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SOLUTIONS OF NONHOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH EXCEPTIONALLY FEW ZEROS

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Abstract. We investigate equations of the form (1.1) that possess solutions which have a Borel exceptional value at zero.

1. Introduction

For $n \geq 1$ consider the nonhomogeneous linear differential equation

$$f^{(n)} + P_{n-1}(z) f^{(n-1)} + \cdots + P_0(z) f = H(z) \tag{1.1}$$

where $P_0(z)$, $P_1(z), \ldots, P_{n-1}(z)$ are polynomials ($P_0(z) \not\equiv 0$), and $H(z) \not\equiv 0$ is an entire function. It is well known that every solution f of equation (1.1) is an entire function.

For an entire function f, we let $\varrho(f)$ denote the order of f, and when $f \not\equiv 0$, we let $\lambda(f)$ denote the exponent of convergence of the sequence of zeros of f. Several authors have recently investigated the possible values of $\varrho(f)$ and $\lambda(f)$ when f is a solution of an equation of the form (1.1).

In this paper we investigate equations of the form (1.1) which possess solutions f satisfying

$$\lambda(f) < \varrho(f),$$

i.e., solutions which have a Borel exceptional value at zero.

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2. Main results

Most of our results concern equations of the form (1.1) where $H(z)$ satisfies $\lambda(H) < \varrho(H) < \infty$. These are equations of the form

$$f^{(n)} + P_{n-1}(z)f^{(n-1)} + \cdots + P_0(z)f = h(z)e^{Q(z)} \tag{2.1}$$

where $n \geq 1$, $P_0(z)$, $P_1(z), \ldots, P_{n-1}(z)$ are polynomials ($P_0(z) \not\equiv 0$), $h(z) \not\equiv 0$ is entire, and $Q(z)$ is a nonconstant polynomial, such that

$$\lambda(h) = \varrho(h) < \deg Q. \tag{2.2}$$

We need some notation for equation (2.1). Set $\beta = \deg Q \geq 1$, and let $b \neq 0$ be the constant satisfying

$$Q(z) = \frac{1}{\beta} b z^{\beta} + \cdots. \tag{2.3}$$

Let $d_k = \deg P_k$ for $0 \leq k \leq n$, where we set $P_n(z) \equiv 1$. Let A_k be the leading coefficient of P_k:

$$P_k(z) = A_k z^{d_k} + \cdots.$$

Set

$$\tau = \max_{0 \leq k \leq n} \{d_k + k(\beta - 1)\}. \tag{2.4}$$

For each $0 \leq k \leq n$, let A_k^* be the constant defined by

$$A_k^* = \begin{cases} A_k & \text{if } d_k + k(\beta-1) = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Now let $S(t)$ be the polynomial defined by

$$S(t) = \sum_{k=0}^{n} A_k^* t^k. \tag{2.5}$$

We use the above notation in the results below.

Theorem 1. *For equation* (2.1), *suppose that the constant b in* (2.3) *is a zero of multiplicity $m \geq 0$ of the polynomial $S(t)$ in* (2.5). *Then equation* (2.1) *can admit at most $m+1$ linearly independent solutions f satisfying $\lambda(f) < \varrho(f)$.*

The number "$m+1$" in Theorem 1 cannot be replaced with any integer less than $m+1$; see Examples 1 and 2 in Section 3. In particular, Example 2 shows that for any given choice of the integers m and n $(0 \le m \le n,\ n \ge 1)$, it is possible to obtain exactly $m+1$ solutions as described in Theorem 1. Hence Theorem 1 is sharp for all possible values of m and n. On the other hand, in the statement of Theorem 1, the words "can admit at most" cannot be improved upon in general because there exist equations of the form (2.1) that do not possess $m+1$ solutions as described in Theorem 1; see Examples 9 and 10 in Section 3 and Corollary 3 below.

A natural question is: which equations of the form (2.1) have the property that every solution f of the equation satisfies $\lambda(f) < \varrho(f)$? Theorem 2 and Corollary 1 below completely characterize these equations.

Theorem 2. *Every solution f of equation* (2.1) *satisfies $\lambda(f) < \varrho(f)$ if and only if $S(t) = (t-b)^n$.*

We obtain the following result from Theorem 2 and the definition of $S(t)$ in (2.5).

Corollary 1. *Every solution f of equation* (2.1) *satisfies $\lambda(f) < \varrho(f)$ if and only if*

$$P_k(z) = \binom{n}{k}(-b)^{n-k} z^{(n-k)(\beta-1)} + \cdots \qquad \text{for each } k = 0, 1, \ldots, n-1,$$

where b and β are given in (2.3).

Thus Corollary 1 gives the necessary and sufficient conditions on the coefficients in (2.1), so that every solution f of (2.1) satisfies $\lambda(f) < \varrho(f)$. Examples 3 and 6 in Section 3 illustrate Theorem 2 and Corollary 1.

We also mention the following result, which is a corollary of Theorem 1, the definition of $S(t)$, and Theorem 2.

Corollary 2. *If equation* (2.1) *possesses $n+1$ linearly independent solutions $f_1, f_2, \ldots, f_{n+1}$, satisfying $\lambda(f_k) < \varrho(f_k)$ for $k = 1, 2, \ldots, n+1$, then every solution f of the equation satisfies $\lambda(f) < \varrho(f)$.*

Corollary 2 can also be proved by using Theorem 4 below.

In the case when $S(b) \ne 0$ for equation (2.1), it follows from Theorem 1 that every solution f of (2.1) satisfies $\lambda(f) = \varrho(f)$ with at most one exceptional solution f_0. The next result gives information about such an exceptional solution f_0.

Theorem 3. *Suppose that $S(b) \ne 0$ for equation* (2.1). *Then every solution f of* (2.1) *satisfies $\lambda(f) = \varrho(f)$ with at most one exceptional solution f_0. For such an exceptional solution f_0, the following statements hold*:

(i) *If $h(z)$ is transcendental, then $\lambda(f_0) = \lambda(h) < \varrho(f_0)$.*

(ii) *If $h(z)$ is a polynomial, then* $\deg h \geq \tau$ *where τ is the constant in* (2.4), *and* f_0 *has the form* $f_0 = ge^Q$ *where Q is the polynomial in* (2.1) *and g is a polynomial satisfying* $\deg g = \deg h - \tau$.

Examples 4, 5, 6, and 7 in Section 3 illustrate Theorem 3. Examples 5, 6, and 7 show that the condition "$S(b) \neq 0$" cannot be deleted from the hypothesis of Theorem 3.

We obtain the following result from Theorem 3(ii), and it gives examples of equations of the form (2.1) which do not possess any solutions with a Borel exceptional value at zero.

Corollary 3. *Suppose that $S(b) \neq 0$ for equation* (2.1) *where $h(z)$ is a polynomial. If* $\deg h < \tau$, *then every solution f of* (2.1) *satisfies* $\lambda(f) = \varrho(f)$.

It is easy to construct examples of Corollary 3. The converse of Corollary 3 does not hold; see Example 11 in Section 3.

In [9, Theorem 3.1], Laine considered equations of the form (2.1) where $h(z)$ is a polynomial, and where the following conditions are satisfied:

$$\frac{d_k}{n-k} \leq \beta - 1 \quad \text{for } k = 1, 2, \ldots, n-1, \qquad \text{and} \qquad \beta < 1 + \frac{d_0}{n}. \tag{2.6}$$

For these equations we have $S(t) \equiv A_0$ where A_0 is the leading coefficient of $P_0(z)$. Hence $S(b) = A_0 \neq 0$, and so we can apply Theorem 3(ii) and Corollary 3 to these equations. By using Theorem 3(ii) and Corollary 3, a different proof of [9, Theorem 3.1] can be given.

Furthermore, by using Theorem 3(ii) and Corollary 3, it can be shown that if, in [9, Theorem 3.1], the condition (2.6) is replaced by the weaker condition

$$\frac{d_k}{n-k} \leq \frac{d_0}{n} \quad \text{for } k = 1, 2, \ldots, n-1, \qquad \text{and} \qquad \beta < 1 + \frac{d_0}{n},$$

then [9, Theorem 3.1] still holds.

Now consider equations of the form (2.1) where the following condition holds:

$$\beta > 1 + \max_{0 \leq k \leq n-1} \frac{d_k}{n-k}. \tag{2.7}$$

For these equations we have $S(t) = t^n$. Thus $S(b) = b^n \neq 0$, and so we can apply Theorem 3 and Corollary 3 to these equations.

Gao [4, Theorem 1] considered equations of the form (2.1) where $h(z)$ is a polynomial and where (2.7) holds. By using Theorem 3(ii) and Corollary 3, a different proof of [4, Theorem 1] can be given.

In [1, Theorem 1(iii)], Chen and Gao showed that if (2.7) holds for equation (2.1), then every solution f of (2.1) satisfies $\lambda(f) = \varrho(f)$ with at most one exceptional solution f_0. For such an exceptional solution f_0, they showed that $\lambda(f_0) = \lambda(h)$. These two statements follow from Theorem 3.

We use the next result as a lemma, although it has independent interest.

Theorem 4. *If f is a solution of* (2.1) *satisfying $\lambda(f) < \varrho(f)$, then f has the form*

$$f = ge^{Q}$$

where Q is the polynomial in (2.1)*, and g is an entire function satisfying $\varrho(g) < \deg Q$.*

Regarding Theorem 4, the result of Frank and Hellerstein [2] gives very specific information when f is a solution of (2.1) that has only finitely many zeros, where $h(z)$ is a polynomial in (2.1), and where $Q(z)$ is allowed to be any entire function.

We now consider equations of the form (1.1) which do not have the form (2.1). The next result is a consequence of Theorems 1 and 2 in [3].

Theorem A. *If $H(z)$ in equation* (1.1) *satisfies $\lambda(H) = \varrho(H) < \infty$, then every solution f of* (1.1) *satisfies $\lambda(f) = \varrho(f)$.*

For the case when $\varrho(H) = \infty$, we prove the following result.

Theorem 5. *If $H(z)$ in equation* (1.1) *satisfies $\varrho(H) = \infty$, then every solution f of* (1.1) *satisfies $\lambda(f) = \varrho(f) = \infty$ with at most one exceptional solution f_0.*

It is possible to have an exceptional solution f_0 in Theorem 5. In fact, it is possible to have an exceptional solution f_0 in Theorem 5 for each of the following three cases: (i) $\lambda(H) = 0 < \varrho(H) = \infty$, (ii) $0 < \lambda(H) < \varrho(H) = \infty$, and (iii) $\lambda(H) = \varrho(H) = \infty$; see Example 8 and the discussion before Example 8 in Section 3. The result of Frank and Hellerstein [2] gives very specific information for the particular case when f_0 is an exceptional solution in Theorem 5 under the conditions that both f_0 and $H(z)$ have only finitely many zeros.

In this paper a meromorphic function always means meromorphic in the whole complex plane. We assume the reader is familiar with the standard definitions and results of Nevanlinna theory (see [8]). Throughout the paper we make the following two notations:

(i) We let $D \subset [0, \infty)$ denote a set of finite linear measure, where the set D may not necessarily be the same set each time it appears.

(ii) We let $E = E_0 \cup [0, 1]$ where $E_0 \subset (0, \infty)$ is a set of finite logarithmic measure, where the set E may not necessarily be the same set each time it appears.

3. Examples

In this section we give examples to illustrate our theorems. Some of these examples show the sharpness of our theorems, while others exhibit some possibilities that can occur. We use the notations for b, β, τ, and $S(t)$ given in Section 2.

Example 1. The three linearly independent functions $f_1 = e^z$, $f_2 = ze^z$, $f_3 = z^2e^z$ are solutions of the equation

$$f^{(4)} - 2f''' + (z^2 - 2z + 1)f'' + (-2z^2 + 2z + 2)f' + z^2 f = 2e^z. \tag{3.1}$$

For equation (3.1) we have $\beta = 1$, $\tau = 2$, $b = 1$, and $S(t) = (t-1)^2$. Thus $b = 1$ is a double zero of $S(t)$. This example shows that it is possible to obtain exactly $m+1$ solutions as described in Theorem 1.

In general, the next example shows that for any given choice of the integers m and n ($0 \le m \le n$, $n \ge 1$), there exist equations of the form (2.1) that possess exactly $m+1$ solutions as described in Theorem 1. Hence Theorem 1 is sharp for all possible values of m and n.

Example 2. First we consider the case when $m = 0$ in Theorem 1. Let $n \ge 1$ be an integer. For some polynomial $P(z) \not\equiv 0$, the function $f_1(z) = \exp\{\frac{1}{2}z^2\}$ will be a solution of the equation

$$f^{(n)} + z^n f = P(z)\exp\{\tfrac{1}{2}z^2\}. \tag{3.2}$$

For equation (3.2) we have $\beta = 2$, $\tau = n$, $b = 1$, and $S(t) = t^n + 1$. Thus $S(b) = 2 \ne 0$ and $m = 0$ in Theorem 1. Hence from Theorem 1, equation (3.2) can admit at most one solution f satisfying $\lambda(f) < \varrho(f)$. Since $f_1(z) = \exp\{\frac{1}{2}z^2\}$ is a solution of (3.2) satisfying $\lambda(f_1) < \varrho(f_1)$, this shows that Theorem 1 is sharp when $m = 0$ and $n \ge 1$.

Next we consider the cases when $m \ge 1$ in Theorem 1. Let m and n be integers satisfying $1 \le m \le n$. Consider the equation

$$f^{(m)} + \sum_{k=0}^{m-1} \binom{m}{k} (-z)^{m-k} f^{(k)} = \exp\{\tfrac{1}{2}z^2\}. \tag{3.3}$$

For equation (3.3) we have $\beta = 2$, $\tau = m$, $b = 1$, and $S(t) = (t-1)^m$. Hence from Theorem 2, every solution f of (3.3) satisfies $\lambda(f) < \varrho(f)$. It follows that there exist $m+1$ linearly independent solutions $f_1, f_2, \ldots, f_{m+1}$ of (3.3) satisfying $\lambda(f_k) < \varrho(f_k)$ for $k = 1, 2, \ldots, m+1$.

By differentiating equation (3.3) $n-m$ times, we obtain an equation of the form

$$f^{(n)} + B_{n-1}(z)f^{(n-1)} + \cdots + B_0(z)f = D(z)\exp\{\tfrac{1}{2}z^2\} \tag{3.4}$$

where $D(z) \not\equiv 0$ is a polynomial, and $B_0(z), B_1(z), \ldots, B_{n-1}(z)$ are polynomials satisfying

$$\deg B_k < n-k \qquad \text{when } 0 \le k \le n-m-1,$$
$$B_k(z) = \binom{m}{k-n+m}(-z)^{n-k} + \cdots \qquad \text{when } n-m \le k \le n-1.$$

For equation (3.4) we have $\beta = 2$, $\tau = n$, $b = 1$, and

$$S(t) = t^{n-m}(t-1)^m.$$

Since $b = 1$ is a zero of multiplicity m of $S(t)$, we obtain from Theorem 1 that equation (3.4) can admit at most $m+1$ linearly independent solutions f satisfying $\lambda(f) < \varrho(f)$. On the other hand, the functions $f_1, f_2, \dots, f_{m+1}$ are exactly $m+1$ linearly independent solutions of (3.4) satisfying $\lambda(f_k) < \varrho(f_k)$ for $k = 1, 2, \dots, m+1$. Hence for equation (3.4), we obtain exactly $m+1$ solutions as described by Theorem 1. This shows that Theorem 1 is sharp for all m and n satisfying $1 \le m \le n$. Since we also showed that Theorem 1 is sharp when $m = 0$ and $n \ge 1$, this proves that Theorem 1 is sharp for all possible values of m and n.

The next example illustrates Theorem 2 and Corollary 1.

Example 3. The function $f_0(z) = \exp\{z + z^4\}$ satisfies the equation

$$f'' + (z - 8z^3)f' + (16z^6 - 4z^4 - 12z^2 + z)f = (2z+1)\exp\{z + z^4\}. \tag{3.5}$$

Now set $w(z) = g(z)e^{z^4}$ where g is any solution of the equation

$$g'' + zg' + zg = 0. \tag{3.6}$$

Then w satisfies the corresponding homogeneous equation to (3.5):

$$w'' + (z - 8z^3)w' + (16z^6 - 4z^4 - 12z^2 + z)w = 0.$$

It follows that every solution f of (3.5) has the form

$$f(z) = w(z) + f_0(z) = \big(g(z) + e^z\big)e^{z^4} \tag{3.7}$$

where g is a solution of (3.6). From Lemma 3 in Section 5, it follows that every solution g of (3.6) satisfies $\varrho(g) \le 2$. Hence from (3.7), every solution f of (3.5) satisfies

$$\lambda(f) \le 2 < \varrho(f) = 4.$$

For equation (3.5) we have $\beta = 4$, $\tau = 6$, $b = 4$, and $S(t) = (t-4)^2$. This is an example of Theorem 2 and Corollary 1.

Examples 4, 5, 6, and 7 below illustrate Theorem 3. Examples 5, 6, and 7 show that the condition "$S(b) \ne 0$" cannot be deleted from the hypothesis of either Theorem 3(i) or Theorem 3(ii). In contrast with Theorem 3(i), Examples 5 and 6 show that there exist equations of the form (2.1) where $S(b) = 0$, which possess a solution f satisfying $0 < \lambda(h) < \lambda(f) < \varrho(f)$.

Example 4. Let g be the transcendental entire function defined by

$$g(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad a_n = \frac{1}{n^{n^2}}.$$

Then $\varrho(g) = 0 = \lim n \log n / \log(1/|a_n|)$ (see [13, p. 253]). Now let h be the entire function defined by

$$h = g'' + (2z+1)g' + (2z^2 + z + 1)g. \tag{3.8}$$

If h were a polynomial, then it would follow from equation (3.8) that [14, pp. 106–108] either $\varrho(g) > 0$ or g is a polynomial, which is a contradiction. Hence h is transcendental. Since $\varrho(g) = 0$, it follows from (3.8) that $\varrho(h) = 0$.

Now consider the equation

$$f'' + f' + z^2 f = h(z) \exp\{\tfrac{1}{2}z^2\} \tag{3.9}$$

For equation (3.9) we have $\beta = 2$, $\tau = 2$, $b = 1$, and $S(t) = t^2 + 1$. Thus $S(b) = 2 \neq 0$.

Now set $f_0(z) = g(z) \exp\{\frac{1}{2}z^2\}$. From (3.8) we obtain that f_0 is a solution of equation (3.9). We have $\lambda(f_0) = 0 < \varrho(f_0) = 2$ and $\lambda(f_0) = \lambda(h) = 0$. This is an example of Theorem 3(i).

Example 5. For any constant c,

$$f_c(z) = \big(ce^{z^2} + e^z + 1\big)e^{z^3}$$

satisfies the equation

$$f'' - (9z^4 + 12z^3 + 4z^2 + 6z + 2)f = \big[(-12z^3 + 2z^2 - 1)e^z - (12z^3 + 4z^2 + 2)\big]e^{z^3}. \tag{3.10}$$

If $c \neq 0$, then (3.10) is an equation of the form (2.1) which possesses the solution f_c, such that

$$0 < \lambda(h) < \lambda(f_c) < \varrho(f_c).$$

For equation (3.10) we have $\beta = 3$, $\tau = 4$, $b = 3$, and $S(t) = t^2 - 9$. Here $b = 3$ is a simple zero of $S(t)$. This example shows that the condition "$S(b) \neq 0$" cannot be deleted from the hypothesis of Theorem 3(i).

Example 6. The function $f(z) = \big(e^{z^2} + e^z + 1\big)e^{z^3}$ satisfies the equation

$$f'' - (6z^2 + 2z + 1)f' + (9z^4 + 6z^3 + 3z^2 - 4z - 2)f = \big(-2e^z + 2z - 2\big)e^{z^3}. \tag{3.11}$$

Thus like Example 5, this is an example of an equation of the form (2.1) which possesses a solution f satisfying $0 < \lambda(h) < \lambda(f) < \varrho(f)$.

However, there are some differences with Example 5. For equation (3.11) we have $\beta = 3$, $\tau = 4$, $b = 3$, and $S(t) = (t-3)^2$. Thus from Theorem 2, every solution f of equation (3.11) satisfies $\lambda(f) < \varrho(f)$. In contrast, it follows from Theorem 2 that this property does not hold for equation (3.10), because $S(t) = t^2 - 9$ for (3.10).

Example 7. The function $f(z) = (z^3 + z)e^z$ satisfies the following two equations:

$$f''' + zf'' + f = (z^4 + 8z^3 + 16z^2 + 22z + 9)e^z, \tag{3.12}$$

$$f''' + zf'' - zf = (7z^3 + 15z^2 + 21z + 9)e^z. \tag{3.13}$$

For equation (3.12) we have $\beta = 1$, $\tau = 1$, $\deg h = 4$, $b = 1$, $S(t) = t^2$, and $S(b) = 1 \neq 0$. This is an example of Theorem 3(ii).

For equation (3.13) we have $\beta = 1$, $\tau = 1$, $\deg h = 3$, $b = 1$, $S(t) = t^2 - 1$, and $S(b) = 0$. In this example, the conclusion in Theorem 3(ii) does not hold. Hence this example shows that the condition "$S(b) \neq 0$" cannot be deleted from the hypothesis of Theorem 3(ii).

We next note [2, p. 409] that

$$f_0(z) = \exp\left\{\int_0^z \exp(t^2)\,dt - z^2\right\}$$

is a solution of the equation

$$f''' + 4(1 - z^2)f' - 4zf = f_0(z)e^{3z^2}.$$

(Note: the term $-4zf$ in this equation seems to be written incorrectly as $-8zf$ on page 409 of [2].) This is an example of an exceptional solution f_0 in Theorem 5 where $\lambda(H) = 0 < \varrho(H) = \infty$. We note that this function $f_0(z)$ also satisfies the equation

$$f''' + 4(1 - z^2)f' - 3zf = f_0(z)\big(e^{3z^2} + z\big),$$

which is an example of an exceptional solution f_0 in Theorem 5 where $0 < \lambda(H) < \varrho(H) = \infty$. The next example gives an exceptional solution f_0 in Theorem 5 where $\lambda(H) = \varrho(H) = \infty$.

Example 8. Let $P(z) \not\equiv 0$ be any polynomial, and consider the equation

$$f'' + P(z)f = H(z) \tag{3.14}$$

where

$$H(z) = \big[P(z) + (e^{2z} + e^z)e^{e^z} + e^{2z}e^{2e^z}\big]\exp\{e^{e^z}\}. \tag{3.15}$$

The function $f_0(z) = \exp\{e^{e^z}\}$ satisfies equation (3.14), and obviously $\lambda(f_0) = 0 < \varrho(f_0) = \infty$.

Since $\varrho(H) = \infty$, f_0 is an example of an exceptional solution in Theorem 5. We now prove that $\lambda(H) = \infty$.

In (3.15) we set

$$G(z)=(e^{2z}+e^{z})e^{e^{z}}+e^{2z}e^{2e^{z}}. \tag{3.16}$$

Since (see [8, p. 7])

$$T\big(r,e^{e^{z}}\big)=\big(1+o(1)\big)\frac{e^{r}}{\sqrt{2\pi^{3}r}} \tag{3.17}$$

as $r\to\infty$, it can be deduced that

$$T(r,G)=\big(2+o(1)\big)\frac{e^{r}}{\sqrt{2\pi^{3}r}} \tag{3.18}$$

as $r\to\infty$. From (3.16) we see that

$$\overline{N}(r,0,G)=\overline{N}(r,0,e^{e^{z}}+1+e^{-z}). \tag{3.19}$$

We now apply Lemma 4 in Section 5 with $F(z)=e^{e^{z}}$, $a_1(z)\equiv 0$, and $a_2(z)=-1-e^{-z}$, and use (3.17) to obtain that

$$\overline{N}\big(r,0,e^{e^{z}}+1+e^{-z}\big)=\big(1+o(1)\big)T\big(r,e^{e^{z}}\big) \tag{3.20}$$

as $r\to\infty$, $r\notin D$. Hence from (3.20), (3.19), (3.18), and (3.17),

$$\overline{N}(r,0,G)=\tfrac{1}{2}\big(1+o(1)\big)T(r,G) \tag{3.21}$$

as $r\to\infty$, $r\notin D$. From (3.15) and (3.16),

$$N(r,0,H)=N(r,0,G+P). \tag{3.22}$$

Since $P(z)\not\equiv 0$ is a polynomial, and since (3.18) holds, we can again apply Lemma 4 with $F(z)=G(z)$, $a_1(z)\equiv 0$, and $a_2(z)=-P(z)$, and use (3.21) to obtain that

$$\overline{N}(r,0,G+P)\geq\tfrac{1}{2}\big(1+o(1)\big)T(r,G) \tag{3.23}$$

as $r\to\infty$, $r\notin D$. From (3.23), (3.22), and (3.18), it follows that $\lambda(H)=\infty$, which is what we wanted to prove.

Since $\lambda(H)=\infty$ and $\lambda(f_0)=0$, this example also shows that the words "finite order" cannot be deleted from Lemma 9 in Section 5.

In contrast to Examples 1 and 2, the next two examples give equations of the form (2.1) that do not possess $m+1$ solutions as described in Theorem 1. This shows that in the statement of Theorem 1, the words "can admit at most" cannot be improved upon in general.

Example 9. Consider the equation

$$f'' + (-z^6 - 3z^2 + z)f = z \exp\{\tfrac{1}{4}z^4\}. \tag{3.24}$$

For equation (3.24) we have $\beta = 4$, $\tau = 6$, $b = 1$, and $S(t) = t^2 - 1$, and so $b = 1$ is a simple zero of $S(t)$. Thus from Theorem 1, it follows that equation (3.24) can admit at most 2 linearly independent solutions f satisfying $\lambda(f) < \varrho(f)$. However, we now show that there exists exactly one solution f of (3.24) that satisfies $\lambda(f) < \varrho(f)$.

First note that $f_1(z) = \exp\{\frac{1}{4}z^4\}$ is a solution of (3.24). We now show that if f is any solution of (3.24) where $f \not\equiv f_1$, then $\lambda(f) = \varrho(f)$.

Assume the contrary. Suppose that there exists a solution f_2 of (3.24) satisfying $\lambda(f_2) < \varrho(f_2)$ and $f_2 \not\equiv f_1$. From Theorem 4 we obtain that

$$f_2 = g(z) \exp\{\tfrac{1}{4}z^4\} \quad \text{where } g \text{ is entire and } \varrho(g) < 4.$$

Set $F(z) = f_1(z) - f_2(z)$. Then $F = (1 - g)\exp\{\frac{1}{4}z^4\} = G \exp\{\frac{1}{4}z^4\}$ where

$$\varrho(G) < 4, \qquad G \not\equiv 0. \tag{3.25}$$

Moreover, F is a solution of the homogeneous equation

$$f'' + (-z^6 - 3z^2 + z)f = 0. \tag{3.26}$$

Substituting $F = G \exp\{\frac{1}{4}z^4\}$ into (3.26), we find that $w = G(z)$ is a solution of the equation

$$w'' + 2z^3 w' + zw = 0. \tag{3.27}$$

Obviously, equation (3.27) does not admit any nontrivial polynomial solutions. Furthermore, all transcendental solutions of (3.27) are of order 4 (see [7, Theorem 1(i)]). Thus either $G \equiv 0$ or $\varrho(G) = 4$, which contradicts (3.25). This contradiction shows that such a solution f_2 of (3.24) cannot exist. Therefore, there exists exactly one solution f of (3.24) satisfying $\lambda(f) < \varrho(f)$.

Example 10. Consider the equation

$$f''' - z^3 f'' + (-z^6 - 9z^2)f' + (z^9 + 3z^5)f = h(z) \exp\{\tfrac{1}{4}z^4\} \tag{3.28}$$

where $h(z) \not\equiv 0$ is any entire function satisfying $\varrho(h) < 4$. For equation (3.28) we have $\beta = 4$, $\tau = 9$, $b = 1$, and $S(t) = t^3 - t^2 - t + 1 = (t-1)^2(t+1)$. Since $b = 1$ is a double zero of $S(t)$, it follows from Theorem 1 that equation (3.28) can admit at most 3 linearly independent solutions f satisfying $\lambda(f) < \varrho(f)$. However, we now show that there cannot exist more than one solution f of (3.28) that satisfies $\lambda(f) < \varrho(f)$.

We proceed as in Example 9 and assume the contrary. Suppose that there exist two solutions f_1, f_2, of (3.28) satisfying $\lambda(f_i) < \varrho(f_i)$ for $i = 1, 2$, and $f_1 \not\equiv f_2$. From Theorem 4 we obtain that

$$f_i = g_i \exp\left\{\tfrac{1}{4}z^4\right\} \quad \text{where } g_i \text{ is entire and} \quad \varrho(g_i) < 4,\ i = 1, 2.$$

Set $F(z) = f_1(z) - f_2(z)$. A contradiction can now be deduced by using the same reasoning as in Example 9. This contradiction shows that there can exist at most one solution f of (3.28) satisfying $\lambda(f) < \varrho(f)$.

The next example shows that the converse of Corollary 3 does not hold.

Example 11. Consider the equation

$$f'' + zf = z^2 e^{z^2}. \tag{3.29}$$

For equation (3.29) we have $\beta = 2$, $\tau = 2$, $\deg h = 2$, $b = 2$, $S(t) = t^2$, and $S(b) = 4 \neq 0$.

Suppose that f_0 is a solution of (3.29) satisfying $\lambda(f_0) < \varrho(f_0)$. Then from (3.29) and Theorem 4, $f_0(z) = g_0(z)e^{z^2}$ where $g = g_0(z)$ is an entire function satisfying the equation

$$g'' + 4zg' + (4z^2 + z + 2)g = z^2, \tag{3.30}$$

and $\varrho(g_0) < 2$. If g_0 is transcendental, then by applying the Wiman–Valiron theory (see [14, pp. 105–108]) to equation (3.30), it can be deduced that $\varrho(g_0) = 2$, which is a contradiction. Hence g_0 must be a polynomial. But this is impossible from inspection of (3.30). This contradiction proves that the assumption $\lambda(f_0) < \varrho(f_0)$ does not hold.

Therefore, every solution f of equation (3.29) satisfies $\lambda(f) = \varrho(f)$. Since $S(b) \neq 0$ and deg $h = \tau$ for equation (3.29), this shows that the converse of Corollary 3 does not hold.

4. Proof of Theorem 4

Let f be a solution of (2.1) satisfying $\lambda(f) < \varrho(f)$. Since $\varrho(f) < \infty$ from Lemma 6 in Section 5, it follows that f has the form

$$f(z) = u(z)e^{R(z)} \tag{4.1}$$

where $u(z) \not\equiv 0$ is an entire function and $R(z)$ is a nonconstant polynomial, such that

$$\varrho(u) < \deg R. \tag{4.2}$$

With $\beta = \deg Q$ in (2.1), we set

$$Q(z) = cz^{\beta} + \cdots \tag{4.3}$$

where $c \neq 0$. Then by substituting (4.1) and (4.3) into (2.1), we obtain an equation of the form

$$A(z)e^{R(z)} = B(z)e^{cz^{\beta}} \tag{4.4}$$

where $A(z)$ is a polynomial in $z, u, u', \ldots, u^{(n)}, R', R'', \ldots, R^{(n)}$, and $B(z) \not\equiv 0$ is an entire function satisfying

$$\varrho(B) < \beta. \tag{4.5}$$

Since $R(z)$ is a polynomial, and since (4.2) holds, it follows that

$$\varrho(A) < \deg R. \tag{4.6}$$

Therefore, from (4.6) and (4.5), we can deduce from (4.4) that $\deg R = \beta$ and $R(z) = cz^{\beta} + \cdots$. Theorem 4 now follows from (4.3), (4.2), and (4.1).

5. Lemmas

In this section we give lemmas which are used in the proofs of Theorems 1, 2, 3, and 5.

In the following lemma we use the definitions of β, b, τ, and $S(t)$ in (2.3), (2.4), and (2.5).

Lemma 1. *Let f be a solution of equation* (2.1), *and let $Q(z)$ be the polynomial in* (2.1). *Then $g = fe^{-Q}$ satisfies an equation of the form*

$$g^{(n)} + a_{n-1}(z)g^{(n-1)} + \cdots + a_0(z)g = h(z) \tag{5.1}$$

where $h(z)$ is the function in (2.1), *and where each $a_k(z)$ is a polynomial satisfying*

$$\deg a_k \leq \tau - k(\beta - 1) \qquad \text{and} \qquad a_k(z) = \frac{S^{(k)}(b)}{k!} z^{\tau - k(\beta-1)} + \cdots. \tag{5.2}$$

Proof. Since $f = ge^{Q}$, we obtain by induction that for each $p = 1, 2, \ldots, n$,

$$f^{(p)} = \left(g^{(p)} + pQ'g^{(p-1)} + \sum_{j=2}^{p} \left[\binom{p}{j} (Q')^{j} + H_{j-1}(Q') \right] g^{(p-j)} \right) e^{Q}, \tag{5.3}$$

where $H_{j-1}(Q')$ is a differential polynomial of total degree at most $j-1$ in $Q', Q'', Q''', \ldots$, with constant coefficients.

Substituting (5.3) into equation (2.1) we obtain that g satisfies an equation of the form (5.1), where the coefficients $a_0(z), \ldots, a_{n-1}(z)$ take the form

$$\begin{aligned} a_{n-1} &= P_{n-1} + nQ', \\ a_k &= P_k + (k+1)Q'P_{k+1} + \sum_{j=2}^{n-k} \left[\binom{j+k}{j} (Q')^j + H_{j-1}(Q') \right] P_{j+k}, \end{aligned} \tag{5.4}$$

for $k = 0, 1, \ldots, n-2$. Here we set $P_n(z) \equiv 1$. Recalling the definition of A_k^* in Section 2, we can write $P_k(z)$ in the form

$$P_k(z) = A_k^* z^{\tau - k(\beta - 1)} + \cdots, \qquad k = 0, 1, \ldots, n. \tag{5.5}$$

By inspection of the leading term of each $a_k(z)$ in (5.4), we obtain (5.2) from (5.5), (2.3), and the definition of $S(t)$ in (2.5). This proves Lemma 1.

Lemma 2 [5]. *Let f be a transcendental meromorphic function of finite order ϱ, let k and j be integers satisfying $k > j \geq 0$, and let $\varepsilon > 0$ be a given constant. Then*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\varrho - 1 + \varepsilon)}, \qquad |z| \notin E.$$

Lemma 3 [10, p. 127]. *If f is a solution of the equation*

$$f^{(n)} + P_{n-1}(z) f^{(n-1)} + \cdots + P_0(z) f = 0$$

where each $P_k(z)$ is a polynomial ($P_0(z) \not\equiv 0$), then

$$\varrho(f) \leq 1 + \max_{0 \leq k \leq n-1} \frac{\deg P_k}{n-k}.$$

Lemma 4 below is essentially the three small functions theorem of Nevanlinna, because it can be proved by using the proof of Theorem 2.5 on pp. 47–48 of [8] together with a Möbius transformation. Although there are more general forms of the small functions theorem (see [11, 12]), Lemma 4 is in a form that is suitable for our purposes.

Lemma 4. *Let $F(z)$ be a nonconstant meromorphic function, and suppose that $a_1(z)$ and $a_2(z)$ are two distinct meromorphic functions satisfying*

$$\lim_{\substack{r \to \infty \\ r \in I}} \frac{T(r, a_k)}{T(r, F)} = 0, \qquad k = 1, 2,$$

where I is some set of infinite linear measure. Then

$$\big(1 + o(1)\big) T(r, F) \leq \overline{N}(r, F) + \overline{N}(r, 0, F - a_1) + \overline{N}(r, 0, F - a_2)$$

as $r \to \infty$, $r \in I \setminus D$.

The next lemma combines the method of reduction of order for linear differential equations with estimates of logarithmic derivatives. See Lemmas 6.4 and 6.5 in [7].

Lemma 5 [7]. *Let $f_{0,1},\dots,f_{0,m},f_{0,m+1}$ $(m\ge 1)$ be $m+1$ linearly independent meromorphic solutions of an equation of the form*

$$y^{(n)}+A_{0,n-1}(z)y^{(n-1)}+\cdots+A_{0,0}(z)y=0,\qquad n\ge m+1,$$

where $A_{0,0}(z),\dots,A_{0,n-1}(z)$ are meromorphic functions. For $q=1,2,\dots,m$, set

$$f_{q,j}=\left(\frac{f_{q-1,j+1}}{f_{q-1,1}}\right)',\qquad j=1,2,\dots,m+1-q. \tag{5.6}$$

Then for each $q=1,2,\dots,m$, the function $f_{q,1}$ is not identically zero and is a solution of the equation

$$y^{(n-q)}+A_{q,n-q-1}(z)y^{(n-q-1)}+\cdots+A_{q,0}(z)y=0, \tag{5.7}$$

where

$$A_{q,j}(z)=\sum_{k=j+1}^{n-q+1}\binom{k}{j+1}A_{q-1,k}(z)\frac{f_{q-1,1}^{(k-j-1)}(z)}{f_{q-1,1}(z)}$$

for $j=0,1,\dots,n-q-1$. Here we set $A_{k,n-k}(z)\equiv 1$ for $k=0,1,\dots,q$.

Moreover, suppose that for each j, $j=0,1,\dots,n-1$, there exists a real number $\tau_{0,j}$ such that

$$|A_{0,j}(z)|\le |z|^{\tau_{0,j}},\qquad |z|\notin E. \tag{5.8}$$

Suppose further that $\varrho(f_{0,j})<\infty$ for each j, and set $\varrho_0=\max_{1\le j\le m+1}\{\varrho(f_{0,j})\}$. Let $\varepsilon>0$ be any given constant. Considering equation (5.7) when $q=m$, we have for $j=1,2,\dots,n-m-1$,

$$|A_{m,j}(z)|\le |z|^{\tau_{m,j}},\qquad |z|\notin E, \tag{5.9}$$

where

$$\tau_{m,j}=\max_{m+j\le k\le n}\{\tau_{0,k}+(k-m-j)(\varrho_0-1)+\varepsilon\}, \tag{5.10}$$

while for $A_{m,0}(z)$ we have

$$A_{m,0}(z)=A_{0,m}(z)+G_m(z), \tag{5.11}$$

where $G_m(z)$ satisfies

$$|G_m(z)|\le |z|^{\tau_m},\qquad |z|\notin E, \tag{5.12}$$

where

$$\tau_m=\max_{m+1\le k\le n}\{\tau_{0,k}+(k-m)(\varrho_0-1)+\varepsilon\}. \tag{5.13}$$

Lemma 6. *Every solution f of equation (1.1) satisfies*

$$\varrho(H)\le\varrho(f)\le\max\left\{\varrho(H),\ 1+\max_{0\le k\le n-1}\frac{\deg P_k}{n-k}\right\}. \tag{5.14}$$

Theorem 1(i) and Theorem 2(i) in [1] are corollaries of Lemma 6.

Proof of Lemma 6. The first inequality $\varrho(H) \le \varrho(f)$ in (5.14) follows easily from an elementary order consideration on both sides of equation (1.1).

To prove the second inequality in (5.14), we let $f_1, f_2, \ldots, f_n$ be a fundamental set of solutions of the corresponding homogeneous equation of (1.1). Then from Lemma 3,

$$\varrho(f_j) \le 1 + \max_{0 \le k \le n-1} \frac{\deg P_k}{n-k}, \qquad j = 1, 2, \ldots, n. \tag{5.15}$$

From the well-known method of variation of parameters, there exist n entire functions $A_1(z), A_2(z), \ldots, A_n(z)$, satisfying

$$\varrho(A_j) \le \max\{\varrho(H), \varrho(f_1), \ldots, \varrho(f_n)\}, \qquad j = 1, 2, \ldots, n, \tag{5.16}$$

such that

$$f_0 = A_1 f_1 + A_2 f_2 + \cdots + A_n f_n \tag{5.17}$$

is a solution of equation (1.1); see [10, pp. 144–145]. From (5.17) and (5.16),

$$\varrho(f_0) \le \max\{\varrho(H), \varrho(f_1), \ldots, \varrho(f_n)\}. \tag{5.18}$$

Now let f be any solution of equation (1.1). Then f can be represented in the form

$$f = f_0 + c_1 f_1 + \cdots + c_n f_n, \tag{5.19}$$

where $c_1, c_2, \ldots, c_n$ are constants. Thus from (5.19), (5.18), and (5.15), we obtain

$$\varrho(f) \le \max\left\{\varrho(H),\ 1 + \max_{0 \le k \le n-1} \frac{\deg P_k}{n-k}\right\},$$

which is the second inequality in (5.14). This proves Lemma 6.

Remark. The Wiman–Valiron theory can be used to give an alternate proof of Lemma 6.

Lemma 7 [6]. *Suppose that $U(r)$ and $W(r)$ are monotone nondecreasing functions on $0 \le r < \infty$ such that $U(r) \le W(r)$ for $r \notin E$. Then for any given constant $\alpha > 1$, there exists a constant $R = R(\alpha) > 0$ such that $U(r) \le W(\alpha r)$ for all $r \ge R$.*

Lemma 8. *If g and h are entire functions satisfying $\varrho(g) > \varrho(h)$, then there exists a set $J \subset (0, \infty)$ that has infinite logarithmic measure such that*

$$M(r, h) < M(r, g), \qquad r \in J. \tag{5.20}$$

Here $M(r, h)$ denotes the usual maximum modulus function: $M(r, h) = \max_{|z|=r} |h(z)|$.

Proof. Suppose that (5.20) is not true. Then

$$M(r,g) \le M(r,h), \qquad r \notin E.$$

By applying Lemma 7 with $U(r) = M(r,g)$ and $W(r) = M(r,h)$, we obtain that there exists a constant $R > 0$ such that

$$M(r,g) \le M(2r,h), \qquad r \ge R.$$

But this implies $\varrho(g) \le \varrho(h)$, which contradicts $\varrho(g) > \varrho(h)$. This proves Lemma 8.

The next result was proved by Gao in [3]. We provide a different proof.

Lemma 9. *Let $H(z) \not\equiv 0$ be an entire function of finite order. Then any solution f of equation* (1.1) *satisfies $\lambda(f) \ge \lambda(H)$.*

Proof. Since $\varrho(H) < \infty$, we obtain from Lemma 6 that $\varrho(f) < \infty$. Hence

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r), \qquad k = 1, 2, \dots, n, \tag{5.21}$$

as $r \to \infty$. Since f satisfies (1.1), we have

$$\frac{H}{f} = P_0(z) + P_1(z)\frac{f'}{f} + \cdots + P_{n-1}(z)\frac{f^{(n-1)}}{f} + \frac{f^{(n)}}{f}. \tag{5.22}$$

It follows from (5.22) and (5.21) that

$$m\left(r, \frac{H}{f}\right) = O(\log r).$$

Then by using the first fundamental theorem, we obtain

$$\begin{aligned} O(1) &= T\left(r, \frac{H}{f}\right) - T\left(r, \frac{f}{H}\right) \le T\left(r, \frac{H}{f}\right) - N\left(r, \frac{f}{H}\right) \\ &= N\left(r, \frac{H}{f}\right) - N\left(r, \frac{f}{H}\right) + m\left(r, \frac{H}{f}\right) = N\left(r, \frac{1}{f}\right) - N\left(r, \frac{1}{H}\right) + O(\log r), \end{aligned}$$

which gives

$$N\left(r, \frac{1}{f}\right) \ge N\left(r, \frac{1}{H}\right) - O(\log r).$$

Hence $\lambda(f) \ge \lambda(H)$, which proves Lemma 9.

Remark. The words “finite order” cannot be deleted from Lemma 9; see Example 8 in Section 3.

6. Proof of Theorem 1

Assume the contrary, i.e., suppose that an equation of the form (2.1) possesses $m+2$ linearly independent solutions $f_1, f_2, \ldots, f_{m+2}$, satisfying $\lambda(f_k) < \varrho(f_k)$ for $k = 1, 2, \ldots, m+2$. Then from Theorem 4, we have

$$f_k = g_k e^Q, \quad \text{where } \varrho(g_k) < \deg Q = \beta, \quad k = 1, 2, \ldots, m+2. \tag{6.1}$$

It is easy to see that $g_1, g_2, \ldots, g_{m+2}$ are also linearly independent. From Lemma 1 we obtain that $g_1, g_2, \ldots, g_{m+2}$ are $m+2$ solutions of one particular equation of the form

$$g^{(n)} + a_{n-1}(z) g^{(n-1)} + \cdots + a_0(z) g = h(z), \tag{6.2}$$

where $a_0(z), \ldots, a_{n-1}(z)$ are polynomials satisfying $\deg a_j \leq \tau - j(\beta - 1)$ and

$$a_j(z) = \frac{S^{(j)}(b)}{j!} z^{\tau - j(\beta-1)} + \cdots, \qquad j = 0, 1, \ldots, n-1. \tag{6.3}$$

Now set

$$f_{0,j}(z) = g_j(z) - g_{m+2}(z), \qquad \text{for } j = 1, 2, \ldots, m+1. \tag{6.4}$$

It follows that $f_{0,1}, \ldots, f_{0,m+1}$ are $m+1$ linearly independent solutions of the corresponding homogeneous equation of (6.2):

$$g^{(n)} + a_{n-1}(z) g^{(n-1)} + \cdots + a_0(z) g = 0. \tag{6.5}$$

Since, by hypothesis, b is a zero of multiplicity m of $S(t)$, we obtain from (6.3) that

$$\begin{aligned} \alpha_m &:= \deg a_m = \tau - m(\beta - 1), \\ \alpha_j &:= \deg a_j \leq \tau - j(\beta - 1), \qquad j \neq m. \end{aligned} \tag{6.6}$$

For convenience, we set $a_n(z) \equiv 1$ and $\alpha_n = 0$. Denote $A_{0,j}(z) = a_j(z)$ for $j = 0, 1, \ldots, n$. Then by applying Lemma 5 to equation (6.5), we obtain a function $f_{m,1}(z) \not\equiv 0$ that satisfies equation (5.7) when $q = m$. This gives

$$-A_{m,0}(z) = \frac{f_{m,1}^{(n-m)}}{f_{m,1}} + \sum_{j=1}^{n-m-1} A_{m,j}(z) \frac{f_{m,1}^{(j)}}{f_{m,1}}. \tag{6.7}$$

From (5.6) we see that

$$\varrho(f_{m,1}) \leq \varrho_0 \tag{6.8}$$

where $\varrho_0 = \max_{1\le j\le m+1}\{\varrho(f_{0,j})\}$. Also, from (6.1) and (6.4) we have

$$\varrho_0 < \beta. \tag{6.9}$$

Now let $\varepsilon > 0$ be a given small constant. Then

$$|A_{0,j}(z)| \le |z|^{\alpha_j+\varepsilon}, \qquad |z| \notin E,\ j = 0, 1, \ldots, n. \tag{6.10}$$

We now obtain an estimate for $|A_{m,0}(z)|$ by using equation (6.7), and applying the estimates in Lemma 2 and (5.9). Specifically, by using (6.8) and Lemma 2, and also (6.10), (5.8), (5.9), and (5.10), we deduce that

$$|A_{m,0}(z)| \le |z|^{\eta}, \qquad |z| \notin E, \tag{6.11}$$

where

$$\eta = \max_{1\le j\le n-m}\{\tau_{m,j} + j(\varrho_0 - 1 + \varepsilon)\}, \tag{6.12}$$

where $\tau_{m,n-m} = 0$, and where for $1 \le j \le n - m - 1$, $\tau_{m,j}$ is given by (5.10) with $\tau_{0,k} = \alpha_k + \varepsilon$ for $m + j \le k \le n$ in (5.10). From (6.12) and (5.10), it can be deduced that

$$\eta \le \max_{m+1\le k\le n}\{\alpha_k + (k - m)(\varrho_0 - 1) + (n + 2)\varepsilon\}. \tag{6.13}$$

Next, from (5.11) we have

$$|A_{0,m}(z)| \le |A_{m,0}(z)| + |G_m(z)| \tag{6.14}$$

where $G_m(z)$ satisfies (5.12) and (5.13). From (6.14), (5.12), (5.13), (6.11), and (6.13), we obtain that

$$|A_{0,m}(z)| \le |z|^{\mu}, \qquad |z| \notin E, \tag{6.15}$$

where

$$\mu = \max_{m+1\le k\le n}\{\alpha_k + (k - m)(\varrho_0 - 1) + (n + 3)\varepsilon\}. \tag{6.16}$$

Now we estimate μ in (6.16). By using (6.6) and (6.9), we obtain for each $k = m + 1,\ m + 2, \ldots, n$,

$$\begin{aligned} \alpha_k + (k - m)(\varrho_0 - 1) + (n + 3)\varepsilon &\le \tau - k(\beta - 1) + (k - m)(\varrho_0 - 1) + (n + 3)\varepsilon \\ &= \alpha_m + (k - m)(\varrho_0 - \beta) + (n + 3)\varepsilon \\ &< \alpha_m, \end{aligned} \tag{6.17}$$

when ε is chosen sufficiently small. Thus from (6.17) and (6.16),

$$\mu < \alpha_m. \tag{6.18}$$

But $A_{0,m}(z) = a_m(z)$, and so $\deg A_{0,m} = \alpha_m$ from (6.6). Thus (6.18) and (6.15) give a contradiction. This contradiction proves Theorem 1.

7. Proof of Theorem 2

Proof of necessity. Suppose that every solution f of (2.1) satisfies $\lambda(f) < \varrho(f)$. Then equation (2.1) possesses $n+1$ linearly independent solutions f, each satisfying $\lambda(f) < \varrho(f)$. Hence from Theorem 1, b must be a zero of $S(t)$ of multiplicity at least n. But $S(t)$ is a polynomial of degree at most n. Thus from (2.5), $S(t) = (t-b)^n$.

Proof of sufficiency. Suppose that $S(t) = (t-b)^n$. Then from the definitions of $S(t)$ and τ in (2.5) and (2.4), we obtain

$$d_k + k(\beta - 1) = \tau, \qquad k = 0, 1, \dots, n. \tag{7.1}$$

It follows that

$$\tau = n(\beta - 1) \quad \text{and} \quad d_k = (n-k)(\beta - 1) \quad \text{for } k = 0, 1, \dots, n-1. \tag{7.2}$$

Let f be any solution of (2.1). From Lemma 6 and (7.2) we obtain

$$\beta \le \varrho(f) \le \max\left\{\beta,\ 1 + \max_{0 \le k \le n-1} \frac{d_k}{n-k}\right\} = \beta,$$

which gives

$$\varrho(f) = \beta. \tag{7.3}$$

Set

$$g = fe^{-Q}. \tag{7.4}$$

Then from Lemma 1, g is a solution of an equation of the form

$$g^{(n)} + a_{n-1}(z)g^{(n-1)} + \cdots + a_0(z)g = h(z), \tag{7.5}$$

where $a_0(z), \dots, a_{n-1}(z)$ are polynomials satisfying $\deg a_k \le \tau - k(\beta - 1)$ and

$$a_k(z) = \frac{S^{(k)}(b)}{k!} z^{\tau - k(\beta - 1)} + \cdots, \qquad k = 0, 1, \dots, n-1. \tag{7.6}$$

Since $S(t) = (t-b)^n$, we obtain from (7.6) that

$$\deg a_k < \tau - k(\beta - 1), \qquad k = 0, 1, \dots, n-1. \tag{7.7}$$

From (7.7) and (7.2) we have

$$\deg a_k < (n-k)(\beta - 1), \qquad k = 0, 1, \dots, n-1. \tag{7.8}$$

We now apply Lemma 6 to equation (7.5), and use (7.8) and the fact that $\varrho(h) < \beta$, to obtain

$$\varrho(g) \le \max\left\{\varrho(h),\ 1 + \max_{0 \le k \le n-1} \frac{\deg a_k}{n-k}\right\} < \beta. \tag{7.9}$$

However, from (7.4) we have

$$\lambda(f) = \lambda(g) \le \varrho(g). \tag{7.10}$$

Hence from (7.10), (7.9), and (7.3), it follows that $\lambda(f) < \varrho(f)$.

8. Proof of Theorem 3

Since $S(b) \neq 0$, it follows from Theorem 1 that every solution f of (2.1) satisfies $\lambda(f) = \varrho(f)$ with at most one exceptional solution.

Now suppose that f_0 is such an exceptional solution. Then $\lambda(f_0) < \varrho(f_0)$, and from Theorem 4, f_0 has the form

$$f_0 = ge^Q, \tag{8.1}$$

where

$$\varrho(g) < \deg Q = \beta. \tag{8.2}$$

From Lemma 1, g is a solution of an equation of the form

$$g^{(n)} + a_{n-1}(z)g^{(n-1)} + \cdots + a_0(z)g = h(z), \tag{8.3}$$

where $a_0(z), \ldots, a_{n-1}(z)$ are polynomials satisfying $\deg a_k \leq \tau - k(\beta - 1)$ and

$$a_k(z) = \frac{S^{(k)}(b)}{k!} z^{\tau - k(\beta-1)} + \cdots, \qquad k = 0, 1, \ldots, n-1. \tag{8.4}$$

Since $S(b) \neq 0$, we obtain from (8.4) that

$$\deg a_0 = \tau \quad \text{and} \quad \deg a_k \leq \tau - k(\beta - 1) \quad \text{for } k = 1, 2, \ldots, n-1. \tag{8.5}$$

From equation (8.3) we see that $\varrho(g) \geq \varrho(h)$. We now prove that $\varrho(g) = \varrho(h)$. Assume the contrary, and suppose that $\varrho(g) > \varrho(h)$. Then from Lemma 8, there exists a set $J \subset (0, \infty)$ that has infinite logarithmic measure such that

$$M(r, h) < M(r, g), \qquad r \in J. \tag{8.6}$$

For each $r \in J$, we now choose a point z_r that satisfies $|z_r| = r$ and $|g(z_r)| = M(r, g)$. Then from (8.6) we obtain

$$\left| \frac{h(z_r)}{g(z_r)} \right| < 1, \qquad r \in J. \tag{8.7}$$

Since g is a solution of equation (8.3), we have

$$a_0(z_r) = \frac{h(z_r)}{g(z_r)} - \frac{g^{(n)}(z_r)}{g(z_r)} - \sum_{k=1}^{n-1} a_k(z_r) \frac{g^{(k)}(z_r)}{g(z_r)}, \qquad r \in J. \tag{8.8}$$

We now estimate the right side of (8.8). Set $\mu = \varrho(g)$, and let $\varepsilon > 0$ be a small fixed constant. Then from Lemma 2, it follows that

$$\left|\frac{g^{(n)}(z_r)}{g(z_r)}\right| + \sum_{k=1}^{n-1} |a_k(z_r)| \left|\frac{g^{(k)}(z_r)}{g(z_r)}\right| \le |z_r|^{\sigma} \qquad \text{for } r \in J \setminus E, \tag{8.9}$$

where

$$\sigma = \max_{1\le k\le n} \{\deg a_k + k(\mu - 1) + \varepsilon\}. \tag{8.10}$$

Here we set $a_n(z) \equiv 1$. Since $\mu < \beta$ from (8.2), it can be seen from (8.10) and (8.5) that

$$\sigma \le \max\{n(\mu - 1) + \varepsilon, \tau + \mu - \beta + \varepsilon\}. \tag{8.11}$$

Note that $\tau \ge n(\beta - 1)$ from (2.4), and $\mu < \beta$ from (8.2). Thus if ε is chosen sufficiently small, then we see from (8.11) that

$$\sigma < \tau. \tag{8.12}$$

Now we use (8.7) and (8.9) in (8.8), and obtain

$$|a_0(z_r)| < 1 + |z_r|^{\sigma}, \qquad r \in J \setminus E, \tag{8.13}$$

where σ satisfies (8.12). However, from (8.5), $a_0(z)$ is a polynomial of degree τ, which contradicts (8.13) and (8.12). This contradiction proves that

$$\varrho(g) = \varrho(h). \tag{8.14}$$

We consider two cases.

Case (i): *$h(z)$ is transcendental.* From Lemma 9 and (2.2) we have

$$\lambda(f_0) \ge \varrho(h).$$

On the other hand, from (8.1) and (8.14) we obtain

$$\lambda(f_0) = \lambda(g) \le \varrho(g) = \varrho(h).$$

Thus $\lambda(f_0) = \varrho(h)$. Hence from (2.2), $\lambda(f_0) = \lambda(h)$. This proves Theorem 3(i).

Case (ii): *$h(z)$ is a polynomial.* In this case, $\varrho(g) = 0$ from (8.14). Since any transcendental solution of equation (8.3) has positive order (see [14, pp. 106–108]), it follows that g is a polynomial. Since $\beta \ge 1$ from (2.3), by inspection of (8.3) and (8.5) we see that

$$\deg h \ge \tau \qquad \text{and} \qquad \deg g = \deg h - \tau. \tag{8.15}$$

From (8.1) and (8.15) we obtain Theorem 3(ii).

9. Proof of Theorem 5

Suppose that f_0 is a solution of equation (1.1) satisfying $\lambda(f_0) < \infty$. Since $\varrho(H) = \infty$, we have $\varrho(f_0) = \infty$ from (1.1). Thus

$$f_0(z) = u(z)e^{w(z)} \tag{9.1}$$

where $u(z) \not\equiv 0$ and $w(z)$ are entire functions satisfying $\varrho(u) < \infty$ and $\varrho(e^w) = \infty$.

Now suppose that f is a solution of (1.1) where $f \not\equiv f_0$. We now show that $\lambda(f) = \infty$, which will prove Theorem 5. From (9.1) and (1.1) it follows that

$$f(z) = g(z) + u(z)e^{w(z)} \tag{9.2}$$

where $g \not\equiv 0$ is a solution of the homogeneous equation

$$g^{(n)} + P_{n-1}(z)g^{(n-1)} + \cdots + P_0(z)g = 0. \tag{9.3}$$

From Lemma 3 and (9.3), we have $\varrho(g) < \infty$. Since g and u are entire, we see from (9.2) that

$$N\Big(r, -1, \frac{u}{g}e^w) \le N(r, 0, f\Big). \tag{9.4}$$

We now make the assumption that $\lambda(f) < \infty$. Set

$$B(z) = \frac{u(z)}{g(z)}e^{w(z)}. \tag{9.5}$$

From the second fundamental theorem and (9.5) we obtain

$$\begin{aligned}\big(1 + o(1)\big)T(r, B) &\le N(r, B) + N(r, 0, B) + N(r, -1, B)\\ &\le N(r, 0, g) + N(r, 0, u) + N\Big(r, -1, \frac{u}{g}e^w\Big)\end{aligned} \tag{9.6}$$

as $r \to \infty$, $r \notin D$. Then from (9.6), (9.4), and the first fundamental theorem, we obtain

$$\big(1 + o(1)\big)T(r, B) \le T(r, g) + T(r, u) + N(r, 0, f) \tag{9.7}$$

as $r \to \infty$, $r \notin D$. Since $\varrho(g) < \infty$, $\varrho(u) < \infty$, and $\lambda(f) < \infty$, it follows from (9.7) that there exists a constant $\alpha > 0$ such that

$$T(r, B) \le r^\alpha, \qquad r \notin D. \tag{9.8}$$

Since a set of finite linear measure is also a set of finite logarithmic measure, we can apply Lemma 7 to (9.8). We obtain that there exists a constant $R > 0$ such that

$$T(r, B) \le (2r)^\alpha \quad \text{for all } r \ge R,$$

which implies that $B(z)$ has finite order. Since g and u also have finite order, it follows from (9.5) that $\varrho(e^w) < \infty$, which is a contradiction. This contradiction proves that our assumption $\lambda(f) < \infty$ must be false. Hence $\lambda(f) = \infty$, which is what we wanted to prove.

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