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MAPPING PROPERTIES OF FATOU COMPONENTS

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Abstract. If $f(z)$ is meromorphic in $\mathbf{C}$ and N_1 and N_2 are components of the Fatou set of $f(z)$ such that $f(z)\colon N_1 \to N_2$, it is shown that $D = N_2 \setminus f(N_1)$ is a set which contains at most two points. If $f(z)$ is entire then D contains at most one point. Examples show that these results are sharp and also that the points of D are in general neither Picard exceptional nor Nevanlinna deficient.

1. Introduction

Let $f(z)\colon \mathbf{C}$ (or $\widehat{\mathbf{C}}$) $\to \widehat{\mathbf{C}}$ be a non-constant meromorphic function which is not a Möbius transformation. We define the the n^{th} iterate of $f(z)$ at z as $f^n(z)$, where $f^1 = f(z)$ and $f^{n+1}(z) = f\big(f^n(z)\big)$. The subset of the sphere such that $\{f^n\}$, $n \in \mathbf{N}$, is defined meromorphically on some open neighbourhood, U of z and forms a normal family in U is termed the Fatou set of $f(z)$, $F(f)$, and consists of a countable union of connected open components. The complement of $F(f)$ in $\widehat{\mathbf{C}}$, $J(f)$ is termed the Julia set of $f(z)$ and is non-empty and perfect.

Non-rational functions are not defined continuously at transcendental singularities and so these points must lie in $J(f)$.

The Fatou set has the property of complete invariance, that is $z \in F(f)$ if and only if $f(z) \in F(f)$; for rational $f(z)$, $J(f)$ has this property whilst for non-rational $f(z)$, $z \in J(f) \setminus \{\infty\}$ if and only if $f(z) \in J(f)$.

Let N_1 be any component of $F(f)$. Then $f(N_1)$ is a connected subset of $F(f)$ and so lies inside some component of $F(f)$, say N_2.

If $f(z)$ is rational then there are at worst finitely many algebraic singularities of f^{-1} in N_2, and so $f(z)\colon N_1 \to N_2$ is a finite branched cover. However this need not be the case if $f(z)$ is non-rational. For instance, consider the function $f(z) = e^z - 1$. Let $H = \{z : \operatorname{Re}(z) < 0\}$. Then $f(H) \subset H$ and so by Montel's theorem $\{f^n\}$, $n \in \mathbf{N}$, forms a normal family in H which is then contained in some (forward invariant) component of $F(f)$, say N. Now $-1 \in H$ but -1 is a Picard omitted value of $f(z)$ and so $F(N) \subset N$ but -1 is not in $f(N)$.

It is already known (e.g. [4]) that if $f(z)\colon N_1 \to N_2$, then $f(N_1)$ is open and dense in N_2, but nothing further seems to be known about the set $N_2 \setminus f(N_1)$. Our main result is

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Theorem 1. *If $f(z)$ is meromorphic in* $\mathbf{C}$ *and* $f(z)\colon N_1 \to N_2$*, where* N_i *are components of* $F(f)$*, then* $|N_2 \setminus f(N_1)| \le 2$.

Theorem 2. *If* $a \in N_2 \setminus f(N_1)$*, then* a *is admitted as an asymptotic value of* $f(z)$ *along some path* $\gamma \subset N_1$ *which runs to* ∞.

We then have two obvious corollaries.

Corollary 1. *If* N_1 *is bounded, then* $f(z)\colon N_1 \to N_2$ *is a finite branched cover.*

Corollary 2. *If* $f(z)$ *does not admit asymptotic values, then for all components* N_1 *of* $F(f)$, $f(z)\colon N_1 \to N_2$ *is surjective.*

In [7, pp. 7–11], a family of meromorphic functions which do not admit asymptotic values is introduced.

Theorem 3. *If* $f(z)$ *is transcendental and* $F(f)$ *is connected, then each point of* $F(f) \setminus f\bigl(F(f)\bigr)$ *is a Picard exceptional value of* $f(z)$*, and so the latter set contains at most two points.*

We are able to prove more under assumptions on the structure of N_1:

Theorem 4. *Let* $f(z)$ *be transcendental meromorphic in* $\mathbf{C}$ *such that* $F(f)$ *contains at least two components, with* N_1 *unbounded and let* Γ *be the component of* ∂N_1 *which contains* ∞*. Then if* $\partial N_1 \setminus \Gamma$ *is bounded in* $\mathbf{C}$, $|N_2 \setminus f(N_1)| \le 1$.

Corollary 3. *If* $f(z)$ *is transcendental entire and* N_1 *is unbounded, then* N_1 *and* N_2 *are simply-connected and* $|N_2 \setminus f(N_1)| \le 1$.

Corollary 4. *If* $f(z)$ *is a non-entire transcendental meromorphic function, and* N_1 *is of finite connectivity* ≥ 2*, then* $|N_2 \setminus f(N_1)| \le 1$.

It has been shown in [5] that for non-entire meromorphic functions $F(f)$ may have unbounded components of arbitrary connectivity.

Theorem 5. *Let* $f(z)$ *be a non-entire transcendental meromorphic function, with* N_1 *of connectivity at most two. Then if* N_2 *is simply-connected,* $|N_2 \setminus f(N_1)| \le 1$ *whilst if* N_2 *is multiply-connected then* $f(N_1) = N_2$.

We may also investigate this problem for holomorphic self maps of the punctured plane. We need only consider transcendental non-entire functions which as in [3] are conjugate to

$$f(z) = z^k \exp\left(g(z) + h\left(\frac{1}{z}\right)\right)$$

where g, h are entire, $k \in \mathbf{Z}$ (if k is non-negative, then h is non-constant). It is shown in the same paper that then the connectivity of any component of $F(f)$ is at most two, there being at most one multiply-connected component.

We then have:

Theorem 6. *Let $f(z)$ be a transcendental non-entire self map of $\mathbf{C}^* = \mathbf{C}\setminus\{0\}$ and $f(z)\colon N_1 \to N_2$ where N_1 is a component of $F(f)$. Then if N_2 is simply-connected $|N_2 \setminus f(N_1)| \le 1$ whilst if N_2 is multiply-connected (doubly) then $f(N_1) = N_2$.*

Note that if $f(z)$ is as above and $F(f)$ consists of a single component, then by Theorem 3, $f(z)\colon F(f) \to F(f)$ must be surjective as the two Picard exceptional values of $f(z)$ lie in $J(f)$.

In Section 4 we shall give various examples to show the sharpness of Theorems 1, 3, and 4 (with its corollaries). The obvious way to construct examples is by using functions, such as $\tan z$, which have Picard exceptional values. We shall show, however, that a value in $N_2 \setminus f(N_1)$, although asymptotic need not be a Picard exceptional value, nor even Nevanlinna deficient. In particular, in Section 4, Example 4 for the entire function

$$f(z) = \int_0^z \exp(-e^t)\,dt,$$

$F(f)$ contains a sequence of different components N_i, $i = 1, 2, \ldots$, such that $f(N_i) \subset N_i$, and each $N_i \setminus f(N_i)$ is a singleton $\{a_i\}$, where the value a_i has Nevanlinna deficiency $\delta(a_i, f) = 0$.

2. Results needed for the proofs

Consider a domain D and let $f(z)$ be a function single valued and meromorphic in D, where we assume that the complement of D in $\widehat{\mathbf{C}}$ has non-empty interior. By a spherical rotation, this is equivalent to supposing that D is bounded. When we refer to the capacity of a given set this should always be taken to mean the logarithmic capacity. For a non-isolated point z_0 of $\partial D = \Gamma$ we define

$$C_D(f, z_0) = \{\alpha : \exists z_n \in D,\ z_n \to z_0,\ f(z_n) \to \alpha\}.$$

Suppose further that E is a closed subset of Γ of capacity zero and $z_0 \in E$ is a limit point of $\Gamma \setminus E$. We define

$$C_{\Gamma\setminus E}(f, z_0) = \bigcap_{r>0} \overline{\bigcup_{z\in\Gamma\cap D_r,\, z\notin E} C_D(f, z)},$$

where D_r is a disc of spherical radius r centred at z_0. We denote by $\Omega_{D,E}(f, z_0)$ the open set

$$C_D(f, z_0) \setminus C_{\Gamma\setminus E}(f, z_0).$$

With these assumptions we have:

Lemma 1 [14, Theorem VIII.39, p. 332]. *If* $z_0 \in E$, *then* $\partial C_D(f, z_0) \subset C_{\Gamma \setminus E}(f, z_0)$.

Lemma 2 [14, Theorem VIII.41, p. 335]. *Let* Ω *be a connected component of* $\Omega_{D,E}(f, z_0)$, *where* $z_0 \in E$. *Then in any neighbourhood of* z_0 *in* D, $f(z)$ *takes any value of* Ω *infinitely often with the possible exception of at most a set of capacity zero. If the set* $E = \{z_0\}$, *then the exceptional set above contains at most two points.*

Lemma 3 [12, Theorem 10, p. 25]. *Let* D *be a simply-connected domain which is of hyperbolic type and* $z_0 \in E$ *as above. If* $C_D(f, z_0)$ *is not the entire sphere then for any connected component* Ω *of* $\Omega_{D,E}(f, z_0)$, $f(z)$ *takes any value of* Ω *infinitely often in a neighbourhood of* z_0 *with at most one exception.*

Lemma 4 [8, Theorem 3.5, p. 56]. *Let* $w = f(z)$ *be analytic and univalent in the disc* D. *Then the radial limits* $f(e^{i\theta}) = \lim_{r\to 1-} f(re^{i\theta})$ *exist on* ∂D *outside a set of capacity zero, and can be constant only on sets of capacity zero.*

Lemma 5 [8, Theorem 5.14, p. 109]. *Let* $w = f(z)$ *be analytic and bounded,* $|w| < 1$ *in* $|z| < 1$ *and suppose that*

$$\lim_{r\to 1-} |f(re^{i\theta})| = 1$$

outside a set of capacity zero. Then either $f(z)$ *reduces to a finite Blaschke product or* $f(z)$ *assumes every value in* $|w| < 1$ *infinitely often with at most one exception.*

Lemma 6. *If* N *is a simply or doubly-connected domain,* D *is the unit disc and* $\Phi\colon D \to N$ *is the uniformizing map then* $\Phi(e^{i\theta}) = \lim_{r\to 1-} \Phi(re^{i\theta})$ *exists outside a set of capacity zero on* ∂D. *Given a value* α, *then* $\Phi(e^{i\theta})$ *takes the value* α *at most on a set of capacity zero.*

Proof. Because of Lemma 4 we may assume that N is doubly-connected. Let α, β be finite values which lie in different components of ∂N and put

$$\Psi = \log \frac{\Phi - \alpha}{\Phi - \beta} \colon D \to \mathbf{C},$$

where we begin by choosing some $z \in D$ and some branch of $\log\{(\Phi(z) - \alpha)/(\Phi(z) - \beta)\}$. By the monodromy theorem the continuation of this branch gives a single-valued analytic function Ψ in D. In fact, Φ gives a conformal map of D to the universal cover of N and the log maps this universal cover univalently to a simply-connected region B. Thus $\Psi\colon D \to B$ is univalent. Hence by Lemma 4 the radial limit $\Psi(e^{i\theta})$ exists except for a set E of capacity zero, and for any $c \in \widehat{\mathbf{C}}$, $E_c = \{e^{i\theta} : \Psi(e^{i\theta})$ exists and $= c\}$ is a set of capacity zero for $e^{i\theta} \notin E \cup E_\infty$,

$e^{i\theta}$ lies in E_c for some finite c which implies that $\Phi(e^{i\theta}) = T(e^c) \neq \alpha, \beta$, where $T(w) = (\alpha - \beta w)/(1-w)$, and thus $e^c \neq 0, \infty$.

For a given u the set $\{e^{i\theta} : \Phi(e^{i\theta}) = u\}$ is a subset of $\cup E_{c_n} \cup E \cup E_\infty$, where $e^{c_n} = T^{-1}(u)$, which is a union of countably many sets of capacity zero and hence has capacity zero.

3. Proofs

Proof of Theorem 2. (See [4, p. 242]). A brief sketch may be useful. This follows from the Star theorem of Gross (see e.g. [11, p. 287]). If $z_0 \in N_1$, $w_0 = f(z_0) \in N_2$ and g is the branch of f^{-1} such that $g(w_0) = z_0$, the Star theorem ensures that we can continue g analytically along some polygonal path δ in N_2 up to a point $\beta \in D(\alpha, \epsilon) \subset N_2$ and then along some circular arc δ' in $D(\alpha, \epsilon)$ which goes from β up to α. Then $g(\delta \cup \delta') = \gamma$ is in N_1 by the complete invariance of $F(f)$ and γ goes to ∞ since $\alpha \notin f(N_1)$. i.e. the continuation is not possible over α.

Proof of Theorem 3. Suppose that $F(f)$ is a single connected component. Then if $U_r = \{z : r < |z| < \infty\}$, by Picard's theorem $f(U_r)$ covers each point of the complex sphere infinitely often with the exception of at most two points. By the complete invariance of $F(f)$, $f\big(U_r \cap F(f)\big)$ then covers each point of $F(f)$ infinitely often with at most two exceptions and the result follows.

Proof of Theorem 1. Since Theorem 3 already covers the case when $F(f)$ has a single component we assume that there are at least two components. Thus the complement of N_1 has a non-empty interior and we may apply Lemmas 1 and 2 with $D = N_1$.

Suppose that $\alpha \in N_2 \setminus f(N_1)$. Then α is admitted as an asymptotic value of $f(z)$ along a path γ to ∞ in N_1. Thus $\alpha \in C_{N_1}(f, \infty)$. Now

$$\begin{aligned} C_{\partial N_1 \setminus \{\infty\}}(f, \infty) &= \bigcap_{r>0} \overline{\bigcup_{v \in \partial N_1 \cap D_r,\, v \neq \infty} C_{N_1}(f, v)}, \quad D_r = \{z : |z| > r\} \\ &= \bigcap_{r>0} \overline{\bigcup_{v \in \partial N_1 \cap D_r,\, v \neq \infty} f(v)}, \end{aligned}$$

which is contained in ∂N_2. Thus

$$\Omega_{N_1, \infty}(f, \infty) = C_{N_1}(f, \infty) \setminus C_{\partial N_1 \setminus \{\infty\}}(f, \infty)$$

is non-empty since it contains $\alpha \in N_2$. From Lemma 1

$$\partial C_{N_1}(f, \infty) \subset C_{\partial N_1 \setminus \{\infty\}}(f, \infty) \subset J(f)$$

and so there must be a connected component Ω of $\Omega_{N_1, \{\infty\}}(f, \infty)$ which contains N_2. From the second part of Lemma 2 we see that each point of Ω is assumed infinitely often by $f(z)$ in N_1 with at most two exceptions. Thus $|N_2 \setminus f(N_1)| \leq 2$ as claimed.

Proof of Theorem 4. Let Γ be the component of ∂N_1 which contains ∞. If Γ were a point then it would have to be a limit point of bounded components of ∂N_1, which contradicts the hypothesis, and so Γ must be a non-degenerate continuum. If we let M be the component of $\widehat{\mathbf{C}} \setminus \Gamma$ which contains N_1, then M must be simply-connected, and as $J(f)$ is infinite M must be of hyperbolic type.

Suppose that $\alpha \in N_2 \setminus f(N_1)$, which is then admitted as an asymptotic value on some path to ∞ in N_1, say γ. Then $\alpha \in C_M(f,\infty)$. As in the previous proof we see that $C_{\Gamma\setminus\{\infty\}}(f,\infty) \subset \partial N_2$, and $\partial C_M(f,\infty) \subset C_{\Gamma\setminus\{\infty\}}(f,\infty)$ so that $\Omega_{M,\{\infty\}}(f,\infty)$ contains α and has a connected component Ω which must contain N_2.

Now $C_M(f,\infty) \subset \overline{N_2} \neq \widehat{\mathbf{C}}$. By using Lemma 3 we see that with the possible exception of a singleton, each value in Ω is taken infinitely often by $f(z)$ in any neighbourhood of ∞ in M. Under the hypothesis we may choose such a neighbourhood which lies entirely within N_1 and so $|N_2 \setminus f(N_1)| \le 1$.

Proof of Corollary 3. If $f(z)$ is transcendental entire and $F(f)$ is connected, then this follows from Theorem 3. Otherwise $F(f)$ has at least two components, N_1 is unbounded and so is simply-connected (cf. [1]), in which case $\partial N_1 \setminus \Gamma$ is empty and the result follows from Theorem 4. Note that if N_2 were multiply-connected then by [2, Theorem 3.1] N_1 would be bounded which is a contradiction.

Proof of Corollary 4. If N_1 is of finite connectivity ≥ 2 then by [6, Theorem 3.1] N_1 is not completely invariant and so there must be other components of $F(f)$. The result then follows directly from Theorem 4.

Proof of Theorem 5. Let $\psi_i\colon D \to N_i$ be the uniformizing maps of N_1, N_2; then by Lemma 6, $\psi_1(e^{i\theta})$ exists and is finite outside a set E of capacity zero. Pick $e^{i\theta} \notin E$, then $\psi_1(e^{i\theta}) = \alpha \in \partial N_1 \setminus \{\infty\}$ and $f\psi_1(e^{i\theta}) = f(\alpha) \in \partial N_2$. If λ is any branch of ψ_2^{-1} then we may continue $F = \lambda f \psi_1$ to an analytic function in D, and by the properties of the map ψ_2 we see that $|F(re^{i\theta})| \to 1$ as $r \to 1^-$. Then by Lemma 5 we see that F is either a finite Blaschke product, in which case $F(D) = D$ and $f(N_1) = N_2$, or $F(z)$ takes every value D infinitely often with at most one exceptional point, so that $N_2 \setminus f(N_1)$ is at most a singleton. If however N_2 is multiply-connected then given any $w \in N_2$, there are infinitely many points p_n such that $\psi_2(p_n) = w$ with at most one of the p_n not taken by $F(z)$, so that $f(N_1) = N_2$.

Proof of Theorem 6. If $f(z)$ is a self map of the punctured plane then the components of $F(f)$ are of connectivity at most two and so we may use the proof of Theorem 5 to show the claim.

This concludes the proofs of the theorems. There now follows a series of examples which demonstrate the sharpness of Theorem 3 and Corollaries 3 and 4.

4. Examples

Example 1. Consider the meromorphic function $f(z) = \tan z$. Then from [9, p. 61] we see that $J(f) = \mathbf{R} \cup \{\infty\}$ and $F(f)$ consists of two completely invariant components H^+ and H^-, the upper and the lower planes respectively. Now $f(z) \neq \pm i$ in $\mathbf{C}$ and so $H^+ \setminus f(H^+) = \{i\}$ and so the bound in Corollary 4 is sharp.

Example 2. Consider $f(z) = \frac{1}{2}\tan z$. From [9, pp. 62–63] we see that $J(f)$ is an unbounded totally disconnected subset of $\mathbf{R}$, so $F(f)$ is a single component. Also $f(z) \neq \pm\frac{1}{2}i$ in $\mathbf{C}$, so that $F(f) \setminus f\big(F(f)\big) = \big\{\pm\frac{1}{2}i\big\}$ and the bounds in Theorems 1 and 3 are sharp.

The question of whether the case $|N_2 \setminus f(N_1)| = 2$ can occur when $F(f)$ is not connected remains open.

Example 3. Consider

$$f_p(z) = \int_0^z e^{-t^p}\,dt = z - \frac{z^{p+1}}{p+1} + \cdots, \qquad p \geq 2. \tag{1}$$

From (1), zero is a parabolic fixed point of f with multiplicity $p+1 \geq 3$. It is then easy to see that for $x \in \mathbf{R}^+$, $z \in \mathbf{C}$ the following properties of f_p hold:

$$f_p(e^{2\pi \upsilon i z/p}) = e^{2\pi \upsilon i/p} f_p(z), \quad \upsilon = 1, 2, \dots, p, \tag{2}$$

$$f_p(x) \leq x, \tag{3}$$

$$f_p(\mathbf{R}^+) \subseteq \mathbf{R}^+. \tag{4}$$

Note also that $f'(z) = e^{-z^p}$ which is non-zero in the finite plane, so that f^{-1} has no finite branch points.

Define sets A_υ, $\upsilon = 1, 2, \dots, p$, by

$$A_\upsilon = \left\{ z : \left| \arg z - \frac{2\upsilon\pi}{p} \right| \leq \frac{\pi}{2p} \right\},$$

and non-zero constants

$$\alpha_\upsilon = e^{2\pi\upsilon i/p} \int_0^\infty \exp(-r^p)\,dr.$$

Then by [11, p. 267] we have that as $z \to \infty$, $z \in A_\upsilon$, $f_p(z) \to \alpha_\upsilon$ whilst as $z \to \infty$ in the rest of the plane $|f(z)|$ is unbounded, so the only finite transcendental singularities of f_p^{-1} are $\alpha_1, \dots, \alpha_p$.

The points ∞ and α_υ have Nevanlinna deficiencies $\delta(\infty) = 1$ and $\delta(\alpha_\upsilon) = 1/p$, so that each α_υ is non-Picard exceptional.

We now claim that there are p disjoint, simply-connected components of $F(f_p)$, say N_v, $v = 1, 2, \dots, p$, such that:

$$f(N_v) \subseteq N_v, \qquad e^{2\pi vi/p}\mathbf{R}^+ \subseteq N_v.$$

For, by using the Leau–Fatou Flower Theorem (cf. [10, Theorem 7.2, p. 45] we see that there are p disjoint forward invariant components N_v of $F(f)$ such that zero lies in each ∂N_v and that a segment $[0, \epsilon]$, $\epsilon > 0$ of $\mathbf{R}^+$ lies in one N_v, say N_p. From (3) and (4) we then see that $f_p^n|_{\mathbf{R}^+} \to 0$ as $n \to \infty$ and $\mathbf{R}^+$ is contained in N_p. Using this fact and (2) we may label each N_v so that N_v contains $e^{2\pi vi/p}\mathbf{R}^+$, which is then unbounded and so simply-connected (cf. [1]).

Now each $\alpha_v \in e^{2\pi vi/p}\mathbf{R}^+ \subset N_v$. Suppose that $\alpha_v \in f_p(N_v)$. Then there exists some $\beta \in N_v$ with $f_p(\beta) = \alpha_v$ and some branch g of f_p^{-1} defined analytically on some open neighbourhood of β with $g(\alpha_v) = \beta$.

Continuation of g along any path γ in N_v which does not return to α is clearly possible. If $U = D(\alpha_v, r) \subset N_v$ so that g is analytic in U, suppose γ leaves $\overline{U}$ at z_1 and returns a first time to $\overline{U}$ at z_2. If σ is the circumference of U and τ is the arc from z_1 to z_2 on ∂U then γ is homotopic in $N_v \setminus \{\alpha_v\}$ with fixed ends z_1, z_2 to $\tau\sigma^n$ for some integer n. Thus on analytic continuation on γ, g returns to the same branch and remains analytic at α_v. But this is impossible, since in a parabolic basin such as N_v the function f_p cannot be univalent.

So $\alpha_v \notin f(N_v)$, and we have a function f_p demonstrating the sharpness of Corollary 3. Further the exceptional points of α_v are not Picard exceptional.

We now show that exceptional points do not have to be even Nevanlinna exceptional.

Example 4. Consider the transcendental entire function $f(z)$ defined as

$$f(z) = \int_0^z \exp(-e^t)\,dt = \int_0^z \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} e^{nt}\,dt = z + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!n}(e^{nz} - 1). \tag{6}$$

From (6), we see that for all $z \in \mathbf{C}$

$$f(z + 2\pi i) = f(z) + 2\pi i,$$

and so in particular, $f(2k\pi i) = 2k\pi i + f(0) = 2k\pi i$, $k \in \mathbf{Z}$. Thus each point $2k\pi i$ is a fixed point of $f(z)$, and as $f'(z) = \exp(-e^z)$, $f'(2k\pi i) = e^{-1} < 1$, it is actually an attracting fixed point of $f(z)$. As such, each $2k\pi i$ lies in $F(f)$.

Suppose that $z = x + iy$, $x > 0$, $|y| \le \frac{1}{2}\pi$. Evaluating $f(z)$ along the paths $\gamma_1 : \operatorname{Im} z = 0$, $\operatorname{Re} z = t \in [0, x]$; $\gamma_2 : \operatorname{Re} z = x$, $\operatorname{Im} z = t \in [0, y]$ we get

$$f(z) = \int_0^x \exp(-e^t)\,dt + \int_0^y i\exp(-e^{x+it})\,dt.$$

As $x > 0$ we have that

$$\alpha(x) = \int_0^x \exp(-e^t)\,dt < \int_0^x e^{-t}\,dt,$$

so that $\alpha(x) \to \alpha \in [0,1]$ as $x \to \infty$. As $|y| \le \frac{1}{2}\pi$, if $0 \le t \le |y|$, then $\sin \ge 2t/\pi$, so that

$$\begin{aligned}\left|\int_0^y i\exp(-e^{x+it})\,dt\right| &\le \left|\int_0^y \exp(-e^{x\cos t})\,|dt|\right| = \int_0^{|y|} \exp(-e^{x\cos t})\,dt \\ &\le \int_0^{\pi/2} \exp(-e^{x\cos t})\,dt = \int_0^{\pi/2} \exp(-e^{x\sin s})\,ds \\ &\le \int_0^{\pi/2} \exp\left(\frac{-2t}{\pi}e^x\right)dt \\ &= \tfrac{1}{2}\pi(e^{-x} - e^{-x}e^{-e^x}) \to 0 \quad \text{as } x \to \infty.\end{aligned}$$

Thus if $x \to \infty$, $|y| \le \frac{1}{2}\pi$ then

$$f(z) = f(x+iy) \to \int_0^\infty \exp(-e^t)\,dt = \alpha\ (\ne 0),\ \text{say}.$$

Now for $k \in \mathbf{Z}$, set

$$A_k = \{z : \operatorname{Re} z > 0,\ |\operatorname{Im} z - 2k\pi| \le \tfrac{1}{2}\pi\}.$$

We have from (7) that as $z \to \infty$, with z lying in some A_k, that

$$f(z) \to \alpha + 2k\pi i = \alpha_k,\ \text{say}.$$

We claim that $|f(z)|$ is unbounded as $z \to \infty$ on the following paths:
(8) $\gamma_k : \mathbf{R}^+ + (2k+1)\pi i$, $k \in \mathbf{Z}$.
(9) Any path γ: $[0,1] \to \widehat{\mathbf{C}}$ such that $\operatorname{Im}\gamma(t)$ is unbounded.
(10) Any path γ: $[0,1] \to \widehat{\mathbf{C}}$ such that $\operatorname{Re}\gamma(t) < x_0$ for sufficiently large t.

To show (8), if $z = x + (2k+1)\pi i$, $k \in \mathbf{Z}$, $x > 0$, then

$$\begin{aligned}f(z) &= \int_0^{(2k+1)\pi} i\exp(-e^{it})\,dt + \int_0^x \exp\bigl(-e^{t+(2k+1)\pi i}\bigr)\,dt \\ &= O(1) + \int_0^x \exp(e^t)\,dt\end{aligned}$$

so that $|f(z)| \to \infty$ as $|x| \to \infty$, and (8) is shown.

To show (9), suppose that γ is any such path to ∞. We may find values $z_k = x_k + 2\pi ki \in \gamma$ for an unbounded sequence of integers k and $x_k \in \mathbf{R}$. Then $f(z_k) = f(x_k) + 2\pi ki$. As $f(z)$ is real on the real line, $|f(z)| \geq 2\pi|k| \to \infty$ as $|k| \to \infty$.

In proving (10) we may now assume because of (9) that $\operatorname{Im}\gamma(s)$ is bounded with $\operatorname{Re}\gamma(s) \to -\infty$, and $|\operatorname{Im}\gamma(s)| < y_0$, say. Then if $z = x + iy$ lies on γ,

$$f(z) = \int_0^x \exp(-e^t)\,dt + \int_0^y i\exp(-e^{x+it})\,dt.$$

Now $t < 0$, so $e^t < 1$, $e^{-e^t} > e^{-1}$ and then

$$\left|\int_0^x \exp(-e^t)\,dt\right| > |x|e^{-1} \to \infty$$

and

$$\begin{aligned}\left|\int_0^y \exp(-e^{x+it})\,dt\right| &\leq \left|\int_0^y \exp(-e^{x\cos t})\,|dt|\right| \\ &\leq \left|\int_0^y \exp(e^x)\,|dt|\right| \leq |y_0|\exp(e^x) \to |y_0| \quad \text{as } x \to -\infty.\end{aligned}$$

Thus as $z \to \infty$ on γ, $|f(z)| \to \infty$, and the claims (8)–(10) above are shown. It is clear that f' is non-zero in the finite plane, and so f^{-1} has no finite algebraic singularities. We now claim that the finite asymptotic values of $f(z)$ are the points $\{a_k\}$. Suppose not, and so there is a path $\gamma \to \infty$ so that as $z \to \infty$ on γ, $f(z) \to a \in \mathbf{C} \setminus \{\alpha_k\}$, $k \in \mathbf{Z}$. By claims (8)–(10) we see that γ must eventually lie in the strip S in the right half-plane between an A_k and one of the lines l_k, l_{k-1}; without loss of generality we may assume that it lies between A_k and l_{k-1}, so $S = \operatorname{Re}(z) > 0$, $(2k-1)\pi \leq \operatorname{Im}(z) \leq \bigl(2k - \frac{1}{2}\bigr)\pi$. We may then take the line $\delta : \operatorname{Im}(z) = \bigl(2k - \frac{1}{2}\pi\bigr)$ contained in A_k and join γ and δ by a vertical line l, so that these paths together with ∞ enclose a region R contained inside the horizontal strip S.

Note that there exists $A > 0$ such that $|f(z)| \leq \exp(e^{|z|})|z| < \exp(Ae^{|z|})$ for all large enough $|z|$.

Clearly $|f|$ is bounded on ∂R, $|f| < M$, say. Consider the function g, where

$$g(z) = \exp\bigl(\epsilon\exp\bigl(m\{z - (2k - \tfrac{3}{4}\pi i)\}\bigr)\bigr), \qquad \epsilon > 0,\ \sqrt{2} < m < 2.$$

Then for $z \in R$,

$$\begin{aligned}|g(z)| &= \exp\bigl(\epsilon(\exp mx)\cos\bigl(my - m(2k - \tfrac{3}{4})\bigr)\bigr) \\ &> \exp\bigl(\epsilon\bigl(\exp(mx)\cos(m\tfrac{1}{4}\pi)\bigr)\bigr) > 1,\end{aligned}$$

since $\sqrt{2} < m < 2$. Thus $|f/g| < M$ on ∂R, and as $x \to \infty$ in R,

$$|f/g| < \exp\{Ae^{x(1+v(x))} - \epsilon \cos(\tfrac{1}{4}m\pi)e^{mx}\}$$

where $v(x) = (|z|/x) - 1 \to 0$ as $x \to \infty$. Hence we see that $f/g \to 0$ as z goes to ∞ in R. We now choose a $z \in R$ and take a curve $\Gamma \subseteq R$ which connects γ and δ so that z lies in the finite part R_0 of R cut off by Γ. Using the above and the maximum principle we see that, so long as Γ is chosen far enough to the right then $|f/g| < M$ on Γ, and ∂R_0, so $|f/g| < M$ in R_0.

So $|f(z)| < M|g(z)|$, and as ϵ is arbitrary and z fixed, we see that $|f(z)| < M$. However the choice of z was arbitrary and so this is true of all $z \in R$. We then use [13, Theorem 340, p. 172] to see that the asymptotic values a, α_k are equal, and so the only finite asymptotic values of f are the points α_k which are then exactly the finite transcendental singularities of f^{-1}.

The positive real axis, $\mathbf{R}^+$, is mapped into itself by $f(z)$ which has an attracting fixed point at 0, and on $\mathbf{R}$, $f' < 1$, so by Rolle's theorem, $f(x) = x$ has no real solutions other than 0. Also, for $x > 0$ (< 0), $f(x) < 0$ (> 0), and so we may then see that $f^n \to 0$ on $\mathbf{R}$ as $n \to \infty$, so $\mathbf{R}$ lies in an unbounded (and so simply-connected) forward invariant component N_0 of $F(f)$.

Using (7) we see that each $\mathbf{R} + 2k\pi i$ lies in some simply connected, forward invariant component N_k of $F(f)$ which has an attracting fixed point of $f(z)$ at $2k\pi i$, so that the N_k must be disjoint. We then use exactly the same reasoning as in Example 3, to see that each $\alpha_k \in N_k \setminus f(N_k)$.

It is shown by [11, p. 290] that the points α_k are non-Nevanlinna exceptional points of $f(z)$, and so the claim at the end of the introduction is shown.

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