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A FUNDAMENTAL DOMAIN FOR THE MODULAR GROUP OF RIEMANN SURFACES OF TYPE (0,n)

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Abstract. In this paper we introduce the concept of extreme loop structures, and basing on it we present an explicit fundamental domain for the action of modular group in the deformation space of Riemann surfaces of genus 0 with n $(n \geq 4)$ boundary components.

Introduction

There have been several well-known ways of parametrizing the deformation space of a surface: for example, Fricke cooordinates, Teichmüller coordinates and Fenchel–Nielsen coordinates. Though these parametrizations provide an overview of all possible Riemann surfaces, some basic questions remain unsolved. We cannot decide whether two Riemann surfaces, when expressed by the parameters, are isometric or not. This is mainly a problem of marking a surface in different ways, then read off the parameters to get a hold of the action of the modular group on the deformation space.

The present work is an attempt towards an understanding of this problem. We present an explicit fundamental domain for the action of the modular group on the deformation space of Riemann surfaces of genus zero with $n = m+1$ $(m \geq 3)$ boundary components. Similar exploration for the topological types $(0,4)$ and $(1,1)$ were considered in [M1] and [M2].

In Section 2, we introduce the concept of extreme loop structures. An extreme loop structure is mapped to an extreme loop structure under a conformal mapping, and with certain restricting conditions, it is unique. Even though an extreme loop structure is defined only by finitely many length inequalities, it has a certain global control ability (Theorem 1). Moreover, one can write an algorithm to search all extremal loop structures on a Riemann surface of type $(0,n)$.

As a natural corollary, the inequalities that define our extreme loop structure yield the required fundamental domain.

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1. Preliminaries

Throughout this paper G will denote a Fuchsian group acting on the upper half plane H and $S=H/G$ a Riemann surface of genus zero with $m+1$ ($m\geq 3$) boundary components $A_0, A_1, \dots, A_m$.

Assume that α and β are two closed curves on S. Their geometric intersection number $\#(\alpha,\beta)$ is the infimum of the cardinality of $\alpha'\cap\beta'$ for all $\alpha'\sim\alpha$, $\beta'\sim\beta$ where $\sim$ denotes homotopy. This infimum is realized by $\alpha'\cap\beta'$ if $\alpha'\sim\alpha$, $\beta'\sim\beta$ and both α' and β' are distinct closed geodesics.

Let l be a closed curve on S. If $g\in G$ has the property that any curve in H with z and $g(z)$ as its terminal points is projected to a closed curve on S which is freely homotopic to l, then we say that g covers l. Suppose that g covers l and $l'\sim l$, then g also covers l'. Moreover, the set $\{g'\circ g\circ g'^{-1} : g'\in G\}$ contains all elements in G which cover l. Assume that

$$g(z)=\frac{az+b}{cz+d}, \qquad ad-bc=1.$$

Define $\mid g\mid=\mid a+d\mid$. Note that $|l|=2\log\big(\sqrt{|g|^2-4}+|g|\big)-\log 4$ is the hyperbolic length of the shortest geodesic in the homotopy class of l.

Provide l with a certain orientation, and let x and y be two distinct points on l. We denote by $l(x,y)$ the path on l which starts at x, follows the orientation, and ends at y. Suppose that l_1 and l_2 are two oriented curves such that l_2 starts at the terminal point of l_1. Then l_2+l_1 will represent the curve which starts at the starting point of l_1, goes along l_1 and l_2, and ends at the terminal point of l_2.

Fix a point $\tilde{o}$ on S and choose a set of generators $\{\alpha_i : i=0,1,\dots,m\}$ of the fundamental group $\pi(S,\tilde{o})$ with the following properties: for each i, α_i is freely homotopic to A_i, $\alpha_j\circ\alpha_{j-1}\circ\cdots\circ\alpha_0$ ($0<j<m$) are simple, and $\alpha_m\circ\cdots\circ\alpha_1\circ\alpha_0\sim\tilde{o}$. By a well-known lifting procedure, we get the corresponding generators g_0, $g_1,\dots,g_m$ of G, with g_j ($0\leq j\leq m$) covering α_j, $g_0\circ g_1\circ\cdots\circ g_j$ ($0<j<m$) covering $\alpha_j\circ\alpha_{j-1}\circ\cdots\circ\alpha_0$, and $g_0\circ g_1\circ\cdots\circ g_m=\mathrm{id}$. The generators g_j ($j=0,1,\dots$) are called standard generators. See Figure 1, where $m=3$.

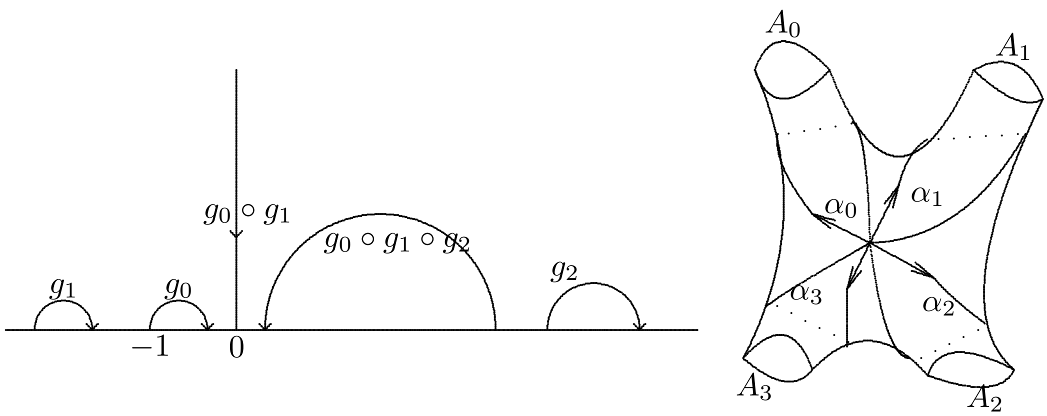

Figure 1.

Let $\mathscr{M}$ be the set of all Möbius transformations fixing H. A deformation of G is a monomorphism ψ: $G \mapsto \mathscr{M}$ for which there is an orientation preserving homeomorphism φ: $H \mapsto H$, with $\varphi \circ g \circ \varphi^{-1} = \psi(g)$, for all $g \in G$. We may write $\psi = \bigl(\psi(g_j)\bigr)$. Two deformations ψ_1 and ψ_2 are equivalent if there exists an element $A \in \mathscr{M}$, with $A \circ \psi_1(g) \circ A^{-1} = \psi_2(g)$ for all $g \in G$. The set of equivalence classes is the deformation space F. A deformation ψ is called modular if $\psi(G) = G$ and it acts on F in the following way: if ψ_1 is any deformation, then define $\psi(\psi_1) = \psi_1 \circ \psi$. The action defined above induces a group operation on the set M of all modular deformations, which is called the *modular group.* The quotient F/M is the moduli space, or Riemann space of conformally distinct Riemann surfaces of genus zero, with $m+1$ boundary components.

2. Extreme loop structures and the fundamental domain

Definition. Suppose that for $1 \le i \le 3$, C_i is either a boundary component of S or a simple closed curve on S. If there exist pants $P(C_1, C_2, C_3)$ on S such that C_1, C_2 and C_3 are exactly its border components, then we write $(C_1, C_2, C_3) = 1$.

Let l_j $(j = 1, 2, \dots, m)$ be simple closed curves on S with the following properties:

(1) $(A_0, A_j, l_j) = 1$, $j = 1, 2, \dots, m$;
(2) $\#(l_i, l_j) = 2$ for $i \ne j$.

Imaging A_j $(j = 1, 2, \dots, m)$ as points on a circle centered at the point A_0 with l_j as intervals lying inside it, we can give a clockwise order for $\{A_j : j = 1, 2, \dots, m\}$, provided we choose one boundary component, say A_1, as the initial element. Without loss of generality, we may assume that the indexes of $\{A_j\}$ are consistent with the order, i.e.,

$$A_1 \prec A_2 \prec \cdots \prec A_m,$$

and call $\{A_0, A_j, l_j\}_{j=1}^m$ a loop structure with the center A_0 and the initial A_1.

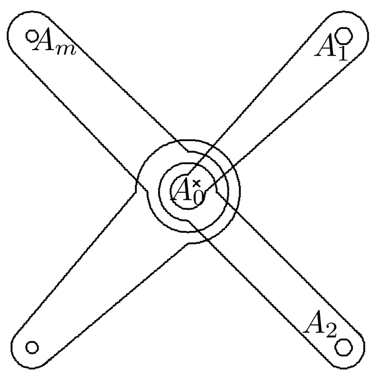

Figure 2.

Let $\{A_0, A_j, l_j\}_{j=1}^m$ and $\{A_0, A_j, l'_j\}_{j=1}^m$ be two loop structures with the center A_0 and initial A_1. We define $\{A_0, A_j, l_j\}_{j=1}^m = \{A_0, A_j, l'_j\}_{j=1}^m$ if l_j or $l_j^{-1} \sim l'_j$, $j = 1, 2, \dots, m$.

Definition. A loop structure $\{A_0, A_j, l_j\}_{j=1}^m$ is extreme if

$$(*) \qquad |l_j| \le \{|l| : (A_0, A_j, l) = 1,\ \#(l, l_i) = 2,\ i \ne j\}, \qquad \text{for } j = 1, 2, \dots, m.$$

It is easy to see that we can choose the α_j in Section 1 in such a way that they satisfy $\alpha_j \circ \alpha_0 \sim l_j$ or l_j^{-1}. We call such a set $\{\alpha_j\}$ of generators compatible with the loop structure $\{A_0, A_j, l_j\}_{j=1}^m$. Thus, if $\{\alpha_j\}$ is compatible with $\{A_0, A_j, l_j\}_{j=1}^m$, the inequalities $(*)$ can be written in the following way:

$$\begin{aligned}
|\alpha_1 \circ \alpha_0| \le \min\{&|\alpha_2 \circ \alpha_1^{-1} \circ \alpha_2^{-1} \circ \alpha_0^{-1}|, |\alpha_3 \circ \alpha_2 \circ \alpha_1^{-1} \circ \alpha_2^{-1} \circ \alpha_3^{-1} \circ \alpha_0^{-1}|,\\
&\dots, |\alpha_{m-1} \circ \dots \circ \alpha_2 \circ \alpha_1^{-1} \circ \alpha_2^{-1} \circ \dots \circ \alpha_{m-1}^{-1} \circ \alpha_0^{-1}|\};\\
&\dots\\
|\alpha_m \circ \alpha_0| \le \min\{&|\alpha_1 \circ \alpha_m^{-1} \circ \alpha_1^{-1} \circ \alpha_0^{-1}|, |\alpha_2 \circ \alpha_1 \circ \alpha_m^{-1} \circ \alpha_1^{-1} \circ \alpha_2^{-1} \circ \alpha_0^{-1}|,\\
&\dots, |\alpha_{m-2} \circ \dots \circ \alpha_1 \circ \alpha_m^{-1} \circ \alpha_1^{-1} \circ \dots \circ \alpha_{m-2}^{-1} \circ \alpha_0^{-1}|\}.
\end{aligned}$$

Lemma 1. *There exists an extreme loop structure on S with center A_0 and initial A_1.*

Proof. Since the length spectrum of closed curves on S is discrete, we see that there are $l(A_j)$ $(j = 1, 2, \dots, m)$ on S such that
(1) $\big(A_0, A_j, l(A_j)\big) = 1$, $\#\big(l(A_j), l(A_i)\big) = 2$ for $i \ne j$;
(2) $|l(A_j)| = \min\{|l| : (A_0, A_j, l) = 1,\ \#\big(l, l(A_i)\big) = 2,\ 1 \le i < j\}$.
In the way we used in defining our loop structure, we can give an order for $\{A_j\}$. If we change the indexes of $\{A_j\}$ according to the order, we see that the lemma follows.

Theorem 1. *Let $\{A_0, A_j, l_j\}_{j=1}^m$ be an extreme loop structure. Then*

$$|l_j| = \min\{|l| : (A_0, A_j, l) = 1\}, \qquad j = 1, 2, \dots, m.$$

Moreover, if all inequalities in $(*)$ hold strictly, then $\{A_0, A_j, l_j\}_{j=1}^m$ is the unique extreme loop structure on S with the center A_0 and initial A_1.

Proof. For any $1 \le j \le m$, let l be a closed geodesic satisfying $l \not\sim l_j$ and $(A_0, A_j, l) = 1$. We will show that $|l_j| \le |l|$. The assertion holds trivially if $\#(l, l_i) = 2$ for all $i \ne j$. Suppose that $\#(l, l_i) > 2$ for some $i \ne j$. Since all A_k $(1 \le k, k \ne j)$ do not lie in the pants $P(A_0, A_j, l)$, we see immediately that $P(A_0, A_j, l) \cap P(A_0, A_i, l_i)$ has $p+1$ connected components for some integer $p \ge 1$. It is not difficult to see that i can be chosen so that we may assume that the intersection points p_{k1}, p_{k2}, p_{k3}, p_{k4} $\in \partial D_k \cap l \cap l_i$ $(k = 1,2)$ shown by Figure 3 have the following properties:

1. the inner part of the curve $l(P_{11}, P_{12})$ meets $l_{i'}$ twice for all $1 \le i' \le m$;
2. among all intersection components D_2 is the nearest to A_0 in $P(A_0, A_i, l_i)$.

It may happen that $D_1 = D_2$.

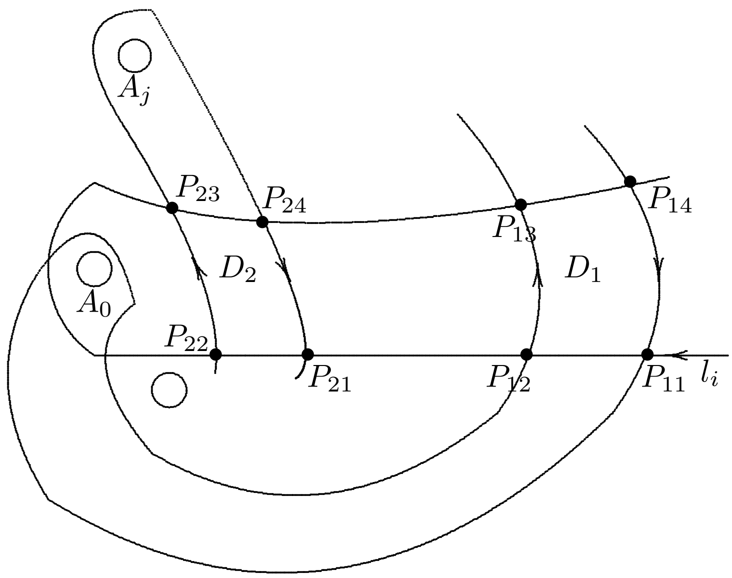

Figure 3.

Then

$$|l| + |l_i| > |l(p_{11}, p_{13}) + l_i(p_{13}, p_{11})| + |l(p_{23}, p_{21}) + l_i(p_{21}, p_{23})|.$$

Since $|l(p_{11}, p_{13}) + l_i(p_{13}, p_{11})| \ge |l_i|$ by our assumption, we get $|l| > |l(p_{23}, p_{21}) + l_i(p_{21}, p_{23})|$. Denoting by l' the geodesic in the homotopy class of $l(p_{23}, p_{21}) + l_i(p_{21}, p_{23})$, we see that

$$(A_0, A_j, l') = 1, \qquad \sum_{i=1}^{m} \#(l', l_i) < \sum_{i=1}^{m} \#(l, l_i), \qquad \text{and} \qquad |l'| < |l|.$$

Repeating the same procedure on l', we get $|l| > |l_j|$ after finitely many steps. Suppose the inequalities in the definition of our extreme loop structure hold strictly. Let $\{A_0, A_{k(i)}, l'\}_{i=1}^{m}$ be an extreme loop structure on S with the center A_0 and initial A_1, where $k(i) \in \{1, 2, \ldots, m\}$ and $k(1) = 1$. Then $l'_i \sim l_{k(i)}$ or $l_{k(i)}^{-1}$ $(i = 1, 2, \ldots, m)$, and the assertion follows.

Since conformal mappings preserve length, the following corollary is obvious.

Corollary. *Let* $\{A_0, A_j, l_j\}_{j=1}^m$ *be an extreme loop structure on* S *and* $f\colon S \mapsto S_0$ *a conformal homeomorphism. Then* $\{f(A_0), f(A_j), f(l_j)\}_{j=1}^m$ *is an extreme loop structure on* S_0. *Furthermore, if all inequalities in* $(*)$ *hold strictly, then* $\{f(A_0), f(A_j), f(l_j)\}_{j=1}^m$ *is the unique extreme structure on* S_0 *with the center* $f(A_0)$ *and initial* $f(A_1)$.

Remark. Let S' be a Riemann surface of topological type $(0, m+1)$, and $\{A_0, A_j, l_j\}_{j=1}^m$ and $\{B_0, B_j, l'_j\}_{j=1}^m$ be extreme loop structures on S and S', respectively. Suppose that $\{\alpha_j\}$ and $\{\alpha'_j\}$ are sets of generators compatible with $\{A_0, A_j, l_j\}_{j=1}^m$ and $\{B_0, B_j, l'_j\}_{j=1}^m$, respectively. Then by Theorem 4.1 in [O], we see that S is conformally equivalent to S', provided the following equalities are true:

$$\begin{aligned} |\alpha_j| &= |\alpha'_j|, && 0 \le j \le m; \\ |l_j| &= |l'_j|, && 1 \le j \le m; \\ |\alpha_j \circ \alpha_1| &= |\alpha'_j \circ \alpha'_1|, && 2 \le j \le m. \end{aligned}$$

Since there are only finitely many extreme loop structures on S and S', it is easy to see that the above discussion leads to an algorithm of comparing two Riemann surfaces of topological type $(0, m+1)$.

Theorem 2. *The fundamental domain for the action of the modular group* M *on* F *can be realized by the following inequalities*:

$$(*')\quad \begin{aligned} &|(\psi(g_0)| \le |\psi(g_1)| \le |(\psi(g_j)|, \qquad 2 \le j \le m; \\ &|(\psi(g_0 \circ g_1)| \le \min\big\{|(\psi(g_0^{-1} \circ g_2^{-1} \circ g_1^{-1} \circ g_2)|, \\ &\qquad |\psi(g_0^{-1} \circ g_3^{-1} \circ g_2^{-1} \circ g_1^{-1} \circ g_2 \circ g_3)|, \\ &\qquad \dots, |(\psi(g_0^{-1} \circ g_{m-1}^{-1} \circ \cdots \circ g_2^{-1} \circ g_1^{-1} \circ g_2 \circ \cdots \circ g_{m-1})|\big\}; \\ &\quad \cdots \\ &|\psi(g_0 \circ g_m)| \le \min\big\{|(\psi(g_0^{-1} \circ g_1^{-1} \circ g_m^{-1} \circ g_1)|, \\ &\qquad |\psi(g_0^{-1} \circ g_2^{-1} \circ g_1^{-1} \circ g_m^{-1} \circ g_1 \circ g_2)|, \\ &\qquad \dots, |(\psi(g_0^{-1} \circ g_{m-2}^{-1} \circ \cdots \circ \circ g_1^{-1} \circ g_m^{-1} \circ g_1 \circ \cdots \circ g_{m-2})|\big\}. \end{aligned}$$

Proof. Let $q = \big(\psi(g_i)\big) \in F$ and denote by S_q the Riemann surface attached to q. Choose $0 \le i_0$, $i_1 \le m$, $i_0 \ne i_1$, such that $|\psi(g_{i_0})| \le |\psi(g_{i_1})| \le |\psi(g_j)|$ for all $j \ne i_0, i_1$. According to Lemma 1, there exists an extreme loop structure on S_q with the center and initial covered by $\psi(g_{i_0})$ and $\psi(g_{i_1})$, respectively. Thus, applying the lifting procedure, we get a set of standard generators $\{\psi'(g_j) : 0 \le j \le m\}$ in the covering group such that $(*')$ hold, which implies that q is modular equivalent to a point satisfying our restricting condition $(*')$. Now let $q_1 = \big(\psi_1(g_i)\big)$ and $q_2 = \big(\psi_2(g_i)\big)$ be two distinct points in F. Suppose that for these two points, all inequalities in $(*')$ hold strictly. Assume $\psi_i(g) = \varphi_i \circ g \circ \varphi_i^{-1}$, $g \in G$, where φ_i $(i = 1, 2)$ are the associated orientation preserving homeomorphisms. Denote by $\widetilde{\varphi}_i$ the projections of φ_i $(i = 1, 2)$, i.e., $\widetilde{\varphi}_i\colon S \to S_{q_i}$. It is easy to see that $\psi_i(g_j)$ cover $\widetilde{\varphi}_i(\alpha_j)$ $(0 \le j \le m)$ and $\psi_i(g_0 \circ g_k)$ cover $\widetilde{\varphi}_i(\alpha_k \circ \alpha_0)$ $(1 \le k \le m)$. Hence the inequalities in $(*')$ guarantee that for $i = 1, 2$, $E_i = \{\widetilde{\varphi}_i(A_0), \widetilde{\varphi}_i(A_j), \widetilde{\varphi}_i(\alpha_j \circ \alpha_0)\}_{j=1}^m$ is an extreme loop structure on S_{q_i} with the center $\widetilde{\varphi}_i(A_0)$ and initial $\widetilde{\varphi}_i(A_1)$. Suppose there is a conformal homeomorphism $f\colon S_{q_1} \to S_{q_2}$. Then from the corollary of Theorem 1 and $(*')$, it is easy to verify that $f\bigl(\widetilde{\varphi}_1(\alpha_j)\bigr) \sim \widetilde{\varphi}_2(\alpha_j)$, $0 \le j \le m$, and $f\bigl(\widetilde{\varphi}_1(\alpha_k \circ \alpha_0)\bigr) \sim \widetilde{\varphi}_2(\alpha_k \circ \alpha_0)$, $1 \le k \le m$. It follows that $f \circ \widetilde{\varphi}_1 \sim \widetilde{\varphi}_2$, which implies that for all $g \in G$, $\psi_1(g)$ differ from $\psi_2(g)$ only by a conjugation of an element in $\mathscr{M}$. Thus $q_1 = q_2$, which is a contradiction, and the theorem is proved.

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