Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 23, 1998, 475–494

POINT SHIFT DIFFERENTIALS AND EXTREMAL QUASICONFORMAL MAPPINGS

Kurt Strebel

Freiestraße 14, CH-8032 Zürich, Switzerland; kstrebel@math.unizh.ch

Abstract. We consider quasiconformal selfmappings f of the disk with given boundary values. For fixed z_0 the set of image points $\{f_0(z_0)\}$ for all extremal mappings f_0 is called the variability set $V[z_0]$ of z_0. Extremal mappings f_m which take z_0 into points $w_m \notin V[z_0]$ are called point shift mappings. They are Teichmüller mappings associated with quadratic differentials φ_m of norm $\|\varphi_m\| = 1$, called point shift differentials. It is shown that the φ_m form Hamilton sequences for the extremal mappings which take z_0 into boundary points of $V[z_0]$. From that it follows, using the frame mapping criterion, that the φ_m depend continuously in norm of the point w_m. Their constant dilatation $K[w_m]$ is called the dilatation function. Based on a variational method for point shift mappings, it is shown that the level lines of $K[w_m]$ are Jordan curves separating $V[z_0]$ from $\partial\mathbf{D}_w$. Thus, $V[z_0]$ is a compact, connected set without holes.

1. Introduction

1. A holomorphic quadratic differential $\varphi(z)\,dz^2$ induces a conformally invariant metric $|\varphi(z)|^{1/2}\,|dz|$. It is Euclidean in the neighborhood of the non critical points of φ (the critical points are the zeroes and poles): if we apply the conformal mapping

$$\zeta = \Phi(z) = \int^z \sqrt{\varphi(z)}\,dz \tag{1}$$

we find $d\zeta^2 = \varphi(z)\,dz^2$, which is the Euclidean line element in the ζ-plane. This metric and its geodesics (for details see [12]) lead to certain inequalities which constitute a fundamental tool in the theory of extremal quasiconformal mappings.

2. The first of these inequalities is the so-called Main Inequality (see [6] for the disk, and [11] for arbitrary Riemann surfaces).

Let $w = f(z)$ and $\widetilde{w} = \tilde{f}(z)$ be quasiconformal mappings of the unit disk $\mathbf{D}_z$ onto $\mathbf{D}_w$ which agree on the boundary of $\mathbf{D}_z$. Let $dw = p(z)\,dz + q(z)\,d\bar{z}$ be the differential of f, $\varkappa = q/p$ its complex dilatation and K its maximal dilatation. The inverse of $\tilde{f}$ is denoted by $f_1 = \tilde{f}^{-1}$, its complex dilatation by $\varkappa_1$. Of course, its maximal dilatation K_1 is equal to the maximal dilatation $\widetilde{K}$ of $\tilde{f}$. The combination $f_1 \circ f = \tilde{f}^{-1} \circ f$ is a qc selfmapping of $\mathbf{D}_z$ which is equal to the identity on $\partial\mathbf{D}_z$.

1991 Mathematics Subject Classification: Primary 30C70; Secondary 30C75.

Let now $\varphi \not\equiv 0$ be any holomorphic quadratic differential in $\mathbf{D}_z$ of norm $\|\varphi\| = \iint |\varphi(z)|\,dx\,dy = 1$. It serves as a test quantity. Let α be any non critical trajectory of φ. It has two different end points on $\partial\mathbf{D}_z$, and the curve $\widetilde{\alpha} = f_1 \circ f(\alpha)$ has the same end points as α. Since α is a geodesic, we have the length inequality, in terms of the φ-metric, $|\widetilde{\alpha}|_\varphi \geq |\alpha|_\varphi$. Using the conformal parameter Φ along the trajectories of φ this length inequality can be integrated over the whole disk $\mathbf{D}_z$, which leads to the "Main Inequality"

$$1 \leq \iint |\varphi(z)| \frac{\bigl|1 - \varkappa(\varphi/|\varphi|)\bigr|^2}{1 - |\varkappa|^2} \{\cdots\}\,dx\,dy, \tag{2}$$

with

$$\{\cdots\} = \frac{\left|1 - \varkappa_1 \dfrac{\overline{p}}{p} \dfrac{\varphi}{|\varphi|} \dfrac{1 - \overline{\varkappa}(\overline{\varphi}/|\varphi|)}{1 - \varkappa(\varphi/|\varphi|)}\right|^2}{1 - |\varkappa_1|^2} \tag{3}$$

(see [6], ineq. 1.5.1, and [11] for Riemann surfaces; the method first appears in [8].)

The term (3) in brackets has the upper bound $K_1 = \widetilde{K}$. If we set $\tilde{f} = f_0$, where f_0 is an extremal quasiconformal mapping for the given boundary values, i.e. one with smallest maximal dilatation K_0, we have $\{\cdots\} \leq K_0$. This turns (2) into the simpler form

$$\frac{1}{K_0} \leq \iint |\varphi(z)| \frac{\bigl|1 - \varkappa(\varphi/|\varphi|)\bigr|^2}{1 - |\varkappa|^2}\,dx\,dy, \tag{4}$$

which we call the reduced form of the Main Inequality. Its contents is the following: Let f be a qc selfmapping of $\mathbf{D}$ and let K_0 be the smallest maximal dilatation in the class of all qc selfmappings of $\mathbf{D}$ with the boundary values of f. Then, for every holomorphic quadratic differential φ of norm one in $\mathbf{D}$ we have (4). (See [6], and [11] for Riemann surfaces.)

3. The holomorphic quadratic differentials of finite norm give rise to a very interesting necessary and sufficient condition for the complex dilatation $\varkappa$ of a qc mapping to be extremal. It reads as follows: Let $\varkappa$ be the complex dilatation of a qc mapping f. Then, f is extremal for its boundary values (and in its homotopy class) if and only if

$$\sup_{\|\varphi\|=1} \operatorname{Re} \iint \varkappa\varphi\,dx\,dy = \|\varkappa\|_\infty. \tag{5}$$

The necessity was shown by R.S. Hamilton [1] and simultaneously (for complex dilatations of constant absolute value) by S. Krushkal [2]. The sufficiency is a consequence of the reduced form (4) of the main inequality and was shown, for selfmappings of the disk, by E. Reich and K. Strebel [6] and, for the general case, by K. Strebel [11].

A sequence of holomorphic quadratic differentials φ_n of norm one such that

$$\operatorname{Re} \iint \varkappa \varphi_n \, dx \, dy \to \|\varkappa\|_\infty \tag{6}$$

is called a Hamilton sequence. If there exists such a sequence which converges to a quadratic differential φ in the sense that $\|\varphi_n - \varphi\| \to 0$, it is easily seen that

$$\varkappa = k \frac{\overline{\varphi}}{|\varphi|}. \tag{7}$$

This is the complex dilatation of a Teichmüller mapping. The crucial situation is when there is no such sequence, i.e. every Hamilton sequence for $\varkappa$ converges locally uniformly to zero (what we call a "degenerating Hamilton sequence").

The Main Inequality provides a criterion to the effect that there is no degenerating Hamilton sequence for $\varkappa$. It is called the Frame Mapping Criterion ([9], [10]) and reads as follows (we only formulate it for the disk here, but it is true in full generality): Let K_0 be the smallest maximal dilatation for all qc selfmappings of $\mathbf{D}$ with a given quasisymmetric boundary homeomorphism h. Assume that h can be continued quasiconformally into a neighborhood of $\partial\mathbf{D}$ with a maximal dilatation $H < K_0$. Then, $\varkappa$ does not admit a degenerating Hamilton sequence and thus has the Teichmüller form $\varkappa = k\overline{\varphi}/|\varphi|$, $\|\varphi\| = 1$. Moreover, every Hamilton sequence (φ_n) for $\varkappa$ tends in norm to φ ([10], Theorem 1). According to an earlier theorem (originally in [7], with a new proof in [6]) the mapping f is then uniquely extremal.

4. The existence of a Hamilton sequence (φ_n) for every extremal $\varkappa$ was proved by R. Hamilton in an abstract way. It is however possible, for the disk, to give Hamilton sequences in a more concrete manner, namely as differentials of certain Teichmüller mappings.

For a given extremal mapping f with complex dilatation $\varkappa$ pick a finite set of boundary points ζ_i, $i = 1, \dots, n$. The disk $\mathbf{D}$ together with these points is called a polygon Π_n with vertices ζ_i. According to Teichmüller's theorem there is a unique extremal selfmapping of $\mathbf{D}$ which takes the vertices ζ_i into their images by f, $\zeta_i' = f(\zeta_i)$. It is a Teichmüller mapping associated with a holomorphic quadratic differential φ, $\|\varphi\| = 1$, which is real along the sides of the polygon, i.e. $\varphi(z)\, dz^2$ real for tangential dz between the vertices. (The vertices themselves are at worst first order poles.)

Call the φ "polygon differentials". One can show that, if the vertices of the polygons Π_n become more and more dense, the associated polygon differentials form a Hamilton sequence for $\varkappa$ ([6], Theorem 6).

5. The basis of this article is a new kind of Hamilton sequences, namely point shift differentials. They were introduced in ([10], Theorem 7). Pick a point $z_0 \in \mathbf{D}$. Consider again the qc selfmappings f of $\mathbf{D}$ with boundary values h. The set of image points of z_0 by the extremal mappings f_0 is called the set of variability of z_0 and denoted by $V[z_0]$ (see [10], p. 479). Of course, if h has a unique extremal solution, $V[z_0]$ consists of a single point for every $z_0 \in \mathbf{D}$. In general it is a compact subset of $\mathbf{D}$. Choose $w_n \notin V[z_0]$. Then, the extremal qc mapping f_n with boundary values h on $\partial\mathbf{D}$ and taking z_0 into w_n is a Teichmüller mapping associated with a quadratic differential φ_n of norm one which has a first order pole at z_0. (This is an immediate consequence of the frame mapping criterion.) φ_n is called a point shift differential, f_n a point shift mapping.

Let now w_0 be a boundary point of $V[z_0]$, and let $f_0(z_0) = w_0$. It will be shown that for $w_n \to w_0$ the point shift differentials φ_n form a Hamilton sequence for the complex dilatation $\varkappa_0$ of f_0.

It should be noticed that the polygon differentials provide Hamilton sequences for every extremal mapping, i.e. also for extremal mappings which take z_0 into an interior point of $V[z_0]$. This is not so for point shift differentials, because they only exist if the shifted point w_n is outside $V[z_0]$. In the case of unique extremality this is of course no restriction. The obvious advantage of the point shift differentials is the fact that they only depend on one complex parameter, namely the shifted point w_n. On the other hand the polygon differentials depend on an increasing number of parameters, namely the vertices of the polygons. This makes them geometrically extremely hard to control.

Clearly, point shift differentials exist in the most general case, in the same generality as the frame mapping criterion.

6. The simplest but quite formal definition of a Teichmüller mapping is by its complex dilatation, which is $\varkappa = k\overline{\varphi}/|\varphi|$, $0 < k < 1$, where φ is a meromorphic quadratic differential. For our purposes it is however necessary to express this in geometric terms. It is done here for the disk (see [8]), but most generally true for arbitrary Riemann surfaces.

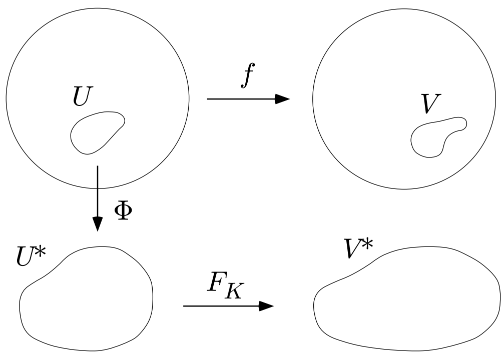

Figure 1.

Let $f\colon \mathbf{D}_z \to \mathbf{D}_w$ be a qc mapping with complex dilatation (7). Pick a neighborhood $U \subset \mathbf{D}_z$ such that φ is holomorphic in U without zeroes (Figure 1). Let

$$\zeta = \xi + i\eta = \Phi(z) = \int^z \sqrt{\varphi(z)}\, dz \tag{8}$$

in U. By restriction of U, if necessary, we may assume that Φ is schlicht in U. Set $U^* = \Phi(U)$ and let F_K be the horizontal stretching of U^* by $K = (1+k)/(1-k)$. We have

$$\zeta^* = F_K(\zeta) = K\xi + i\eta = \tfrac{1}{2}(K+1)\zeta + \tfrac{1}{2}(K-1)\bar{\zeta} = \tfrac{1}{2}(K+1)\big(\Phi(z) + k\overline{\Phi}(z)\big). \tag{9}$$

From this expression follows that the complex dilatation of the composition $F_K \circ \Phi$ is given by (7). Therefore the mappings f and $F_K \circ \Phi$ are related by a conformal mapping Ψ: $V = f(U) \to V^* = F_K(U^*)$ such that

$$\Psi \circ f = F_K \circ \Phi. \tag{10}$$

It is easy to see that $\psi = \Psi'^2$ is single valued in $\mathbf{D}_w$. It is the well defined quadratic differential which is associated to φ. We realize that the horizontal and the vertical trajectories of φ through a point $z \in U$ go into the horizontal and vertical trajectories of ψ through $w = f(z)$ respectively. Horizontal lengths are stretched by K, vertical lengths are preserved.

We will call φ and ψ the quadratic differentials associated with the Teichmüller mapping f. Clearly, if $\|\varphi\| = 1$, $\|\psi\| = K$.

II Point shift differentials

7. Let $T[w_1, w_2]$ be the extremal qc selfmapping of the unit disk $\mathbf{D}$ which takes w_1 into w_2 and is equal to the identity on $\partial\mathbf{D}$. The Frame Mapping Criterion assures that it is a Teichmüller mapping associated with a pair of quadratic differentials φ and ψ of finite norm. φ has a first order pole at w_1, ψ has one at w_2, and if we set $\|\varphi\| = 1$ and use the horizontal stretching version for Teichmüller mappings (as we always do) we have $\|\psi\| = K_T$ where K_T is the dilatation of T. The mapping T is conjugate, by a Möbius transformation of the disk, to the mapping in Teichmüller's "Verschiebungssatz", where $w_1 = 0$ and $w_2 = -\varrho$. It is not hard to see that $K_T \to 1$ for $w_2 \to w_1$. Because of this connection we call $T[w_1, w_2]$ the Teichmüller shift.

8. Denote the unit disks of the z- and the w-plane by $\mathbf{D}_z$ and $\mathbf{D}_w$ respectively. Let h be a fixed quasisymmetric mapping of $\partial\mathbf{D}_z$ onto $\partial\mathbf{D}_w$. Choose a fixed point $z_0 \in \mathbf{D}_z$ and, for $w \in \mathbf{D}_w$, let f be an extremal qc mapping of $\mathbf{D}_z$ onto $\mathbf{D}_w$ which is equal to h on $\partial\mathbf{D}_z$ and takes z_0 into w. Let K be its maximal dilatation.

Theorem 1. *The maximal dilatation K is, for fixed z_0, a continuous function of w which tends to infinity for $|w| \to 1$. It is called the dilatation function and denoted by $K[w]$.*

Proof. To prove the first statement, choose $\widetilde{w} \neq w$ and let $\widetilde{K}$ be the maximal dilatation of an extremal mapping $\widetilde{f}$, $\widetilde{f}(z_0) = \widetilde{w}$, $\widetilde{f} \mid \partial \mathbf{D}_z = h$. The composition $T[w, \widetilde{w}] \circ f$ takes z_0 into $\widetilde{w}$ and has therefore a maximal dilatation which is at least equal to $\widetilde{K}$. On the other hand it is smaller or equal to $K_T \cdot K$, thus $K_T \cdot K \geq \widetilde{K}$, which gives

$$\varlimsup_{\widetilde{w} \to w} \widetilde{K} \leq K.$$

On the other hand, since f is extremal,

$$\varliminf_{\widetilde{w} \to w} \widetilde{K} \geq K,$$

which gives

$$\lim_{\widetilde{w} \to w} \widetilde{K} = K,$$

as claimed.

To prove the second statement, let $w_n \to \omega \in \partial \mathbf{D}_w$. Let $\zeta = h^{-1}(\omega)$ and choose two disjoint closed intervals δ_1 and δ_2 on $\partial \mathbf{D}_z$ none of which contains the point ζ. The set of cross cuts γ joining δ_1 to δ_2 in $\mathbf{D}_z$ and separating z_0 from ζ goes over by f_n into a set of curves γ' connecting $\delta_1' = h(\delta_1)$ and $\delta_2' = h(\delta_2)$ and separating w_n from ω. Here, f_n is an extremal mapping, with maximal dilatation K_n, taking z_0 into w_n. We have the extremal length inequality $\lambda\{\gamma'\} \leq K_n \lambda\{\gamma\}$, and since the left hand side evidently tends to infinity, we find $K_n \to \infty$, as claimed.

Corollary. *The set of variability of z_0,*

$$V[z_0] = \{w = f(z_0)\,; f \text{ extremal for the boundary values } h\}$$

is compact.

This is so because all the extremal mappings f_0 for all points $w_0 \in V[z_0]$ have the same maximal dilatation K_0. Therefore $V[z_0]$ is bounded away from $\partial \mathbf{D}_w$, and because of the continuity of the maximal dilatation it contains all its accumulation points.

9. Let w_n be a point of $\mathbf{D}_w$ outside the variablity set $V[z_0]$. Then, any extremal qc mapping f_n of $\mathbf{D}_z$ onto $\mathbf{D}_w$ with the boundary values h and $f_n(z_0) = w_n$ has a maximal dilatation $K_n > K_0$, K_0 being the smallest maximal dilatation without the additional condition that $z_0 \to w_n$. We can therefore use any extremal mapping f_0 near $\partial \mathbf{D}_z$ as a frame mapping (near z_0 we just use the translation $z_0 \to w_n$, which is of course conformal). The frame mapping condition applied to the punctured disks tells us that f_n is a Teichmüller mapping associated with a quadratic differential φ_n of finite norm ($\|\varphi_n\| = 1$, say). φ_n has a first order pole at z_0 and is otherwise holomorphic (without the pole, f_n would be uniquely extremal for its boundary values $f_n \mid \partial \mathbf{D}_z = h$, see [8] and [6]). We call f_n a point shift mapping and φ_n the associated point shift differential (see [10], p. 479).

The next result is basic for the whole paper.

10.

Theorem 2. *Let w_0 be a boundary point of the variablity set $V[z_0]$, and let f_0, with complex dilatation $\varkappa_0$ and maximal dilatation K_0 be an extremal qc mapping which takes z_0 into w_0 (for brevity, we call such a mapping a "boundary element", see also* [5], p. 290). *Let $w_n \in \mathbf{D}_w \setminus V[z_0]$, and let f_n, φ_n and K_n be as in Section* 9. *Then, for $w_n \to w_0$ the sequence (φ_n) is a Hamilton sequence for the complex dilatation $\varkappa_0$ of f_0, i.e.* $\operatorname{Re} \iint \varkappa_0 \varphi_n \, dx\, dy \to k_0 = \|\varkappa_0\|_\infty$.

Proof. Let $T_n \equiv T[w_0, w_n]$ be the Teichmüller shift which takes w_0 into w_n. Let $\tilde{f}_n = T_n \circ f_0$. This is a qc mapping of $\mathbf{D}_z$ onto $\mathbf{D}_w$, with the same boundary values as f_n, sending z_0 into w_n, and with a maximal dilatation $\widetilde{K}_n$ which tends to K_0 for $w_n \to w_0$.

Consider a regular trajectory α_n of φ_n, i.e. one which does not go to a zero or the simple pole of φ_n. It connects two different boundary points of the disk $\mathbf{D}_z$. The image α_n' of α_n by f_n is a well defined regular trajectory of the image differential ψ_n connecting the images of the end points of α_n. Let $\widetilde{\alpha}_n$ be the image of α_n by $\tilde{f}_n$; it has the same end points as α_n'. Since $f_n(z_0) = \tilde{f}_n(z_0) = w_n$, the two mappings f_n and $\tilde{f}_n$ are homotopic in $\mathbf{D}_w \setminus \{w_n\}$ modulo the boundary. Therefore α_n and $\widetilde{\alpha}_n$ are in the same homotopy class in $\mathbf{D}_w \setminus \{w_n\}$ and thus satisfy the length inequality with respect to the ψ_n-metric, namely

$$|\widetilde{\alpha}_n|_{\psi_n} \geq |\alpha_n'|_{\psi_n} = K_n |\alpha_n|_{\varphi_n}. \tag{11}$$

This inequality can now be integrated over the disk $\mathbf{D}_z$ (see [8], or [6] p. 384–385 for polygon differentials).

The image of α_n by $\zeta = \Phi_n(z)$ is an open horizontal interval α_n^* in the $\zeta = \xi + i\eta$-plane. We have, with $\widetilde{w}_n = \tilde{f}_n(z)$,

$$K_n |\alpha_n|_{\varphi_n} = K_n \int_{\alpha_n^*} d\xi \leq \int_{\widetilde{\alpha}_n} |d\Psi_n(\widetilde{w}_n)| = \int_{\widetilde{\alpha}_n} \left| \frac{d\Psi_n(\widetilde{w}_n)}{d\widetilde{w}_n} \right| |d\widetilde{w}_n|. \tag{12}$$

With the notations $d\widetilde{w}_n = \tilde{p}_n(z)\, dz + \tilde{q}_n(z)\, d\bar{z}$, $d\zeta = \Phi_n'(z)\, dz$, we find along α_n^*

$$d\widetilde{w}_n = \left\{ \tilde{p}_n(z) \frac{1}{\Phi_n'(z)} + \tilde{q}_n(z) \frac{1}{\overline{\Phi_n'(z)}} \right\} d\xi = \frac{1}{\Phi_n'(z)} \left\{ \tilde{p}_n(z) + \tilde{q}_n(z) \frac{\varphi_n(z)}{|\varphi_n(z)|} \right\} d\xi. \tag{13}$$

We put this expression into the equation (12) and integrate it over a short vertical interval in the $\zeta = \xi + i\eta$-plane. The domain of integration is a (Euclidean) horizontal strip Σ in the ζ-plane. Going back to the z-plane with

$d\xi\, d\eta = |\Phi_n'(z)|^2\, dx\, dy = |\varphi_n(z)|\, dx\, dy$ the domain of integration is a strip S swept out by horizontal trajectories of φ_n. Finally, summing up over a denumerable set of such strips forming an exhaustion of $\mathbf{D}_z$ (up to a set of measure zero) we get

$$(14) \qquad K_n\|\varphi_n\| \le \iint_{|z|<1} |\psi_n(\widetilde{w}_n)|^{1/2}|\varphi_n(z)|^{1/2}\left|\tilde{p}_n(z) + \tilde{q}_n(z)\frac{\varphi_n(z)}{|\varphi_n(z)|}\right| dx\, dy.$$

Before we apply the Schwarz inequality to this expression we insert the square root of the functional determinant

$$(15) \qquad J(\widetilde{w}_n \mid z) = |\tilde{p}_n(z)|^2 - |\tilde{q}_n(z)|^2$$

and thus have

$$(16) \quad K_n\|\varphi_n\| \le \iint_{|z|<1} |\psi_n(\widetilde{w}_n)|^{1/2} J(\widetilde{w}_n \mid z)^{1/2}\frac{|\varphi_n|^{1/2}\left|\tilde{p}_n + \tilde{q}_n(\varphi_n/|\varphi_n|)\right|}{J(\widetilde{w}_n \mid z)^{1/2}}\, dx\, dy.$$

Finally, the Schwarz inequality yields
(17)

$$K_n^2\|\varphi_n\|^2 \le \iint_{|z|<1} |\psi_n(\widetilde{w}_n)| J(\widetilde{w}_n \mid z)\, dx\, dy \cdot \iint_{|z|<1} |\varphi_n|\frac{\left|1 + \widetilde{\varkappa}_n(\varphi_n/|\varphi_n|)\right|^2}{1 - |\widetilde{\varkappa}_n|^2}\, dx\, dy,$$

with $\widetilde{\varkappa}_n = \tilde{q}_n(z)/\tilde{p}_n(z)$ the complex dilatation of $\tilde{f}_n$.

Taking into account that

$$(18) \qquad \iint_{|z|<1} |\psi_n(\widetilde{w}_n)| J(\widetilde{w}_n \mid z)\, dx\, dy = \|\psi_n\| = K_n\|\varphi_n\|$$

and normalizing $\|\varphi_n\| = 1$ we get

$$(19) \qquad K_n \le \iint_{|z|<1} |\varphi_n(z)|\frac{\left|1 + \widetilde{\varkappa}_n(z)(\varphi_n(z)/|\varphi_n(z)|)\right|^2}{1 - |\widetilde{\varkappa}_n(z)|^2}\, dx\, dy.$$

We call this the "point shift inequality". It resembles very much the "polygon inequality" (see [6], p. 385).

11. To evaluate the point shift inequality, we write

$$(20) \qquad \left|1 + \widetilde{\varkappa}_n\frac{\varphi_n}{|\varphi_n|}\right|^2 = (1 + |\widetilde{\varkappa}_n|)^2 - 2\left\{|\widetilde{\varkappa}_n| - \operatorname{Re}\widetilde{\varkappa}_n\frac{\varphi_n}{|\varphi_n|}\right\}.$$

Since

$$(21) \qquad \frac{1 + |\widetilde{\varkappa}_n|}{1 - |\widetilde{\varkappa}_n|} \le \frac{1 + \tilde{k}_n}{1 - \tilde{k}_n} = \widetilde{K}_n \qquad \text{a.e.}$$

the point shift inequality (19) gives

$$(22)\quad \begin{aligned} K_n &\le \iint_{|z|<1} |\varphi_n(z)| \frac{1+|\widetilde{\varkappa}_n|}{1-|\widetilde{\varkappa}_n|}\,dx\,dy - 2\iint_{|z|<1} \frac{|\widetilde{\varkappa}_n|\,|\varphi_n| - \operatorname{Re}\widetilde{\varkappa}_n\varphi_n}{1-|\widetilde{\varkappa}_n|^2}\,dx\,dy \\ &\le \iint_{|z|<1} |\varphi_n| \frac{1+|\widetilde{\varkappa}_n|}{1-|\widetilde{\varkappa}_n|}\,dx\,dy \le \widetilde{K}_n. \end{aligned}$$

Let now $w_n \to w_0 \in \partial V[z_0]$. Then, because of the continuity of $K[w_n]$ and since $K[w_0] = K_0$ we have $K_n \to K_0$. On the other hand, we have $\widetilde{K}_n \le K_{T_n} \cdot K_0$ and thus

$$\overline{\lim_{w_n \to w_0}}\ \widetilde{K}_n \le K_0.$$

From (22) we conclude that $\widetilde{K}_n \to K_0$ for $w_n \to w_0$.

Another use of (22) shows that for $w_n \to w_0$

$$(23)\qquad \iint_{|z|<1} |\varphi_n(z)| \frac{1+|\widetilde{\varkappa}_n|}{1-|\widetilde{\varkappa}_n|}\,dx\,dy \to K_0$$

and

$$(24)\qquad \iint_{|z|<1} \frac{|\widetilde{\varkappa}_n|\,|\varphi_n| - \operatorname{Re}\widetilde{\varkappa}_n\varphi_n}{1-|\widetilde{\varkappa}_n|^2}\,dx\,dy \to 0.$$

Therefore necessarily

$$(25)\qquad \iint_{|z|<1} \bigl\{|\widetilde{\varkappa}_n|\,|\varphi_n| - \operatorname{Re}\widetilde{\varkappa}_n\varphi_n\bigr\}\,dx\,dy \to 0.$$

The quantity $\widetilde{\varkappa}_n$ is the complex dilatation of the composition $\tilde{f}_n = T_n \circ f_0$, with $T_n = T[w_0, w_n]$. Denote the complex dilatation of T_n by $\varkappa_{T_n}$; since it is a Teichmüller mapping, its absolute value is constant, $|\varkappa_{T_n}| = k_{T_n}$. The formula for the complex dilatation of the composition gives

$$(26)\qquad \widetilde{\varkappa}_n = \frac{\varkappa_0 + \varkappa_{T_n}\tau_0}{1 + \varkappa_{T_n}\overline{\varkappa}_0\tau_0}, \qquad \tau_0 = \frac{\overline{p}_0}{p_0},$$

(the differential of the extremal mapping $w = f_0(z)$ is $dw = p_0(z)\,dz + q_0(z)\,d\bar{z}$). From this is follows that

$$(27)\qquad \widetilde{\varkappa}_n - \varkappa_0 = \varkappa_{T_n} \frac{\tau_0\{1 - |\varkappa_0|^2\}}{1 + \varkappa_{T_n}\overline{\varkappa}_0\tau_0}.$$

Since $|\varkappa_{T_n}(z)| \equiv k_{T_n} \to 0$, $|\varkappa_0(z)| \le k_0 < 1$ and $|\tau_0| = 1$ we conclude that

$$(28)\qquad \bigl|\,|\widetilde{\varkappa}_n| - |\varkappa_0|\bigr| \le |\widetilde{\varkappa}_n - \varkappa_0| \le k_{T_n} \frac{1}{1 - k_{T_n}k_0} \to 0$$

uniformly in z with $n \to \infty$. We can therefore replace $\widetilde{\varkappa}_n$ by $\varkappa_0$ in relations (23) and (25) and get

$$(23') \qquad \iint_{|z|<1} |\varphi_n(z)| \frac{1 + |\varkappa_0(z)|}{1 - |\varkappa_0(z)|} \, dx \, dy \to K_0$$

and

$$(25') \qquad \iint_{|z|<1} \bigl\{ |\varkappa_0(z)| \, |\varphi_n(z)| - \operatorname{Re} \varkappa_0(z) \varphi_n(z) \bigr\} \, dx \, dy \to 0$$

for $w_n \to w_0$.

If $|\varkappa_0(z)| = k_0$ a.e., relation $(25')$ gives

$$(29) \qquad \operatorname{Re} \iint \varkappa_0(z) \varphi_n(z) \, dx \, dy \to k_0,$$

which is the Hamilton relation (6) for $\varkappa_0$.

If, however, $|\varkappa_0(z)|$ is not equal to k_0 a.e., it follows from relation $(23')$ that $\iint_E |\varphi_n(z)| \, dx \, dy \to 0$ on every measurable set E on which $|\varkappa_0(z)| \le k_0 - \varepsilon$, for every $\varepsilon > 0$. Since the φ_n are analytic and the sequence (φ_n) is locally bounded in $\mathbf{D}_z$, it must be degenerating. Writing

$$(30) \qquad \iint_{|z|<1} |\varkappa_0(z)| \, |\varphi_n(z)| \, dx \, dy = \iint_E + \iint_{\mathbf{D}_z \setminus E}$$

and considering that on $\mathbf{D}_z \setminus E$ we have $k_0 - \varepsilon < |\varkappa_0(z)| \le k_0$ we get

$$\iint_{|z|<1} |\varkappa_0(z)| \, |\varphi_n(z)| \, dx \, dy \to k_0$$

as before. $(25')$ thus leads to (29) again, which proves the Theorem.

12. Summing up, we have considered the following situation. The qc mapping f_0 has the boundary values h on $\partial \mathbf{D}_z$ and takes z_0 into w_0. Its maximal dilatation is K_0, and it is extremal for the above data. Moreover there is a sequence of points $w_n \in \mathbf{D}_z$, $w_n \to w_0$, such that the corresponding extremal mappings f_n, $f_n(z_0) = w_n$, are Teichmüller mappings associated with quadratic differentials φ_n of norm one. The φ_n are defined and holomorphic in $\mathbf{D}_z \setminus \{z_0\}$. They have a first order pole at z_0, because otherwise the mapping f_n would be uniquely extremal for its boundary values on $\partial \mathbf{D}_z$, without any further condition. Then, we have shown that these φ_n form a Hamilton sequence for the complex dilatation $\varkappa_0$ of f_0.

Let us now consider a point w_m in $\mathbf{D}_w$ outside the variability set $V[z_0]$. The mapping f_m: $z_0 \to w_m$, with boundary values h on $\partial \mathbf{D}_z$ is a uniquely extremal

Teichmüller mapping with a (constant) dilatation $K_m > K_0$. Let $w_n \to w_m$. Then, the mappings f_n, $f_n(z_0) = w_n$, tend to f_m uniformly in $\mathbf{D}_z$ (because f_m is uniquely extremal) and the associated normalized quadratic differentials φ_n, all defined and holomorphic in $\mathbf{D}_z \setminus \{z_0\}$, form a Hamilton sequence for the complex dilatation $\varkappa_m = k_m(\overline{\varphi}_m/|\varphi_m|)$ of f_m. But now the frame mapping criterion says: there is no degenerating Hamilton sequence for $\varkappa_m$ (as a frame mapping we can use any extremal mapping f_0 near $\partial\mathbf{D}_z$ and the translation $z \to w_m + z - z_0$ near z_0, since $K_m > K_0$). Therefore the φ_n tend to φ_m in norm. We thus have proved the

Theorem 3. *Let $w_m \in \mathbf{D}_z \setminus V[z_0]$, and let $w_n \to w_m$. Then, the normalized point shift differentials φ_n tend in norm to the Teichmüller differential φ_m of f_m, i.e. $\|\varphi_n - \varphi_m\| \to 0$.*

13. Actually, every Teichmüller mapping is associated with a pair of quadratic differentials, φ on the original domain and ψ on the image domain, and the mapping sends the trajectories and orthogonal trajectories of φ into the trajectories and orthogonal trajectories respectively of ψ. We have seen, that, for $w_n \to w_m$, $\varphi_n \to \varphi_m$ in norm, and hence, because of analyticity, locally uniformly; we will need the analogue for the ψ_n.

To fix the ideas, take, for a given w_m, a closed, regular φ_m-rectangle S_m and the corresponding ψ_m-rectangle S'_m. S_m is bounded by two opposite horizontal and two opposite vertical intervals, and there are no zeroes of φ_m on S_m. The same holds for S'_m with respect to ψ_m. $\zeta = \Phi_m(z)$, $\zeta^* = \Psi_m(w)$ take S_m and S'_m into horizontal Euclidean rectangles R_m, R'_m respectively. By a proper choice of the additive constants we can achieve that $\zeta = \zeta^* = 0$ in the lower left corners of the rectangles. We express everything in terms of these parameters ζ and ζ^* respectively. Since $d\zeta^2 = \varphi_m(z)\,dz^2$, we have $\varphi_m(\zeta) \equiv 1$, and analogously $\psi_m(\zeta^*) \equiv 1$. Moreover, $\Phi_m(\zeta) \equiv \zeta$, $\Psi_m(\zeta^*) \equiv \zeta^*$, and the Teichmüller mapping f_m is

$$\zeta^* = f_m(\zeta) = K_m\xi + i\eta \equiv F_{K_m}(\zeta), \qquad \zeta = \xi + i\eta.$$

F_{K_m} denotes the horizontal stretching of R_m onto R'_m by K_m.

We have seen, in the preceding section, that, with $w_n \to w_m$ and the normalization $\|\varphi_n\| = \|\varphi_m\| = 1$, we have $\|\varphi_n - \varphi_m\| \to 0$. In terms of the parameter ζ we therefore have $\varphi_n(\zeta) \to 1$ uniformly in R_m. Therefore $\Phi_n(\zeta) \to \zeta$, also uniformly in R_m, and clearly $f_n(\zeta) \to f_m(\zeta)$, also uniformly.

The mapping f_n has the representation, locally

$$f_n = \Psi_n^{-1} \circ F_{K_n} \circ \Phi_n,$$

and thus

$$\Psi_n(\zeta^*) = F_{K_n}\bigl(\Phi_n\bigl(f_n^{-1}(\zeta^*)\bigr)\bigr).$$

Taking into account that, with $w_n \to w_m$, $f_n^{-1}(\zeta^*) \to \zeta$, $\Phi_n(\zeta) \to \zeta$, $K_n \to K_m$, hence $F_{K_n}(\zeta) \to \zeta^*$ uniformly, we find

$$\Psi_n(\zeta^*) \to \zeta^*$$

uniformly in R'_m. This gives, because of the analyticity, $\psi_n(\zeta^*) \to 1$ uniformly in R'_m.

The natural normalization of the ψ-differentials is, according to the Euclidean geometry of the Ψ-planes and the horizontal stretching by the dilatation K, $\|\psi_m\| = K_m$, $\|\psi_n\| = K_n$. With this we have, in any finite collection $B = \bigcup_j S_m^j$ of ψ_m-rectangles,

$$\|\psi_n - \psi_m\|_B \to 0.$$

On the other hand we can clearly approximate $\mathbf{D}_w$, for any positive ε, by a finite system B of non-overlapping ψ_m rectangles, such that $\|\psi_m\|_{\mathbf{D}_w \setminus B} < \varepsilon$ and $\|\psi_n\|_{\mathbf{D}_w \setminus B} < \varepsilon$ for all w_n which are lying sufficiently close to w_m. This gives the result $\|\psi_n - \psi_m\| \to 0$. We have proved the following

Theorem 4. *Let $w_m \notin V[z_0]$, $w_n \to w_m$, and let f_n, f_m be a corresponding point shift mappings. Let ψ_n, ψ_m be the image differentials of the φ_n, φ_m in the normalization $\|\psi_n\| = K_n$, $\|\psi_m\| = K_m$. Then, $\|\psi_n - \psi_m\| \to 0$.*

We can interpret that as continuity of the ψ_n with respect to the norm topology. By the mean value theorem we get $\psi_n \to \psi_m$ locally uniformly in $\mathbf{D}_w \setminus \{w_m\}$.

III A method of variation for point shift mappings

14. The quadratic differential

$$\varphi(z)\, dz^2 = \frac{1}{z}\, dz^2 \tag{31}$$

is basic in the following considerations. Its regular (horizontal) trajectories are the parabolas with focus $z_0 = 0$ and with vertex on the negative real axis: this is geometrically evident, since the field of line element $(1/z)\, dz^2 > 0$, i.e. $\arg dz = \frac{1}{2} \arg z \pmod{\pi}$, is equal to the field of tangential elements of the parabolas. The only critical trajectory is the positive real half line. The vertical trajectories are the orthogonal parabolas and the negative real half line. Clearly, the field of line elements as well as the trajectories and orthogonal trajectories are invariant with respect to homotheties with center z_0.

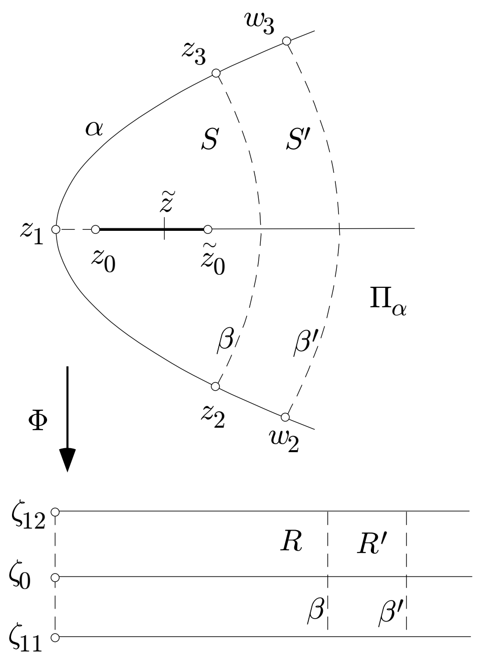

Figure 2.

Let α be one of the parabolas. The interior of it is denoted by $\Pi_\alpha = \Pi$. It is mapped by $\zeta = \Phi(z) = \int \sqrt{\varphi(z)}\,dz = 2z^{1/2}$ onto a horizontal parallel half strip (Figure 2). The interval between the vertex z_1 and the focus z_0 is opened up into a vertical interval with center $\zeta_0 = 0 = \Phi(0)$, whereas the two branches of α go over into the two horizontal half lines which bound the strip. This again is geometrically evident because the tangential elements of the parabolas are turned into horizontal position by Φ. A Teichmüller mapping f of Π_α associated with the quadratic differential φ and any constant dilatation $K > 1$ is represented by the horizontal stretching $\zeta = \xi + i\eta \to K\xi + i\eta$ of the half strip. It is therefore a self-mapping of Π_α with fixed point z_0. Actually all the points of the horizontal interval $[z_1, z_0]$ are fixed points of f. Each trajectory of φ is mapped onto itself, whereas every vertical trajectory is transformed into another one, further away from z_0.

15. Let S be a segment of the parabola domain Π_α, bounded by a symmetric interval of α and a symmetric interval of an orthogonal parabola β. For short, we call these intervals α and β again. Let us denote the restriction of f to S by f_0, its dilatation by K_0. $S' = f_0(S)$ is a segment of the same parabola domain Π_α. In the ζ-plane the corresponding quantities are rectangles R and R' and a horizontal stretching of R onto R' by K_0 (Figure 2). The mapping $f_0\colon S \to S'$ is uniquely extremal for its boundary values on ∂S and with z_0 as fixed point. This follows from the fact that it is a Teichmüller mapping associated with a holomorphic quadratic differential of finite norm, namely the restriction of φ to $S \setminus \{z_0\}$.

We now look at the extremal mapping $\tilde{f}_0: S \to S'$ with the boundary values of f_0 but without the assumption that z_0 be fixed. The boundary dilatation of f_0 on ∂S is determined by the local boundary dilatation at the two points z_2 and z_3 on ∂S, because f_0 is analytic on the boundary intervals in-between these two points. However, at these two points the local boundary dilatation is smaller than K_0 (see [7]). Therefore, $H < K_0$ (see [10], p. 475). If the maximal dilatation $\widetilde{K}_0$ of $\tilde{f}_0$ were known to be greater than H, an application of the frame mapping condition would tell us that $\tilde{f}_0$ is a uniquely extremal Teichmüller mapping associated with a holomorphic quadratic differential $\widetilde{\varphi}_0$ of finite norm. However, the only thing which is known is $\widetilde{K}_0 \le K_0$. This is the reason for an additional step in the argument.

Move the vertical interval β to the right. Then, the segment S tends to Π_α. In the ζ-plane again, this corresponds to a shift of the right hand vertical boundary interval of R to the right. $\tilde{f}_0$ goes over into a $\widetilde{K}_0$-qc mapping of a half strip, equal to the stretching by K_0 on its horizontal boundaries. It is then well known (see [7]) that the maximal dilatation $\widetilde{K}_0$ tends to K_0. Therefore there is a segment S such that $\widetilde{K}_0 > H$, and for this S the above conclusion holds. Also, because of the unique extremality of $\tilde{f}_0$ and the symmetry of the boundary values, the point $\tilde{z}_0 = \tilde{f}_0(z_0)$ lies on the real axis. Again because of the unique extremality and since $\widetilde{\varphi} \ne \varphi$, $\tilde{f}_0(z_0) \ne z_0$ and $\widetilde{K}_0 < K_0$.

An extremal length argument shows that $\tilde{z}_0$ cannot lie between z_0 and the vertex z_1 of α. To this end, look at the upper half of the parabola segment S. It is mapped by $\tilde{f}_0$ onto the upper half of S'. The extremal distance of the interval (z_1, z_0) from the upper half of the orthogonal parabola segment β is $2a/b$, a being the length of the rectangle R and b its height. The analogous quantity in S' is $K_0 \cdot 2a/b$. Assume that $\tilde{z}_0 \in (z_1, z_0)$. Then, the extremal distance of $(z_1, \tilde{z}_0)$ from the upper half of β' is $> K_0 2a/b$. On the other hand, since $\tilde{f}_0$ is $\widetilde{K}_0$-qc, it is $\le \widetilde{K}_0 2a/b$ which is smaller than $K_0 2a/b$, a contradiction.

16. Next, we claim that for all points $\tilde{z} \in (z_0, \tilde{z}_0)$ on the real axis the extremal mapping $\tilde{f}: z_0 \to \tilde{z}$, $\tilde{f} \mid \partial S = f_0 \mid \partial S$ is a Teichmüller mapping with a dilatation $\widetilde{K}$, $\widetilde{K}_0 < \widetilde{K} < K_0$. Its quadratic differential has finite norm and a first order pole at z_0.

To this end we use a homothety with center z_0, retracting the segments S, S' into inner parabola segments $\widetilde{S}$, $\widetilde{S}'$. The extremal mapping $\tilde{f}_0: S \to S'$ goes over into the extremal mapping of $\widetilde{S}$ onto $\widetilde{S}'$, with the boundary values of f_0 on $\partial\widetilde{S}$. It takes z_0 into the point $\tilde{z} \in (z_0, \tilde{z}_0)$ which is the image of $\tilde{z}_0$ by the homothety. Of course, its dilatation is again $\widetilde{K}_0$. This retracted mapping can be continued, by the orginal mapping f_0, to a qc mapping of S onto S'. Its dilatation is $\widetilde{K}_0$ in $\widetilde{S}$ and K_0 in $S \setminus \widetilde{S}$, and it takes z_0 into $\tilde{z}_0$. The extremal mapping $\tilde{f}$ with these conditions must have a dilatation $\widetilde{K} > \widetilde{K}_0$, because it is subject to the additional condition $z_0 \to \tilde{z}$, whereas $\tilde{f}_0$ was free on z_0. It is therefore a uniquely extremal Teichmüller mapping. Because of that, clearly $\widetilde{K} < K_0$, because the latter is the maximal dilatation of a competing mapping.

The construction with the homothety is possible for all $\tilde{z} \in (z_0, \tilde{z}_0)$, just by changing the homothety factor. We have thus shown the claim.

The final step is the following. The parabola segment S we were forced to take in the first step and in which we have constructed the variations of f_0 could be quite long. However, by means of a homothety of S we can make it as small as we want. We have thus proved the following

Parabola Lemma. *Consider the quadratic differential $\varphi(z)\,dz^2 = (1/z)\,dz^2$ and a number $K_0 > 1$. Let f_0 be the corresponding Teichmüller mapping of the plane. Then, in any neighborhood of $z_0 = 0$ there are parabola segments S, S', associated with φ, and mappings $\tilde{f}\colon S \to S'$ with maximal dilatations $\widetilde{K} < K_0$, $\tilde{f}(z_0) = \tilde{z}$, $\tilde{f} \mid \partial S = f_0 \mid \partial S$. The points $\tilde{z}$ fill out an interval $(z_0, \tilde{z}_0]$ of positive length δ. The extremal mappings satisfying the above boundary conditions are actually Teichmüller mappings, the quadratic differentials of which have finite norm and, if $\tilde{z} \neq \tilde{z}_0$, a first order pole at z_0.*

17. Application to the point shift mappings.

Theorem 5. *Let $f_m\colon \mathbf{D}_z \to \mathbf{D}_w$ be extremal for its boundary values, with $f_m(z_0) = w_m \notin V[z_0]$. Let the associated quadratic differentials be φ_m in $\mathbf{D}_z$ and ψ_m in $\mathbf{D}_w$, and let K_m be the dilatation of f_m. Denote the critical trajectory of ψ_m emanating from w_m by γ_m. Then, there exists a closed subinterval $\Delta\gamma_m$ of γ_m with w_m one of its endpoints such that for every $w_n \in \Delta\gamma_m$, $w_n \neq w_m$, the point shift mapping f_n, $f_n(z_0) = w_n$, has a dilatation $K_n < K_m$* (Figure 3).

If the point w_m varies on a compact set $C \subset \mathbf{D}_w$, $C \cap V[z_0] = \emptyset$, the Euclidean length of the intervals $\Delta\gamma_m$ has a positive lower bound.

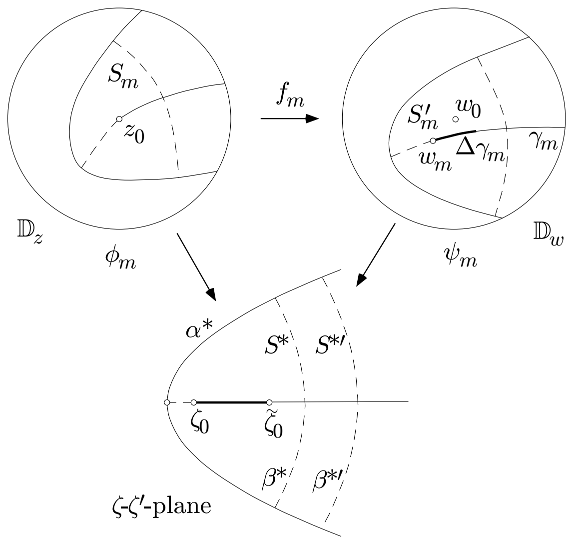

Figure 3.

Proof. We introduce the normal parameter for φ_m near z_0 and for ψ_m near w_m. That means that we apply local conformal mappings $z \leftrightarrow \zeta$ and $w \leftrightarrow \zeta$

such that φ_m and ψ_m both have the representation, in terms of ζ, $\varphi_m(z)\,dz^2 = (1/\zeta)\,d\zeta^2$ and $\psi_m(w)\,dw^2 = (1/\zeta)\,d\zeta^2$ (see [12]).

The mapping f_m is locally represented in the ζ-plane, by a parabola mapping f_m^*, taking a parabola segment S^* into a parabola segment $S^{*\prime}$ and with fixed point $\zeta_0 = 0$. The segments are bounded by subintervals of the same parabola α^* and two different vertical parabola segments β^* and $\beta^{*\prime}$ respectively.

The corresponding domains in the z- and w-planes are denoted by S_m and S'_m respectively. The points z_0 and w_m go over, by the parameter transformations, into $\zeta_0 = 0$.

The Parabola Lemma tells us that there exists a closed interval $[\zeta_0, \tilde{\zeta}_0]$ on the positive real axis of the ζ-plane such that for every $\tilde{\zeta} \in (\zeta_0, \tilde{\zeta}_0]$ the extremal mapping $\tilde{f}\colon S^* \to S^{*\prime}$, with the induced boundary values and with $\tilde{f}(\tilde{\zeta}_0) = \tilde{\zeta}$ has a dilatation $\widetilde{K} < K_m$. The mapping $f_m\colon S_m \to S'_m$ can now be replaced by the mapping $\tilde{f}$ composed with the parameter transformations $z \to \zeta$ and $\zeta \to w$. This local mapping can be continued to the whole disks $\mathbf{D}_z$, $\mathbf{D}_w$ outside S_m, S'_m by f_m. Its maximal dilatation is K_m but it takes z_0 into a point $w_n \in \Delta\gamma_m$ which corresponds to $\tilde{\zeta}$. The extremal mapping with this property is f_n. Because of the unique extremality of f_n we have $K_n < K_m$. Taking into account that the interval $[\zeta_0, \tilde{\zeta}_0]$ corresponds to a subinterval $\Delta\gamma_m$ of γ_m of positive length, we have proved the theorem. The last statement follows because, due to the locally uniform convergence $\varphi_n \to \varphi_m$, $\psi_n \to \psi_m$ for $w_n \to w_m$ we can use the same parabola segments S^*, $S^{*\prime}$ in the natural parameter plane for all n and m.

18. Whereas in the previous section we have found points w_n with $K[w_n] < K[w_m]$ we are now going to show the existence of points w_n, in every neighborhood of w_m, with $K[w_n] > K[w_m]$.

Theorem 6. *Let $w_m \notin V[z_0]$ and let $K[w_m]$ be the dilatation of the point shift mapping f_m, $f_m(z_0) = w_m$. Then, in every neighborhood of w_m there is a point w_n such that the point shift mapping f_n, $f_n(z_0) = w_n$, has a dilatation $K[w_n] > K[w_m]$.*

Proof. The proof consists in finding points $w_n \neq w_m$ such that $w_m \in \Delta\gamma_n$. We can then apply the previous result to get the desired inequality.

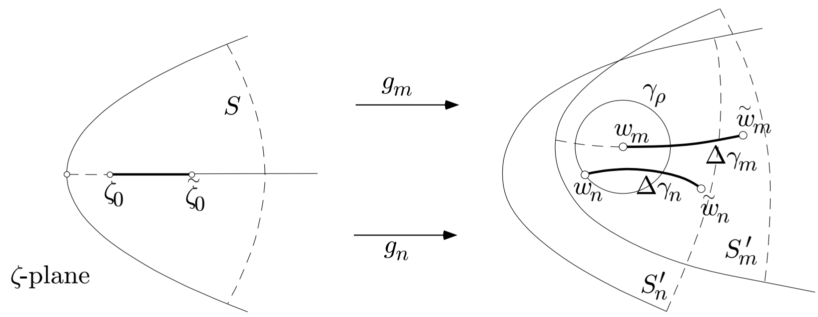

Figure 4.

We start with a fixed parabola segment S in the ζ-plane (Figure 4). Let g_m be the inverse of the parameter mapping for ψ_m, with $g_m(S) = S'_m$, and $g_m([\zeta_0, \tilde{\zeta}_0]) = \Delta\gamma_m$ the distinguished critical trajectory interval of ψ_m, with end points w_m, $\widetilde{w}_m$. Likewise, let g_n be the inverse of the parameter mapping for ψ_n, $g_n(S) = S'_n$, $g_n(\zeta_0) = w_n$, $g_n([\zeta_0, \tilde{\zeta}_0]) = \Delta\gamma_n$ the distinguished critical parameter interval of ψ_n. Because of the convergence $\|\psi_n - \psi_m\| \to 0$ for $w_n \to w_m$ the mappings g_n tend to g_m uniformly in S. In particular, the images $g_n(\zeta)$ and $g_m(\zeta)$, $\zeta \in [\zeta_0, \tilde{\zeta}_0]$, have a distance $|g_n(\zeta) - g_m(\zeta)|$ which is arbitrarily small, and the directions, in which the critical trajectories γ_n and γ_m leave the points w_n and w_m respectively are arbitrarily close. This is so because this direction is the same as minus the arg of the residue, and the residues are arbitrarily close by the locally uniform convergence $\psi_n \to \psi_m$ away from the pole w_m and the Cauchy representation of the residue.

Choose now $\varrho > 0$ small enough and let w_n turn around w_m on the circle $|w_n - w_m| = \varrho$. Then, it is evident that the interval $\Delta\gamma_n$ must pass through the point w_m at least once, and its origin is the point w_n we are looking for.

IV The level sets of $K[w]$

19. Given a quasisymmetric boundary homeomorphism $h\colon \partial\mathbf{D}_z \to \partial\mathbf{D}_w$ and a fixed point $z_0 \in \mathbf{D}_z$. Recall that for every $w \in \mathbf{D}_w$ we define $K[w]$ to be the smallest maximal dilatation of all quasiconformal mappings $f\colon \mathbf{D}_z \to \mathbf{D}_w$ which agree with h on $\partial\mathbf{D}_z$ and satisfy $f(z_0) = w$. The smallest maximal dilatation of all qc mappings f with $f \mid \partial\mathbf{D}_z = h$ but without a given value at a fixed point is called K_0. Every extremal qc mapping $f_0\colon \mathbf{D}_z \to \mathbf{D}_w$, $f_0 \mid \partial\mathbf{D}_z = h$, has maximal dilatation K_0, and the set of images of z_0 by all extremal qc mappings f_0 is called the variability set of z_0 and denoted by $V[z_0]$. Clearly, if the boundary homeomorphism h admits a unique extremal solution, $V[z_0]$ consists of a single point for every z_0. In general, this is however not the case, but for every $w_0 \in V[z_0]$ there exists at least one extremal mapping f_0 with $f_0(z_0) = w_0$, by definition.

If $w_m \notin V[z_0]$ the value $K[w_m]$ is realized as the (constant) dilatation of a unique extremal Teichmüller mapping f_m associated with a pair of quadratic differentials φ_m, ψ_m of norm $\|\varphi_m\| = 1$, $\|\psi_m\| = K[w_m]$. φ_m has a first order pole at z_0, while ψ_m has one at w_m; both differentials are holomorphic otherwise.

We already know that $V[z_0]$ is compact. But in addition to that we can now prove the following

Theorem 7. *The variability set $V[z_0]$ is simply connected, i.e. connected and without holes.*

The proof of the second statement is immediate. For, let G be a component of $\mathbf{D}_w \backslash V[z_0]$ which is bounded away from $\partial\mathbf{D}_w$. Then, the boundary ∂G lies in $V[z_0]$ and the continuous function $K[w]$ must attain its maximum in a point $w_m \in G$, $K[w_m] > K_0$. But by Theorem 6 there is a point w in every neighborhood of w_m

with $K[w] > K[w_m]$, contradicting that $K[w_m]$ is the maximum value of $K[w]$ in G.

The first statement will follow from the theorem in the next section.

20. For every $w_m \in \mathbf{D}_w \setminus V[z_0]$ we consider the following three disjoint subsets of $\mathbf{D}_w$. We set $K[w_m] = K_m$.

$$\begin{aligned} \Gamma^-_{K_m} &:= \{w\,; K[w] < K_m\} \\ \Gamma_{K_m} &:= \{w\,; K[w] = K_m\} \\ \Gamma^+_{K_m} &:= \{w\,; K[w] > K_m\} \end{aligned}$$

Theorem 8. *The level set Γ_{K_m} is a Jordan curve, $\Gamma^-_{K_m}$ its interior and $\Gamma^+_{K_m}$ its extrerior (in $\mathbf{D}_w$). The sets $\Gamma^-_{K_m}$ are decreasing with $K_m \searrow K_0$ and $V[z_0] = \bigcap_{K_m > K_0} \Gamma^-_{K_m}$.*

Proof. (1) Since $K[w] \to \infty$ for $|w| \to 1$ the set $\Gamma^+_{K_m}$ contains an annulus $r < |w| < 1$ for sufficiently large r. As an open set, $\Gamma^+_{K_m}$ decomposes into domains, i.e. maximal connected open subsets. Let G be such a domain and assume that it is bounded away from $\partial\mathbf{D}_w$. Then, its boundary consists of points w with $K[w] = K_m$, whereas for $w \in G$ $K[w] > K_m$. Therefore the function $K[w]$ assumes its maximum at a point $\widehat{w} \in G$. But this is impossible, since in every neighborhood of $\widehat{w}$ there are points w with $K[w] > K[\widehat{w}]$. We conclude that every component of $\Gamma^+_{K_m}$ contains points arbitrarily close to $\partial\mathbf{D}_w$. This means that all components contain the annulus $r < |w| < 1$ and are thus connected themselves, hence $\Gamma^+_{K_m}$ is a domain.

(2) Let C be a component of its complement. It consists of points w with $K[w] \leq K_m$. Every boundary point of C is a point $w_m \in \Gamma_{K_m}$. The trajectory interval $\Delta\gamma_m$ lies in $\Gamma^-_{K_m}$, except for its initial point w_m. On the other hand, as $\Delta\gamma_m$ is connected, it is contained in C. We conclude that C has interior points, and in fact that every boundary point of C is a boundary point of its interior.

Let G^ν be a component of its interior, hence a subdomain of C. For every point $w \in G^\nu K[w] < K_m$, whereas on $\partial G^\nu K[w] = K_m$. Therefore $K[w]$ must assume its minimum at a point $\widehat{w} \in G^\nu$, and by the same reason as above $K[\widehat{w}] = K_0$. We conclude that every component G^ν of int C contains points of $V[z_0]$. From this it follows, because of the uniform continuity of $K[w]$ in C, that int C consists of finitely many components G^ν only. Otherwise, we could pick a point $w_0^\nu \in V[z_0]$ out of every component G^ν. An accumulation point w_0 of these points would lie in $V[z_0]$ and in ∂C, an impossibility. Every G^ν is clearly simply connected.

(3) The next step is to show that there can indeed be only one component of int C. For, let G^0 be a maximal subdomain of int C and assume there is more than one. The boundary ∂G^0 is a subset of the connected set ∂C. It is now easy to see that there exists a point $w_m \in \partial G^0$ which is also a boundary point of at

least one other subdomain of int C, because otherwise we could find a point in ∂C which is not a boundary point of int C. Assume that the points $w \in \Delta\gamma_m \setminus \{w_m\}$ lie in G^0, and let G^1 be another subdomain of int C which has w_m as boundary point. For $w^1 \in G^1$, sufficiently close to w_m, the interval $\Delta\gamma^1$ with initial point w^1 would be arbitrarily close to $\Delta\gamma_m$, and it would thus have to cut ∂G^0. But since $K[w^1] < K_m$ and hence $K[w] < K_m$ for every $w \in \Delta\gamma^1$ this would give a contradiction. We conclude that int C consists of a single simply connected domain G.

(4) We now proceed to show that G is a Jordan domain. Consider a boundary point w_m of G. The interval $\Delta\gamma_m$ with initial point w_m lies in G, except for w_m. This means that every boundary point of G is the end point of an end-cut (the same as "erreichbarer Randpunkt", see [3], pp. 357–363). Consider two end-cuts with the same end point w_m. Their initial points, which lie in G, can be joined by an arc in G. We thus have a Jordan curve which lies in G, except for the point w_m itself. If the two end-cuts were not equivalent, the domain interior to this Jordan curve would contain boundary points of G, i.e. points of Γ_{K_m}. But in every neighborhood of such a point there is a point of $\Gamma^+_{K_m}$, which is impossible.

Therefore the equivalence classes of end-cuts are in one-to-one correspondence with the boundary points of G. By standard arguments the Riemann mapping of G onto the disk has a homeomorphic extension to ∂G, which shows that ∂G is a Jordan curve.

(5) The last step is to show that $\Gamma^-_{K_m}$ consists of only one Jordan domain and hence the level set Γ_{K_m} is a single Jordan curve for every $K_m > K_0$. We have seen that the set $\Gamma^-_{K_m}$ consists of finitely many Jordan domains each of which contains points of $V[z_0]$, and the boundary curves of these Jordan domains are the components of Γ_{K_m}. For $K_n > K_m$ the set $\Gamma^-_{K_n}$ contains $\Gamma^-_{K_m}$ and therefore Γ_{K_m} has at least as many components as Γ_{K_n}. Choose $r < 1$ such that the set $V[z_0]$ is contained in the disk $|w| < r$. For $K_m > \min\{K[w]\,;|w| = r\}$ the circle $|w| = r$ lies in $\Gamma^-_{K_m}$ and hence in one of its components. This component contains all of $V[z_0]$ and is hence the only one. We conclude that Γ_{K_m} consists of a single Jordan curve for all sufficiently large K_m.

Let $K > K_m$, and denote the finitely many components of Γ_{K_m} by $\Gamma^\nu_{K_m}$, hence $\Gamma_{K_m} = \cup\Gamma^\nu_{K_m}$, $\nu = 1,\dots,N$. The level set Γ_K is composed of finitely many Jordan curves Γ^μ_K, $\mu = 1,\dots,M$, $M \le N$. They contain the curves $\Gamma^\nu_{K_m}$ in their interior, and it is easy to see, by contradiction, that for $K \searrow K_m$ the level set Γ_K tends to Γ_{K_m}. Therefore there cannot be fewer curves, one Γ^μ_K for each $\Gamma^\nu_{K_m}$, thus $M = N$, as soon as K is sufficiently close to K_m.

Assume now that $K < K_m$. Again, for $K \nearrow K_m$ the limit set of Γ_K is $\Gamma_{K_m} = \cup\Gamma_{K_m}$. The Jordan curves Γ^μ_K tend to the Jordan curves $\Gamma^\mu_{K_m}$, and eventually in a neighborhood of each $\Gamma^\nu_{K_m}$ there can be only one Γ^μ_K, $\mu = 1,\dots,M$. We conclude that $M = N$ for all $K < K_m$ which are sufficiently close to K_m.

The number of components of Γ_{K_m} is thus a continuous function of K_m, and being entire, it is constant, q.e.d.

The set $V[z_0]$ is evidently equal to the intersection $V[z_0] = \bigcap_{K_m > K_0} \Gamma^-_{K_m}$. This proves the theorem.

Added in proof. In a forthcoming paper, Clifford Earle and Nikola Lakic are going to generalize most of the theorems of this article to the case of Riemann surfaces, using Teichmüller space methods.

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Received 4 February 1997