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ON THE DYNAMICS OF PSEUDO-ANOSOV HOMEOMORPHISMS ON REPRESENTATION VARIETIES OF SURFACE GROUPS

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Abstract. We study the action of pseudo-Anosov homeomorphisms $f\colon R \to R$ on the character varieties of $\mathrm{SL}(2,\mathbf{C})$-representations of the fundamental groups $\pi_1(R)$ of closed orientable hyperbolic surfaces R. We prove that the representation $\pi_1(R) \hookrightarrow \mathrm{SL}(2,\mathbf{C})$ corresponding to the holonomy representation of the hyperbolic structure on the mapping torus of f is a hyperbolic fixed point for the action of f on the character variety $X(\pi_1(R))$.

1. Introduction

Suppose that R is a closed oriented hyperbolic surface and $f\colon R \to R$ an orientation-preserving pseudo-Anosov homeomorphism inducing an automorphism ϕ of the fundamental group $\pi_1(R)$. According to Thurston's hyperbolization theorem the mapping torus M_f of f admits a hyperbolic structure (see [O]). The fundamental group $\pi_1(M_f)$ is the semidirect product $G = \pi_1(R) \rtimes \mathbf{Z}$, where a generator t of $\mathbf{Z}$ acts on $\pi_1(R)$ by the conjugation $x \mapsto t^{-1}xt$, this action is the same as the automorphism ϕ. We use the hyperbolic structure on M_f to realize the group G as a discrete group of isometries of the hyperbolic 3-space $\mathbf{H}^3$. Let $X\bigl(\pi_1(R)\bigr) = \mathrm{Hom}\bigl(\pi_1(R), \mathrm{SL}(2,\mathbf{C})\bigr) /\!/ \mathrm{SL}(2,\mathbf{C})$ be the character variety. The isomorphism $\phi\colon \pi_1(R) \to \pi_1(R)$ induces a holomorphic automorphism

$$\Phi\colon X\bigl(\pi_1(R)\bigr) \to X\bigl(\pi_1(R)\bigr), \qquad \Phi : [\rho] \mapsto [\rho \circ \phi]$$

where $[\rho]$ denotes the $\mathrm{SL}(2,\mathbf{C})$-equivalence class of a representation $\rho\colon \pi_1(R) \to \mathrm{SL}(2,\mathbf{C})$. The equivalence class of the identity embedding $\iota\colon \pi_1(R) \hookrightarrow G \subset \mathrm{SL}(2,\mathbf{C})$ is a fixed point of Φ.

The main goal of this paper is to prove the following theorem that was conjectured by Curt McMullen in [Mc]:

Theorem 1.1. *The point $[\iota]$ is a hyperbolic fixed point of Φ, i.e. the derivative*

$$d\Phi\colon T_{[\iota]}X\bigl(\pi_1(R)\bigr) \longrightarrow T_{[\iota]}X\bigl(\pi_1(R)\bigr)$$

has no eigenvalues λ such that $|\lambda| = 1$.

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Remark 1.2. McMullen proved in [Mc] that there are at least two eigenvalues of $d\Phi$ which do not belong to the unit circle.

Unlike the approach of McMullen, our proof is mostly cohomological, we study the action of ϕ on the cohomology group $\mathrm{H}^1(\pi_1(R), sl(2,\mathbf{C}))$ of $\pi_1(R)$, where $sl(2,\mathbf{C})$ is the Lie algebra of $\mathrm{SL}(2,\mathbf{C})$ and $\pi_1(R)$ acts on $sl(2,\mathbf{C})$ via the adjoint representation $ad\circ\iota$. We prove the following theorem which implies Theorem 1.1:

Theorem 1.3. *The action* $\phi^*\colon \mathrm{H}^1(\pi_1(R), sl(2,\mathbf{C})) \to \mathrm{H}^1(\pi_1(R), sl(2,\mathbf{C}))$ *has no eigenvalues* λ *such that* $|\lambda| = 1$.

Note however that not every fixed point of Φ is hyperbolic. For instance, we can choose ϕ that acts trivially on $\mathrm{H}_1(R,\mathbf{Z})$, thus the trivial representation of $\pi_1(R)$ to $\mathrm{SL}(2,\mathbf{C})$ is not a hyperbolic fixed point of the corresponding mapping Φ. More generally, for each $m \geq 1$ there are hyperbolic 3-manifolds M fibered over circles so that the character variety $X(\pi_1(M))$ contains a smooth complex m-dimensional submanifold Y^m. Let $\pi_1(R)$ denote a normal surface subgroup in $\pi_1(M)$ and $\Phi\colon X(\pi_1(R)) \to X(\pi_1(R))$ the automorphism corresponding to the fibration of M over $\mathbf{S}^1$ with the fiber R. Thus, for each $[\rho] \in Y^m$ the point $[\rho|_{\pi_1(R)}]$ is a fixed point of Φ and the derivative of Φ at $[\rho|_{\pi_1(R)}]$ has at least m-dimensional fixed subspace.

As a warm-up to the proof of Theorem 1.3 we prove that ϕ^* has no roots of unity as eigenvalues. Indeed, if λ is an eigenvalue of ϕ^* such that $\lambda^n = 1$, then $(\phi^n)^*$ has a nonzero invariant vector in $\mathrm{H}^1(\pi_1(R), sl(2,\mathbf{C}))$. Note that for every $n \geq 1$ the group $G_n := \pi_1(R) \rtimes \langle t^n \rangle$ acts on the hyperbolic 3-space as a uniform lattice. Now the Serre–Hochschild exact sequence (see [Br, Corollary 6.4]) implies that $\mathrm{H}^1(G_n, sl(2,\mathbf{C})) \neq 0$ since it is isomorphic to the space of $(\phi^n)^*$-invariants on $\mathrm{H}^1(\pi_1(R), sl(2,\mathbf{C}))$; this contradicts the Calabi–Weil infinitesimal rigidity theorem [Ra].

Below is an outline of our proof in the general case: Suppose that we have a cocycle $\sigma \in Z^1(\pi_1(R), sl(2,\mathbf{C}))$ such that $\phi^{\#}(\sigma) = \lambda\sigma$, $\lambda \in \mathbf{C}^*$. Then there is a smooth vector-field ξ on $\mathbf{H}^3$ and its lift $\tilde{\xi}$ to the group $\mathrm{SL}(2,\mathbf{C})$ such that:

(a) $ad(\gamma)\xi - \xi = \sigma_\gamma$ for all $\gamma \in \pi_1(R)$.

(b) $ad(t)\tilde{\xi} = \lambda^{-1}\xi$.

If $|\lambda| = 1$ it follows that ξ is a *quasiconformal vector-field* on $\mathbf{H}^3$ in the sense of Ahlfors. Then we construct a *tangential extension* ξ_∞ of ξ to the ideal boundary $\mathbf{S}^2$ of $\mathbf{H}^3$ so that ξ_∞ is a quasiconformal vector-field on $\mathbf{S}^2 \cong \mathbf{C}\cup\{\infty\}$ which still satisfies the property (a) with respect to the action of $\pi_1(R)$ on $\mathbf{S}^2$. Then $\mu = \bar{\partial}\xi_\infty$ is a $\pi_1(R)$-invariant Beltrami differential, hence Sullivan's rigidity theorem [Su] implies that $\mu = 0$ almost everywhere.[1] Therefore ξ_∞ is actually a Moebius

[1] Note that we have to use Sullivan's theorem since it is the only rigidity theorem which deals with discrete groups of infinite covolume such as $\pi_1(R) \subset \mathrm{SL}(2,\mathbf{C})$.

vector-field and the property (a) implies that the cocycle σ is a coboundary. This proves that the only solution τ of the equation

$$\phi^*(\tau) = \lambda\tau, \quad |\lambda| = 1, \quad \tau \in \mathrm{H}^1\big(\pi_1(R), sl(2, \mathbf{C})\big)$$

is the trivial cohomology class which concludes the proof.

The reader will notice that our arguments in the proof of Theorem 1.3 provide an alternative proof of the Calabi–Weil infinitesimal rigidity theorem à la Mostow. Namely, suppose that $\Gamma \subset \mathrm{Isom}(\mathbf{H}^n)$ is a uniform lattice ($n \geq 3$). Then the Calabi–Weil infinitesimal rigidity theorem states that $\mathrm{H}^1(\Gamma, ad) = 0$. The usual way to prove this is to take a harmonic representative α of a class $[\alpha] \in \mathrm{H}^1(\Gamma, ad)$ and then verify that $\alpha = 0$ via Bochner's technique. Instead we take a quasiconformal vector field ξ on $\mathbf{H}^n$ representing $[\alpha]$, extend ξ (tangentially) to the sphere at infinity and then check (using for instance Sullivan's rigidity theorem) that this extension is actually a Moebius vector field. This proves that α is a coboundary.

The main technical difficulty in our proof is to establish existence of a "continuous extension" of quasiconformal vector fields from the open ball in $\mathbf{R}^n$ to its boundary. We prove that under some extra condition[2] there is a *tangential* continuous extension. This extension will suffice for our purposes. The nontrivial analytical ingredient of our construction of the *tangential* extension is *Semenov's stability theorem* (see Theorem 3.5), which is a part of the general stability theory for the spatial quasiconformal mappings. This theorem proves that k-quasiconformal vector fields in the unit open ball in $\mathbf{R}^n$, $n \geq 3$, are bounded. We discuss this and some basic facts about quasiconformal vector fields in Section 3. In Section 4 we establish existence of tangential extensions of quasiconformal vector fields. Theorem 1.3 is proven in Section 5. In Section 2 we show how to deduce Theorem 1.1 from Theorem 1.3.

The proof of Theorem 1.3 given here is not entirely satisfactory: it does not tell how eigenvalues of ϕ^* are related to combinatorial invariants of the pseudo-Anosov homeomorphism $f\colon R \to R$ (like its Perron–Frobenius matrix, see [FLP]). Another interesting question is to describe stable and unstable manifolds of the fixed point $[\iota]$ of the mapping Φ. Note that Φ preserves the natural symplectic structure on $X\big(\pi_1(R)\big)$ (see [Go]). Thus ϕ^* has $3g-3$ eigenvalues whose absolute value is less than 1, and the same number of eigenvalues outside the unit disc (where g is the genus of R). In particular, the (complex) dimension of stable and unstable manifolds E^s, E^u of Φ at $[\iota]$ is $3g-3$. McMullen in [Mc] proved the following theorem about E^s and E^u:

Theorem 1.4. *Take a pair of singly degenerate Kleinian groups* F^+, $F^- \subset \mathrm{SL}(2, \mathbf{C})$, *whose ending laminations* L^+, L^- *are those of the discrete doubly*

[2] The vector field must be *automorphic* under the action of a discrete group of Moebius transformations whose limit set is the boundary of the ball, see Definition 4.4.

degenerate group $\pi_1(R) \subset G \subset \mathrm{SL}(2,\mathbf{C})$. *Then the Teichmuller spaces* $T(F^+)$, $T(F^-)$ *are open subsets in* E^s *and* E^u.

However it is unclear how E^s, E^u behave outside of the domains $T(F^+)$, $T(F^-)$.

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2. Local dynamics on the representation variety

Most of the material of this section is fairly well known, we present it here for the sake of completeness. Let $\pi_1(R)$, ϕ be as in the introduction. Let $ad(g)\xi = g^{-1}\xi g$ denote the adjoint action of the group $\mathrm{SL}(2,\mathbf{C})$ on its Lie algebra $sl(2,\mathbf{C})$. We consider the action of the automorphism ϕ on the variety $\mathrm{Hom}\big(\pi_1(R),\mathrm{SL}(2,\mathbf{C})\big)$ given by:

$$\phi : \rho \mapsto \rho\circ\phi, \qquad \rho \in \mathrm{Hom}\big(\pi_1(R),\mathrm{SL}(2,\mathbf{C})\big).$$

This action of ϕ projects to a holomorphic automorphism Φ of the *character variety* $X\big(\pi_1(R)\big) = \mathrm{Hom}\big(\pi_1(R),\mathrm{SL}(2,\mathbf{C})\big)/\!/\,\mathrm{SL}(2,\mathbf{C})$.

Remark 2.1. The character variety is the quotient of $\mathrm{Hom}\big(\pi_1(R),\mathrm{SL}(2,\mathbf{C})\big)$ in the sense of the geometric invariants theory and in general is different from the “naive” set-theoretic quotient $\mathrm{Hom}\big(\pi_1(R),\mathrm{SL}(2,\mathbf{C})\big)/\,\mathrm{SL}(2,\mathbf{C})$. However if we restrict to the open subvariety $\mathrm{Hom}\big(\pi_1(R),\mathrm{SL}(2,\mathbf{C})\big)^-$ consisting of Zariski dense representations then the projection $X\big(\pi_1(R)\big)^-$ of $\mathrm{Hom}\big(\pi_1(R),\mathrm{SL}(2,\mathbf{C})\big)^-$ to $X\big(\pi_1(R)\big)$ is naturally isomorphic to the set-theoretic quotient $\mathrm{Hom}\big(\pi_1(R),\mathrm{SL}(2,\mathbf{C})\big)^-/\,\mathrm{SL}(2,\mathbf{C})$, see [JM, Theorem 1.1]. Moreover $X\big(\pi_1(R)\big)^-$ is a smooth complex manifold of the dimension $6g-6$, see [W], [Go].

Suppose that $[\rho_0] \in X\big(\pi_1(R)\big)$ is a fixed point for Φ and the image of ρ_0 is Zariski dense in $\mathrm{SL}(2,\mathbf{C})$. Then there is an element $t \in \mathrm{SL}(2,\mathbf{C})$ such that

$$t\rho_0\big(\phi(\gamma)\big)t^{-1} = \rho_0(\gamma), \qquad \text{for all } \gamma \in \pi_1(R).$$

We consider the induced action $T_0(\Phi)$ of Φ on the tangent space $T_{[\rho_0]}$ to $X\big(\pi_1(R)\big)$ at $[\rho_0]$. Our goal is to identify this action with the natural action ϕ^* of ϕ on the first cohomology group

$$\mathrm{H}^1\big(\pi_1(R), sl(2,\mathbf{C})\big) = Z^1\big(\pi_1(R), sl(2,\mathbf{C})\big)/B^1\big(\pi_1(R), sl(2,\mathbf{C})\big).$$

Recall that the action ϕ^* is defined by

$$\phi^{\#}(c)_\gamma = ad(t^{-1})c_{\phi(\gamma)}, \qquad \phi^{\#} : Z^1\big(\pi_1(R), sl(2,\mathbf{C})\big) \longrightarrow Z^1\big(\pi_1(R), sl(2,\mathbf{C})\big)$$
$$\phi^*[c] = [\phi^{\#}(c)], \qquad [c] \in \mathrm{H}^1\big(\pi_1(R), sl(2,\mathbf{C})\big)$$

where $c\colon \gamma \mapsto c_\gamma \in sl(2,\mathbf{C})$ are cocycles in $Z^1\big(\pi_1(R), sl(2,\mathbf{C})\big)$.

Remark 2.2. Here and in what follows we use the notation c_γ for the value of the cocycle c on the element $\gamma \in \pi_1(R)$. Note that c_γ belongs to $sl(2,\mathbf{C})$ and is a vector field on $\mathrm{SL}(2,\mathbf{C})$ and $\mathbf{H}^3$. Thus given $g \in \mathrm{SL}(2,\mathbf{C})$ or $x \in \mathbf{H}^3$ the evaluations $c_\gamma(g)$ and $c_\gamma(x)$ are tangent vectors in $T_g\big(\mathrm{SL}(2,\mathbf{C})\big)$ and $T_x\mathbf{H}^3$ respectively.

It was first noticed by Andre Weil in [W] (see also [Go]) that the (Zariski) tangent space $T_{[\rho_0]}$ is naturally isomorphic to $\mathrm{H}^1\big(\pi_1(R), sl(2,\mathbf{C})\big)$.

Lemma 2.3. *Under the natural isomorphism between* $T_{[\rho_0]}$ *and* $\mathrm{H}^1\big(\pi_1(R), sl(2,\mathbf{C})\big)$ *we have*: $T_0(\Phi) = \phi^*$.

Proof. Recall (see [Go]) that curves of representations ρ_ε through the point ρ_0 can be described as

$$\rho_\varepsilon(g) = \exp\big(\varepsilon c_\gamma + O(\varepsilon^2)\big)\rho_0(\gamma), \qquad \varepsilon \to 0$$

where c belongs to $Z^1\big(\pi_1(R), sl(2,\mathbf{C})\big)$. The isomorphism

$$T_{\rho_0}\operatorname{Hom}\big(\pi_1(R), \mathrm{SL}(2,\mathbf{C})\big) \cong Z^1\big(\pi_1(R), sl(2,\mathbf{C})\big)$$

is given by:

$$\frac{d}{d\varepsilon}\rho_\varepsilon\Big|_{\varepsilon=0} \mapsto c.$$

Thus

$$\rho_\varepsilon\big(\phi(\gamma)\big) \approx \exp\big(\varepsilon\ c_{\phi(\gamma)}\big)\rho_0\big(\phi(\gamma)\big) = t^{-1}\exp\big(\varepsilon\, ad(t^{-1})c_{\phi(\gamma)}\big)\rho_0(\gamma)t.$$

The last has the same projection to $X\big(\pi_1(R)\big)$ as $\exp\big(\varepsilon ad(t^{-1})c_{\phi(\gamma)}\big)\rho_0(\gamma)$, which has the tangent vector $\phi^{\#}(c)$. Thus

$$T_0(\Phi)(c) - \phi^{\#}(c) \in B^1\big(\pi_1(R), sl(2,\mathbf{C})\big)$$

which concludes the proof. □

Corollary 2.4. *Theorem* 1.3 *is equivalent to Theorem* 1.1.

3. Quasiconformal vector fields and the S-operator

For the proof of Theorem 1.3 we will need definitions and properties of *quasiconformal vector fields* that were introduced by Lars Ahlfors in [Ah1] under the name of *quasiconformal deformations.* Our discussion here will follow [Ah1] and [Ah2].

Let U denote the upper half-space $\mathbf{R}^n_+ = \{(x_1, \ldots, x_n) : x_n > 0\}$, we will identify it with the hyperbolic n-space $\mathbf{H}^n$ with the hyperbolic metric $|dx|/x_n$. Let Mob_n denote the group of Moebius transformations of $\overline{\mathbf{R}^n}$ and $\mathrm{Mob}_n(U)$ denote the stabilizer of U in Mob_n. The group $\mathrm{Mob}_n(U)$ acts as the group of isometries $\mathrm{Isom}(\mathbf{H}^n)$ of $\mathbf{H}^n$. We shall use the notations mob_n and $isom(\mathbf{H}^n)$ for the Lie algebras of the groups Mob_n and $\mathrm{Isom}(\mathbf{H}^n)$. We will realize mob_n and $isom(\mathbf{H}^n)$ as subalgebras in the space of all vector fields on $\mathbf{R}^n$ and U.

If X is a smooth manifold and $A: X \to X$ is a diffeomorphism of X then we shall denote by $A_*: \xi \mapsto A_*(\xi)$ the action of A on vector fields on X. In particular, the action of Moebius transformations A on vector fields ξ is given by the formula:

$$A_*\big(\xi(x)\big) = (DA_x)^{-1}\xi(Ax)$$

where DA_x is the Jacobian matrix of A at x. Then $A_*(\xi) = ad(A)(\xi)$ for all $\xi \in mob_n$. Similarly, if we identify mob_n with the Lie algebra of left-invariant vector fields on Mob_n then $A_*(\xi) = ad(A)(\xi)$. Elements ξ of mob_n have the form of “second degree polynomials”:

$$\xi(\vec{x}) = \vec{u} + (Q + aI)\vec{x} + |\vec{x}|^2\vec{b} - 2\langle \vec{x}, \vec{b}\rangle \vec{x}.$$

Here $\langle \cdot , \cdot \rangle$ is the scalar product in $\mathbf{R}^n$, $\vec{u}, \vec{b}$ are vectors in $\mathbf{R}^n$, $Q = -Q^T$, $a \in \mathbf{R}$ and I is the identity matrix. If $\xi \in isom(\mathbf{H}^n)$, then:

$$\langle \vec{u}, \vec{e}_n\rangle = \langle \vec{b}, \vec{e}_n\rangle = 0, \qquad Q\vec{e}_n = \vec{0}.$$

We define the norm $\|A\|$ for $n\times n$-matrices as $\|A\| = \sqrt{\mathrm{Tr}(AA^T)}$. Define the supremum-norm $\|h\|_\infty$ for $h: \Omega \subset \mathbf{R}^n \to \mathbf{R}^n$ by $\|h\|_\infty = \sup_{x\in\mathbf{R}^n} |f(x)|$ using the Euclidean metric on $\mathbf{R}^n$ (the issue here is that we have to consider hyperbolic metrics on domains in $\mathbf{R}^n$ as well). We shall denote by $B_x(r)$ the metric ball with the radius r and center x in $\mathbf{R}^n$, let $B(1) = B_0(1)$ and $S(1) = \partial B(1)$. We shall identify $S(1)$ with the sphere $\mathbf{S}^{n-1}$.

Consider a domain Ω in $\mathbf{R}^n$ $(n \geq 2)$ and a continuous vector field $f: \Omega \to \mathbf{R}^n$ on Ω. Suppose that f has locally integrable distributional partial derivatives. Define the matrix $S = S_n[f](x)$ by the formula:

$$S = \frac{1}{2}(Df + Df^T) - \frac{1}{n}\,\mathrm{Tr}(Df).$$

The operator $S[f]$ is called the *Ahlfors S*-operator, it was introduced by Lars Ahlfors in [Ah1]. If $n = 2$ then the S-operator becomes the $\bar\partial$-operator:

$$S[f] = \begin{bmatrix} \mathrm{Re}(\partial f/\partial \bar z) & \mathrm{Im}(\partial f/\partial \bar z) \\ \mathrm{Im}(\partial f/\partial \bar z) & -\mathrm{Re}(\partial f/\partial \bar z) \end{bmatrix}$$

under the standard matrix realization of $\mathbf{C}$.

Definition 3.1. A vector-field f above is called a *k-quasiconformal vector-field* if $\|S[f]\| \in L_\infty(\Omega)$ and $\|S[f]\|_\infty \le k\sqrt{n}$.

Quasiconformal vector-fields are infinitesimal analogues of quasiconformal mappings in $\mathbf{R}^n$. More precisely, suppose that h_t, $t \in [0,1]$, is a smooth family of quasiconformal homeomorphisms such that $h_0 = \mathrm{id}$. Then $(d/dt)h_t|_{t=0} = f$ is a quasiconformal vector field. If $M_t = M(h_t)$ is the *Beltrami differential* of h_t then $(d/dt)M_t|_{t=0} = S[f]$, see [Ah1], [Ah2] for more details. Similarly to quasiconformal mappings quasiconformal vector fields are differentiable a.e. in their domains.

Assume that $n = 2$, $f(z)$ transforms as a vector-field under Moebius mappings α. Then $\mu = S[f] = \bar\partial(f)$ transforms as a Beltrami differential under α:

$$\alpha_*\colon \mu(z) \mapsto \mu(\gamma z)\overline{\gamma'(z)}/\gamma'(z).$$

More generally, if $n \ge 2$ then

$$A_*S[f] := S[A_*(f)] = (DA)^{-1}(S[f] \circ A)DA \tag{1}$$

(see [Ah1]). This implies:

$$\|S[A_*(f)]\| = \|S[f] \circ A\|. \tag{2}$$

Therefore Moebius transformations send k-quasiconformal vector-fields to k-quasiconformal vector-fields. This allows us to define quasiconformal vector fields on the extended Euclidean space $\overline{\mathbf{R}^n} = \mathbf{R}^n \cup \{\infty\}$.

Note that the matrix $S[f]$ is symmetric and traceless. Suppose that $S(z) = S[f](z) \neq 0$, let φ_z be the corresponding quadratic form. Then $S(z)$ determines a *nontrivial* splitting of $\mathbf{R}^n$ (which we identify with the tangent space at z):

$$\mathbf{R}^n = P_{S,z} \oplus N_{S,z} \tag{3}$$

where the restriction of φ_z to P_z is a positive-definite quadratic form and the restriction of φ_z to N_z is a non-positive quadratic form. Thus the formula (1) implies

Lemma 3.2. *Let $\Omega \subset \mathbf{R}^n$ be a domain where $S \neq 0$ almost everywhere. Suppose that $A\colon \Omega \to \Omega$ is a Moebius transformation and $A_*S = S$. Then a.e. defined splitting $P_{S,z} \oplus N_{S,z}$ of the tangent bundle $T\Omega$ is invariant under A.*

The following lemma is the usual Weyl lemma if $n = 2$, for $n \ge 3$ it was proven by Ahlfors in [Ah2]:

Lemma 3.3. *If f is a quasiconformal vector field on a connected domain $\Omega \subset \overline{\mathbf{R}^n}$ and $S[f] = 0$ a.e. in Ω, then f is a conformal vector field. If $n \ge 3$ or $\Omega = \overline{\mathbf{R}^n}$ this implies that f is a Moebius vector field , i.e. $f \in mob_n|_\Omega$.*

J. Sarvas proved in [Sar] that k-quasiconformal vector-fields have the following "convergence property" similar to one for quasiconformal mappings:

Lemma 3.4 (J. Sarvas, [Sar]). *Suppose that f_j is a sequence of k-quasiconformal vector fields in a domain $\Omega \subset \mathbf{R}^n$ such that $\|f_j\| \le C$ for all j. Then $\{f_j\}$ contains a subsequence which is convergent uniformly on compacts in Ω to a k-quasiconformal vector field on Ω.*

The following theorem was proven by V. Semenov in [Se]:

Theorem 3.5 (Semenov's stability theorem, [Se, Lemma 1]). *Suppose that $n \ge 3$. Then there exists a universal constant c_n depending only on the dimension n of $\mathbf{R}^n$ so that the following holds*:

Let f be a k-quasiconformal vector field on the open *unit ball $B(1) \subset \mathbf{R}^n$. Then there exists $\lambda \in mob_n$ such that $\|f - \lambda\|_\infty \le c_n k$.*

In particular, every quasiconformal vector field in $B(1)$ is bounded (this obviously fails if $n = 2$).

In what follows we will need to establish a relation between the operators S_n and S_{n-1} in $\mathbf{R}^n$ and $\mathbf{R}^{n-1}$. Let $n \ge 2$ and $\xi(x) = (\xi_1, \dots, \xi_n)$ be a differentiable vector field in a domain $\Omega \subset \mathbf{R}^n$, take the hyperplane $L = \{x_n = 0\}$ which intersects Ω along a domain $\Omega' \ne \emptyset$. Define a vector field $\hat{\xi}$ on the domain Ω' by

$$\hat{\xi}(x') = (\xi_1, \dots, \xi_{n-1}), \qquad x' = (x_1, \dots, x_{n-1}).$$

Lemma 3.6. *For every vector field ξ we have: $\|S_n[\xi]\| \ge \|S_{n-1}[\hat{\xi}]\|$.*

Proof. Let $\mathrm{diag}(D\xi)$ denote the diagonal matrix whose diagonal entries are the diagonal entries of $D\xi$. Then

$$\|S_n[\xi]\|^2 = \left\|\frac{1}{2}(D\xi + D\xi^T) - \mathrm{diag}(D\xi)\right\|^2 + \left\|\mathrm{diag}(D\xi) - \frac{1}{n}\,\mathrm{Tr}\big(\mathrm{diag}(D\xi)\big)\right\|^2.$$

Let $q_n := \|\,\mathrm{diag}(D\xi) - (1/n)\,\mathrm{Tr}\big(\mathrm{diag}(D\xi)\big)\|^2$. It is obvious that

$$\|\tfrac{1}{2}(D\xi + D\xi^T) - \mathrm{diag}(D\xi)\|^2 \ge \|\tfrac{1}{2}(D\hat{\xi} + D\hat{\xi}^T) - \mathrm{diag}(D\hat{\xi})\|^2.$$

Thus what we need to prove is: $q_n \ge q_{n-1}$. We introduce the notations: $z_i = \partial\xi_i/\partial x_i$, $\tau_n = z_1 + \dots + z_n$, then $\tau_{n-1} = \tau_n - z_n$. Clearly

$$q_n = \sum_{j=1}^{n} z_j^2 - \frac{1}{n}\tau_n^2$$

$$q_{n-1} = \sum_{j=1}^{n-1} z_j^2 - \frac{1}{n-1}\tau_{n-1}^2$$

and

$$q_n - q_{n-1} = z_n^2 - \frac{1}{n}\tau_n^2 + \frac{1}{n-1}\tau_{n-1}^2 = \frac{1}{n-1}\Big(nz_n^2 + \frac{1}{n}\tau_n^2 - 2\tau_n z_n\Big).$$

The determinant of the matrix of the quadratic form $Q(z_n, \tau_n) = nz_n^2 + (1/n)\tau_n^2 - 2\tau_n z_n$ is zero and diagonal entries are positive, thus $q_n - q_{n-1} = Q(z_n, \tau_n) \geq 0$. ◻

4. Tangential extension of quasiconformal vector fields

It will be crucial for us to construct continuous extensions of quasiconformal vector fields from the open unit ball $B(1)$ to its boundary $S(1)$. The extension that we will construct *is not* a vector field on $\mathbf{R}^n$ but rather a tangent vector field on the sphere $S(1) \cong \mathbf{S}^{n-1}$. The only extension results for quasiconformal vector fields that I know are theorems of L. Ahlfors [A3] and J. Sarvas [Sar] below:

Theorem 4.1 (L. Ahlfors, [A3]). *Suppose that f is a quasiconformal vector field on the open unit ball $B(1) \subset \mathbf{R}^n$, such that*:

(a) $|f(x)|$ *is uniformly bounded in* $B(1)$,

(b) $\lim_{|x|\to 1}\langle f(x), x\rangle = 0$, *i.e. f is* asymptotically tangential.

Then f admits a continuous extension to the unit sphere $S(1)$ and, by reflection in $S(1)$, an extension to a quasiconformal vector field on $\overline{\mathbf{R}^n}$.

Theorem 4.2 (J. Sarvas, [Sar]). *Isolated singularities of quasiconformal vector fields in $\overline{\mathbf{R}^n}$ ($n \geq 3$) are removable.*

Remark 4.3. To understand the difficulty note that for any holomorphic function $f(z)$ defined in the upper half-plane $U := \{\mathrm{Im}(z) > 0\} \subset \mathbf{C}$ the vector field $f(z)\partial/\partial z$ is 0-quasiconformal in U, but usually it cannot be continuously extended to the boundary.

As far as I can see, Ahlfors' theorem is not strong enough to suffice for our purposes. Namely, quasiconformal vector fields that we consider are not apriori *asymptotically tangential*. On the other hand, if ξ is a quasiconformal vector field on $B(1)$, then its projection $\mathscr{P}_p(\xi)$ (defined below) is not in general a quasiconformal vector field on $B(1)$.

We will prove existence of a continuous *tangential* extension under the following technical assumption:

Definition 4.4. We say that a continuous vector field f in $B(1)$ is *automorphic* if the following holds:

There exists a discrete group F of Moebius transformations of $B(1)$ whose limit set equals $S(1)$ and a 1-cocycle σ: $F \to \mathit{isom}(\mathbf{H}^n)$ so that

$$A_* f - f = \sigma_A$$

for all $A \in F$.

To describe the extension we will need several preliminary constructions. Pick a point $p \in S(1)$ and consider the family $\mathscr{U}_p$ of horospheres U_p in $\mathbf{H}^n = B(1)$ centered at p. Let $\nu_{x,p}$ denote the unit (with respect to the Euclidean metric) normal vector to the sphere U_p at the point $x \in U_p$. We assume that $\nu_{x,p}$ is directed outside U_p. Let ξ be a vector field on $B(1)$. We define another vector field $\mathscr{P}_p(\xi)$ in $B(1)$ by the formula

$$\mathscr{P}_p(\xi)(x) = f(x) - \langle f(x), \nu_{x,p} \rangle \nu_{x,p}$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbf{R}^n$. Clearly $\mathscr{P}_p(f)(x)$ is tangent to the sphere U_p at the point x and is obtained via the orthogonal projections of $f(x)$ to $T_x(U_p)$.

Proposition 4.5. *Suppose that f is a smooth automorphic k-quasiconformal vector field on the open unit ball $B(1)$ in $\mathbf{R}^n$, $n \geq 3$. Then for any $p \in S(1)$ the field $\mathscr{P}_p(f)$ admits a continuous extension to $[B(1) \cup S(1)] - \{p\}$ whose restriction to the unit sphere $S(1) \cong \mathbf{S}^{n-1}$ is a vector field $f_{p,\infty}$ which is tangent to $S(1)$ and is k-quasiconformal on $\mathbf{S}^{n-1} - \{p\} \cong \mathbf{R}^n$.*

Proof. Let F denote the discrete group and σ the cocycle so that f is automorphic with respect to F and σ.

Step 1. We first prove that the field f has well-defined limits (with respect to the conical topology) on a dense countable subset in $S(1)$, which consists of fixed points of loxodromic elements of the group F.

Lemma 4.6. *Let $L \subset S(1)$ be the collection of fixed points of loxodromic elements of the group F. Then the vector field f admits a continuous* (*with respect to the conical topology*) *extension to L.*

Proof. It is enough to consider the upper half-space model $U = \mathbf{R}^n_+$ of the hyperbolic n-space $\mathbf{H}^n$, assume that the origin O is a fixed point of a loxodromic element A: $\vec{x} \mapsto \lambda W \vec{x}$ of the group F. We assume that $0 < \lambda < 1$, the orthogonal matrix W belongs to $O(n-1)$ (the stabilizer of ∂U in $SO(n)$).

Pick a sequence of points $z_k \in U$ that is convergent to the origin O in the conical topology, i.e. there is $r > 0$ so that z_k belong to the hyperbolic r-neighborhood of the geodesic $\{(0, 0, \dots, 0, x_n), x_n > 0\} \subset \mathbf{H}^n$. Let K denote the metric ball in $\mathbf{H}^n$ with the center at $\vec{e_n}$ and hyperbolic radius $r + \exp(\lambda)$. Then each z_k belongs to the $\langle A \rangle$-orbit of K:

$$A^{-k}(z_k) = y_k \in K, \qquad \text{for some } k \in \mathbf{Z}.$$

Let $\sigma = \sigma_A \in isom(\mathbf{H}^n)$ be the value of the cocycle σ at A, then

$$(A^k_* f)(y) = \sum_{j=0}^{k-1} (A^j_* \sigma)(y) + f(y)$$

for each $y \in K$. This means:

$$f(A^k y) = \lambda^k W^k f(y) + \lambda^k W^k \sum_{j=0}^{k-1} (A_*^j \sigma)(y).$$

Since $0 < \lambda < 1$ it is clear that $\lim_{k\to\infty} \lambda^k U^k f(y) = 0$ uniformly for $y \in K$. Note that

$$|(A_*^j \sigma)(y)| = \lambda^{-j} |\sigma(A^j y)|.$$

To estimate $|\sigma(A^j y)|$ we let

$$\sigma(x) = \vec{v} + (T + sI)\vec{x} + (|x|^2 \vec{b} - 2\langle \vec{x}, \vec{b}\rangle \vec{x}) = \sigma_{(-1)}(x) + \sigma_{(0)}(x) + \sigma_{(1)}(x).$$

Then

$$\sigma_{(0)}(A^j \vec{y}) = (T + sI)A^j \vec{y} = A^j (T_j + sI)\vec{y} = \lambda^j W^j (T_j + sI)\vec{y}$$

where T_j is a sequence of skew-symmetric matrices which have uniformly bounded norm ($T_j = A^{-j} T A^j$). Thus

$$|(T + sI)A^j \vec{y}| \le \text{Const} \cdot \lambda^j$$

and

$$\left| \lambda^k W^k \sum_{j=0}^{k-1} A_*^j \sigma_{(0)}(y) \right| = \left| \lambda^k W^k \sum_{j=0}^{k-1} \lambda^{-j} (T + sI) A^j \vec{y} \right| \le \text{Const} \cdot k \cdot \lambda^k$$

which tends to zero as $k \to \infty$. The estimate for $|\sigma_{(1)}(A^j y)|$ is:

$$|\sigma_{(1)}(A^j y)| \le 3\lambda^{2j} |\vec{b}| \cdot |\vec{y}|^2.$$

Thus

$$\left| \lambda^k W^k \sum_{j=0}^{k-1} \lambda^{-j} \sigma_{(1)}[A^j \vec{y}] \right| \le 3|\vec{b}| \cdot |\vec{y}|^2 \lambda^k \frac{\lambda^{2k} - 1}{\lambda^2 - 1}$$

which again tends to zero as $k \to \infty$ uniformly for $y \in K$. We conclude that

$$\begin{aligned} \lim_{k\to\infty} f(A^k y) &= \lim_{k\to\infty} \sum_{j=0}^{n-1} \lambda^k W^k A_*^j \sigma_{(-1)}(y) \\ &= \lim_{k\to\infty} \sum_{j=0}^{k-1} \lambda^{k-j} W^{k-j} \vec{v} = -\vec{v} + \sum_{j=0}^{\infty} \lambda^j W^j \vec{v} = [1 - A]^{-1} \vec{v} - \vec{v}. \end{aligned}$$

Therefore

$$\lim_{k\to\infty} f(A^k y) = [1 - A]^{-1} \vec{v} - \vec{v} = ([1 - A]^{-1} - 1)\sigma(0)$$

uniformly for $y \in K$. Hence, for the sequence z_k convergent to O in the conical topology we get:

$$\lim_{k\to\infty} f(z_k) = \lim_{k\to\infty} f(A^k y_k) = ([1 - A]^{-1} - 1)\sigma(0)$$

where $A^{-k} z_k = y_k \in K$. □

Step 2. Now we change the coordinates by an isometry of $\mathbf{H}^n = U$ so that the point p becomes the point $\infty \in \overline{\mathbf{R}^n}$. Then $f\colon U \to \mathbf{R}^n$ is a k-quasiconformal deformation whose sup-norm is bounded in $U \cap B(R)$, $0 < R < \infty$.

We will prove that $\mathscr{P}_p(f)$ has a continuous extension $f_{p,\infty}$ to $\mathrm{cl}(U) \cap B(R/2)$, so that $f_{p,\infty}|_{B(R/2)\cap\{x_n=0\}}$ is a k-quasiconformal vector field in $\mathbf{R}^{n-1}$.

For each $0 < t < 1$ we consider the restriction f_t of $\mathscr{P}_p(f)$ to the unit $(n-1)$-disc $B_{C_t}(1) \subset \{x_n = t\}$ with the center at $C_t = (0, \dots, 0, t)$. We regard f_t as a vector field on $B^{n-1}(R/2) \subset \mathbf{R}^{n-1}$ by "forgetting" the last coordinate x_n. Lemma 3.6 implies that f_t is a k-quasiconformal vector field.

Note that sup-norms of the k-quasiconformal vector fields f_t are uniformly bounded (Theorem 3.5). Hence, according to Lemma 3.4, up to a subsequence t_j, f_t are convergent to a k-quasiconformal vector field $f_{p,\infty}$ on $B(R/2) \cap \mathbf{R}^{n-1}$. The limit $f_{p,\infty}$ is supposed to be the boundary value of the extension of $\mathscr{P}_p(f)$ to the hyperplane $\{x_n = 0\}$. The problem is that for different sequences $t_j \to 0$ we may get distinct limits. However we know that different limits must coincide on a dense set of points (Step 1). Thus the vector field $\mathscr{P}_p f$ admits a continuous extension to $B(R/2) \cap \{x_n = 0\}$, which is k-quasiconformal in $\mathbf{R}^{n-1}$. □

Note however that $f_{p,\infty}$ is not defined at the point $p = \infty$. Thus we have to consider extensions corresponding to another family of horospheres. Let $q \in \mathbf{S}^{n-1} - \{p\}$. From Theorem 4.5 we know that both vector fields $\mathscr{P}_p(f)$, $\mathscr{P}_q(f)$ admit continuous extensions $f_{p,\infty}$, $f_{q,\infty}$ to the sphere $\mathbf{S}^{n-1}$.

Proposition 4.7. *For every pair of points* $p, g \in \mathbf{S}^{n-1}$ *we have:* $f_{p,\infty} = f_{q,\infty}$ *on* $\mathbf{S}^{n-1} - \{p, q\}$.

Proof. We go back to the unit ball model of the hyperbolic space. Pick a pair of horospheres $U_p \in \mathscr{U}_p$, $U_q \in \mathscr{U}_q$, let ρ denote the minimum of their Euclidean radii. Take a point $x \in U_p \cap U_q$. Then obviously

$$\lim_{\rho\to 1} |\nu_{x,p} - \nu_{x,q}| = 0$$

where $|\cdot|$ denotes the Euclidean norm.

$$\lim_{\rho\to 1} |\mathscr{P}_p(f)(x) - \mathscr{P}_q(f)(x)| \le \lim_{\rho\to 1} \mathrm{Const}\, |\nu_{x,p} - \nu_{x,q}| = 0. \ \square$$

Thus for each automorphic vector field f there is a k-quasiconformal vector field f_∞ tangent to $\mathbf{S}^{n-1}$ which coincides with each $f_{p,\infty}$ on $\mathbf{S}^{n-1} - \{p\}$. Therefore we have proved

Theorem 4.8. *Suppose that* f *is a smooth automorphic* k*-quasiconformal vector field on the open unit ball* $B(1)$ *in* $\mathbf{R}^n$, $n \ge 3$. *Then* f *admits a continuous* tangential extension f_∞ *to* $S(1)$. *The vector field* f_∞ *is again a* k*-quasiconformal vector field on the sphere* $S(1) = S^{n-1}$.

Proposition 4.9. *Let f be a quasiconformal vector field on $B = B(1)$ which is automorphic with respect to a group $F \subset \mathrm{Mob}_n$ and a cocycle σ: $A_* f - f = \sigma_A$ for all $A \in F$. Assume that F is torsion-free. Then*

$$A_* f_\infty - f_\infty = \sigma_A|_{\mathbf{S}^{n-1}} \qquad \textit{for all } A \in F.$$

Proof. Let $A \in F - \{1\}$, then A has a fixed point $p \in \mathbf{S}^{n-1}$. Thus the family of horospheres $\mathscr{U}_p$ is invariant under A. Therefore

$$A_* \mathscr{P}_p(f) = \mathscr{P}_p(A_* f).$$

Since $\mathscr{P}_p(f)$ has continuous extension f_∞ to $\mathbf{S}^{n-1}$ we conclude that

$$A_* f_\infty - f_\infty = (\sigma_A)_\infty = \sigma_A|_{\mathbf{S}^{n-1}}. \ \square$$

Remark 4.10. It is easy to check that the above proposition is also valid for groups with torsion.

5. Proof of the main theorem

In what follows we will identify the hyperbolic 3-space with the quotient space $\mathrm{SU}(2)\backslash \mathrm{SL}(2,\mathbf{C})$; let $\mathscr{P} : \mathrm{SL}(2,\mathbf{C}) \to \mathbf{H}^3$ denote the projection. Let *Vec* be the space of C^∞-vector fields on $\mathrm{SL}(2,\mathbf{C})$ which are invariant under the left $\mathrm{SU}(2)$-action. Clearly we have the well-defined projection P_*: $Vec \to \mathrm{H}^0(T\mathbf{H}^3)$ to the space of vector-fields on the hyperbolic 3-space. This projection commutes with the (right) action of the group $\mathrm{SL}(2,\mathbf{C})$ on vector-fields. Note that *Vec* is invariant under the multiplication by complex numbers (since the left action of $\mathrm{SU}(2)$ on $T\big(\mathrm{SL}(2,\mathbf{C})\big)$ is $\mathbf{C}$-linear). We shall identify the Lie algebra $sl(2,\mathbf{C})$ of $\mathrm{SL}(2,\mathbf{C})$ with the algebra of left-invariant vector-fields on $\mathrm{SL}(2,\mathbf{C})$.

The following simple lemma is critical for proving quasiconformality of the vector field ξ that we will construct in Theorem 5.2(c). Note that Lemma 5.1 clearly fails without the assumption $|\lambda| = 1$.

Lemma 5.1. *Let $\tilde{\xi} \in Vec$, $U \subset \mathbf{H}^3$ is a metric ball, $\widetilde{U} = P^{-1}(U)$. Then there exists a constant $C = C(\xi, U)$ such that*:

$$\|S[P_*(\lambda \cdot \tilde{\xi}|_{\widetilde{U}})]\| \le C$$

for all $\lambda \in \mathbf{C}$ such that $|\lambda| = 1$.

Proof. The family of vector fields $\lambda \cdot \tilde{\xi}|_{\widetilde{U}}$, $|\lambda| = 1$, is compact (with respect to the C^1-topology of uniform convergence). Thus the assertion follows from continuity of the S-operator. $\square$

We consider the action of the group $G = \pi_1(R) \rtimes \langle t \rangle$ on the hyperbolic 3-space as in the introduction. This action extends to the (right) action of G on $\mathrm{SL}(2,\mathbf{C})$. Our main goal is to prove the following

Theorem 5.2. *Suppose that $\sigma \in Z^1\big(\pi_1(R), sl(2,\mathbf{C})\big)$ is such that $\phi^{\#}(\sigma) = \lambda\sigma$, $\lambda \in \mathbf{C}^*$. Then there is a smooth vector-field ξ on $\mathbf{H}^3$ and its lift $\tilde{\xi} \in Vec$ to the group $\mathrm{SL}(2,\mathbf{C})$ such that:*

(a) *$ad(\gamma)\xi - \xi = \sigma(\gamma)$ for all $\gamma \in \pi_1(R)$.*
(b) *$ad(t)\tilde{\xi} = \lambda^{-1}\tilde{\xi}$.*
(c) *The vector field ξ is quasiconformal.*

Proof. To construct the vector field ξ we will need *a partition of unity* $\tilde{\eta}$ on the space $\mathrm{SL}(2,\mathbf{C})$ corresponding to the subgroup $\pi_1(R)$ and invariant under the action of t:

Lemma 5.3. *There exists a positive bounded C^∞-function $\tilde{\eta}$ on $\mathrm{SL}(2,\mathbf{C})$ which satisfies the following properties:*

(1) *$\tilde{\eta}$ is invariant under the right action of t and the left action of $\mathrm{SU}(2)$.*
(2) *$\sum_{\gamma\in\pi_1(R)} \tilde{\eta}(g\gamma) = 1$ for all $g \in \mathrm{SL}(2,\mathbf{C})$.*

Proof. Our proof follows [Kr, Chapter V, Lemma 3.1]. Take a finite covering $\{D_j\}$ of the manifold $\mathrm{SU}(2)\backslash \mathrm{SL}(2,\mathbf{C})/G$ by open metric balls D_j which satisfy the condition:

The radii of the balls D_j are smaller than the injectivity radius of $\mathbf{H}^3/G$.

Let m denote the multiplicity of the covering $\{D_j\}$ (i.e. the maximal number of balls which have nonempty intersection). The group G acts on the lift of this covering to $\mathbf{H}^3$ with trivial stabilizers. Let $\{\bar{\eta}_j\}$ denote the partition of unity on $\mathbf{H}^3/G$ corresponding to the covering $\{D_j\}$. For each D_j choose a connected component V_j of its lift to $\mathbf{H}^3$ and let η_j be the lift of $\bar{\eta}_j$ to V_j. Extend the function η_j to $\mathbf{H}^3$ by:

$$\eta_j(x) = \begin{cases} \eta_j(t^n x) & \text{if } x \in t^{-n}(V_j), \\ 0 & \text{otherwise.} \end{cases}$$

Finally let

$$\eta(x) = \sum_j \eta_j(x).$$

It is clear that η is invariant under the action of t, $\sum_{\gamma\in\pi_1(R)} \eta\big(\gamma(x)\big) = 1$ for all x and $0 \le \eta \le 1$. Then we lift the function η to $\tilde{\eta} = \eta \circ P$: $\mathrm{SL}(2,\mathbf{C}) \to \mathbf{R}$. □

Suppose that $\sigma \in Z^1\big(\pi_1(R), sl(2,\mathbf{C})\big)$ is such that $\phi^{\#}(\sigma) = \lambda\sigma$, $\lambda \in \mathbf{C}^*$ (at this stage we do not need the assumption $|\lambda| = 1$).

Lemma 5.4. *Under the assumptions above there exists a vector field* $\tilde{\xi} \in$ *Vec which satisfies the following properties*:

(i) $t_* \tilde{\xi} = \lambda^{-1} \tilde{\xi}$;

(ii) $\alpha_* \tilde{\xi} - \tilde{\xi} = \sigma_\alpha$, *for all* $\alpha \in \pi_1(R)$.

Proof. Our proof again follows [Kr, Chapter V, Theorem 3.2] (see [KM] for more general constructions). Let

$$\tilde{\xi}(g) = - \sum_{\gamma \in \pi_1(R)} \tilde{\eta}(g\gamma) \cdot \sigma_\gamma(g).$$

(Here $\sigma_\gamma(g)$ is a tangent vector to $\mathrm{SL}(2, \mathbf{C})$ at g, see Remark 2.2.) Note that for every $g \in \mathrm{SL}(2, \mathbf{C})$ not more than m terms of this series are different from zero (where m is the multiplicity of the covering $\{D_j\}$). Hence the infinite series which we use to define $\tilde{\xi}$ is convergent to a smooth vector field on $\mathrm{SL}(2, \mathbf{C})$. To show that $\tilde{\xi}$ is invariant under the (left) $\mathrm{SU}(2)$-action we note that the elements σ_γ of the Lie algebra are left-invariant under $\mathrm{SL}(2, \mathbf{C})$ and the function $\tilde{\eta}$ on $\mathrm{SL}(2, \mathbf{C})$ is invariant under the left action of $\mathrm{SU}(2)$.

Now we verify (i). We have:

$$\tilde{\xi}(g) = - \sum_{\phi(\gamma) \in \pi_1(R)} \tilde{\eta}\big(g\phi(\gamma)\big) \cdot \sigma_{\phi(\gamma)}(g),$$

$$t_*^{-1} \tilde{\xi}(g) = - \sum_{\gamma \in \pi_1(R)} \tilde{\eta}(g\gamma) \cdot t_*^{-1} \sigma_{\phi(\gamma)}(g)$$

since $\tilde{\eta}$ is t-invariant.

$$- \sum_{\gamma \in \pi_1(R)} \tilde{\eta}(g\gamma) \cdot t_*^{-1} \sigma_{\phi(\gamma)}(g) = - \sum_{\gamma \in \pi_1(R)} \tilde{\eta}(g\gamma) \cdot (\phi^{\#}\sigma)_\gamma(g).$$

The latter equals

$$-\lambda \sum_{\gamma \in \pi_1(R)} \tilde{\eta}(g\gamma) \cdot \sigma_\gamma(g) = \lambda \tilde{\xi}(g)$$

which proves (i). The second assertion of Lemma 5.4 again follows from the direct computation:

$$(\alpha_* \tilde{\xi})(g) = - \sum_{\gamma \in \pi_1(R)} \tilde{\eta}(g\alpha\gamma)(\alpha_* \sigma_\gamma)(g) =$$

and hence, by the cocycle condition $\sigma_{\gamma\circ\alpha} - \sigma_\alpha = \alpha_* \sigma_\gamma$,

$$= \sum_{\gamma \in \pi_1(R)} \tilde{\eta}(g\alpha\gamma)\sigma_\alpha(g) - \sum_{\gamma \in \pi_1(R)} \tilde{\eta}(g\alpha\gamma)\sigma_{\gamma\circ\alpha}(g) = \sigma_\alpha(g) + \tilde{\xi}(g). \quad \square$$

Thus $\xi := P_*(\tilde{\xi})$ is a smooth vector-field on $\mathbf{H}^3$, where $\tilde{\xi}$ is the vector field constructed in the previous lemma.

Lemma 5.5. *The field ξ satisfies the following properties*:
(i) $\alpha_*\xi - \xi = \sigma_\alpha$ *for all* $\alpha \in \pi_1(R)$.
(ii) $\|S[\xi(x)]\| \le \mathrm{Const}$ *for all* $x \in \mathbf{H}^3$.

Proof. The first assertion follows from the statement (ii) of Lemma 5.4. Let us prove the second assertion. Recall that the quotient $\mathbf{H}^3/G$ is compact, pick a compact $K \subset \mathbf{H}^3$ which is a fundamental domain for the action of G. Let $C_1 = \max \|S[\xi](x)\|$, $x \in K$. The equality $\gamma_*\xi - \xi = \sigma_\gamma \in sl(2,\mathbf{C})$ implies:

$$S[\gamma_*\xi] = S[\xi]$$

for all $\gamma \in \pi_1(R)$. Hence it follows from the equality (2) in Section 3 that

$$\|S[\xi]\| = \|S[\xi] \circ \gamma\|$$

and $\|S[\xi]\|$ is bounded by C_1 along the $\pi_1(R)$-orbit of K. Similarly,

$$\|S[\xi](t^n x)\| = \|S[P_*(\lambda^{-n}\tilde{\xi})](x)\|, \qquad n \in \mathbf{Z},$$

thus Lemma 5.1 implies that $\|S[\xi](t^n x)\| \le C_2 = C(\xi, K)$ for all $n \in \mathbf{Z}$. Therefore $\|S[\xi(x)]\| \le \mathrm{Const} = \max(C_1, C_2)$ for all $x \in \mathbf{H}^3$. ▫

This concludes the proof of Theorem 5.2. ▫

Now we can start proving Theorem 1.3. We assume that ϕ^* has an eigenvalue λ such that $|\lambda| = 1$, let $\tau \in \mathrm{H}^1(\pi_1(R), sl(2,\mathbf{C})) - \{0\}$ be an eigenvector: $\phi^*(\tau) = \lambda\tau$. As we proved in the introduction, $\lambda \ne 1$.

Lemma 5.6. *Under the conditions above there exists a cocycle* $\sigma \in Z^1(\pi_1(R), sl(2,\mathbf{C}))$ *such that* $\tau = [\sigma]$ *and* $\phi^\#(\sigma) = \lambda\sigma$.

Proof. Choose bases in $\mathrm{H}^1(\pi_1(R), sl(2,\mathbf{C}))$ and $B^1(\pi_1(R), sl(2,\mathbf{C}))$; the automorphism $\phi^*\colon \mathrm{H}^1(\pi_1(R), sl(2,\mathbf{C})) \to \mathrm{H}^1(\pi_1(R), sl(2,\mathbf{C}))$ is represented by a matrix D. Hence, $\phi^\#\colon Z^1(\pi_1(R), sl(2,\mathbf{C})) \to Z^1(\pi_1(R), sl(2,\mathbf{C}))$ is represented by a matrix:

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}.$$

Note that A is the matrix of the action of $\phi^\#$ on the complex 3-dimensional space $B^1(\pi_1(R), sl(2,\mathbf{C}))$. The element $t \in \mathrm{SL}(2,\mathbf{C})$ is loxodromic, the eigenvalues of its adjoint representation are: 1, μ^2, μ^{-2}, where $|\mu| \ne 1$, hence none of these numbers equals λ. The spectrum of the action of ϕ on $B^1(\pi_1(R), sl(2,\mathbf{C}))$ is the same as the spectrum of the adjoint representation of t^{-1} (since the subgroup $\pi_1(R) \subset \mathrm{SL}(2,\mathbf{C})$ is Zariski dense). Now existence of the eigenvector σ follows from the linear algebra. ▫

We apply Theorem 5.2 to the cocycle σ (given by Lemma 5.6) and construct a quasiconformal vector field ξ on $\mathbf{H}^3$ such that

$$ad(\gamma)\xi - \xi = \sigma_\gamma \qquad \text{for all } \gamma \in \pi_1(R).$$

Hence by Theorem 4.8 the vector field ξ admits a tangential extension $\zeta = \xi_\infty$ to $\mathbf{S}^2 = \partial B$, which is a quasiconformal tangent vector field on $\mathbf{S}^2$ such that

$$\gamma_*\zeta - \zeta = \sigma_\gamma \qquad \text{for all } \gamma \in \pi_1(R) \tag{4}$$

(see Proposition 4.9). Recall that $S[\chi] = 0$ for any conformal vector field χ. Thus, by applying the operator S to both sides of the equation (4), we get:

$$\gamma_* S[\zeta] - S[\zeta] = 0 \qquad \text{for all } \gamma \in \pi_1(R).$$

The matrix-valued function $S[\zeta]$ is measurable and belongs to $L_\infty(\overline{\mathbf{C}})$. Let L_S be the subset of $\overline{\mathbf{C}}$ where $S[\zeta] \neq 0$. This subset is measurable, $\pi_1(R)$-invariant and the field $S[\zeta]$ determines a nontrivial $\pi_1(R)$-invariant measurable splitting of the tangent bundle TL_S (see Lemma 3.2). Now we can apply

Theorem 5.7 (Sullivan's rigidity theorem, [Su], see also [O]). *Suppose that $F \subset \mathrm{PSL}(2,\mathbf{C})$ is a discrete finitely-generated group whose limit set is $\overline{\mathbf{C}}$ and $L \subset \overline{\mathbf{C}}$ is an F-invariant measurable subset of nonzero measure. Then there are no F-invariant measurable line-fields defined on L.*

We let $F = \pi_1(R)$, then Sullivan's theorem implies that the set L_S has zero measure. This means that $S[\zeta] = 0$ almost everywhere and, according to Lemma 3.3, ζ is a Moebius vector field : $\zeta \in mob_2 = sl(2,\mathbf{C})$. We conclude that the cocycle σ is a coboundary:

$$\delta_\zeta(\gamma) = ad(\gamma)\zeta - \zeta = \sigma_\gamma$$

which contradicts our assumption that the class $[\sigma] = \tau \in \mathrm{H}^1\big(\pi_1(R), sl(2,\mathbf{C})\big)$ is nontrivial. This proves Theorem 1.3. □

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