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BLOCH SPACE ON THE UNIT BALL OF $\mathbf{C}^n$

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Abstract. In this paper we prove that for $0 < p < \infty$, the norm of a function f in the Bergman space L_a^p on the unit ball B of $\mathbf{C}^n$, $n \geq 1$, is equivalent to the quantity

$$\int_B |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2} h(|z|)\, d\tau(z),$$

where $\widetilde{\nabla}$ and τ denote the invariant gradient and invariant measure on B, respectively, and

$$h(|z|) = \int_{|z|}^1 \frac{(1-t^2)^{n-1}(1-t^{2n})}{t^{2n-1}}\, dt.$$

If $n > 1$, this result allows us to extend the characterization J_2 of the Bloch space obtained in [OYZ, Theorem 2] to the range $0 < p < \infty$. We also get this kind of description of Bloch functions for $n = 1$.

Moreover, we generalize the result obtained in [CKP] and show that $f \in H(B)$ is a Bloch function if and only if for some p, $0 < p < \infty$, $|\widetilde{\nabla} f(z)|^p dv(z)$ is a Bergman–Carleson measure. Finally, we get some results for spaces H^p and BMOA, e.g. an extension of the classical Littlewood–Paley inequality to the case of the unit ball.

1. Introduction

Let B denote the unit ball in C^n, dv the normalized Lebesgue measure on B and $d\sigma$ the normalized surface measure on the boundary S of B. Let $H(B)$ be the space of all holomorphic functions on B. For $0 < p < \infty$, the Bergman space L_a^p and the Hardy space H^p are defined respectively as

$$L_a^p = \left\{ f : f \in H(B),\ \|f\|_{L_a^p}^p = \int_B |f(z)|^p\, dv(z) < \infty \right\}$$

and

$$H^p = \left\{ f : f \in H(B),\ \|f\|_{H^p}^p = \sup_{0<r<1} \int_S |f(r\zeta)|^p\, d\sigma(\zeta) < \infty \right\}.$$

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For $f \in H(B)$, set

$$Q_f(z) = \sup\left\{ \frac{|\langle \nabla f(z), \bar{x}\rangle|}{\big(H_z(x,x)\big)^{1/2}} : 0 \neq x \in \mathbf{C}^n \right\},$$

where $\nabla f(z) = (\partial f/\partial z_1, \dots, \partial f/\partial z_n)$ is the complex gradient of f and $H_z(x,x)$ is the Bergman metric on B, i.e.

$$H_z(x,x) = \frac{n+1}{2} \frac{(1-|z|^2)|x|^2 + |\langle x, z\rangle|^2}{(1-|z|^2)^2}.$$

The Bloch space $\mathscr{B}$ (introduced by R. Timoney, [T]) is the set of holomorphic functions f on B for which

$$\|f\|_{\mathscr{B}} = \sup\{Q_f(z) : z \in B\} < \infty.$$

R. Timoney [T] has proved that the norms

$$\|f\|_1 = \sup\{|\nabla f(z)|(1-|z|^2) : z \in B\},$$
$$\|f\|_2 = \sup\{|\langle \nabla f(z), \bar{z}\rangle|(1-|z|^2) : z \in B\}$$

and $\|f\|_{\mathscr{B}}$ are equivalent.

Let $\widetilde{\nabla}$ denote the invariant gradient on B, that is,

$$(\widetilde{\nabla} f)(z) = \nabla(f \circ \varphi_z)(0),$$

where φ_z denotes the involutive automorphism of B satisfying $\varphi_z(0) = z$, $\varphi_z(z) = 0$.

Recently C. Ouyang, W. Yang and R. Zhao [OYZ] gave the following characterization of the Bloch space.

Theorem A. *Let $n > 1$, $p \geq 2$; then the following quantities are equivalent:*

(a) $\|f\|_{\mathscr{B}}^p$,

(b) $J_2 = \sup\limits_{a \in B} \int_B |\widetilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} \big(1 - |\varphi_a(z)|^2\big)^{n+1} |\varphi_a(z)|^{-2n+2}\, d\tau(z)$,

(c) $J_3 = \sup\limits_{a \in B} \int_B |\widetilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} [G(z,a)]^{1+(1/n)}\, d\tau(z)$,

where $G(z,a)$ is the Green function of B and $d\tau(z) = (n+1)dv(z)/(1-|z|^2)^{n+1}$.

The authors [OYZ, p. 4310, Remark 3] conjectured that Theorem A holds for all $p > 0$, that is, also for $0 < p < 2$. Here we show that the quantities given by formulas (a) and (b) are equivalent for all positive p. Moreover, we prove

Theorem 1. *Let $0 < p < \infty$ and $n > 1$; then $\|f\|_{\mathscr{B}}^p$ and J_2, defined in Theorem A, and the following quantities are equivalent*

$$J_4 = \sup_{a\in B} \int_B |\widetilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} G(z,a)\big(1 - |\varphi_a(z)|^2\big)\, d\tau(z),$$

$$J_5 = \sup_{a\in B} \int_B |\widetilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} h\big(|\varphi_a(z)|\big)\, d\tau(z),$$

where the function h is given by the formula

$$h(|z|) = \int_{|z|}^1 \frac{(1-t^2)^{n-1}(1-t^{2n})}{t^{2n-1}}\, dt.$$

For the case $n = 1$ we get

Theorem 2. *Let $0 < p < \infty$ and f be a holomorphic function on the unit disc $\mathbf{D}$ in the complex plane. Then f is a Bloch function if and only if*

$$\sup_{a\in \mathbf{D}} \int_{\mathbf{D}} |f(z) - f(a)|^{p-2} |f'(z)|^2 g(z,a)\big(1 - |\varphi_a(z)|^2\big)\, dv(z) < \infty \tag{1}$$

where $g(z,a)$ is the Green function $\log\big|(1-\bar{a}z)/(z-a)\big|$ of $\mathbf{D}$ with logarithmic singularity at $a \in \mathbf{D}$.

Our next characterization of Bloch functions is connected with a Bergman–Carleson measure. For $\eta \in S$ and $\delta > 0$, let

$$D_\delta(\eta) = \{z \in B : |1 - \langle z, \eta\rangle| < \delta\}.$$

A positive measure μ on B is called a Bergman–Carleson measure (Carleson measure) if and only if $\mu\big(D_\delta(\eta)\big) = O(\delta^{n+1})$ ($\mu\big(D_\delta(\eta)\big) = O(\delta^n)$). It has been proved in [CKP] that a holomorphic function f on B is a Bloch function if and only if $|\widetilde{\nabla} f(z)|^2\, dv(z)$ is a Bergman–Carleson measure. Here we obtain the following generalization of this result.

Theorem 3. *Let $0 < p < \infty$. If $f \in H(B)$, then the following statements are equivalent*:

(a) $\|f\|_{\mathscr{B}} < \infty$,
(b) $|\widetilde{\nabla} f(z)|^p\, dv(z)$ *is a Bergman–Carleson measure,*
(c) $|\nabla f(z)|^p (1-|z|^2)^p\, dv(z)$ *is a Bergman–Carleson measure.*

Section 4 of this paper contains the proof of Theorem 3 and some results for the Hardy space H^p and BMOA functions in the unit ball, e.g. a higher dimensional version of the classical Littlewood–Paley result [LP]: if $f \in H^p$, $2 \le p < \infty$, then

$$\int_{\mathbf{D}} |f'(z)|^p (1-|z|)^{p-1}\, dv(z) < \infty.$$

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2. Preliminaries

For $a \in B$, let φ_a be the involutive automorphism of B given by the formula

$$\varphi_a(z) = \frac{a - P_a z - (1-|a|^2)^{1/2} Q_a z}{1 - \langle z, a\rangle}, \tag{2}$$

where P_a is the orthogonal projection into the space spanned by a i.e.

$$P_a z = \frac{\langle z, a\rangle}{|a|^2} a, \qquad P_0 z = 0,$$

and

$$Q_a = I - P_a.$$

It has been shown in [R] that

$$\varphi_a'(0) = -(1-|a|^2)P_a - \sqrt{1-|a|^2}\, Q_a. \tag{3}$$

Now, by the chain rule and the symmetry of matrix (3) we get

$$\begin{aligned} |\widetilde{\nabla} f(z)|^2 &= |\nabla(f \circ \varphi_z)(0)|^2 = |\nabla f(z)\varphi_z'(0)|^2 = |\varphi_{\bar z}'(0)\nabla f(z)|^2 \\ &= (1-|z|^2)^2 |P_{\bar z}\nabla f(z)|^2 + (1-|z|^2)|Q_{\bar z}\nabla f(z)|^2 \\ &= (1-|z|^2)\big(|\nabla f(z)|^2 - |\langle \nabla f(z), \bar z\rangle|^2\big) \\ &= (1-|z|^2)\big(|\nabla f(z)|^2 - |Rf(z)|^2\big), \end{aligned} \tag{4}$$

where Rf is the radial derivative of f.

For $u \in C^2(B)$, let $\widetilde{\Delta} u$ denote the invariant Laplacian on B, i.e.

$$(\widetilde{\Delta} u)(z) = \Delta(u \circ \varphi_z)(0),$$

where Δ is the ordinary Laplacian.

K. Hahn and E. Youssfi [HY] proved that for $f \in H(B)$

$$Q_f(z) = \frac{1}{\sqrt{2(n+1)}}\sqrt{\widetilde{\Delta}|f|^2(z)} = \sqrt{\frac{2}{n+1}}\,|\widetilde{\nabla} f(z)|.$$

Hence the following statements are equivalent:

(i) $f \in \mathscr{B}$,

(ii) $\sup\{\widetilde{\Delta}|f|^2 : z \in B\} < \infty$,

(iii) $\sup\{|\widetilde{\nabla} f(z)| : z \in B\} < \infty$.

We will say that the quantities A_f and B_f that depend on f are equivalent if there exists a positive constant C independent of f such that

$$A_f/C \le B_f \le CA_f \qquad \text{for every } f \in H(B).$$

In this paper C and C_j will always denote positive constants, independent of f, not necessarily the same at each occurrence.

The next lemma is the generalization of the result due to S. Axler (see also [S]) to the several variables case. For $p \ge 1$, this lemma has been proved in [O2].

Lemma 1. *Let* $0 < p < \infty$. *A holomorphic function* f *is in the Bloch space* $\mathscr{B}$ *if and only if*

$$\sup_{a \in B} \|f \circ \varphi_a - f(a)\|_{L_a^p} < \infty. \tag{5}$$

Proof. Assume that $f \in \mathscr{B}$. Then we have

$$\begin{aligned} |f(z) - f(0)| &= \left| \int_0^1 \langle \nabla f(tz), \bar{z} \rangle \, dt \right| \le \int_0^1 |\nabla f(tz)| \, |z| \, dt \\ &\le C \|f\|_{\mathscr{B}} \int_0^1 \frac{|z|}{(1 - |zt|)} \, dt = C \|f\|_{\mathscr{B}} \ln \frac{1}{1 - |z|}. \end{aligned}$$

The Möbius invariance property of the Bloch space implies

$$|f \circ \varphi_a(z) - f(a)| \le C \|f\|_{\mathscr{B}} \ln \frac{1}{1 - |z|}.$$

This gives

$$\int_B |f \circ \varphi_a(z) - f(a)|^p \, dv(z) \le C \|f\|_{\mathscr{B}}^p \int_B \left(\ln \frac{1}{1 - |z|} \right)^p dv(z) \le C \|f\|_{\mathscr{B}}^p < \infty.$$

Now, suppose that $\sup_{a \in B} \|f \circ \varphi_a - f(a)\|_{L_a^p} < \infty$. It was shown in [Shi, p. 623 (9)] that for $f \in H(B)$ and $0 < p < \infty$

$$|\nabla f(0)|^p \le C \int_B |f(z)|^p \, dv(z). \tag{6}$$

Replacing f by $f \circ \varphi_a - f(a)$ we obtain

$$|\widetilde{\nabla} f(a)|^p = |\nabla (f \circ \varphi_a)(0)|^p \le C \int_B |f \circ \varphi_a(z) - f(a)|^p \, dv(z),$$

which implies $\sup_{a \in B} |\widetilde{\nabla} f(a)|^p < \infty$.

3. Proof of Theorems 1 and 2

For $f \in H(B)$, let

$$f_p^{\#}(z) = p^2 |f(z)|^{p-2} |\widetilde{\nabla} f(z)|^2.$$

The proof of Theorem 1 is based on the following

Lemma 2. *If* $0<p<\infty$, $f\in L^p_a$ *and* $f(0)=0$, *then*

$$\|f\|^p_{L^p_a}=\frac{1}{(2n)^2}\int_B f^\#_p(z)h(|z|)\,d\tau(z),$$

where

$$h(|z|)=\int_{|z|}^1\frac{(1-t^2)^{n-1}(1-t^{2n})}{t^{2n-1}}\,dt.$$

Proof. Combining formulas (3) and (4) of [OYZ, p. 4305] we get

$$\int_S|f(r\zeta)|^p\,d\sigma(\zeta)=\frac{1}{2n}\int_0^r\frac{(1-t^2)^{n-1}}{t^{2n-1}}\Big(\int_{B_t}f^\#_p(z)\,d\tau(z)\Big)\,dt,$$

where $B_t=\{z\in\mathbf{C}^n:|z|<t\}$.

Multiplying both sides of this equality by r^{2n-1} and integrating with respect to r give

$$\int_B|f(z)|^p\,dv(z)=\frac{1}{2n}\int_0^1 r^{2n-1}\,dr\Big(\int_0^r\frac{(1-t^2)^{n-1}}{t^{2n-1}}\,dt\int_{B_t}f^\#_p(z)\,d\tau(z)\Big).$$

By Fubini's theorem

$$\begin{aligned}\int_B|f(z)|^p\,dv(z)&=\frac{1}{2n}\int_0^1\Big(\int_t^1 r^{2n-1}\,dr\frac{(1-t^2)^{n-1}}{t^{2n-1}}\int_{B_t}f^\#_p(z)\,d\tau(z)\Big)\,dt\\&=\frac{1}{(2n)^2}\int_0^1\frac{(1-t^{2n})(1-t^2)^{n-1}}{t^{2n-1}}\int_{B_t}f^\#_p(z)\,d\tau(z)\,dt.\end{aligned}$$

Let

$$\chi_{|z|}(t)=\begin{cases}1, & |z|<t,\\ 0, & \text{otherwise,}\end{cases}$$

then the right side of the last equality can be be written as

$$\begin{aligned}&\frac{1}{(2n)^2}\int_B\int_0^1\frac{(1-t^{2n})(1-t^2)^{n-1}}{t^{2n-1}}\chi_{|z|}(t)f^\#_p(z)\,dt\,d\tau(z)\\&\qquad=\frac{1}{(2n)^2}\int_B h(|z|)f^\#_p(z)\,d\tau(z).\ \square\end{aligned}$$

Proof of Theorem 1. Let $n>1$ and $0<p<\infty$ be fixed. By Lemma 1 $\|f\|^p_{\mathscr{B}}$ is equivalent to $\sup_{a\in B}\|f\circ\varphi_a-f(a)\|_{L^p_a}$. Next, using Lemma 2 we obtain

$$\begin{aligned}&\sup_{a\in B}\|f\circ\varphi_a-f(a)\|_{L^p_a}\\&\qquad=\sup_{a\in B}\frac{p^2}{(2n)^2}\int_B h(|z|)|f\circ\varphi_a(z)-f(a)|^{p-2}|\widetilde{\nabla}(f\circ\varphi_a)(z)|^2\,d\tau(z)\\&\qquad=\sup_{a\in B}\frac{p^2}{(2n)^2}\int_B h\big(|\varphi_a(z)|\big)|f(z)-f(a)|^{p-2}|\widetilde{\nabla}f(z)|^2\,d\tau(z).\end{aligned}$$

Since

$$\lim_{|z|\to 1}\frac{h(|z|)}{(1-|z|^2)^{n+1}|z|^{-2n+2}}=\frac{n}{2(n+1)}$$

and

$$\lim_{|z|\to 0}\frac{h(|z|)}{(1-|z|^2)^{n+1}|z|^{-2n+2}}=\frac{1}{2(n-1)},$$

and the function $h(|z|)$ is continuous in the interval $(0,1)$ there are positive constants C_1 and C_2 such that for all $z\in B\setminus\{0\}$

$$C_1(1-|z|^2)^{n+1}|z|^{-2n+2}\le h(|z|)\le C_2(1-|z|^2)^{n+1}|z|^{-2n+2}.$$

Thus the quantities $\|f\|_{\mathscr{B}}^p$, J_2 and J_5 are equivalent. It follows immediately from Lemma 1 of [OYZ] that also J_4 is equivalent to each of these quantities. □

Proof of Theorem 2. The proof is based on the Hardy–Stein identity [H, p. 42]

$$r\frac{d}{dr}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\right)=\int_{|z|<r}f^{\#}(z)\,dv(z),$$

where

$$f^{\#}(z)=\tfrac{1}{2}p^2|f(z)|^{p-2}|f'(z)|^2.$$

If $f(0)=0$ then we have (by integration)

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta=\int_0^r\left(\int_{|z|<t}\frac{1}{t}f^{\#}(z)\,dv(z)\right)dt.$$

From this

$$\begin{aligned}\int_{\mathbf{D}}|f(z)|^p\,dv(z)&=2\int_0^1 r\left(\int_0^r\left(\int_{|z|<t}\frac{1}{t}f^{\#}(z)\,dv(z)\right)dt\right)dr\\&=2\int_0^1\left(\frac{1}{t}\int_t^1 r\,dr\int_{|z|<t}f^{\#}(z)\,dv(z)\right)dt\\&=\int_0^1\frac{1-t^2}{t}\left(\int_{|z|<t}f^{\#}(z)\,dv(z)\right)dt\\&=\int_{\mathbf{D}}\int_{|z|}^1\frac{1-t^2}{t}\,dt f^{\#}(z)\,dv(z)=\int_{\mathbf{D}}h_1(|z|)f^{\#}(z)\,dv(z),\end{aligned}$$

where $h_1(r)=\ln(1/r)-(1-r^2)/2$, $0<r<1$. Now, applying Lemma 1 and a change of variables we conclude that f is a Bloch function if and only if

$$\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}|f(z)-f(a)|^{p-2}|f'(z)|^2h_1\big(|\varphi_a(z)|\big)\,dv(z)<\infty. \tag{7}$$

Since

$$\lim_{r\to 1^-}\frac{(1-r^2)\ln(1/r)}{h_1(r)}=2\qquad\text{and}\qquad\lim_{r\to 0^+}\frac{(1-r^2)\ln(1/r)}{h_1(r)}=1,$$

condition (7) is equivalent to (1). □

4. Carleson measure characterizations of Bloch and BMOA space

Proof of Theorem 3. The implications: (a) implies (c) and (b) implies (c) follow immediately from the descriptions of the Bloch space presented in the introduction and from Proposition 3.1 of [CKP]. In fact, for example,

$$\sup_{a\in B}\int_B\left(\frac{1-|z|^2}{|1-\langle z,a\rangle|^2}\right)^{n+1}|\widetilde{\nabla}f(z)|^p\,dv(z)$$
$$\le \sup_{z\in B}|\widetilde{\nabla}f(z)|^p\sup_{a\in B}\int_B\left(\frac{1-|z|^2}{|1-\langle z,a\rangle|^2}\right)^{n+1}dv(z)<\infty.$$

Now assume that (for some positive p) $|\widetilde{\nabla}f(z)|^p\,dv(z)$ is a Bergman–Carleson measure. According to Proposition 3.1 of [CKP] this means that

$$\sup_{a\in B}\int_B\left(\frac{1-|z|^2}{|1-\langle z,a\rangle|^2}\right)^{n+1}|\widetilde{\nabla}f(z)|^p\,dv(z)<\infty. \tag{7}$$

Let $0<r<1$ be arbitrarily fixed. The subharmony of $|\partial f/\partial z_i|^p\circ\varphi_a$, $i=1,2,\ldots,n$, implies

$$\left|\frac{\partial f}{\partial z_i}(a)\right|^p\le\frac{1}{r^{2n}}\int_{B_r}\left|\frac{\partial f}{\partial z_i}\right|^p\circ\varphi_a(z)\,dv(z)\le\frac{1}{r^{2n}}\int_{B_r}|\nabla f|^p\circ\varphi_a(z)\,dv(z). \tag{8}$$

In view of the inequality

$$|\nabla f(a)|^p\le\begin{cases}\displaystyle\sum_{i=1}^n\left|\frac{\partial f}{\partial z_i}(a)\right|^p & \text{if } 0<p\le 1,\\ \displaystyle n^{p-1}\sum_{i=1}^n\left|\frac{\partial f}{\partial z_i}(a)\right|^p & \text{if } p>1,\end{cases}$$

(8) yields

$$|\nabla f(a)|^p\le C\frac{1}{r^{2n}}\int_{B_r}|\nabla f|^p\circ\varphi_a(z)\,dv(z).$$

The change of variables $z=\varphi_a(w)$ gives

$$|\nabla f(a)|^p\le C\frac{1}{r^{2n}}\int_{\varphi_a(B_r)}|\nabla f(z)|^p\left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{n+1}dv(z).$$

The equality

$$1-|\varphi_a(z)|^2=\frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2}$$

implies

$$1-|z|^2 > \tfrac{1}{4}(1-r)(1-|a|^2) \qquad \text{for } z \in \varphi_a(B_r).$$

This follows that

$$\begin{aligned}(1-|a|^2)^p|\nabla f(a)|^p &\le \frac{4^p}{(1-r)^p r^{2n}} \int_{\varphi_a(B_r)} |\nabla f(z)|^p (1-|z|^2)^p \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{n+1} dv(z) \\ &\le \frac{4^p}{(1-r)^p r^{2n}} \int_B |\nabla f(z)|^p (1-|z|^2)^p \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{n+1} dv(z) \\ &\le \frac{4^p}{(1-r)^p r^{2n}} \int_B |\widetilde{\nabla} f(z)|^p \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{n+1} dv(z)\end{aligned}$$

which together with (7) imply $f \in \mathscr{B}$. □

Recall that $f \in H^2$ is said to be a BMOA function if its radial limit function $f^\star$ is a function of bounded mean oscillation on S with respect to nonisotropic balls $Q_\delta(\zeta) = \{\eta \in S : |1-\langle \eta,\zeta\rangle| < \delta\}$, $\zeta \in S$, $0 < \delta \le 2$. The following Carleson measure characterizations of BMOA functions have been obtained in [J] and [CC].

Theorem B. *Suppose* $f \in H^2$. *Then the following statements are equivalent*:

(a) $f \in \mathrm{BMOA}$,

(b) $|\widetilde{\nabla} f(z)|^2 (1-|z|^2)^{-1}\, dv(z)$ *is a Carleson measure,*

(c) $|\nabla f(z)|^2 (1-|z|^2)\, dv(z)$ *is a Carleson measure.*

In view of Theorem 3 it is natural to ask if there is also an analogous generalization of this theorem. Here, we will give the following necessary conditions for f to belong to BMOA.

Theorem 4. *If* $p \ge 2$ *and* $f \in \mathrm{BMOA}$ *then*

(a) $|\widetilde{\nabla} f(z)|^p (1-|z|^2)^{-1}\, dv(z)$ *is a Carleson measure,*

(b) $|\nabla f(z)|^p (1-|z|^2)^{p-1}\, dv(z)$ *is a Carleson measure.*

Remark 1. For $n = 1$, the results contained in Theorems 3 and 4 are equivalent to those stated in [S, Theorem 1, D and p. 417]. Moreover, it is shown in [M] that the function $f(z) = \sum_{n=1}^{\infty} n^{-1/2} z^{2^n}$, $z \in \mathbf{D}$, satisfies condition (a) (or equivalently (b)) given in Theorem 4 with $p > 2$ but $f \notin \mathrm{BMOA}$. So, we do not think that in the case $n > 1$ and $p > 2$ these conditions are sufficient for f to belong to BMOA.

To prove Theorem 4 we will need the below stated characterization of the Hardy space H^p, $p \ge 2$, which is an extension of a Littlewood–Paley classical result to the setting of the unit ball B. Similarly to the case $n = 1$ this can be proved by using the M. Riesz–Thorin theorem (see e.g. [Z, pp. 95, 216])

For $F \in C^1(B)$ let

$$DF = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial y_1}, \dots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial y_n}\right), \qquad z_k = x_k + iy_k,\ k = 1,\dots,n,$$

be the real gradient of F and let $\widetilde{D}F(z) = D(F \circ \varphi_z)(0)$. If, as above,

$$\nabla F = \left(\frac{\partial F}{\partial z_1}, \dots, \frac{\partial F}{\partial z_n}\right)$$

is the complex gradient of F then, as one can easily check,

$$2|DF|^2 = |\nabla F|^2 + |\nabla \overline{F}|^2.$$

As in [R] the Poisson integral $P[f]$ of a function $f \in L^1(S, d\sigma)$ is defined, for $z \in B$, by

$$P[f](z) = \int_S \frac{(1-|z|^2)^n f(\zeta)}{|1-\langle z,\zeta\rangle|^{2n}}\, d\sigma(\zeta).$$

The next lemma is an analogue of Theorem 3.24 of [Z, p. 216].

Lemma 3. *If* $2 \le p \le \infty$, $f \in L^p(S, d\sigma)$ *and* $F = P[f]$, *then*

(9) $$\int_B |\widetilde{D}F(z)|^p (1-|z|^2)^{-1}\, dv(z) \le C \int_S |f(\zeta)|^p\, d\sigma.$$

Proof. We assume that $n > 1$. For $f \in L^p(S, d\sigma)$, define the operator T by the formula

$$Tf = \widetilde{D}(F).$$

Inequality (9) can be written in the form

$$\|Tf\|_p \le C\|f\|_p, \qquad 2 \le p \le \infty,$$

where the norm $\|Tf\|_p$ is taken with respect to the measure $(1-|z|^2)^{-1}\, dv(z)$. In view of the Riesz–Thorin theorem it is enough to show that T is both of type $(2,2)$ and of type (∞,∞). Since F is an $\mathscr{M}$-harmonic function in B (see [R, p. 49]) Lemma 3.2 in [CC] and (4) give

(10) $$\begin{aligned}\widetilde{\Delta}|F|^2 &= 4(1-|z|^2)(|\nabla F|^2 - |RF|^2 + |\nabla \overline{F}|^2 - |R\overline{F}|^2)\\ &= 4(|\widetilde{\nabla} F|^2 + |\widetilde{\nabla}\overline{F}|^2) = 8|\widetilde{D}F|^2.\end{aligned}$$

By Theorem A in [CC]

$$\int_B G\widetilde{\Delta}|F|^2\, d\tau = (n+1)^2 \int_S |f - F(0)|^2\, d\sigma \le 4(n+1)^2 \int_S |f|^2\, d\sigma$$

where

$$G(z) = \frac{n+1}{2n} \int_{|z|}^1 r^{-2n+1}(1-r)^{n-1}\, dr.$$

From (10) and Lemma 1 in [OYZ]

$$\begin{aligned}\int_B |\widetilde{D}F(z)|^2(1-|z|^2)^{-1}\,dv(z) &= \frac{1}{8}\int_B \widetilde{\Delta}|F(z)|^2(1-|z|^2)^{-1}\,dv(z)\\ &\le C_1\int_B |\widetilde{\nabla}F(z)|^2 G(z)\,d\tau(z)\\ &\le C_2\int_S |f(\zeta)|\,d\sigma(\zeta),\end{aligned}$$

which proves that T is of type $(2,2)$.

Let

$$P(z,\zeta)=\frac{(1-|z|^2)^n}{|1-\langle z,\zeta\rangle|^{2n}}$$

denote the Poisson kernel. To prove that T is of type (∞,∞) it is enough to show that

$$\int_S |\widetilde{D}P(z,\zeta)|\,d\sigma(\zeta)\le C<\infty. \tag{11}$$

Notice first that since $P(z,\zeta)=\overline{P}(z,\zeta)$

$$|\widetilde{D}P(z,\zeta)|=|\widetilde{\nabla}P(z,\zeta)|. \tag{12}$$

To calculate $|\widetilde{\nabla}P(z,\zeta)|$ we will use formula (4). We have

$$\nabla P(z,\zeta)=-n\bigg(\frac{(1-|z|^2)^{n-1}}{|1-\langle z,\zeta\rangle|^{2n}}\bar{z}-\frac{(1-|z|^2)^n(1-\overline{\langle z,\zeta\rangle}\,)}{|1-\langle z,\zeta\rangle|^{2n+2}}\bar{\zeta}\bigg)$$

and

$$\begin{aligned}P_{\bar{z}}\nabla P(z,\zeta)&=-n\bigg(\frac{(1-|z|^2)^{n-1}}{|1-\langle z,\zeta\rangle|^{2n}}-\frac{(1-|z|^2)^n(1-\overline{\langle z,\zeta\rangle}\,)\langle\bar{\zeta},\bar{z}\rangle}{|z|^2|1-\langle z,\zeta\rangle|^{2n+2}}\bigg)\bar{z},\\ Q_{\bar{z}}\nabla P(z,\zeta)&=-n\frac{(1-|z|^2)^n(1-\overline{\langle z,\zeta\rangle}\,)(|z|^2\bar{\zeta}-\langle\bar{\zeta},\bar{z}\rangle\bar{z})}{|z|^2|1-\langle z,\zeta\rangle|^{2n+2}}\end{aligned}$$

Hence

$$\begin{aligned}|\widetilde{\nabla}P(z,\zeta)|&\le(1-|z|^2)|P_{\bar{z}}\nabla P(z,\zeta)|+(1-|z|^2)^{1/2}|Q_{\bar{z}}\nabla P(z,\zeta)|\\ &\le n\bigg(\frac{(1-|z|^2)^n}{|1-\langle z,\zeta\rangle|^{2n}}+\frac{(1-|z|^2)^{n+1}}{|1-\langle z,\zeta\rangle|^{2n+1}}+\frac{(1-|z|^2)^{n+(1/2)}}{|1-\langle z,\zeta\rangle|^{2n+(1/2)}}\bigg).\end{aligned}$$

Now, Proposition 1.4.10. in [R, p. 17] and [12] imply [11]. □

If $1 \le p < \infty$, then H^p is the space of the Poisson integrals of L^p functions that are holomorphic on B (see e.g. [R, pp. 87–88]). Therefore, Lemma 3 implies immediately the following

Corollary. *If* $f \in H^p$, $2 \le p < \infty$, *then*

$$\text{(a)} \qquad \int_B |\widetilde{\nabla} f(z)|^p (1-|z|^2)^{-1}\, dv(z) < \infty,$$

$$\text{(b)} \qquad \int_B |\nabla f(z)|^p (1-|z|^2)^{p-1}\, dv(z) < \infty.$$

Proof of Theorem 4. The theorem follows from the corollary and from the following characterization of BMOA functions [O1]. For $f \in H^2$ and for any p, $0 < p < \infty$

$$f \in BMOA \qquad \text{if and only if} \qquad \sup_{a\in B} \|f \circ \varphi_a - f(a)\|_{H^p} < \infty.$$

Moreover, the change of variables gives

$$f \in \text{BMOA} \qquad \text{if and only if}$$

$$\sup_{a\in B} \int_B |\widetilde{\nabla}(f\circ\varphi_a)(z)|^p (1-|z|^2)^{-1}\, dv(z) = \sup_{a\in B} \int_B |\widetilde{\nabla} f(z)|^p (1-|z|^2)^{-1} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}}\, dv(z) < \infty.$$

In view of Lemma 4.1 in [CC] the last condition means that $|\widetilde{\nabla} f(z)|^p(1-|z|^2)^{-1}\, dv(z)$ is a Carleson measure and (a) is proved. But (a) implies (b) since, by (4), $|\nabla f(z)|(1-|z|^2) \le |\widetilde{\nabla} f(z)|$. □

Remark 2. As in the case $n = 1$ (see [S]) condition (a) (or (b)) of Theorem 4 with some p, $0 < p \le 2$, is sufficient for $f \in$BMOA. To prove this show first that if $0 < p \le 2$ then

$$\|f\|_{H^p}^p \le C\Big(|f(0)|^p + \int_B |\nabla f(z)|^p (1-|z|)^{p-1}\, dv(z)\Big) \le C\Big(|f(0)|^p + \int_B |\widetilde{\nabla} f(z)|^p (1-|z|)^{-1}\, dv(z)\Big)$$

and next proceed as in the proof of Theorem 4.

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