (6.6) follows from the parabolic maximum principle, see [19].

The proof of (6.5) is similar to the proof of Theorem 2.2 and we only present its main step.

Recall the notation Ω, G, σ, λ, ϱ from Sections 2 and 3. We also write $A^* := \mathbf{P}(A)$. The points of σ, λ, and ϱ will be denoted by s, l, and r respectively.

By the strong Markov property

$$\begin{aligned} Q^t_D(x,A) = Q^t_\Omega(x,A) &+ \int_0^\infty\int_\lambda P^x(B_{\tau_\Omega}\in dl,\tau_\Omega\in dt_1)Q_D^{t-t_1}(l,A)\\ &+ \int_0^\infty\int_\sigma P^x(B_{\tau_\Omega}\in dl,\tau_\Omega\in dt_1)Q_D^{t-t_1}(s,A). \end{aligned} \tag{6.7}$$

We apply the Markov property to $Q_D^{t-t_1}(l,A)$ and to $Q_D^{t-t_1}(s,A)$, and we substitute in (6.7) to obtain

$$\begin{aligned} Q^t_D(x,A) = Q^t_\Omega(x,A) &+ \int_0^\infty\int_\lambda P^x(B_{\tau_\Omega}\in dl,\tau_\Omega\in t_1)\, Q_{D_+}^{t-t_1}(l,A)\\ &+ \int_0^\infty\int_0^\infty\int_\lambda\int_\varrho A(dl,dt_1)P^l(B_{\tau_{D_+}}\in dr,\tau_{D_+}\in dt_2)Q_D^{t-t_1-t_2}(r,A)\\ &+ \int_0^\infty\int_\sigma P^x(B_{\tau_\Omega}\in ds,\tau_\Omega\in dt_1)Q_{D_-}^{t-t_1}(s,A)\\ &+ \int_0^\infty\int_0^\infty\int_\sigma\int_\varrho A(ds,dt_1)P^s(B_{\tau_{D_-}}\in dr,\tau_{D_-}\in dt_2)Q_D^{t-t_1-t_2}(r,A), \end{aligned}$$

where here and below $A(dl,dt_1) = P^x(B_{\tau_\Omega}\in dl,\tau_\Omega\in dt_1)$ and $A(ds,dt_1) = P^x(B_{\tau_\Omega}\in ds,\tau_\Omega\in dt_1)$.

Similarly, for G and A^* we have

$$\begin{aligned} Q^t_G(x,A^*) = Q^t_\Omega(x,A^*) &+ \int_0^\infty\int_\lambda P^x(B_{\tau_\Omega}\in dl,\tau_\Omega\in t_1)Q_{G_+}^{t-t_1}(l,A^*)\\ &+ \int_0^\infty\int_0^\infty\int_\lambda\int_\varrho A(dl,dt_1)P^l(B_{\tau_{G_+}}\in dr,\tau_{G_+}\in dt_2)Q_G^{t-t_1-t_2}(r,A^*)\\ &+ \int_0^\infty\int_{\bar\sigma} P^x(B_{\tau_\Omega}\in ds,\tau_\Omega\in dt_1)\, Q_{G_+}^{t-t_1}(s,A^*)\\ &+ \int_0^\infty\int_0^\infty\int_{\bar\sigma}\int_\varrho A(ds,dt_1)P^s(B_{\tau_{G_+}}\in dr,\tau_{G_+}\in dt_2)Q_G^{t-t_1-t_2}(r,A^*). \end{aligned}$$

We check now that the following "domain monotonicity" inequalities hold for all intervals $I \subset \varrho$, all times t_0 and all time intervals T_o.

(a) $P^l(B_{\tau_{D_+}} \in I, \tau_{D_+} \in T_o) \le P^l(B_{\tau_{G_+}} \in I, \tau_{G_+} \in T_o)$.

(b) $P^s(B_{\tau_{D_-}} \in I, \tau_{D_-} \in T_o) \le P^{\bar{s}}(B_{\tau_{G_+}} \in I, \tau_{G_+} \in T_o)$.

(c) $Q^{t_o}_{D_+}(l, A) \le Q^{t_o}_{G_+}(l, A^*)$.

(d) $Q^{t_o}_{D_-}(s, A) \le Q^{t_o}_{G_+}(\bar{s}, A^*)$.

For example, (a) holds because $\{w \in W : B_{\tau_{D_+}} \in I,\ \tau_{D_-} \in T_o\} \subset \{w \in W : B_{\tau_{G_+}} \in I,\ \tau_{G_+} \in T_o\}$. The proof of (b) is similar. (c) holds because of the inclusion $\{w : B_{t_o} \in A,\ t_o < \tau_{D_+}\} \subset \{w : B_{t_o} \in A^*,\ t_o < \tau_{G_+}\}$ and (d) is proven similarly. We omit the rest of the proof. It proceeds as in Section 3.

Because of identity (6.3), the polarization inequalities of Theorem 6.4 imply the corresponding inequalities for Green function (and essentially all the results of Section 4). Also, by successive polarizations, we can prove known symmetrization results for heat kernels (see [1] and [4]).

We show now that the results on harmonic measure obtained in Section 2 can also be generalized. We consider "time dependent harmonic measures" defined as follows:

Let D and E be as in Section 2. For $z \in D$ and $0 \le \tau_1 < \tau_2 \le +\infty$, let

$$\omega_{\tau_1}^{\tau_2}(z, E, D) = P^z(B_{\tau_D} \in E, \tau_1 < \tau_D < \tau_2). \tag{6.8}$$

Thus $\omega_{\tau_1}^{\tau_2}(z, E, D)$ is the probability that a Brownian motion starting at z exits D through E, in the time interval (τ_1, τ_2). For $\tau_1 = 0$ and $\tau_2 = +\infty$, $\omega_{\tau_1}^{\tau_2}(z, E, D)$ is the usual harmonic measure $\omega(z, E, D)$.

Theorem 6.9. *With the above notation we have*

$$\omega_{\tau_1}^{\tau_2}(x, E, D) \le \omega_{\tau_1}^{\tau_2}\big(x, \mathbf{P}(E), \mathbf{P}(D)\big), \qquad x \in \mathbf{R} \cap D. \tag{6.10}$$

Proof. We again present only the main step of the proof. By the strong Markov property:

$$\begin{aligned} \omega_{\tau_1}^{\tau_2}(x, E, D) = \omega_{\tau_1}^{\tau_2}(x, E, \Omega) &+ \int_0^\infty \int_\lambda A(dl, dt_1)\, \omega_{\tau_1 - t_1}^{\tau_2 - t_1}(l, E, D) \\ &+ \int_0^\infty \int_\sigma A(ds, dt_1)\, \omega_{\tau_1 - t_1}^{\tau_2 - t_1}(s, E, D). \end{aligned} \tag{6.11}$$

Applying again the strong Markov property to $\omega_{\tau_1 - t_1}^{\tau_2 - t_1}(l, E, D)$ as well as to

$\omega_{\tau_1-t_1}^{\tau_2-t_1}(s,E,D)$, and substituting in (6.11) we obtain

$$\omega_{\tau_1}^{\tau_2}(x,E,D)=\omega_{\tau_1}^{\tau_2}(x,E,\Omega)+\int_0^\infty\int_\lambda A(dl,dt_1)\omega_{\tau_1-t_1}^{\tau_2-t_1}(l,E,D_+)$$
$$+\int_0^\infty\int_0^\infty\int_\lambda\int_\varrho A(dl,dt_1)\,P^l(B_{\tau_{D_+}}\in dr,\,\tau_{D_+}\in dt_2)\,\omega_{\tau_1-t_1-t_2}^{\tau_2-t_1-t_2}(r,E,D)$$
$$+\int_0^\infty\int_\sigma A(ds,dt_1)\omega_{\tau_1-t_1}^{\tau_2-t_1}(s,E,D_-)$$
$$+\int_0^\infty\int_0^\infty\int_\sigma\int_\varrho A(ds,dt_1)\,P^s(B_{\tau_{D_-}}\in dr,\,\tau_D\in dt_2)\,\omega_{\tau_1-t_1-t_2}^{\tau_2-t_1-t_2}(r,E,D).$$

We write an analogous formula for $\omega_{\tau_1}^{\tau_2}\big(z,\mathbf{P}(E),\mathbf{P}(D)\big)$ and we check easily that the following "domain monotonicity" inequalities hold for all times $h_1<h_2$, all intervals $I\subset\varrho$ and all time intervals T:

(a) $\omega_{h_1}^{h_2}(l,E,D_+)\le\omega_{h_1}^{h_2}\big(l,\mathbf{P}(E),G_+\big)$.

(b) $\omega_{h_1}^{h_2}(s,E,D_-)\le\omega_{h_1}^{h_2}\big(\bar{s},\mathbf{P}(E),G_+\big)$.

(c) $P^l(B_{\tau_{D_+}}\in I,\tau_{D_+}\in T)\le P^l(B_{\tau_{G_+}}\in I,\tau_{G_+}\in T)$.

(d) $P^s(B_{\tau_{D_-}}\in I,\tau_{D_-}\in T)\le P^{\bar{s}}(B_{\tau_{G_+}}\in I,\tau_{G_+}\in T)$.

The proof proceeds as in Section 3.

From the many possible consequences of Theorems 6.4 and 6.9 we mention only two:

Let D and A be as in Theorem 6.4. Let $E(x,A,D)$ be the expected length of time that a Brownian motion starting at $x\in\mathbf{R}\cap D$ stays in A before it exits D. ("Green measure" is another name of $E(x,A,D)$). This quantity is related to $Q_D^t(x,A)$ via the identity:

$$E(x,A,D)=\int_0^\infty Q_D^t(x,A)\,dt. \tag{6.12}$$

Thus Theorem 6.4 implies:

Corollary 6.13. *The expected length of time $E(x,A,D)\le E\big(x,\mathbf{P}(A),\mathbf{P}(D)\big)$. In particular $E(x,D,D)\le E\big(x,\mathbf{P}(D),\mathbf{P}(D)\big)$.*

Theorem 6.9 implies (successive polarizations again) the following symmetrization result:

Corollary 6.14. *Let K be a compact set in $\mathbf{D}$ and K^* be the circular symmetrization of K with respect to the negative semi-axis. Then for $x\in(0,1)$ and $0\le\tau_1<\tau_2\le\infty$,*

$$\omega_{\tau_1}^{\tau_2}(x,K^*,\mathbf{D})\le\omega_{\tau_1}^{\tau_2}(x,K,\mathbf{D}). \tag{6.15}$$

Remark. The results of Section 5 on the Robin function can be interpreted as results on reflected Brownian motion and can be accordingly generalized using the methods of this section.

7. Further applications of polarization

The proposition that follows is the harmonic measure analog of a result of Solynin [20] for the conformal radius of a domain:

Proposition 7.1. *Let D be a domain in the right half-plane, symmetric with respect to the real axis. Let $E \subset \partial D$ be a closed interval on the imaginary axis, symmetric with respect to the real axis. Assume that every vertical line $l_x = \{x + iy : y \in \mathbf{R}\}$ intersects D in a single vertical interval D_x. For $x \in \mathbf{R}^+$, let $h_x(y) = \omega(x+iy, E, D)$. Then for all $x \in \mathbf{R} \cap D$, $h_x(y)$ is a strictly decreasing function of $y \in [0, |D_x|/2)$, except for two cases*:

(i) *When D is the right half plane and E is the imaginary axis.*

(ii) *When D is a vertical strip and E is the imaginary axis. In these two cases h_x is a constant function of $y \in \mathbf{R}$.*

Proof. Let $x \in \mathbf{R} \cap D$ and $-y_2 < -y_1 < 0$. A polarization with respect to the line $l = \{z : \operatorname{Im} z = -(y_1 + y_2)/2\}$ together with (2.4) shows that

$$\omega(x - iy_1, E, D) \geq \omega(x - iy_2, E, D). \tag{7.2}$$

The equality statement for (2.4) implies that

$$\omega(x - iy_1, E, D) > \omega(x - iy_2, E, D) \tag{7.3}$$

unless $D = \overline{\mathbf{P}_l(D)}$ and E is symmetric with respect to l. If this is the case then it is easy to prove that E is the imaginary axis and D is either the right half plane or a vertical strip.

Proposition 7.4. *Let $0 < s \leq t \leq 1$ and $0 \leq \phi \leq \theta \leq 2\pi$. Define $K_1 = \{re^{i\theta} : s \leq r \leq t\}$, $K_2 = \{re^{i\phi} : s \leq r \leq t\}$, $K_1^* = [-t, -s]$, $K_2^* = [s, t]$. Then*

$$\omega(0, K_1 \cup K_2, \mathbf{D}) \leq \omega(0, K_1^* \cup K_2^*, \mathbf{D}). \tag{7.5}$$

This is a special case of the main result in [3].

Proof. We may assume that $\theta = \pi$ and $\phi \in [0, \pi)$. Consider the line $l = \{re^{i\phi/2} : r \in \mathbf{R}\}$ oriented from $e^{i\phi/2}$ to $-e^{i\phi/2}$. Then

$$\begin{aligned}\omega(0, K_1^* \cup K_2^*, \mathbf{D}) &= 1 - \omega(0, \partial\mathbf{D}, \mathbf{D} \setminus K_1^* \setminus K_2^*)\\ &\leq 1 - \omega\big(0, \partial\mathbf{D}, \mathbf{P}(\mathbf{D} \setminus K_1^* \setminus K_2^*)\big)\\ &= 1 - \omega(0, \partial\mathbf{D}, \mathbf{D} \setminus K_1 \setminus K_2) = \omega(0, K_1 \cup K_2, \mathbf{D}).\end{aligned}$$

Next we prove a polarization result for certain extremal distances.

Theorem 7.6. *Let Ω be a domain lying in the strip $S = \{z : 0 < \operatorname{Re} z < 1\}$. Assume that $\partial\Omega$ is the union of smooth curves or arcs. Let E_0 be the union of a finite number of closed vertical segments on $\partial\Omega \cap \{z : \operatorname{Re} z = 0\}$ and E_1 be the union of a finite number of closed vertical segments on $\partial\Omega \cap \{z : \operatorname{Re} z = 1\}$. Then*

$$\lambda(E_0, E_1, \Omega) \geq \lambda\big(\mathbf{P}(E_0), \mathbf{P}(E_1), \mathbf{P}(\Omega)\big) \tag{7.7}$$

Proof. Let V be the family of all Lipschitz functions v on Ω with $v = 0$ on E_0 and $v = 1$ on E_1. Let V^* be the family of all Lipschitz functions v^* on $\mathbf{P}(\Omega)$ with $v^* = 0$ on $\mathbf{P}(E_0)$ and $v^* = 1$ on $\mathbf{P}(E_1)$. Then (see [6, p. 368])

$$\lambda = \lambda(E_0, E_1, \Omega) = \min_{v \in V} \int_\Omega |\nabla v|^2, \tag{7.8}$$

$$\lambda^* = \lambda\big(\mathbf{P}(E_0), \mathbf{P}(E_1), \mathbf{P}(\Omega)\big) = \min_{v^* \in V^*} \int_{\mathbf{P}(\Omega)} |\nabla v^*|^2, \tag{7.9}$$

Let $u \in V$ and set $u = 0$ outside Ω. Define the function

$$u_1(z) = \begin{cases} \min\{u(z), u(\bar z)\}, & \text{if } z \in \mathbf{C}_-, \\ \max\{u(z), u(\bar z)\}, & \text{if } z \in \mathbf{C}_+. \end{cases} \tag{7.10}$$

Then $u_1 \in V^*$ and therefore (see [10])

$$\lambda^* \leq \int_{\mathbf{P}(\Omega)} |\nabla u_1|^2 = \int_\Omega |\nabla u|^2. \tag{7.11}$$

Taking minimum over all $u \in V$ we obtain (7.7).

The method of the above proof is due to Dubinin [10]. Successive applications of Theorem 7.6 lead to a symmetrization result for extremal distances.

Corollary 7.12. *Let E_0, E_1, Ω be as in the previous theorem. Then*

$$\lambda(E_0, E_1, \Omega) \geq \lambda(E_0^*, E_1^*, \Omega^*), \tag{7.13}$$

where $*$ denotes Steiner symmetrization with respect to the real axis.

Remarks. (1) Solynin [20] defined a geometric transformation in the plane called continuous symmetrization and proved theorems on the behaviour of capacity and conformal radius under continuous symmetrization. The main tool in his proofs is the fact that successive polarizations with respect to appropriate oriented lines approximate continuous symmetrization. Later Brock [7] gave a new definition of continuous symmetrization and proved Dirichlet integral inequalities in [8]. Using successive polarizations and the results of the previous sections one can prove theorems that describe the behaviour of harmonic measure and other conformal invariants under continuous symmetrization.

(2) Higher dimension analogs of our results hold with the same proofs.

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