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ON THE FIXPOINTS, MULTIPLIERS AND VALUE DISTRIBUTION OF CERTAIN CLASSES OF MEROMORPHIC FUNCTIONS

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Abstract. We consider the frequency of fixpoints of meromorphic functions for which the set of finite singular values of the inverse function is bounded, and we prove fairly sharp estimates for the multipliers of these fixpoints. We go on to consider the critical values of, and linear differential polynomials in, composite meromorphic functions.

1. Introduction

Let B denote the class of meromorphic functions (this term being used throughout to mean functions meromorphic in the plane) f for which the set of finite singular values of the inverse function f^{-1}, that is, finite asymptotic and critical values of f, is bounded. This class has been considered extensively in iteration theory (see, for example, [4], [12]), as has the sub-class S, for which this singular set is finite. We discuss some value-distribution properties of these functions, in particular with reference to their fixpoints, as well as some general questions which arise in the study of compositions of meromorphic and entire functions.

It is clear that functions in the class S can have Nevanlinna deficient values (the terminology throughout being that of [16]). However, the following theorem shows, in particular, that a non-constant rational function cannot be a deficient function of a transcendental meromorphic function in the class S.

Theorem 1. *Let f be transcendental and meromorphic, in the class B. If h is rational, with $h(\infty)=\infty$, then we have*

$$(1)\qquad m\big(r,1/(f-h)\big)\le m\left(r,z\big(f'(z)-h'(z)\big)/(f(z)-h(z))\right)=O\big(\log rT(r,f)\big)$$

as $r\to\infty$ outside a set of finite measure. If h is transcendental and meromorphic in the plane, with only finitely many poles and with

$$(2)\qquad T(r,h)=o(\log r)^2$$

as $r\to\infty$, then (1) holds as $r\to\infty$ through a set of infinite linear measure.

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We remark that the growth condition (2) assumed on h in this theorem seems unlikely to be sharp. However, when h is rational and f is in B but not in S, the condition $h(\infty)=\infty$ cannot be deleted, as the example $f(z)=e^z+1/z$ shows.

Theorem 1 implies that if f is transcendental and meromorphic, in the class B, then the Nevanlinna counting function $N\big(r,1/(f-z)\big)$ of the fixpoints of f cannot satisfy $N\big(r,1/(f-z)\big)=o\big(T(r,f)\big)$ as $r\to\infty$. The authors thank the referee for bringing the following points to their attention. First, it was proved in the preprint [11] that every transcendental entire function in the class S has infinitely many fixpoints. Further, it follows from Theorem 1 that, if f is a transcendental entire function in the class S and n is a positive integer, then f has a lot of periodic points of exact period n, that is, the iterate f_n has fixpoints which are not fixpoints of any f_k with $1\le k<n$. To see this, we have

$$N\big(r,(f_k-z)^{-1}\big)\le T(r,f_k)+O(\log r)=o\big(T(r,f_n)\big)$$

if $1\le k<n$ while, since f_n is also in the class S, Theorem 1 gives

$$N\big(r,(f_n-z)^{-1}\big)=\big(1+o(1)\big)T(r,f_n)$$

as r tends to infinity outside a set of finite measure.

We turn our attention now to the multipliers of fixpoints of a transcendental meromorphic function f in the class B. It follows from a lemma of Eremenko, Lyubich and Bergweiler [4], [5], [12] (see Section 2) that if z is a fixpoint of f with $|z|$ large then the multiplier $f'(z)$ corresponding to this fixpoint satisfies $|f'(z)|>c\log|z|$, with c a positive absolute constant. The possibility of improving this estimate is suggested by the following observations.

A recent result of Bergweiler and Eremenko [6] implies that if g is meromorphic of finite order, then all but finitely many asymptotic values of g must be limit points of critical values of g and, in particular, if g' has only finitely many zeros, then g is in the class S. It was proved in [24] that if H is meromorphic in the plane with very few multiple points and with order $\rho(H)$ satisfying $\infty\ge\rho(H)>\sigma>0$, then H has infinitely many fixpoints z with $|H'(z)|>|z|^\sigma$. We obtain here comparable estimates for multipliers of functions in the class B.

Theorem 2. *There is a positive constant c such that if f is a transcendental entire function in the class B and $0<\alpha<1$ then f has infinitely many fixpoints z satisfying*

$$f(z)=z,\qquad |f'(z)|>c\log M(\alpha|z|,f).$$

Examples such as $\cos\sqrt{z}$ and $e^{P(z)}$, with P a polynomial, show that Theorem 2 is close to being sharp.

Of course it is well known that entire functions, not in the class B, can fail to have fixpoints, or can have only super-attracting fixpoints, as the simple examples

$z+e^z$, $z+1+e^z$ show at once. However, it was proved by Whittington [31] that if f is entire and transcendental of lower order less than $\frac{1}{2}$ then f has infinitely many fixpoints z_j with multipliers satisfying $f'(z_j)=1$ or $|f'(z_j)|>1$. Further, Bergweiler [3] confirmed a conjecture of Baker by showing that if f is any transcendental entire function and $n\geq 2$ is an integer then the iterate f_n has infinitely many repelling periodic points of exact order n.

For meromorphic functions in the class B, we have the following estimate for multipliers.

Theorem 3. *Let f be a meromorphic function in the class B, with order $\rho(f)$ satisfying $\infty\geq\rho(f)>\sigma>0$. Then f has infinitely many fixpoints z with*

$$f(z)=z, \qquad |f'(z)|>|z|^{\sigma/2}. \tag{3}$$

The example $f(z)=p(z)^n$, with p the Weierstrass doubly periodic function and n a large positive integer, shows that $\frac{1}{2}\sigma$ in Theorem 3 cannot in general be replaced by any constant τ with $\tau>\frac{1}{2}\rho(f)$. This follows from the differential equation

$$(p')^2=4(p-e_1)(p-e_2)(p-e_3),$$

in which the e_j are distinct complex constants.

Of course, Theorem 3 does not apply to functions of order 0, and it is well known that meromorphic functions in the class B may have arbitrarily slow growth: to see this, take the reciprocal of the function h in Lemma 2 below. Further, there is a meromorphic function H [1], [21] in the class S, which satisfies a differential equation

$$(z^2-4)H'(z)^2=4\big(H(z)-e_1\big)\big(H(z)-e_2\big)\big(H(z)-e_3\big) \tag{4}$$

and the growth condition

$$T(r,H)=O(\log r)^2, \qquad r\to\infty. \tag{5}$$

Indeed, this function is extremal for the growth of meromorphic functions in the class S in the sense that, if g is transcendental and meromorphic with

$$\limsup_{r\to\infty}\frac{T(r,g)}{(\log r)^2}=0,$$

then g has infinitely many critical values and so is not in S [23]. We refer the reader to [21, p. 234] and to the proof of Theorem 5 below for further discussion of this function. Taking $f=H^n$ with n a large positive integer shows that there is no positive absolute constant σ such that every meromorphic function f in the class B must have fixpoints satisfying (3). However, if f has a lot of poles of a given multiplicity, then more can be said than (3), and our last result on multipliers is the following.

Theorem 4. *Suppose that f is a transcendental meromorphic function in the class B, and that m is a positive integer. Then there is a constant d depending on f and m such that if r is large and f has poles of multiplicity m at $n \geq 1$ points in $\{z : \frac{1}{2}r \leq |z| \leq r\}$, then f has at least p fixpoints z_j in $\{z : \frac{1}{4}r < |z| < 2r\}$, with $p > \frac{1}{2}n$, each satisfying*

$$|f'(z_j)| > dr^{1/m}n^{1/2}.$$

The examples above show that Theorem 4 is essentially sharp.

The same function H as in (4) and (5) plays a role in our next result, in which we consider the question of when a composition of transcendental functions can be in the class S. Suppose that F is defined by $F = f \circ g$, with g a transcendental entire function and f transcendental and meromorphic.

If f is entire and F is in the class B, then F must have infinite order. To see this, suppose that F has finite order. Then it is well known [16, p. 53] that f has order 0. Consequently, ∞ is a limit point of critical values of f [6], [23], [26], so that there are sequences w_n and z_n tending to infinity such that $f(z_n) = w_n$ and $f'(z_n) = 0$, and at most one of these z_n can be a Picard value of g.

It is also easy to show that if f is meromorphic and F is in the class S, then $T(r,F) \neq O(\log r)^2$ as $r \to \infty$. To see this, suppose that $T(r,F) = O(\log r)^2$, and let a be finite. Then, applying Theorem 1 of [9], or using minimum modulus results for g, we obtain

$$n\big(M(r,g), 1/(f-a)\big) \leq n\big(r^2, 1/(F-a)\big) + O(1) = O(\log r) = o\big(\log M(r,g)\big)$$

as $r \to \infty$. Thus $T(r,f) = o(\log r)^2$ as $r \to \infty$, which implies that f has infinitely many critical values [23], and therefore so has F.

In view of these observations, the following theorem seems to be of some interest.

Theorem 5. *Let $\phi(r)$ be an increasing positive function such that $\phi(r) \to \infty$ as $r \to \infty$. Then there exist a transcendental entire function g and a function f transcendental and meromorphic in the plane, such that the composition $F = f \circ g$ is in the class S, while $T(r,F) = O\big(\phi(r)\log^2 r\big)$ as $r \to \infty$.*

The connection has already been noted between the class S and functions for which the first derivative has only finitely many zeros. There is an extensive literature concerning the zeros of derivatives of meromorphic functions and, more generally, of linear differential polynomials

$$F^{(k)} + \sum_{j=0}^{k-1} a_j F^{(j)}$$

in a meromorphic function F, in which the coefficients a_j are normally small functions satisfying $T(r,a_j) = o\big(T(r,F)\big)$ as $r \to \infty$ (see Chapter 3 of [16] as well as, for example, [13], [21], [22], [24]). We consider here the value distribution of such a linear differential polynomial, when F is a composition of transcendental functions, motivated in part by the following simple observation. Suppose that g is a transcendental entire function and f is transcendental and meromorphic, such that $F = f \circ g$ has finite order. Then f has order zero [9], so that f' has infinitely many zeros [10], and $\overline{N}(r,1/F') \neq O\big(T(r,g)\big)$ as $r \to \infty$. Our methods below do not seem to work when f is meromorphic but, for entire f, we have the following result. The standard abbreviation $(n.e.)$ ("nearly everywhere") will be used to denote "as $r \to \infty$ outside a set of finite measure".

Theorem 6. *Suppose that k is a positive integer, and that f and g are transcendental entire functions of finite order. Define F and H by*

$$F = f \circ g, \qquad H = F^{(k)} + \sum_{j=0}^{k-1} A_j F^{(j)} + A^*, \tag{6}$$

in which $A^*, A_0, \ldots, A_{k-1}$ are meromorphic functions with

$$T(r,A^*) + T(r,A_j) = O\big(T(r,g)\big) \qquad (n.e.). \tag{7}$$

Suppose in addition that there exist a meromorphic function S and a rational number c with

$$A_{k-1} = cS'/S, \qquad T(r,S) = O\big(T(r,g)\big) \qquad (n.e.), \tag{8}$$

and suppose finally that

$$\overline{N}(r,1/H) = O\big(T(r,g)\big) \qquad (n.e.). \tag{9}$$

Then

$$T(r,f) \neq o(r^{1/k}) \qquad as \qquad r \to \infty. \tag{10}$$

Theorem 6 may also be regarded as in the spirit of work on the value distribution of compositions $F = f \circ g$ of transcendental meromorphic f and transcendental entire g, such as Goldstein's theorem [14] that if f is entire and F has finite order then $\delta(0,F) < 1$, and a substantial literature (including [2], [8], [20], [32], [33]) on the fixpoints of such functions. It is not clear whether the extra hypothesis (8) on A_{k-1} is really necessary, but we will show below that no stronger conclusion than (10) on the upper growth of f is possible.

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2. Proof of Theorem 1

The lemma of Eremenko, Lyubich and Bergweiler referred to above is the following [4], [5], [12].

Lemma A. *Suppose that f is a transcendental meromorphic function in the class B. Then there are positive constants c, R, S such that we have*

$$|zf'(z)/f(z)| \geq c\log^+ |f(z)/R| \tag{11}$$

for $|z| > S$.

The lemma is proved by noting that since f is in the class B we have, for some $R > 0$, the estimate $|f(z)| \leq R$ on the union of a circle $|z| = S > 0$ and a path Γ joining $|z| = S$ to infinity. Define a path γ by $e^\gamma = \Gamma$. If R is large enough and $|z| > S$ and $|f(z)| > R$, then a branch of $u = \phi(w) = \log f^{-1}(e^w)$ may be analytically continued without restriction in the half plane $\mathrm{Re}(w) > \log R$, taking values in a domain bounded by $\mathrm{Re}(u) = \log S$ and the paths $k2\pi i + \gamma$, $(k+1)2\pi i + \gamma$, for some integer k. The estimate (11) follows from an application of Bloch's theorem to $\phi(w)$, so that the constant c does not depend on f although R and S in general do.

Assume now that f and h are as in the statement of Theorem 1. We consider first the case where h is rational. Suppose that $|z|$ is large, and that $|f(z) - h(z)| \leq 1$. Then $|h(z)|$ and $|f(z)|$ are large and (11) gives

$$|zf'(z)| \geq c|f(z)| \log |f(z)/R| \geq (c/2)|h(z)| \log |h(z)|. \tag{12}$$

Thus

$$\bigl|z\bigl(f'(z) - h'(z)\bigr)\bigr| \geq (c/2)|h(z)|\bigl(\log |h(z)| - (2/c)|zh'(z)/h(z)|\bigr). \tag{13}$$

Hence we have, writing $g = f - h$,

$$m(r, 1/g) \leq m(r, zg'/g) \leq O\bigl(\log rT(r, f)\bigr) \qquad \text{(n.e.)}. \tag{14}$$

Now suppose that h is transcendental, with only finitely many poles, satisfying (2). Then $g = f - h$ cannot vanish identically, since h cannot be in the class B [6], [23], [26]. There exist positive sequences S_n, T_n, v_n tending to infinity, such that we have [23]

$$\begin{gathered} h(z) = a_n z^{v_n}\bigl(1 + o(1)\bigr), \\ h'(z)/h(z) = v_n/z + o(1/|z|), \\ S_n/T_n \leq |z| \leq S_n T_n, \end{gathered} \tag{15}$$

with each a_n a non-zero constant. For

$$|z| = r, \qquad S_n(T_n)^{-1/4} \le r \le S_n(T_n)^{1/4}, \tag{16}$$

we have

$$v_n \le n(S_n/T_n, 1/h) = o\big(N(r,1/h)\big) = o\big(T(r,h)\big) = o(\log|h(z)|). \tag{17}$$

Therefore, for z satisfying (16) and with $|g(z)| \le 1$, we have (12) and so (13), and (15) and (17) give

$$|zg'(z)| \ge (c/4)|h(z)| \log|h(z)| \ge 1.$$

For such r, we then have (14) again, using the fact that $\log T(r,g) \le \log T(r,f) + O(\log r)$. This proves Theorem 1.

3. Proof of Theorems 2, 3 and 4

We need the following lemma.

Lemma 1. *Suppose that f is a transcendental meromorphic function in the class B, and define G by*

$$G(z) = f(z)/z, \qquad 1 + zG'(z)/G(z) = zf'(z)/f(z). \tag{18}$$

Suppose that δ is a positive constant. Then there exists a positive constant ε such that the following is true. If $|z_1|$ is large and $|G(z_1) - 1| < \frac{1}{4}\varepsilon$ then z_1 lies in a component C_1 of the set $\{z : |G(z) - G(z_1)| < \frac{1}{2}\varepsilon\}$, such that C_1 is contained in $B(z_1, \delta|z_1|)$ and is mapped conformally onto $B\big(G(z_1), \frac{1}{2}\varepsilon\big)$ by G. Further, $|zG'(z)|$ is large on C_1.

Proof. We choose a large positive R_1 and a positive constant ε so small that $|G(z) - 1| > \varepsilon$ on $|z| = R_1$. Suppose that $|z| > R_1$, and that $|G(z) - 1| < \varepsilon$. Then we have $|f(z)| > \frac{1}{2}|z|$, so that Lemma A gives

$$|zf'(z)/f(z)| > c\log|z|.$$

Henceforth c will denote a positive constant, not necessarily the same at each occurrence, but not depending on R_1 or ε. For z as above, we thus have, using (18),

$$|zG'(z)| > R_2 = c\log R_1. \tag{19}$$

Suppose now that z_1 is as in the statement of the lemma, with $|z_1| > 2R_1$. We set $H(z) = G(z) - G(z_1)$. By (19) we have $H'(z_1) \ne 0$, and we define

$$\psi(w) = \sum_{k=0}^{\infty} c_k w^k, \qquad c_0 = z_1, \tag{20}$$

to be that branch of the inverse function H^{-1} which maps 0 to z_1. Let r_1 be the radius of convergence of the power series in (20). Then a standard compactness argument shows that there is some w^*, with $w^* = r_1 e^{i\theta^*}$ for some real θ^*, such that ψ has no analytic continuation to a neighbourhood of w^*. Consequently the image of the path $w = te^{i\theta^*}$, $0 \le t < r_1$, under ψ must either tend to infinity or to a multiple point z^* of H with $H(z^*) = w^*$.

Set $r_2 = \min\{r_1, \frac{1}{2}\varepsilon\}$ and let γ be the path $\gamma(t) = te^{i\theta^*}$, $0 \le t < r_2$. Since $|H(z)| > \frac{3}{4}\varepsilon$ on $|z| = R_1$ the image path $\psi(\gamma)$ lies in $|z| > R_1$ and so (19) implies that

$$\psi(w)/\psi'(w) = zH'(z) = zG'(z)$$

is large on γ. So for w on γ we have

$$\bigl|\log\bigl(\psi(w)/z_1\bigr)\bigr| \le \int_0^{|w|} |\psi'(se^{i\theta^*})/\psi(se^{i\theta^*})|\, ds \le \varepsilon/2R_2.$$

Provided ε was chosen small enough, the path $\psi(\gamma)$ thus lies in $B(z_1, \delta|z_1|)$. On this path we have $|G(z) - 1| < \frac{3}{4}\varepsilon$ and, using (19), $|H'(z)| = |G'(z)| \ge R_2/2|z_1|$. In particular, $\psi(\gamma)$ is bounded and does not tend to a critical point of H, and so we must have $r_1 > \frac{1}{2}\varepsilon$. This proves the lemma.

We now prove Theorem 2. Suppose that f is a transcendental entire function in the class B. With G as in (18), we may write

$$G(z) = f(z)/z = 1 + f_1(z) + f(0)/z, \tag{21}$$

with f_1 a transcendental entire function. Let δ be small and positive, and choose ε as in Lemma 1. By Theorem 4 of [27] we may choose z_1 with $|z_1|$ arbitrarily large and with

$$|f_1(z_1)| < \tfrac{1}{8}\varepsilon, \qquad |z_1 f_1'(z_1)| > b_1 \log M(|z_1|, f_1). \tag{22}$$

Here we are using b_j to denote positive absolute constants. By (18) and (21) we thus have

$$|z_1 f'(z_1)/f(z_1)| > b_2 \log M(|z_1|, f). \tag{23}$$

Now, z_1 lies in a component C of the set $\{z : |G(z) - G(z_1)| < \frac{1}{2}\varepsilon\}$ which by Lemma 1 itself lies in $B(z_1, \delta|z_1|)$ and contains a zero z_2 of $G(z) - 1$. This point z_2 is a fixpoint of f. Further, C lies in a component D of the set $\{z : |f(z)| > R\}$, where R is a large constant as in Lemma A, and

$$z = h(w) = f^{-1}(e^w) \tag{24}$$

maps $E = \{w : \mathrm{Re}(w) > \log R\}$ univalently onto D [26, p. 287]. In addition, the function $\phi(w) = \log h(w)$ is univalent on E, and we have, if z and w are related by (24),

$$\phi'(w)^{-1} = zf'(z)/f(z). \tag{25}$$

Let $h_1\colon D \to E$ be the inverse function of h, and define $w_j = h_1(z_j)$ for $j = 1, 2$. For z in C, we have $|G(z) - 1| < \varepsilon$ and so

$$e^w = \exp\bigl(h_1(z)\bigr) = f(z) = z\bigl(1 + \eta(z)\bigr), \tag{26}$$

where $|\eta(z)| < \varepsilon$. Thus, since C lies in $B(z_1, \delta|z_1|)$,

$$|h_1(z) - w_1| < c(\varepsilon + \delta), \qquad z \in C.$$

In particular, $|w_2 - w_1| < \frac{1}{4}$, provided δ was chosen small enough. But, using (26), $\phi(w)$ is univalent on $B(w_1, 2)$ and so applying standard estimates [28, p. 9] for β''/β' to $\beta(u) = \phi(w_1 + u)$ on $B(0, 1)$ gives

$$\log|\phi'(w_2)| \le \log|\phi'(w_1)| + \int_{w_1}^{w_2} |\phi''(t)/\phi'(t)|\,|dt| \le -\log\log M(|z_1|, f) + b_3,$$

using (23) and (25). Applying (25) again, this gives

$$|z_2 f'(z_2)/f(z_2)| > b_4 \log M(|z_1|, f) > b_4 \log M(\alpha|z_2|, f),$$

provided δ was chosen small enough in Lemma 1. This completes the proof of Theorem 2.

We turn now to the proof of Theorem 3. Suppose that f is a transcendental meromorphic function in the class B, and that $\infty \ge \rho(f) > \rho > 0$. Using Theorem 1, we may choose arbitrarily large r such that there are at least $2r^\rho$ fixpoints z_j of f lying in $\frac{1}{2}r \le |z| \le r$. To each such fixpoint z_j corresponds a component C_j of the set $\{z : |G(z) - 1| < \frac{1}{2}\varepsilon\}$ lying in $\frac{1}{4}r \le |z| \le 2r$, and these disjoint components C_j are simple islands mapped conformally onto $B(1, \frac{1}{2}\varepsilon)$ by G. At least r^ρ of these z_j must be such that C_j has area at most $cr^{2-\rho}$, and for these j we have, using (18) and [28, p. 4],

$$|f'(z_j)| > \tfrac{1}{2}|z_j G'(z_j)| > c|z_j| r^{\rho/2 - 1}.$$

This proves Theorem 3.

We close this section by proving Theorem 4. Suppose that f is a transcendental meromorphic function in the class B, that R is a large positive constant as in Lemma A, and that m is a positive integer. Let T be a positive constant, possibly depending on m, such that T/R is large.

Suppose that r is large, and that $f(z)$ has n poles $w_1, \ldots, w_n$, each of multiplicity m, in $\{z : \frac{1}{2}r \leq |z| \leq r\}$. Set $G(z) = T^{1/m} f(z)^{-1/m}$, so that G is univalent on a neighbourhood of each w_j. Since the inverse function f^{-1} has no singularities in $\{w : T \leq |w| < \infty\}$, each w_j lies in a component E_j of the set $\{z : |f(z)| > T\}$, mapped univalently onto $B(0,1)$ by G. Let $\phi_j \colon B(0,1) \to E_j$ be that branch of the inverse function G^{-1} which maps 0 to w_j. Since

$$w\phi_j'(w)/\phi_j(w) = G(z)/zG'(z)$$

and since

$$-mzG'(z)/G(z) = zf'(z)/f(z),$$

it follows using Lemma A that $\phi_j'(w)/\phi_j(w)$ is small on $|w| = \frac{1}{2}$. Consequently,

$$\phi_j(w) = \phi_j(\tfrac{1}{2})\big(1 + \eta(w)\big), \qquad |w| = \tfrac{1}{2},$$

in which $\eta(w)$ is small. Thus $\phi_j\big(B(0, \frac{1}{2})\big)$ is contained in some disc $B(v_j, \delta|v_j|)$ with δ a small positive constant, and $w_j = \phi_j(0)$ lies in this disc.

Now, each w_j lies in a component D_j of the set $\{z : |G(z)| < \frac{1}{2}\}$ which is also a component of the set $\{z : |f(z)| > 2^m T\}$. Further, D_j lies in $\{z : \frac{1}{4}r < |z| < 2r\}$ and is mapped conformally onto $B(0, \frac{1}{2})$ by G. At least $p > \frac{1}{2}n$ of the D_j each have area at most $cr^2 n^{-1}$ and for these j we thus have $|G'(w_j)|^{-1} = |\phi_j'(0)| < cn^{-1/2} r$, by [28, p. 4] again.

Since the boundary of each of these D_j is a simple closed curve lying in $\{z : |z| \geq \frac{1}{4}r\}$ and on which $|f(z)| = 2^m T$, an application of Rouché's theorem shows that each D_j contains a zero of $f(z)^{-1} - z^{-1}$ and so a fixpoint z_j of f. Further, we have $u_j = G(z_j) = o(1)$. But the branch ϕ_j of the inverse function of G is univalent on $B(0,1)$. We estimate $\phi_j'(u_j)$ using the same application of [28, p. 9] as in the proof of Theorem 2 and deduce that, for these p points z_j, we have $|G'(z_j)| = |\phi_j'(u_j)|^{-1} > cn^{1/2} r^{-1}$. Finally, this gives

$$|f'(z_j)| \geq mT^{-1/m} |z_j|^{1+1/m} |G'(z_j)| \geq d|z_j|^{1/m} n^{1/2}$$

for these j, and Theorem 4 is proved.

4. Proof of Theorem 5

To establish Theorem 5, we will set $F = f \circ g$, with g entire and to be defined below, and $f = H$, where H is the function appearing in (4) and (5). This function H is defined by [21, p. 234]

$$H(z) = p(\log v), \qquad v + v^{-1} = z, \tag{27}$$

in which p is the Weierstrass doubly periodic function, with primitive periods 1 and $2\pi i$. Because of the equation (4), the only multiple values of H are ∞ and the distinct finite values e_1, e_2, e_3. By [21, p. 236, (11.33)], the set of points at which $H(z) = e_1$ is $\{\exp(m+\frac{1}{2}) + \exp(-m-\frac{1}{2}) : m \in \mathbf{Z}\}$. Consequently, the theorem will follow from (5) and the next lemma, with $w_m = \exp(m+\frac{1}{2}) + \exp(-m-\frac{1}{2})$, and $R = 2$.

Lemma 2. *Suppose that R is a constant with $R > 1$ and that $\psi(r)$ is an increasing positive function such that $\psi(r) \to \infty$ as $r \to \infty$. Suppose that (w_m) is a complex sequence such that, for all large r, the annulus $\{z : R^{-1}r \le |w| \le Rr\}$ contains at least one element of the sequence (w_m). Then there exists a transcendental entire function g such that $T(r,g) = O\big(\psi(r)\log r\big)$ as $r \to \infty$ and such that all but finitely many critical values of g are elements of the sequence (w_m).*

To prove Theorem 5, assuming Lemma 2, we note that $F = f \circ g$ has finite order and only finitely many critical values, and so is in the class S, using the theorem of Bergweiler and Eremenko [6] cited above. Alternatively, we may observe that f has no asymptotic values and g has no finite asymptotic values.

Proof of Lemma 2. We first define

$$h(z) = \prod_{k=1}^{\infty} (1 - z/a_k)$$

with each a_k positive, such that $a_{k+1} > (a_k)^8$, while a_k tends to infinity so rapidly that $T(r^M, h) = o\big(\psi(r)\log r\big)$ as $r \to \infty$, for every positive constant M. We have

$$h'(z)/h(z) = \sum_{k=1}^{\infty} 1/(z - a_k), \qquad (h'/h)'(z) = -\sum_{k=1}^{\infty} 1/(z-a_k)^2. \tag{28}$$

Because of the slow growth of h, all zeros b_m of h' are real and positive, and each interval (a_k, a_{k+1}) contains precisely one of the b_m [30, p. 266]. We choose a large zero b_m of h' and assert that $h(b_m)$ is large.

Suppose that $a_N < b_m < a_{N+1}$. Then the restriction of h to (a_N, a_{N+1}) has either a maximum or a minimum at b_m. By a theorem of Hayman [19], we have

$$\log|h(z)| = \big(1 + o(1)\big)\log M(|z|, h) \tag{29}$$

for all large z outside a family of discs $B(a_n,\varepsilon_n a_n)$, where $\sum \varepsilon_n < \infty$. Consequently there is at least one z with $(a_N)^3 < z < (a_N)^4$ such that (29) holds. By a standard convexity argument, we have

$$\begin{aligned}\log M\big((a_N)^2,h\big) &< \big(1-o(1)\big)\log M\big((a_N)^3,h\big)\\ &\le \log|h(b_m)| < \log M(a_{N+1},h).\end{aligned} \tag{30}$$

In particular, $|h(b_m)|$ is large, as asserted.

Put $c_m = h(b_m)$, and choose w_p lying in the annulus $\{z : R^{-1}|c_m| \le |w| \le R|c_m|\}$. Further, choose t in $[-\pi,\pi]$ such that c_m and w_p both lie in $|\arg w - t| \le \frac{1}{2}\pi$. We define $A_m = \{w : R^{-2}|c_m| < |w| < R^2|c_m|,\ t-\frac{3}{4}\pi < \arg w < t+\frac{3}{4}\pi\}$. These simply connected domains A_m have pairwise disjoint closures, by a standard convexity argument and (30).

Let $H(w)$ map A_m conformally onto the unit disk $B(0,1)$, with $H(|c_m|e^{it}) = 0$. It is clear that $d_m = H(c_m)$ and $v_p = H(w_p)$ both lie in the disc $B(0,r_0)$, with $r_0 < 1$ depending on R but not on c_m or w_p. Further, H extends continuously and univalently up to the boundary of A_m.

For $|a| < 1$, the function [7], [25]

$$Q(u) = Q_a(u) = \frac{u+a}{1+a\overline{u}}, \qquad |u| < 1,$$
$$Q_a(u) = u, \qquad |u| \ge 1,$$

is a quasiconformal homeomorphism of the extended plane onto itself. Further, for $|u| < 1$ we have $\partial Q/\partial\overline{u} = \sigma(u)\partial Q/\partial u$, with $|\sigma(u)| \le |a|$. Hence Q_a is $(1+|a|)/(1-|a|)$ quasiconformal and so is its inverse function.

We can now modify h as follows, first defining a quasiconformal homeomorphism H_1 of the extended plane by

$$H_1(w) = (H^{-1}Q_{v_p}Q_{d_m}^{-1}H)(w), \qquad w \in A_m, \tag{31}$$

with

$$H_1(w) = w, \qquad w \notin \bigcup A_m.$$

To verify that H_1 is indeed quasiconformal it is useful to note that (31) holds on a neighbourhood of the closure of A_m, since H has a quasiconformal extension to the extended plane and each Q_a is the identity outside $B(0,1)$. We then write $G(z) = H_1\big(h(z)\big)$. For $|z|$ large, the function G is one-one on a neighbourhood of z unless $G(z) = w_p$, for some p. Consequently, we can write $G(z) = g\big(\omega(z)\big)$ with $\omega(z)$ quasiconformal, and g entire, such that all but finitely many critical values of g are elements of the sequence (w_m). Further, since $|G(z)| \le R^4|h(z)|$ and since there is a positive constant M such that $\omega^{-1}(z) = O(|z|^M)$ as z tends to infinity [25], the function g has the required growth.

5. Proof of Theorem 6

We require the following theorem of Steinmetz [15], [29].

Theorem A. *Suppose that* g *is a non-constant entire function and that* $F_0, F_1, \ldots, F_m$ *are meromorphic functions which do not vanish identically, while* $h_0, h_1, \ldots, h_m$ *are meromorphic functions, not all identically zero, and satisfying*

$$\sum_{j=0}^{m} T(r, h_j) = O\big(T(r, g)\big)$$

as $r \to \infty$ *in a set of infinite measure. Suppose that*

$$F_0(g)h_0 + F_1(g)h_1 + \cdots + F_m(g)h_m \equiv 0.$$

Then there are polynomials $P_0, P_1, \ldots, P_m$, *not all identically zero, as well as polynomials* $Q_0, Q_1, \ldots, Q_m$, *again not all identically zero, such that*

$$P_0(g)h_0 + P_1(g)h_1 + \cdots + P_m(g)h_m \equiv 0, \qquad Q_0F_0 + Q_1F_1 + \cdots + Q_mF_m \equiv 0.$$

To prove Theorem 6, we assume that f, g, F, H are as in the hypotheses, but with

$$T(r, f) = o(r^{1/k}) \tag{32}$$

as $r \to \infty$. We note that, for each positive integer n,

$$F^{(n)} = f^{(n)}(g)(g')^n + \tfrac{1}{2}n(n-1)f^{(n-1)}(g)(g')^{n-2}g'' + \sum_{j=0}^{n-2} f^{(j)}(g)Q_j(g), \tag{33}$$

in which each $Q_j(g)$ is a differential polynomial in g. We note further that H cannot vanish identically and indeed that we cannot have $T(r, H) = O\big(T(r, g)\big)$ (n.e.) because, if so, applying Theorem A to (6) would yield a linear differential equation

$$\sum_{j=0}^{k} B_j(w)f^{(j)}(w) = B^*(w), \tag{34}$$

with polynomial coefficients $B_0, \ldots, B_k, B^*$, not all identically zero, and a standard application of the Wiman–Valiron theory [18] to (34) would show that f has at least order $1/k$, mean type.

Now, (33) and the definition (6) of H give

$$f^{(k+1)}(g)(g')^{k+1} + f^{(k)}(g)\big(\tfrac{1}{2}k(k+1)(g')^{k-1}g'' + A_{k-1}(g')^k\big) + \sum_{j=0}^{k-1} f^{(j)}(g)C_j + C^* = (H'/H)\bigg(f^{(k)}(g)(g')^k + \sum_{j=0}^{k-1} f^{(j)}(g)D_j + D^*\bigg), \tag{35}$$

with the coefficients satisfying

$$T(r, C_j) + T(r, D_j) + T(r, C^*) + T(r, D^*) = O\big(T(r, g)\big) \qquad \text{(n.e.)}.$$

Since f and g have finite order, we may write, using [17],

$$\begin{aligned} m(r, H'/H) &= O\big(\log rT(r, H)\big) \le O\big(\log T(r, F)\big) + O(\log r) \\ &\le O\big(\log\log M(r, F)\big) + O(\log r) \\ &\le O\big(\log\log M\big(M(r, g), f\big)\big) + O(\log r) \\ &= O\big(\log M(r, g)\big) = O\big(T(2r, g)\big) = O\big(T(r, g)\big), \qquad r \in E_1, \end{aligned} \tag{36}$$

with E_j henceforth denoting subsets of $(1, \infty)$ of positive lower logarithmic density. Now (7), (9) and (36) together give

$$T(r, H'/H) = O\big(T(r, g)\big), \qquad r \in E_2. \tag{37}$$

Applying Theorem A again, there are polynomials P_j and P^*, not all identically zero, such that

$$\sum_{j=0}^{k+1} P_j(w) f^{(j)}(w) = P^*(w). \tag{38}$$

Notice that a standard application of the Wiman–Valiron theory to (38) immediately gives $T(r, f) \ne o(r^{1/(k+1)})$ as $r \to \infty$, without requiring the extra hypothesis (8) on A_{k-1}.

We assert that the two equations (35) and (38) together lead to an equation

$$\sum_{j=0}^{k} T_j f^{(j)}(g) = T^*, \qquad T(r, T_j) + T(r, T^*) = O\big(T(r, g)\big), \qquad r \in E_3, \tag{39}$$

in which the coefficients T_j, T^* do not all vanish identically. Assuming (39), a further application of Theorem A leads to an equation of form (34), and so to the conclusion of the theorem.

Suppose then that no such equation (39) exists. We may substitute $w = g$ in (38) and, if this equation and (35) fail to give (39), we must have

$$k(k+1)g''/g' + 2A_{k-1} - 2H'/H = g'R(g), \tag{40}$$

with R a rational function. We identify two sub-cases.

Case 1. Suppose that all poles of $R(w)$ in the finite plane are simple and have rational residues. Then we may use (8) to obtain

$$(g')^{c_1} S^{c_2} H^{c_3} = e^{S_1(g)} S_2(g), \tag{41}$$

with S_1 a polynomial, S_2 a rational function and c_1, c_2, c_3 integers, c_3 non-zero. We write (41) in the form

$$e^{d_1 S_1(g)} = H^{d_2} G, \tag{42}$$

in which $d_1 = \pm 1$ and d_2 is a positive integer, while G is meromorphic, of finite order. We proceed to show that S_1 is constant. Assume then that S_1 is non-constant, with

$$d_1 S_1(w) = \big(1 + o(1)\big) s_1 w^{n_1}$$

when $|w|$ is large. Routine estimates based on the Poisson–Jensen formula [16] give a constant $M_1 > 0$ such that

$$\log^+ |G(z)| + \sum_{j=0}^{k-1} \log^+ |A_j(z)| + \log^+ |A^*(z)| \le |z|^{M_1}, \qquad |z| \notin F_1, \tag{43}$$

in which F_1 has finite measure. We choose r_0 large, normal for g with respect to the Wiman–Valiron theory, and not in the exceptional set F_1. Choosing z_0 with $|z_0| = r_0$ and $|g(z_0)| = M(r_0, g)$, we have

$$g(z) = g(z_0)(z/z_0)^{\nu}\big(1+o(1)\big), \quad d_1 S_1\big(g(z)\big) = s_1 g(z_0)^{n_1} (z/z_0)^{n_1 \nu}\big(1+o(1)\big), \tag{44}$$

for $|\log(z/z_0)| \le \nu^{-1/3}$, where $\nu = \nu(r_0, g)$ is the central index of g. On an arc Ω given by $z = z_0 e^{it}$, $-\nu^{-1/3} \le t \le \nu^{-1/3}$, the variation of $\arg\big(d_1 S_1\big(g(z)\big)\big)$ is greater than 2π, and (44) implies that Ω contains a point z_1 with

$$\operatorname{Re}\big(d_1 S_1\big(g(z_1)\big)\big) \ge \tfrac{1}{2} |s_1| M(r_0, g)^{n_1}. \tag{45}$$

On the other hand, since we certainly have $\log M(r, f^{(j)}) = o(r^{1/k})$ as $r \to \infty$, for $0 \le j \le k$, by (32), the estimate (43) gives, for some positive constant M_2,

$$\log |H^{d_2}(z_1) G(z_1)| \le O(r_0)^{M_2} + o\big(M(r_0, g)\big)^{1/k},$$

contradicting (42) and (45), and establishing our assertion that S_1 is constant.

The equation (41) now gives

$$T(r, H) = O\big(T(r, g)\big) \qquad \text{(n.e.)},$$

so that an application of Theorem A to (6) leads to (39), which we assumed impossible. Case 1 is thus disposed of.

Case 2. Suppose that, at some finite α, the rational function $R(w)$ has a multiple pole, or a simple pole with non-rational residue. Then (40) implies that α must be an omitted value of g, and so we can write $g=\alpha+e^q$, with q a polynomial. Thus (40) leads to an equation

$$R_1(g)(g')^{c_1}S^{c_2}H^{c_3}=\exp\bigl(\mu q+S_1(e^q)+S_2(e^{-q})\bigr),$$

with S_1 and S_2 polynomials and μ a constant. The same argument as in Case 1 shows that S_1 and S_2 are constant and again we can write (42), with G of finite order, which as before gives a contradiction.

We show now that the conclusion on the upper growth of f in Theorem 6 is sharp. For $k=1$ this is trivial, setting $F=e^g$ and $H=F'$. Now let $k\geq 2$ be an integer. The equation

$$w^{k-1}f^{(k)}(w)=f(w)+1 \tag{46}$$

has an entire solution f of order $1/k$, mean type. Writing $F=f\circ g$, we may choose coefficients A_j, each a rational function in g and its derivatives, with

$$A_{k-1}=\frac{-k(k-1)g''}{2g'},$$

such that, defining H by (6) and using (33) and (46), we have

$$H=(g')^k\bigl(f^{(k)}(g)-f(g)g^{1-k}\bigr)=(g')^k g^{1-k}.$$

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