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THE QUASIHYPERBOLIC METRIC, GROWTH, AND JOHN DOMAINS

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Abstract. A result of Hardy and Littlewood relates Hölder continuity of analytic functions in the unit disk with a bound on the derivative. Gehring and Martio extended this result to the class of uniform domains. We further extend this result to the class of John domains.

1. Introduction

The research in this paper stems from two elements of classical function theory. The first is a criterion due to Hardy and Littlewood for a function to be Hölder continuous in the unit disk $\mathbf{B} \subset \mathbf{C}$. The second is a class of domains first considered by Fritz John in his studies of plane elasticity and rigidity of local quasi-isometries.

Suppose that f is a function analytic in the unit disk $\mathbf{B} \subset \mathbf{C}$ and that $0 < \alpha \leq 1$. Then the theorem of Hardy and Littlewood mentioned above asserts that

$$|f'(z)| \leq m \operatorname{dist}(z, \partial\mathbf{B})^{\alpha-1} \tag{1.1}$$

for all $z \in \mathbf{B}$ if and only if

$$|f(z_1) - f(z_2)| \leq \frac{M}{\alpha}|z_1 - z_2|^{\alpha} \tag{1.2}$$

for all $z_1, z_2 \in \mathbf{B}$, where m and M depend only on each other [HL]. By integration along hyperbolic geodesics and by using the Cauchy integeral formula, we obtain extensions of this result to the cases when $\alpha = 0$ and $\alpha < 0$ in (1.1) above. If $\alpha = 0$, then (1.2) becomes

$$|f(z_1) - f(z_2)| \leq m \log\left(1 + \frac{2|z_1 - z_2|}{\operatorname{dist}(z_2, \partial\mathbf{B})}\right) \leq 2m \log\left(1 + \frac{|z_1 - z_2|}{\operatorname{dist}(z_2, \partial\mathbf{B})}\right).$$

If $\alpha < 0$, then (1.2) becomes

$$|f(z_1) - f(z_2)| \leq M\left(\min_{j=1,2} \operatorname{dist}(z_j, \partial\mathbf{B})\right)^{\alpha}.$$

The goal of this paper is to find geometric criteria for the validity of these extensions of the theorem of Hardy and Littlewood in a simply-connected plane domain. This goal is achieved by the following two main results, proved in Section 4.

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Theorem 1.1. *If $D \subset \mathbf{C}$ is simply-connected, then D is a b-John domain if and only if f analytic and satisfying*

$$|f'(z)| \leq \operatorname{dist}(z, \partial D)^{-1}$$

in D implies

$$|f(z_1) - f(z_2)| \leq a \log\Big(1 + \frac{\lambda_D(z_1, z_2)}{\min_{j=1,2} \operatorname{dist}(z_j, \partial D)}\Big)$$

for all $z_1, z_2 \in D$ where a is a constant which depends only on the constant b.

Theorem 1.2. *If $D \subset \mathbf{C}$ is bounded and simply-connected, then D is a b-John domain if and only if f analytic and satisfying*

$$|f'(z)| \leq \operatorname{dist}(z, \partial D)^{\alpha - 1}$$

in D implies f is in $\operatorname{Ord}_\alpha(D)$ with

$$|f(z_1) - f(z_2)| \leq c\Big(\min_{j=1,2} \operatorname{dist}(z_j, \partial D)\Big)^{\alpha}$$

for all $z_1, z_2 \in D$ where $\alpha < 0$ and c is a constant which depends only on the constants b and α.

A domain $D \subset \mathbf{R}^n$ is a *b-John domain* if each pair of points $x_1, x_2 \in D$ can be joined by an arc $\gamma \subset D$ for which

$$\min_{j=1,2} l\big(\gamma(x_j, y)\big) \leq b \operatorname{dist}(y, \partial D)$$

for all $y \in \gamma$, where $\gamma(x_j, y)$ is the subarc of γ with endpoints x_j and y. A domain is *John* if it is b-John for some constant b. John domains appear naturally in many areas of analysis, including complex dynamics, approximation theory, and elasticity. (See [NV], [MS].)

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2. Preliminary results

We let $\operatorname{dist}(A, B)$ denote the euclidean distance from a set $A \subset \overline{\mathbf{R}}^n$ to a set $B \subset \overline{\mathbf{R}}^n$. The euclidean distance between two points $x, y \in \mathbf{R}^n$ is denoted by $|x - y|$. Also, $l(\alpha)$ denotes the euclidean length of a rectifiable path α.

For $x \in \mathbf{R}^n$ and $r > 0$, $B^n(x, r)$ denotes the ball centered at x of radius r. The unit disk in $\mathbf{C}$, $\{z : |z| < 1\}$, is denoted by $\mathbf{B}$.

Unless stated otherwise, D will always be a simply-connected domain in $\mathbf{R}^n$ with at least two boundary points taken with respect to the usual topology in $\overline{\mathbf{R}}^n$, and $D^* = \overline{\mathbf{R}}^n \setminus \overline{D}$ is the exterior of D. A domain $D \subset \overline{\mathbf{C}}$ is a *conformal disk* if it is conformally equivalent to $\mathbf{B}$; i.e., D is a conformal disk if and only if ∂D is a non-degenerate continuum.

If c is a constant depending only on another constant b, we write "$c = c(b)$".

For $\alpha < 1$, we define the α*-quasihyperbolic metric* k_D^α in a domain $D \subset \mathbf{R}^n$ by

$$k_D^\alpha(x_1, x_2) = \inf_\gamma \int_\gamma \frac{ds}{\operatorname{dist}(x, \partial D)^{1-\alpha}} \tag{2.1}$$

where the infimum is taken over all rectifiable arcs γ joining x_1 and x_2 in D. When $\alpha = 0$, we have $k_D^0 = k_D$, the usual quasihyperbolic metric (see [GO], e.g.).

We define the *inner metric* λ_D in a domain D by

$$\lambda_D(x_1, x_2) = \inf_\beta l(\beta)$$

where β is any path joining x_1, x_2 in D.

We say that D is a b*-John domain*, $1 \le b < \infty$, if each pair of points $x_1, x_2 \in D$ can be joined by an arc $\gamma \subset D$ for which

$$\min_{j=1,2} l\big(\gamma(x_j, y)\big) \le b \operatorname{dist}(y, \partial D) \tag{2.2}$$

for all $y \in \gamma$. Here $\gamma(x_j, y)$ denotes the part of γ between x_j and y. We say that D is *John* if it is b-John for some b. It follows that a bounded domain D is a John domain if there exists a point $x_0 \in D$ such that each point $x \in D$ can be joined to x_0 by an arc $\gamma \subset D$ for which

$$l\big(\gamma(x, y)\big) \le b \operatorname{dist}(y, \partial D) \tag{2.3}$$

for all $y \in \gamma$. The point x_0 is called a *John center*; we can take x_0 such that $\operatorname{dist}(x_0, \partial D) = \sup_{x \in D} \operatorname{dist}(x, \partial D)$. Finally when a John domain D is also a conformal disk, we say that D is a *John disk.* (See [NV].)

An arc satisfying (2.2) is called a *double* b*-cone arc.* An arc satisfying (2.3) is called a b*-cone arc.* Quasihyperbolic and hyperbolic geodesics in b-John disks are double b_1-cone arcs, where $b_1 = b_1(b)$ [GHM], [NV].

A domain D is said to be b*-uniform*, $1 \le b < \infty$, if each pair of points x_1, x_2 in $D \setminus \infty$ can be joined by a rectifiable arc $\gamma \subset D$ which in addition to (2.2) satisfies

$$l(\gamma) \le b|x_1 - x_2|.$$

We say that D is *uniform* if it is b-uniform for some b. Since uniform domains satisfy (2.2), a uniform domain is a John domain. (See [G2], e.g.)

Given a set A in $\mathbf{R}^n$, we let $\mathrm{Lip}_\alpha(A)$, $0<\alpha\le 1$, denote the Lipschitz class of mappings $f\colon A\to\mathbf{R}^p$ satisfying for some $m<\infty$

$$|f(x_1)-f(x_2)|\le m|x_1-x_2|^\alpha \tag{2.4}$$

for all x_1, x_2 in A. If D is a domain in $\mathbf{R}^n$, then $f\colon D\to\mathbf{R}^p$ belongs to the local Lipschitz class $\mathrm{loc\,Lip}_\alpha(D)$ if there exists a constant $m<\infty$ such that (2.4) holds whenever x_1 and x_2 lie in any open ball B which is contained in D.

In $\mathrm{Lip}_\alpha(D)$ and $\mathrm{loc\,Lip}_\alpha(D)$ we shall use the seminorms $\|f\|_\alpha$ and $\|f\|_\alpha^{\mathrm{loc}}$, respectively,

$$\|f\|_\alpha=\inf\{m:|f(x_1)-f(x_2)|\le m|x_1-x_2|^\alpha,\ x_1,x_2\in D\}$$

$$\|f\|_\alpha^{\mathrm{loc}}=\inf\{m:|f(x_1)-f(x_2)|\le m|x_1-x_2|^\alpha,\ x_1,x_2\in B\subset D\},$$

where B ranges over all balls contained in D.

Given a set A in $\mathbf{R}^n$, we let $\mathrm{Ord}_\alpha(A)$, $\alpha<0$, denote the class of mappings $f\colon A\to\mathbf{R}^p$ satisfying for some $m<\infty$

$$|f(x_1)-f(x_2)|\le m\Big(\min_{j=1,2}\mathrm{dist}(x_j,\partial A)\Big)^\alpha \tag{2.5}$$

for all x_1, x_2 in A. We use this notation to parallel that of Lip_α, and because we are examining the order of growth of $f(x)$ as $x\to\partial D$. If D is a domain in $\mathbf{R}^n$, then $f\colon D\to\mathbf{R}^p$ belongs to the class loc $\mathrm{Ord}_\alpha(D)$ if there exists a constant $m<\infty$ such that (2.5) holds whenever x_1 and x_2 lie in any open ball which is contained in D.

In $\mathrm{Ord}_\alpha(D)$ and loc $\mathrm{Ord}_\alpha(D)$ we shall use the seminorms $\|f\|_\alpha^*$ and $\|f\|_\alpha^{*\,\mathrm{loc}}$, respectively,

$$\|f\|_\alpha^*=\inf\Big\{m:|f(x_1)-f(x_2)|\le m\Big(\min_{j=1,2}\mathrm{dist}(x_j,\partial D)\Big)^\alpha,\ x_1,x_2\in D\Big\}$$

$$\|f\|_\alpha^{*\,\mathrm{loc}}=\inf\Big\{m:|f(x_1)-f(x_2)|\le m\Big(\min_{j=1,2}\mathrm{dist}(x_j,\partial D)\Big)^\alpha,\ x_1,x_2\in B\subset D\Big\},$$

where B ranges over all balls in D.

3. John domains and the quasihyperbolic metric

Gehring and Osgood essentially showed (up to an additive constant) that a domain D is uniform if and only if it satisfies

$$k_D(x_1,x_2)\le c j_D(x_1,x_2)$$

for all $x_1,x_2\in D$ and some constant c, where

$$j_D(x_1,x_2)=\frac12\log\bigg(\frac{|x_1-x_2|}{\mathrm{dist}(x_1,\partial D)}+1\bigg)\bigg(\frac{|x_1-x_2|}{\mathrm{dist}(x_2,\partial D)}+1\bigg).$$

(See [GO], [G2, p. 97].) We define a similar metric j_D' by

$$j_D'(x_1,x_2)=\frac12\log\bigg(\frac{\lambda_D(x_1,x_2)}{\mathrm{dist}(x_1,\partial D)}+1\bigg)\bigg(\frac{\lambda_D(x_1,x_2)}{\mathrm{dist}(x_2,\partial D)}+1\bigg).$$

We find that k_D and j_D' are related in John disks.

Theorem 3.1. *A simply-connected proper subdomain $D \subset \mathbf{C}$ is a b-John disk if and only if there exists a constant c such that*

$$k_D(z_1, z_2) \le c j'_D(z_1, z_2)$$

for all $z_1, z_2 \in D$, with $c = c(b)$ and $b = b(c)$.

This result is a planar version of [KL, Theorem 4.1].

4. Domains in the plane

Throughout this section, we will take $D \subset \mathbf{C}$.

In the previous section, we discussed analytic functions in the unit disk with various bounds on the derivative. The result of Hardy and Littlewood mentioned at the beginning of the section can be generalized to certain domains in $\mathbf{C}$. Gehring and Martio extended this result to uniform domains [GM1, Corollary 2.2], and called the property that (1.1) implies (1.2) in D for every $0 < \alpha \le 1$ the *Hardy–Littlewood property.* This property does not characterize uniform domains; a more general geometric condition than uniformity which implies the Hardy–Littlewood property was introduced in [L]. The Hardy–Littlewood property was further studied by Astala, K. Hag, P. Hag, and Lappalainen [AHHL], in which the relationship between several types of domains with various geometric and extension properties was established. They called the property that (1.1) implies (1.2) in D for some α, $0 < \alpha \le 1$, the *Hardy–Littlewood property of order α.*

Gehring and Martio did find that a simply-connected domain $D \subset \mathbf{C}$ has the Hardy–Littlewood property of order α for some $0 < \alpha \le 1$ only if D is LLC_1 (see (5.1.i)).

We will repeatedly use a generalization of a result by Kaufman and Wu [KW]. The proof is essentially identical, so we omit it here. We first introduce an extension of a distance function used by Kaufman and Wu. We define the distance function δ_D^α on a domain $D \subset \overline{\mathbf{C}}$ for $\alpha \le 1$ by

$$\delta_D^\alpha(z_1, z_2) = \sup |f(z_1) - f(z_2)|$$

where the supremum is taken over all analytic functions f on D with

$$|f'(z)| \le \operatorname{dist}(z, \partial D)^{\alpha - 1}. \tag{4.1}$$

We see that δ_D^α is connected to the metric k_D^α, introduced in Section 2 and defined in (2.1).

Lemma 4.1 (See [KW, Theorem 1]). *In a conformal disk D in $\mathbf{C}$,*

$$k_D^\alpha(z_1, z_2) \le b_\alpha \delta_D^\alpha(z_1, z_2)$$

for all $z_1, z_2 \in D$, where b_α is a constant depending only on α.

First, we examine the Hardy–Littlewood property, and some analogues, in John disks.

Theorem 4.2 ([GM1, Theorem 2.1]. *If D is uniform and if f is defined and satisfies*

$$|\partial f(z)| = \limsup_{|h|\to 0} \frac{|f(z+h)-f(z)|}{|h|} \le m \operatorname{dist}(z,\partial D)^{\alpha-1}$$

in D, for some $0<\alpha\le 1$, then f is in $\mathrm{Lip}_\alpha(D)$ with

$$\|f\|_\alpha \le \frac{cm}{\alpha}$$

where c is a constant which depends only on the uniformity constant b.

This implies that a uniform domain has the Hardy–Littlewood property with constant depending only on b. The same is not true for John domains, as the following example shows.

Example 4.3. Let $D=\mathbf{B}\setminus(-1,0]$. Then D is a John disk, but D does not have the Hardy–Littlewood property. For let

$$f(z)=z^{1/2}, \qquad z_n=\tfrac14 e^{i\pi n/(n+1)}, \qquad w_n=\tfrac14 e^{-i\pi n/(n+1)},$$

$n=1,2,\dots$. Then f is analytic and

$$|f'(z)|=\tfrac12|z^{-1/2}|\le \tfrac12 \operatorname{dist}(z,\partial D)^{(1/2)-1}$$

in D, since $0\in\partial D$, but

$$\lim_{n\to\infty}|f(z_n)-f(w_n)|=\tfrac12+\tfrac12=1$$

while

$$\lim_{n\to\infty}|z_n-w_n|^{1/2}=0.$$

If we take $f(z)=z^\alpha$ for any $\alpha\in(0,1)$, we will have

$$|f'(z)|=\alpha|z^{\alpha-1}|\le\alpha\operatorname{dist}(z,\partial D)^{\alpha-1}$$

in D, but

$$\lim_{n\to\infty}|f(z_n)-f(w_n)|>0$$

while

$$\lim_{n\to\infty}|z_n-w_n|^\alpha=0.$$

An analogue of the Hardy–Littlewood property does hold in John disks, however. If we replace the euclidean metric with the inner metric, we get the following result.

Theorem 4.4. *If D is a b-John disk and if f is defined and satisfies*

$$|\partial f(z)| \le m \operatorname{dist}(z, \partial D)^{\alpha-1} \tag{4.2}$$

in D for some $0 < \alpha \le 1$, then

$$|f(z_1) - f(z_2)| \le \frac{cm}{\alpha} \lambda_D(z_1, z_2)^\alpha,$$

where c is a constant which depends only on b.

Proof. Fix $z_1, z_2 \in D$ and let γ be the hyperbolic geodesic joining z_1, z_2 in D. Next let s denote arclength measured along γ from z_1, let $l = l(\gamma)$, and let $z(s)$ denote the corresponding representation for γ.

Set $g(s) = f\big(z(s)\big)$. Then

$$|\partial g(s)| = \limsup_{h \to 0} \frac{|g(s+h) - g(s)|}{|h|} \le \big|\partial f\big(z(s)\big)\big|.$$

Since D is a b-John disk,

$$\min(s, l-s) \le b_1 \operatorname{dist}\big(z(s), \partial D\big),$$

$b_1 = b_1(b)$. Thus by (4.2),

$$|\partial g(s)| \le m \operatorname{dist}\big(z(s), \partial D\big)^{\alpha-1} \le m\Big(\frac{\min(s, l-s)}{b_1}\Big)^{\alpha-1}$$

for $0 < s < l$, and g is absolutely continuous. The Gehring–Hayman inequality gives a constant $c_0 > 0$ such that any curve $\delta \subset D$ with endpoints z_1, z_2 satisfies

$$l(\gamma) \le c_0 l(\delta)$$

[GH, Theorem 2], [Ja]. So we have

$$\begin{aligned} |f(z_1) - f(z_2)| &\le \int_0^l |\partial g(s)|\, ds \le 2m{b_1}^{1-\alpha} \int_0^{l/2} s^{\alpha-1}\, ds \\ &\le \frac{2{b_1}^{(1-\alpha)} m}{\alpha} \Big(\frac{c_0\, \lambda_D(z_1, z_2)}{2}\Big)^\alpha \le \frac{cm}{\alpha} \lambda_D(z_1, z_2)^\alpha, \end{aligned}$$

$c = c(b)$. □

We now have

Corollary 4.5. *If D is a b-John disk and if f is analytic and satisfies*

$$|f'(z)| \leq m \operatorname{dist}(z, \partial D)^{\alpha-1}$$

for z in D, then

$$|f(z_1) - f(z_2)| \leq \frac{cm}{\alpha} \lambda_D(z_1, z_2)^{\alpha},$$

for all z_1, z_2 in D, where c is a constant which depends only on b.

Next, we examine the case $\alpha = 0$.

Theorem 4.6. *A conformal disk $D \subset \mathbf{C}$ is b-uniform if and only if f analytic and satisfying*

$$|f'(z)| \leq \operatorname{dist}(z, \partial D)^{-1} \tag{4.3}$$

in D implies

$$|f(z_1) - f(z_2)| \leq a \log\left(1 + \frac{|z_1 - z_2|}{\min_{j=1,2} \operatorname{dist}(z_j, \partial D)}\right) \tag{4.4}$$

for all $z_1, z_2 \in D$ where a is a constant which depends only on the constant b.

Proof. First, suppose D is b-uniform. Then by (3.1),

$$\begin{aligned} k_D(z_1, z_2) &\leq c \log\left(1 + \frac{|z_1 - z_2|}{\operatorname{dist}(z_1, \partial D)}\right)\left(1 + \frac{|z_1 - z_2|}{\operatorname{dist}(z_2, \partial D)}\right) \\ &\leq a \log\left(1 + \frac{|z_1 - z_2|}{\min_{j=1,2} \operatorname{dist}(z_j, \partial D)}\right) \end{aligned}$$

for all $z_1, z_2 \in D$, where a depends only on b. If f is analytic and satisfies (4.3) in D, then

$$|f(z_1) - f(z_2)| \leq k_D(z_1, z_2) \leq a \log\left(1 + \frac{|z_1 - z_2|}{\min_{j=1,2} \operatorname{dist}(z_j, \partial D)}\right)$$

as desired.

Now suppose that every f analytic and satisfying (4.3) in D also satisfies (4.4). By Lemma 4.1,

$$k_D(z_1, z_2) = k_D^0(z_1, z_2) \leq c_0 \delta_D^0(z_1, z_2) \leq a \log\left(1 + \frac{|z_1 - z_2|}{\min_{j=1,2} \operatorname{dist}(z_j, \partial D)}\right)$$

for all $z_1, z_2 \in D$. So by (3.1), D is uniform. □

Theorem 4.7. *A conformal disk $D \subset \mathbf{C}$ is b-John if and only if f analytic and satisfying*

$$|f'(z)| \leq \operatorname{dist}(z, \partial D)^{-1} \tag{4.5}$$

in D implies

$$|f(z_1) - f(z_2)| \leq a \log\left(1 + \frac{\lambda_D(z_1, z_2)}{\min_{j=1,2} \operatorname{dist}(z_j, \partial D)}\right) \tag{4.6}$$

for all $z_1, z_2 \in D$ where a is a constant which depends only on the constant b.

Proof. First, suppose D is b-John. Then by Theorem 3.1,

$$k_D(z_1, z_2) \leq a \log\left(1 + \frac{\lambda_D(z_1, z_2)}{\min_{j=1,2} \operatorname{dist}(z_j, \partial D)}\right)$$

for all $z_1, z_2 \in D$, where a depends only on b. If f is analytic and satisfies (4.5) in D, then

$$|f(z_1) - f(z_2)| \leq k_D(z_1, z_2) \leq a \log\left(1 + \frac{\lambda_D(z_1, z_2)}{\min_{j=1,2} \operatorname{dist}(z_j, \partial D)}\right)$$

as desired.

Now suppose that every f analytic and satisfying (4.5) in D also satisfies (4.6). By Lemma 4.1,

$$k_D(z_1, z_2) = k_D^0(z_1, z_2) \leq c_0 \delta_D^0(z_1, z_2) \leq a \log\left(1 + \frac{\lambda_D(z_1, z_2)}{\min_{j=1,2} \operatorname{dist}(z_j, \partial D)}\right)$$

for all $z_1, z_2 \in D$. So by Theorem 3.1, D is John. □

Finally, we examine the case $\alpha < 0$.

Theorem 4.8. *If $D \subset \mathbf{C}$ is bounded and simply-connected, then D is a b-John disk if and only if f analytic and satisfying*

$$|f'(z)| \leq \operatorname{dist}(z, \partial D)^{\alpha - 1} \tag{4.7}$$

in D implies f is in $\operatorname{Ord}_\alpha(D)$ with

$$\|f\|_\alpha^* \leq c$$

where $\alpha < 0$ and c is a constant which depends only on the constants b and α.

Proof. First suppose D is a b-John disk, and let f be analytic and satisfying (4.7) in D. Then by integration over double b-cone arcs and by the Cauchy integral formula, f is in $\operatorname{Ord}_\alpha(D)$.

Now suppose that whenever f is a function analytic in D satisfying (4.7), then f is in $\operatorname{Ord}_\alpha(D)$. Then, by Lemma 4.1,

$$k_D^\alpha(z_1, z_2) \leq b_\alpha \delta_D^\alpha(z_1, z_2) = b_\alpha \sup |f(z_1) - f(z_2)| \leq b_\alpha c \Big(\min_{j=1,2} \operatorname{dist}(z_j, \partial D)\Big)^\alpha$$

where the supremum is taken over all analytic functions f on D satisfying (4.7). Theorem 6.5 now implies that D is a John domain. □

5. Geometric properties and extension properties

Recall that a simply-connected domain $D \subset \overline{\mathbf{C}}$ is a K-quasidisk if and only if it is uniform with constant b, $K = K(b)$, $b = b(K)$ [MS]. We say an arbitrary set $E \subset \overline{\mathbf{C}}$ is *c-linearly locally connected* ($c-\mathrm{LLC}$), c a constant, if for $z_0 \in \mathbf{C}$ and $0 < r < \infty$,

$$\text{(5.1)} \qquad \begin{array}{l} \text{(i) points in } E \cap \overline{B}(z_0, r) \text{ can be joined in } E \cap \overline{B}(z_0, cr); \text{ and} \\ \text{(ii) points in } E \setminus \overline{B}(z_0, r) \text{ can be joined in } E \setminus \overline{B}(z_0, r/c). \end{array}$$

The set E is $c-\mathrm{LLC}_1$ if it satisfies (5.1.i), and $c-\mathrm{LLC}_2$ if it satisfies (5.1.ii). A simply-connected domain D is a K-quasidisk if and only if it is c-linearly locally connected, $K = K(c)$, $c = c(K)$ [G1], and D is a b-John disk if and only if it satisfies (5.1.ii), $b = b(c)$, $c = c(b)$ [NV, 4.6].

Gehring and Martio found that the geometric property (5.1.i) is necessary for the Hardy–Littlewood property of order α, $0 < \alpha < 1$, in simply-connected domains $D \subset \mathbf{C}$ [GM1, Theorem 3.3]. They go on to use the Hardy–Littlewood property to characterize quasidisks with ∞ in the boundary [GM1, Theorem 4.2]. Combining these results with the work in this paper yields the following corollary.

Corollary 5.1. *Suppose that $D \subset \mathbf{C}$ is bounded and simply-connected. Then D is a quasidisk if and only if for some $\alpha_1 < 0$, $0 < \alpha_2 < 1$,*

$$|f'(z)| \le \operatorname{dist}(z, \partial D)^{\alpha_1 - 1} \tag{5.2}$$

in D implies $f \in \mathrm{Ord}_{\alpha_1}(D)$ and

$$|f'(z)| \le \operatorname{dist}(z, \partial D)^{\alpha_2 - 1} \tag{5.3}$$

in D implies $f \in \mathrm{Lip}_{\alpha_2}(D)$.

Proof. First, suppose D is a quasidisk. Then D is a John disk, so by Theorem 4.8 (5.2) implies $f \in \mathrm{Ord}_\alpha(D)$ for all $\alpha < 0$. Also, D is uniform, and so it has the Hardy–Littlewood property, i.e. (5.3) implies $f \in \mathrm{Lip}_\alpha(D)$ for all $0 < \alpha < 1$.

Now suppose that for some $\alpha_1 < 0$, $0 < \alpha_2 < 1$, (5.2) implies $f \in \mathrm{Ord}_{\alpha_1}(D)$ in D and (5.3) implies $f \in \mathrm{Lip}_{\alpha_2}(D)$ in D. By Theorem 4.8 D is a John disk, and thus satisfies (5.1.ii). By [GM1, Theorem 3.3], D satisfies (5.1.i). Therefore, D is a quasidisk. □

6. Domains in $\mathbf{R}^n$

The Hardy–Littlewood property can be extended to higher dimensions by using the concept of $\operatorname{loc}\operatorname{Lip}_\alpha$, introduced in Section 2. Recall that a function $f\colon D \to \mathbf{R}^p$ belongs to the local Lipschitz class $\operatorname{loc}\operatorname{Lip}_\alpha(D)$, $0 < \alpha \leq 1$, if there exists a constant $m < \infty$ such that (2.4) holds whenever x_1 and x_2 lie in any open ball B which is contained in D. For functions analytic in a domain $D \subset \mathbf{C}$, this is equivalent to the bound on the derivative

$$|f'(z)| \leq m \operatorname{dist}(z, \partial D)^{\alpha-1}.$$

This extension to $\mathbf{R}^n$ was done by Gehring and Martio in the following way. A domain $D \subset \mathbf{R}^n$ is called a $\operatorname{Lip}_\alpha$*-extension domain* if there exists a constant a depending only on D and α such that $f \in \operatorname{loc}\operatorname{Lip}_\alpha(D)$ implies $f \in \operatorname{Lip}_\alpha(D)$ with

$$\|f\|_\alpha \leq a \|f\|_\alpha^{\mathrm{loc}}$$

(see [GM2]).

Gehring and Martio, in their examination of $\operatorname{Lip}_\alpha$-extension domains, first showed that this extension property is equivalent to a geometric condition. They then used this condition to get subsequent necessary and sufficient conditions for D to be a $\operatorname{Lip}_\alpha$-extension domain. Our work in this section parallels the work of Gehring and Martio on $\operatorname{Lip}_\alpha$-extension domains, $0 < \alpha \leq 1$, in a way analogous to the preceding sections. We find local properties which for functions analytic in a domain $D \subset \mathbf{C}$ are equivalent to the bound on the derivative

$$|f'(z)| \leq m \operatorname{dist}(z, \partial D)^{\alpha-1},$$

for $\alpha = 0$ and $\alpha < 0$. We then define and examine extension domains for these properties.

We first examine Bloch-extension domains.

Theorem 6.1. *A domain $D \subset \mathbf{R}^n$ is a uniform domain if and only if*

$$|f(x_1) - f(x_2)| \leq \log\left(1 + \frac{|x_1 - x_2|}{\min_{j=1,2} \operatorname{dist}(x_j, \partial D)}\right) \tag{6.1}$$

for all $x_1, x_2 \in D$ with $|x_1 - x_2| < \operatorname{dist}(x_1, \partial D)$ implies

$$|f(x_1) - f(x_2)| \leq a \log\left(1 + \frac{|x_1 - x_2|}{\min_{j=1,2} \operatorname{dist}(x_j, \partial D)}\right) \tag{6.2}$$

for all $x_1, x_2 \in D$.

Proof. Suppose D is uniform. Then by (3.1),

$$k_D(x_1,x_2) \le a_1 \log\left(1+\frac{|x_1-x_2|}{\min_{j=1,2}\operatorname{dist}(x_j,\partial D)}\right)$$

for all $x_1,x_2 \in D$. Let f satisfy (6.1). Fix $x_1,x_2 \in D$ and let $\gamma \subset D$ be the quasihyperbolic geodesic with endpoints x_1 and x_2. Let $\gamma(s)$ be the parameterization of γ with respect to arc length measured from x_1, $l = l(\gamma)$. Let $y_1 = x_1$. We choose positive numbers r_i and l_i, and points $y_i \in \gamma$ as follows:

$$r_1 = \tfrac{1}{2}\operatorname{dist}(y_1,\partial D), \qquad l_1 = \max\{s : \gamma(s) \in \overline{B}^n(y_1,r_1)\}, \qquad y_2 = \gamma(l_1);$$

$$r_2 = \tfrac{1}{2}\operatorname{dist}(y_2,\partial D), \qquad l_2 = \max\{s : \gamma(s) \in \overline{B}^n(y_2,r_2)\}, \qquad y_3 = \gamma(l_2);$$

and so on. After a finite number of steps, N, say, $l_N = l$ and the process stops. Let $y_{N+1} = x_2$. So by [GHM, Lemma 2.6],

$$\begin{aligned}|f(x_1)-f(x_2)| &\le \sum_{i=1}^{N} a_2 \log\left(1+\frac{|y_i-y_{i+1}|}{\operatorname{dist}(y_{j+1},\partial D)}\right)\\ &\le a_2 \sum_{i=1}^{n} k_D\bigl(\gamma(y_i,y_{i+1})\bigr) \le a \log\left(1+\frac{|x_1-x_2|}{\min_{j=1,2}\operatorname{dist}(x_j,\partial D)}\right)\end{aligned}$$

as desired.

Now suppose (6.1) implies (6.2) in D. Fix $x_0 \in D$. Let

$$f(x) = k_D(x,x_0).$$

If $x_1,x_2 \in B \subset D$, B an open ball, then

$$|f(x_1)-f(x_2)| \le k_D(x_1,x_2).$$

Let $\gamma \subset B$ be the segment of the circle through x_1, x_2 perpendicular to ∂B with endpoints x_1, x_2. Then

$$l(\gamma) \le \pi |x_1-x_2|$$

and

$$\min_{j=1,2} l\bigl(\gamma(x,x_j)\bigr) \le \pi \operatorname{dist}(x,\partial B) \le \pi \operatorname{dist}(x,\partial D)$$

for all $x \in \gamma$. Following the same argument used in the proof of [KL, Theorem 4.1] we get

$$k_D(x_1,x_2) \le \int_\gamma \frac{ds}{\operatorname{dist}(x,\partial D)} \le c \log\left(1+\frac{|x_1-x_2|}{\min_{j=1,2}\operatorname{dist}(x_j,\partial D)}\right)$$

where c is independent of x_0 and B, i.e. (6.1) holds. So

$$k_D(x,x_0) \le a_1 \log\left(1+\frac{|x-x_0|}{\min\{\operatorname{dist}(x,\partial D),\operatorname{dist}(x_0,\partial D)\}}\right)$$

for all $x \in D$, where a_1 is independent of x_0. Thus

$$k_D(x_1,x_2) \le a_1 \log\left(1+\frac{|x_1-x_2|}{\min_{j=1,2}\operatorname{dist}(x_j,\partial D)}\right)$$

for all $x_1,x_2 \in D$, and hence D is uniform by (3.1). □

A domain $D \subset \mathbf{R}^n$ is called an $\operatorname{Ord}_\alpha$*-extension domain*, $\alpha < 0$, if there exists a constant a depending only on D and α such that $f \in \operatorname{loc}\operatorname{Ord}_\alpha(D)$ implies $f \in \operatorname{Ord}_\alpha(D)$ with

$$\|f\|_\alpha^* \le a\|f\|_\alpha^{*\,\mathrm{loc}}.$$

Theorem 6.2. *Let $\alpha < 0$. A domain $D \subset \mathbf{R}^n$ is an $\operatorname{Ord}_\alpha$-extension domain if and only if there is a constant $M < \infty$ such that each $x_1, x_2 \in D$ can be joined by a rectifiable curve $\gamma \subset D$ with*

$$\int_\gamma \frac{ds}{\operatorname{dist}(x,\partial D)^{1-\alpha}} \le M\Big(\min_{j=1,2} \operatorname{dist}(x_j,\partial D)\Big)^\alpha. \tag{6.3}$$

Proof. First, suppose D is an $\operatorname{Ord}_\alpha$-extension domain. So there is a constant $c > 0$ such that $f \in \operatorname{loc}\operatorname{Ord}_\alpha(D)$ implies $f \in \operatorname{Ord}_\alpha(D)$ with

$$\|f\|_\alpha^* \le c\|f\|_\alpha^{*\,\mathrm{loc}}.$$

Fix $x_0 \in D$ and define a function f by

$$f(x) = k_D^\alpha(x_0,x) = \inf_\gamma \int_\gamma \frac{ds}{\operatorname{dist}(y,\partial D)^{1-\alpha}}$$

where the infimum is taken over all rectifiable arcs γ joining x_0 and x in D. By the triangle inequality,

$$|f(x_1) - f(x_2)| \le k_D^\alpha(x_1,x_2) \tag{6.4}$$

for all $x_1, x_2 \in D$.

Assume that x_1, x_2 belong to an open ball B which is contained in D. Let γ be the subarc between x_1 and x_2 of the circle through x_1, x_2 perpendicular to ∂B. For $x \in \gamma$,

$$\min_{j=1,2} l\big(\gamma(x_j,x)\big) \le \tfrac{1}{2}\pi \operatorname{dist}(x,\partial B) \le \tfrac{1}{2}\pi \operatorname{dist}(x,\partial D).$$

We then parameterize γ with respect to arc-length measured from x_1 and let $d_j = \operatorname{dist}(x_j,\partial D)$, $j = 1,2$. Then

$$\begin{aligned}
\int_\gamma \frac{ds}{\operatorname{dist}(x,\partial D)^{1-\alpha}} &\le \bigg(\int_0^{d_1/2} + \int_{l-d_2/2}^{l} + \int_{d_1/2}^{l-d_2/2}\bigg) \frac{ds}{\operatorname{dist}\big(\gamma(s),\partial D\big)^{1-\alpha}} \\
&\le \frac{d_1^\alpha}{2^\alpha} + \frac{d_2^\alpha}{2^\alpha} + \int_{d_1/2}^{l-(d_2/2)} \frac{(\pi/2)^{1-\alpha}\,ds}{(\min\{s, l-s\})^{1-\alpha}} \\
&\le m\Big(\min_{j=1,2} \operatorname{dist}(x_j,\partial D)\Big)^\alpha,
\end{aligned}$$

where m depends only on α. Together with (6.4) this gives

$$|f(x_1)-f(x_2)| \le m\Big(\min_{j=1,2} \operatorname{dist}(x_j,\partial D)\Big)^{\alpha}$$

and so $f \in \operatorname{loc}\operatorname{Ord}_\alpha(D)$. By the assumption, $f \in \operatorname{Ord}_\alpha(D)$ and $\|f\|_\alpha^*$ has an upper bound which is independent of x_0. The definition of f now yields (6.3).

Now suppose that D satisfies the condition (6.3). Let f be a function with

$$|f(x_1)-f(x_2)| \le \operatorname{dist}(x_1,\partial D)^{\alpha} \tag{6.5}$$

whenever $x_1, x_2 \in D$ with $|x_1-x_2| \le \frac{1}{2}\operatorname{dist}(x_1,\partial D)$. Note that if $f \in \operatorname{loc}\operatorname{Ord}_\alpha(D)$, then f satisfies (6.5) up to a constant.

Fix x_1 and x_2 in D. We can find a rectifiable arc $\gamma \subset D$ joining x_1 and x_2 such that

$$\int_\gamma \frac{ds}{\operatorname{dist}(y,\partial D)^{1-\alpha}} \le M\Big(\min_{j=1,2} \operatorname{dist}(x_j,\partial D)\Big)^{\alpha}. \tag{6.6}$$

Let $\gamma(s)$ be the parameterization of γ with respect to arc length measured from x_1, $l = l(\gamma)$. Let $y_1 = x_1$. We choose positive numbers r_i and l_i, and points $y_i \in \gamma$ as follows:

$$r_1 = \tfrac{1}{4}\operatorname{dist}(y_1,\partial D), \qquad l_1 = \max\{s : \gamma(s) \in \overline{B}^n(y_1,r_1)\}, \qquad y_2 = \gamma(l_1);$$
$$r_2 = \tfrac{1}{4}\operatorname{dist}(y_2,\partial D), \qquad l_2 = \max\{s : \gamma(s) \in \overline{B}^n(y_2,r_2)\}, \qquad y_3 = \gamma(l_2);$$

and so on. After a finite number of steps, N, say, $l_N = l$ and the process stops. Let $y_{N+1} = x_2$. We now have

$$|f(x_1)-f(x_2)| \le \sum_{i=1}^N |f(y_i)-f(y_{i+1})| \le c_\alpha \sum_{i=1}^N \operatorname{dist}(y_i,\partial D)^{\alpha}. \tag{6.7}$$

Let $l_0 = 0$ and for each $i = 1,2,\dots,N-1$, let

$$A_i = \{s \in [l_{i-1}, l_i] : \gamma(s) \in \overline{B}^n(y_i,r_i)\}.$$

Then A_i is closed, and

$$h_1(A_i) \ge r_i = |y_i - y_{i+1}| = \tfrac{1}{4}\operatorname{dist}(y_i,\partial D) \tag{6.8}$$

where h_1 is 1-dimensional Hausdorff measure. Moreover, for $s \in A_i$,

$$\operatorname{dist}\big(\gamma(s),\partial D\big) \le |\gamma(s)-y_i| + \operatorname{dist}(y_i,\partial D) \le 5r_i,$$

and hence

$$\operatorname{dist}\bigl(\gamma(s),\partial D\bigr)^{1-\alpha} \le 5^{1-\alpha} {r_i}^{1-\alpha}.$$

Together with (6.8) this yields

$$\begin{aligned}\int_\gamma \frac{ds}{\operatorname{dist}(y,\partial D)^{1-\alpha}} &\ge \sum_{i=1}^{N-1} \int_{A_i} \frac{ds}{\operatorname{dist}(\gamma(s),\partial D)^{1-\alpha}} \\ &\ge c_\alpha \sum_{i=1}^{N-1} {r_i}^{-(1-\alpha)} h_1(A_i) = c_\alpha \sum_{i=1}^{N-1} \operatorname{dist}(y_i,\partial D)^\alpha.\end{aligned} \tag{6.9}$$

Also, $x_2 \in \overline{B}^n\bigl(y_N, \frac14 \operatorname{dist}(y_N,\partial D)\bigr)$, and so

$$\operatorname{dist}(x_2,\partial D) \le \tfrac54 \operatorname{dist}(y_N,\partial D). \tag{6.10}$$

Then combining (6.7), (6.9), (6.10), and (6.9) yields

$$\begin{aligned}|f(x_1)-f(x_2)| &\le c_\alpha \int_\gamma \frac{ds}{\operatorname{dist}(y,\partial D)^{1-\alpha}} + c_\alpha \bigl(\tfrac54 \operatorname{dist}(x_2,\partial D)\bigr)^\alpha \\ &\le c_\alpha \Bigl(\min_{j=1,2} \operatorname{dist}(x_j,\partial D)\Bigr)^\alpha.\end{aligned}$$

Thus $f \in \operatorname{Ord}_\alpha(D)$ with $\|f\|_\alpha^*$ depending only on α, and D is an $\operatorname{Ord}_\alpha$-extension domain. □

We will need a characterization of John domains in $\mathbf{R}^n$, the proof of which requires two lemmas.

Lemma 6.3. *If there exists an arc $\gamma \subset D$ joining x_1 and x_2 in D with*

$$\int_\gamma \frac{ds}{\operatorname{dist}(x,\partial D)^{1-\alpha}} \le M\Bigl(\min_{j=1,2} \operatorname{dist}(x_j,\partial D)\Bigr)^\alpha,$$

then there is a number $a = a(M,\alpha)$ with

$$\operatorname{dist}(y,\partial D) \ge a\Bigl(\min_{j=1,2} \operatorname{dist}(x_j,\partial D)\Bigr) \qquad \textit{for all } y \in \gamma.$$

Proof. We may assume that $\operatorname{dist}(x_1,\partial D) \le \operatorname{dist}(x_2,\partial D)$. Fix $y \in \gamma$. If $x_1 \in \overline{B}^n\bigl(y, \operatorname{dist}(y,\partial D)\bigr)$, then

$$\operatorname{dist}(x_1,\partial D) \le |x_1 - y| + \operatorname{dist}(y,\partial D) \le 2\operatorname{dist}(y,\partial D)$$

and so $\operatorname{dist}(y,\partial D) \ge \frac12 \operatorname{dist}(x_1,\partial D)$. Likewise, if $x_2 \in \overline{B}^n\bigl(y, \operatorname{dist}(y,\partial D)\bigr)$, then

$$\operatorname{dist}(y,\partial D) \ge \tfrac12 \operatorname{dist}(x_2,\partial D) \ge \tfrac12 \operatorname{dist}(x_1,\partial D).$$

Now assume $x_1, x_2 \notin \overline{B}^n(y, \operatorname{dist}(y, \partial D))$. Let

$$a = \min\left\{\frac{1}{2}, \left(\frac{2^{1+\alpha-2}}{\alpha M}\right)^{-1/\alpha}\right\}$$

so that

$$\frac{-2}{\alpha}(1-2^\alpha)a^\alpha \geq M.$$

Then if $\operatorname{dist}(y, \partial D) < a \operatorname{dist}(x_1, \partial D)$, we get

$$\int_\gamma \frac{ds}{\operatorname{dist}(x, \partial D)^{1-\alpha}} \geq 2\int_0^{\operatorname{dist}(y,\partial D)} \frac{dt}{\big(t + \operatorname{dist}(y, \partial D)\big)^{1-\alpha}} > M \operatorname{dist}(x_1, \partial D)^\alpha,$$

a contradiction. Therefore, $\operatorname{dist}(y, \partial D) \geq a \operatorname{dist}(x_1, \partial D)$ for all $y \in \gamma$. □

Lemma 6.4. *Let x_1, x_2, and γ all be as in Lemma 6.3 above. Then, for $\delta \in (0, \infty)$,*

$$h_1\{y \in \gamma : \operatorname{dist}(y, \partial D) \leq \delta\} \leq \delta^{1-\alpha} M \Big(\min_{j=1,2} \operatorname{dist}(x_j, \partial D)\Big)^\alpha$$

where h_1 is 1-dimensional Hausdorff measure.

Proof. Fix a δ, and let $\gamma' = \{y \in \gamma : \operatorname{dist}(y, \partial D) \leq \delta\}$. Then

$$\int_{\gamma'} \frac{ds}{\operatorname{dist}(x, \partial D)^{1-\alpha}} \geq \frac{h_1(\gamma')}{\delta^{1-\alpha}},$$

and

$$\int_{\gamma'} \frac{ds}{\operatorname{dist}(x, \partial D)^{1-\alpha}} \leq \int_\gamma \frac{ds}{\operatorname{dist}(x, \partial D)^{1-\alpha}}.$$

Thus, by our hypothesis,

$$\frac{h_1(\gamma')}{\delta^{1-\alpha}} \leq M \Big(\min_{j=1,2} \operatorname{dist}(x_j, \partial D)\Big)^\alpha$$

which gives the desired result. □

Theorem 6.5. *A bounded domain D is a b-John domain if and only if there is a constant $M < \infty$ and $\alpha < 0$ such that each $x_1, x_2 \in D$ can be joined by a rectifiable curve $\gamma \subset D$ with*

$$\int_\gamma \frac{ds}{\operatorname{dist}(x, \partial D)^{1-\alpha}} \leq M \Big(\min_{j=1,2} \operatorname{dist}(x_j, \partial D)\Big)^\alpha. \tag{6.11}$$

Proof. First, suppose D is a b-John domain. So for any pair of points $x_1, x_2 \in D$, we can find an arc $\gamma \subset D$ joining x_1 and x_2 with

$$\min_{j=1,2} l\bigl(\gamma(y, x_j)\bigr) \le b \operatorname{dist}(y, \partial D)$$

for all $y \in \gamma$. We may assume $\operatorname{dist}(x_1, \partial D) \le \operatorname{dist}(x_2, \partial D)$. Let $\gamma(s)$ be the parameterization of γ with respect to arc-length measured from x_1 and let $d_j = \operatorname{dist}(x_j, \partial D)$, $j = 1, 2$.

If $l(\gamma) = l \ge \operatorname{dist}(x_1, \partial D)$, we have the following:

$$\begin{aligned} k_D^\alpha(x_1, x_2) &\le \int_0^l \frac{ds}{\operatorname{dist}\bigl(\gamma(s), \partial D\bigr)^{1-\alpha}} \\ &\le \frac{d_1^\alpha}{2^\alpha} + \frac{d_2^\alpha}{2^\alpha} + \int_{d_1/2}^{l-(d_2/2)} \frac{b^{1-\alpha} ds}{(\min\{s, l-s\})^{1-\alpha}} \le c_1 \Bigl(\min_{j=1,2} \operatorname{dist}(x_j, \partial D)\Bigr)^\alpha \end{aligned}$$

where c_1 depends only on α and b.

If $l < \operatorname{dist}(x_1, \partial D)$, then $x_2 \in B^n\bigl(x_1, \operatorname{dist}(x_1, \partial D)\bigr)$ and there is a universal constant c such that $\operatorname{dist}(y, \partial D) \ge c \operatorname{dist}(x_1, \partial D)$ for all y in $[x_1, x_2]$. So

$$k_D^\alpha(x_1, x_2) \le \int_{[x_1, x_2]} \frac{ds}{\operatorname{dist}(y, \partial D)^{1-\alpha}} \le c^{1-\alpha} \operatorname{dist}(x_1, \partial D)^\alpha.$$

Let $M = \sup\{c_1, c^{1-\alpha}\}$.

For the converse, fix $x_0 \in D$ with

$$\operatorname{dist}(x_0, \partial D) = \max_{x \in \gamma} \operatorname{dist}(x, \partial D) = d_0.$$

Fix $x_1 \in D$ and let γ_1 be a rectifiable curve joining x_1 and x_0 in D satisfying

$$\int_{\gamma_1} \frac{ds}{\operatorname{dist}(y, \partial D)^{1-\alpha}} \le M \operatorname{dist}(x_1, \partial D)^\alpha. \tag{6.12}$$

We will construct a rectifiable curve $\gamma \subset D$ joining x_1 and x_0 satisfying

$$l\bigl(\gamma(y, x_1)\bigr) \le b \operatorname{dist}(y, \partial D)$$

for all $y \in \gamma$, where b depends only on α. This will imply that D is John.

Let $y_1 = x_1$.

If $2 \operatorname{dist}(x_1, \partial D) \ge d_0$, then

$$\operatorname{dist}(y, \partial D) \le d_0 \le 2 \operatorname{dist}(x_1, \partial D)$$

for all $y \in \gamma_1$, and so by Lemma 6.4

$$l(\gamma_1) = h_1\{y \in \gamma_1 : \operatorname{dist}(y, \partial D) \le 2\operatorname{dist}(x_1, \partial D)\} \le 2^{1-\alpha} M \operatorname{dist}(x_1, \partial D)$$

and by Lemma 6.3

$$\operatorname{dist}(y, \partial D) \ge a \operatorname{dist}(x_1, \partial D)$$

for all $y \in \gamma_1$. Then we let $y_2 = x_0$, and we stop.

If $2\operatorname{dist}(x_1, \partial D) < d_0$, then by Lemma 6.4,

$$\begin{aligned} h_1\{y \in \gamma_1 : \operatorname{dist}(y, \partial D) \le 2\operatorname{dist}(x_1, \partial D)\} &\le \big(2\operatorname{dist}(x_1, \partial D)\big)^{1-\alpha} M \operatorname{dist}(x_1, \partial D)^{\alpha} \\ &= 2^{1-\alpha} M \operatorname{dist}(x_1, \partial D). \end{aligned}$$

So we can find $y_2 \in \gamma_1$ with $\operatorname{dist}(y_2, \partial D) = 2\operatorname{dist}(x_1, \partial D)$ and $l\big(\gamma_1(y_1, y_2)\big) \le 2^{1-\alpha} M \operatorname{dist}(x_1, \partial D)$. By Lemma 6.3,

$$\operatorname{dist}(y, \partial D) \ge a \operatorname{dist}(x_1, \partial D)$$

for all $y \in \gamma_1(y_1, y_2)$. Let γ_2 be a rectifiable curve joining y_2 and x_0 in D with

$$\int_{\gamma_2} \frac{ds}{\operatorname{dist}(y, \partial D)^{1-\alpha}} \le M \operatorname{dist}(y_2, \partial D)^{\alpha} = M 2^{\alpha} \operatorname{dist}(x_1, \partial D)^{\alpha}.$$

Once again, if $4\operatorname{dist}(x_1, \partial D) \ge d_0$, then

$$\begin{aligned} \operatorname{dist}(y, \partial D) \le d_0 \le 4\operatorname{dist}(x_1, \partial D) \qquad &\text{for all } y \in \gamma_1(y_2, x_0), \\ l(\gamma_2) \le 2^{2-\alpha} M \operatorname{dist}(x_1, \partial D),& \\ \operatorname{dist}(y, \partial D) \ge 2a \operatorname{dist}(x_1, \partial D) \qquad &\text{for all } y \in \gamma_2. \end{aligned}$$

Then we let $y_3 = x_0$, and we stop. Otherwise, we find $y_3 \in \gamma_2$ with $\operatorname{dist}(y_3, \partial D) = 4\operatorname{dist}(x_1, \partial D)$ and $l\big(\gamma_2(y_2, y_3)\big) \le 2^{2-\alpha} M \operatorname{dist}(x_1, \partial D)$, as above. By Lemma 6.3,

$$\operatorname{dist}(y, \partial D) \ge 2a \operatorname{dist}(x_1, \partial D)$$

for all $y \in \gamma_2(y_2, y_3)$.

Continue this process to get points $y_j \in \mathbf{B}$ and curves $\gamma_j(y_j, y_{j+1}) \subset D$, $j = 1, \dots, m$ such that letting $y_{m+1} = x_0$ yields

$$\begin{aligned} \operatorname{dist}(y_j, \partial D) &= 2^{j-1} \operatorname{dist}(x_1, \partial D), \\ l\big(\gamma_j(y_j, y_{j+1})\big) &\le 2^{j-\alpha} M \operatorname{dist}(x_1, \partial D), \end{aligned} \tag{6.13}$$

and if $y \in \gamma_j(y_j, y_{j+1})$, we have

$$\operatorname{dist}(y, \partial D) \ge 2^{j-1} a \operatorname{dist}(x_1, \partial D) \tag{6.14}$$

for $j=1,\dots,m$. Let

$$\gamma = \bigcup_{j=1}^{m} \gamma_j(y_j, y_{j+1}).$$

For any $y \in \gamma$, there is an N such that $y \in \gamma_N[y_N, y_{N+1})$, and so by (6.13)

$$l\big(\gamma(x_1,y)\big) \le \sum_{j=1}^{N} 2^{j-\alpha} M \operatorname{dist}(x_1,\partial D) = (2^N-1)2^{1-\alpha} M \operatorname{dist}(x_1,\partial D) \tag{6.15}$$

and (6.14) implies

$$\operatorname{dist}(y,\partial D) \ge 2^{N-1} a \operatorname{dist}(x_1,\partial D). \tag{6.16}$$

Combining (6.15) and (6.16), we get

$$\frac{2^{2-\alpha}M}{a}\operatorname{dist}(y,\partial D) \ge \frac{(2^N-1)2^{1-\alpha}M}{2^{N-1}a}\operatorname{dist}(y,\partial D) \ge l\big(\gamma(x_1,y)\big)$$

for all $y \in \gamma$. Thus D is a John domain. □

Combining Theorem 6.5 and Theorem 6.2 yields our main result, Theorem 6.6.

Theorem 6.6. *A bounded domain D is a b-John domain if and only if D is an* Ord_α*-extension domain for some $\alpha < 0$.*

Remark. Note that in the proof of Theorem 6.5, we have in fact shown that in a bounded b-John domain,

$$k_D^\alpha(x_1,x_2) \le M\Big(\min_{j=1,2}\operatorname{dist}(x_j,\partial D)\Big)^\alpha$$

in D for *every* $\alpha < 0$ (here M varies with α). So such a John domain is an Ord_α-extension domain for every $\alpha < 0$. In addition, we get the following corollary.

Corollary 6.8. *A bounded domain D is an* Ord_α*-extension domain for some $\alpha < 0$ if and only if D is an* Ord_α*-extension domain for all $\alpha < 0$.*

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