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A REMARK ON THE UNIQUENESS OF QUASI CONTINUOUS FUNCTIONS

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Abstract. We give a simple argument showing that two quasi continuous functions that agree almost everywhere coincide in fact quasi everywhere.

1. Introduction

By a *capacity* on a topological space X we understand a countably subadditive, monotone set function $\mathscr{C}$ defined on the subsets of X with values on the interval $[0,\infty]$ such that $\mathscr{C}(\emptyset)=0$. Moreover, a capacity $\mathscr{C}$ is called an *outer capacity* if

$$\mathscr{C}(A)=\inf\{\mathscr{C}(G): G \text{ is open and } A\subset G\}.$$

Furthermore a function $f\colon X\to\overline{\mathbf{R}}$ is termed $\mathscr{C}$*-quasi continuous* if for each $\varepsilon>0$ there is a set $G\subset X$ such that $\mathscr{C}(G)<\varepsilon$ and the restriction to $X\setminus G$ of f is continuous. Evidently, G can be chosen to be open if $\mathscr{C}$ is outer. See [F] for properties of abstract capacities and quasi topologies.

An example of an outer capacity is the p*-capacity* cap_p on $\mathbf{R}^n$:

$$\operatorname{cap}_p(E)=\inf\int_{\mathbf{R}^n}\bigl(|\nabla\varphi|^p+|\varphi|^p\bigr)\,dx,$$

where the infimum is taken over all Sobolev functions $\varphi\in W^{1,p}(\mathbf{R}^n)$ such that $\varphi\ge 1$ a.e. on an open neighborhood of E. In that case it is well known (see [AH], [DL], [HKM], [MK]) that cap_p-quasi continuous functions enjoy the following uniqueness property: Let f and g be cap_p-quasi continuous functions on an open set $\Omega\subset\mathbf{R}^n$ such that $f=g$ almost everywhere on Ω (w.r.t. Lebesgue measure). Then $f=g$ cap_p-quasi everywhere on Ω. These proofs are based on fine properties of capacitary potentials. Recall that a property is said to hold $\mathscr{C}$*-quasi everywhere* if it holds except on a set E with $\mathscr{C}(E)=0$. Our purpose in this note is to give an elementary and short proof in the abstract setting:

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Theorem. *Suppose that* $\mathscr{C}$ *is an outer capacity on* X *and* μ *is a nonnegative, monotone set function on* X *such that the following compatibility condition is safisfied: if* G *is open and* $\mu(E) = 0$*, then*

$$\mathscr{C}(G) = \mathscr{C}(G \setminus E).$$

Let f *and* g *be* $\mathscr{C}$*-quasi continuous functions on* X *such that*

$$\mu\big(\{x : f(x) \neq g(x)\}\big) = 0.$$

Then $f = g$ $\mathscr{C}$*-quasi everywhere on* X.

Remarks. It is clear from the definition that the usual capacities associated with Sobolev spaces, the p-capacity cap_p in particular, and the Lebesgue measure satisfy the compatibility condition.

Moreover, the proof does not make use of the subadditivity of the capacity.

Proof. Fix $\varepsilon > 0$ and choose an open set G such that $\mathscr{C}(G) < \varepsilon$ and that the restrictions to $X \setminus G$ of f and g are continuous. Thus, by topology, there is an open subset U of X with

$$U \setminus G = \{x \notin G : f(x) \neq g(x)\}.$$

Since both U and G are open and since $\mu(U \setminus G) = 0$, the compatibility condition implies that $\mathscr{C}(G \cup U) = \mathscr{C}(G) < \varepsilon$. The theorem follows since

$$\{x : f(x) \neq g(x)\} \subset G \cup \{x \notin G : f(x) \neq g(x)\} = G \cup U. \quad \square$$

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