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SECOND ORDER OBSTACLE PROBLEMS FOR VECTORIAL FUNCTIONS AND INTEGRANDS WITH SUBQUADRATIC GROWTH

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Abstract. It is shown that the obstacle problem associated with the second order variational integral

$$\int_\Omega (1+|\nabla^2 u|^2)^{p/2}\,dx$$

has a unique solution u in the class $C^{1,\alpha}$ for any $\alpha<1$ $(n=2)$ and for $\alpha=1-1/p$ $(n=3)$.

1. Introduction and main results

Let Ω denote an open bounded set in $\mathbf{R}^n$, $n=2$ or $n=3$, having Lipschitz boundary $\partial\Omega$. For $k\in\mathbf{N}$ and $1\le q\le\infty$ we define the Sobolev space $W_q^k(\Omega,\mathbf{R}^M)$ of vector-valued functions $u\colon\Omega\to\mathbf{R}^M$, $M\ge 1$, in the usual way (see, e.g. [A]). The subspace $\mathring{W}_q^k(\Omega,\mathbf{R}^M)$ is introduced as the closure of $C_0^\infty(\Omega,\mathbf{R}^M)$ in $W_q^k(\Omega,\mathbf{R}^M)$, and we say that a measurable function $u\colon\Omega\to\mathbf{R}^M$ belongs to the local space $W_{q,\mathrm{loc}}^k(\Omega,\mathbf{R}^M)$ if $u\in W_q^k(\Omega',\mathbf{R}^M)$ for any subregion Ω' with compact closure in Ω. Suppose now that $p\in(1,2)$ is a fixed number satisfying $p>\frac32$ in case $n=3$ and that a function $\Phi\in W_2^3(\Omega,\mathbf{R}^M)$ satisfies

$$\Phi\,\big|_{\partial\Omega}<0,\quad\text{i.e.}\quad \Phi^i<0\text{ on }\partial\Omega\text{ for }i=1,\dots,M.$$

We consider the variational integral

$$J(u)=\int_\Omega(1+|\nabla^2u|^2)^{p/2}\,dx$$

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which makes sense for u in the space $W_p^2(\Omega, \mathbf{R}^M)$. Note that $W_p^2(\Omega, \mathbf{R}^M) \subset C^o(\overline{\Omega}, \mathbf{R}^M)$ for any $p > 1$, if $n = 2$, and for $p > \frac{3}{2}$, if $n = 3$. Here $\nabla^2 u$ denotes the matrix $(\partial_\alpha \partial_\beta u^i)_{1 \le \alpha,\beta \le n,\ 1 \le i \le M}$ of all second generalized derivatives. We then look for solutions of the obstacle problem

$$(V) \qquad \begin{cases} \text{to find } u \in \mathbf{K} := \{ w \in \mathring{W}_p^2(\Omega, \mathbf{R}^M) : w^i \ge \Phi^i \text{ a.e., } i = 1, \dots, M \} \\ \text{such that } J(u) = \inf\limits_{\mathbf{K}} J. \end{cases}$$

The scalar case $M = 1$ is of some physical interest: consider a plate whose undeformed state is represented by a region $\Omega \subset \mathbf{R}^2$. If some outer forces are applied acting in vertical direction, then the equilibrium configuration can be found as a minimizer of the energy

$$I(u) = \int_\Omega g(\nabla^2 u)\, dx + \text{potential terms}$$

subject to appropriate boundary conditions. The physical properties of the plate are characterized in terms of the given convex function $g\colon \mathbf{R}^{2\times 2} \to [0, \infty)$. In the case of elastic plates (compare [F]) we have $g(E) = |E|^2$ (up to physical constants), for perfectly plastic plates (see [S]) g is of linear growth near infinity, hence the choice

$$g(E) = (1 + |E|^2)^{p/2}$$

provides some interpolation between these two cases which is a suitable model for plastic plates with power hardening. So, for $n = 2$ and $M = 1$, our variational problem (V) reduces to the obstacle problem for pseudo-plastic plates, i.e. plates with power hardening, where now the plate is forced to lie above some function Φ and, in addition, it is clamped at the boundary.

Let us now state the main result of this paper.

Theorem 1.1. *Problem (V) admits a unique solution $u \in \mathbf{K}$. We have $u \in C^{1,\alpha}(\Omega, \mathbf{R}^M)$ for any $\alpha < 1$, if $n = 2$, and $u \in C^{1,1-1/p}(\Omega, \mathbf{R}^M)$, if $n = 3$.*

The proof of Theorem 1.1 is organized in several steps: in Section 2 we first show that the unique minimizer can be obtained with the help of a suitable approximation. This means that we replace our integrand $g(E) = (1 + |E|^2)^{p/2}$ by the sequence

$$g_\delta(E) = \tfrac{1}{2}\delta |E|^2 + g(E), \qquad \delta > 0,$$

and study the more regular obstacle problem

$$(V)_\delta \qquad \begin{cases} \text{to find } u_\delta \in \mathbf{K}' := \{ w \in \mathring{W}_2^2(\Omega, \mathbf{R}^M) : w \ge \Phi \} \\ \text{such that } J_\delta(u_\delta) := \int_\Omega g_\delta(\nabla^2 u_\delta)\, dx = \inf\limits_{w \in \mathbf{K}'} J_\delta(w) \end{cases}$$

whose solutions converge weakly to the solution of (V).

In Section 3 we use this result to show that the scalar function $(1+|\nabla^2 u|^2)^{p/4}$ is in the space $W^1_{2,\mathrm{loc}}(\Omega)$ which will be deduced in this space from uniform bounds for $(1+|\nabla^2 u_\delta|^2)^{p/4}$. The regularity properties of u are then a consequence of Sobolev's embedding theorem. It should be noted that similar arguments in a different setting have been used in [FR] and [FS].

2. Approximation

First we want to show that our variational problem (V) admits a unique solution which will be immediate as soon as we can show that the class $\mathbf{K}$ is non-empty. This follows from

Lemma 2.1. *Suppose that* $\Phi \in W^3_2(\Omega, \mathbf{R}^M)$ *satisfies* $\Phi|_{\partial\Omega} < 0$. *Then there exists* $\Phi_0 \in \overset{\circ}{W}{}^2_2(\Omega, \mathbf{R}^M)$ *with the property* $\Phi_0 \geq \Phi$ *in* Ω, *in particular* $\Phi_0 \in \mathbf{K}$.

Proof. We may assume that $M = 1$. By Sobolev's embedding theorem (recall $n = 2$ or 3) we have $\Phi \in C^o(\overline{\Omega})$, and since $\Phi|_{\partial\Omega} < 0$, there is a subdomain $\Omega' \subset\subset \Omega$ such that $\Phi < 0$ on $\overline{\Omega} - \Omega'$. Let $c := \max\{0, \max_{\overline{\Omega}} \Phi\}$ and $\Phi_0 := c\eta$, where $\eta \in C^\infty_0(\Omega)$ satisfies $0 \leq \eta \leq 1$ on Ω as well as $\eta = 1$ on Ω'. Clearly $\Phi_0 \in \mathbf{K}' \subset K$. ▫

We recall the following characterization of $\overset{\circ}{W}{}^m_q(G)$; in this lemma the function $f \in W^m_q(\mathbf{R}^n)$ is supposed to be (m,q)-quasicontinuous. In particular, the lemma applies to continuous functions in $W^m_q(\mathbf{R}^n)$.

Lemma 2.2 (see [AH, Theorem 9.1.3]). *Let m be a positive integer,* $1 < q < \infty$ *and* $f \in W^m_q(\mathbf{R}^n)$. *Let G denote an open subset of* $\mathbf{R}^n$, $K = \mathbf{R}^n - G$. *Then the following statements are equivalent*:

(a) $D^\beta f|_K = 0$ *for all multi-indices* β, $|\beta| \leq m-1$,

(b) $f \in \overset{\circ}{W}{}^m_q(G)$,

(c) *for any* $\varepsilon > 0$ *and any compact set* $F \subset G$ *there is a function* $\eta \in C^\infty_0(G)$ *such that* $\eta = 1$ *on* F, $0 \leq \eta \leq 1$ *and* $\|f - \eta f\|_{W^m_q(G)} < \varepsilon$.

Lemma 2.3. *Suppose that* $\Phi, \mathbf{K}$ *and* $\mathbf{K}'$ *are defined as in Section* 1. *Then* $\mathbf{K}'$ *is dense in* $\mathbf{K}$ *with respect to the norm* $\|\cdot\|_{W^2_p(\Omega)}$.

Proof. Again we may assume that $M = 1$.

Step I: Given any $\varepsilon > 0$ and an arbitrary function $v \in \mathbf{K}$, we first show that there exists a function $w \in \mathbf{K}$ with compact support such that $\|v - w\|_{W^2_p(\Omega)} < \varepsilon$.

To this end we notice that there is a subdomain $\Omega' \subset\subset \Omega$ such that $\Phi < 0$ on $\overline{\Omega} - \Omega'$ (recall that Φ is continuous on $\overline{\Omega}$). By letting $v = 0$ on $\mathbf{R}^n - \Omega$ we may assume that $v \in W^2_p(\mathbf{R}^n)$. We then use Lemma 2.2 with $f = v$, $F = \overline{\Omega'}$, $G = \Omega$, $m = 2$, $q = p$ to see that there is $\eta \in C^\infty_0(\Omega)$, $0 \leq \eta \leq 1$, $\eta = 1$ on $\overline{\Omega'}$ such that

$$\|v - \eta v\|_{W^2_p(\Omega)} < \varepsilon. \tag{2.1}$$

We claim that $\eta v \geq \Phi$: in fact, $\eta v = v \geq \Phi$ on $\overline{\Omega'}$. For any $x_0 \in \Omega - \overline{\Omega'}$ we have

$$\eta(x_0)v(x_0) \geq 0 > \Phi(x_0) \qquad \text{if } v(x_0) \geq 0$$

and (observe $0 \leq \eta \leq 1$)

$$\eta(x_0)v(x_0) \geq v(x_0) \geq \Phi(x_0) \qquad \text{if } v(x_0) < 0.$$

Thus $w = \eta v$ belongs to $\mathbf{K}$ with $\operatorname{spt} w \subset\subset \Omega$ and $\|v - w\|_{W_p^2(\Omega)} < \varepsilon$ according to (2.1).

Step II: We show that for any $\varepsilon > 0$ and any $w \in \mathbf{K}$ with compact support there exists a $v_\varepsilon \in \mathbf{K}'$ such that

$$\|w - v_\varepsilon\|_{W_p^2(\Omega)} < \varepsilon. \tag{2.2}$$

This together with the first step will complete the proof of Lemma 2.3. For Step II choose $\eta \in C_0^\infty(\Omega)$ satisfying $0 \leq \eta \leq 1$ and $\eta = 1$ on $\overline{\Omega''}$ for some set $\Omega'' \subset\subset \Omega$ containing $\operatorname{spt}(w)$ and with the property that

$$\Phi < 0 \quad \text{on } \Omega - \overline{\Omega''}.$$

Let w_ρ denote the mollification of w with radius $\rho > 0$ (see, e.g. [A]). Using the fact that $\operatorname{spt} w$ is compact in Ω, we find $\rho_0 = \rho_0(\varepsilon)$ such that for all $0 < \rho < \rho_0$ we have

$$\begin{cases} w_\rho \in C_0^\infty(\Omega), \quad \operatorname{spt} w_\rho \subset \Omega'', \\ \|w - w_\rho\|_{L^\infty(\Omega)} < \frac{1}{2}\varepsilon\|\nabla^2\eta\|_p =: \mu, \\ \|w - w_\rho\|_{W_p^2(\Omega)} < \frac{1}{2}\varepsilon. \end{cases} \tag{2.3}$$

Let us fix $\rho \in (0, \rho_0)$ and let $v_\varepsilon = \eta(w_\rho + \mu)$. Then $v_\varepsilon \in C_0^\infty(\Omega)$ and $v_\varepsilon \geq \Phi$ in Ω which follows from (2.3). It is also immediate that (2.3) implies (2.2). □

Remark 2.1. From the proof of Lemma 2.3 we actually get that $\{w \in C_0^\infty(\Omega, \mathbf{R}^M) : w \geq \Phi \text{ in } \Omega\}$ is dense in $\mathbf{K}$ with respect to $\|\cdot\|_{W_p^2(\Omega)}$.

By Lemma 2.1 we know that $\mathbf{K}' \neq \emptyset$, hence there is a unique solution u_δ of problem $(V)_\delta$. Let $u \in \mathbf{K}$ $(\neq \emptyset)$ denote the unique solution of (V). Then we have

Lemma 2.4. *The solution $u_\delta \rightharpoonup u$ weakly in $W_p^2(\Omega)$, $J_\delta(u_\delta) \to J(u)$ and $\frac{1}{2}\delta\int_\Omega |\nabla^2 u_\delta|^2\,dx \to 0$ as $\delta \downarrow 0$.*

Proof. Let $\Phi_0 \in \mathbf{K}'$ denote the function defined in Lemma 2.1. Then for $0 < \delta \le 1$ we have

$$J_\delta(u_\delta) \le J_\delta(\Phi_0) \le J_1(\Phi_0) =: c_1,$$

hence

$$\int_\Omega (1 + |\nabla^2 u_\delta|^2)^{p/2}\, dx \le c_1,$$

which shows that $\sup_{0<\delta\le 1} \|u_\delta\|_{W^2_p(\Omega)} < \infty$. After passing to a subsequence we may assume that

$$u_\delta \underset{\delta\downarrow 0}{\rightharpoondown} \tilde u \text{ weakly in } W^2_p(\Omega, \mathbf{R}^M)$$

for some function $\tilde u$. Since we may also assume that $u_\delta \to \tilde u$ a.e. (after passing to another subsequence if necessary), we clearly get $\tilde u \in \mathbf{K}$.

For any $w \in \mathbf{K}'$ we have

$$J_\delta(u_\delta) \le J_\delta(w) \underset{\delta\downarrow 0}{\longrightarrow} J(w)$$

and by the lower semicontinuity

$$J(\tilde u) \le \liminf_{\delta\downarrow 0} J(u_\delta) \le \liminf_{\delta\downarrow 0} J_\delta(u_\delta) =: \alpha \le \limsup_{\delta\downarrow 0} J_\delta(u_\delta) =: \beta$$

so that

$$J(\tilde u) \le \alpha \le \beta \le J(w) \quad \text{for all } w \in \mathbf{K}'. \tag{2.4}$$

By Lemma 2.3, $\mathbf{K}'$ is dense in $\mathbf{K}$ with respect to $\|\cdot\|_{W^2_p(\Omega)}$. So (2.4) in fact holds for $w \in \mathbf{K}$ and hence $\tilde u$ coincides with the unique solution u of (V). Therefore $u_\delta \rightharpoondown u$ in $W^2_p(\Omega, \mathbf{R}^M)$ for the whole sequence. Choosing $w = u$ in (2.4) we get $\alpha = \beta = J(u)$, i.e. $J(u) = \lim_{\delta\downarrow 0} J_\delta(u_\delta)$. Using

$$J(u) \le J(u_\delta) \le J_\delta(u_\delta)$$

we see that also

$$J(u_\delta) \underset{\delta\downarrow 0}{\longrightarrow} J(u),$$

in conclusion

$$\tfrac{1}{2}\delta \int_\Omega |\nabla^2 u_\delta|^2\, dx \underset{\delta\downarrow 0}{\longrightarrow} 0,$$

and Lemma 2.4 is established. □

3. Existence of higher order weak derivatives

Lemma 3.1. *Let* $u \in \mathbf{K}$ *denote the unique solution of problem* (V). *Then we have* $(1+|\nabla^2 u|^2)^{p/4} \in W^1_{2,\mathrm{loc}}(\Omega)$.

Accepting Lemma 3.1 for the moment we see by the embedding theorem that

$$|\nabla^2 u| \in L^t_{\mathrm{loc}}(\Omega)$$

for any finite t, if $n=2$, and for $t \le 3p$, if $n=3$, hence $\nabla u \in C^{0,\alpha}(\Omega, \mathbf{R}^{nM})$ for any $\alpha < 1$, if $n=2$, and for $\alpha = 1-1/p$ in case $n=3$. This proves Theorem 1.1.

Proof of Lemma 3.1. We fix a coordinate direction $e_\gamma \in \mathbf{R}^n$, $\gamma = 1,\dots,n$, and define for $h \neq 0$ and a function f

$$\Delta_h f(x) = \frac{1}{h}\big(f(x+he_\gamma) - f(x)\big).$$

Let $\{u_\delta\}$ denote the sequence introduced in Section 2. With δ fixed we consider $\varepsilon > 0$ satisfying $\varepsilon h^{-2} < \frac{1}{2}$ and define

$$v_\varepsilon = u_\delta + \varepsilon \Delta_{-h}(\eta^6 \Delta_h[u_\delta - \Phi])$$

with $\eta \in C^2_0(\Omega)$ such that $0 \le \eta \le 1$. Observing

$$\begin{aligned} v_\varepsilon(x) = \Phi(x) + \Big[1 - \frac{\varepsilon}{h^2}\eta^6(x-he_\gamma) - \frac{\varepsilon}{h^2}\eta^6(x)\Big](u_\delta - \Phi)(x) \\ + \frac{\varepsilon}{h^2}\eta^6(x-he_\gamma)(u_\delta-\Phi)(x-he_\gamma) + \frac{\varepsilon}{h^2}\eta^6(x)(u_\delta-\Phi)(x+he_\gamma) \end{aligned}$$

we see that $v_\varepsilon \ge \Phi$, i.e. $v^i_\varepsilon \ge \Phi^i$ for each component, hence $v_\varepsilon \in \mathbf{K}'$, which gives $J_\delta(u_\delta) \le J_\delta(v_\varepsilon)$, and we deduce

$$\int_\Omega \frac{1}{\varepsilon}\left\{g_\delta\big(\nabla^2 u_\delta + \varepsilon \nabla^2[\Delta_{-h}(\eta^6 \Delta_h(u_\delta - \Phi))]\big) - g_\delta(\nabla^2 u_\delta)\right\} dx \ge 0.$$

Passing to the limit $\varepsilon \to 0$ we infer

$$\int_\Omega Dg_\delta(\nabla^2 u_\delta) : \nabla^2\big(\Delta_{-h}[\eta^6 \Delta_h(u_\delta - \Phi)]\big)\, dx \ge 0,$$

Dg_δ denoting the gradient of g_δ. Using "integration by parts" for Δ_{-h} we end up with the result

$$\int_\Omega \Delta_h\{Dg_\delta(\nabla^2 u_\delta)\} : \nabla^2(\eta^6 \Delta_h[u_\delta - \Phi])\, dx \le 0. \tag{3.1}$$

Introducing $\xi_t := \nabla^2 u_\delta + th\Delta_h(\nabla^2 u_\delta)$ we may write

$$\begin{aligned}\Delta_h\{Dg_\delta(\nabla^2 u_\delta)\} &: \nabla^2(\eta^6\Delta_h[u_\delta-\Phi]) \\ &= \int_0^1 D^2 g_\delta(\xi_t)\big(\Delta_h\nabla^2 u_\delta, \nabla^2(\eta^6\Delta_h[u_\delta-\Phi])\big)\,dt.\end{aligned}$$

Let us further define the bilinear form

$$B_x(X,Y) = \int_0^1 D^2 g_\delta\big(\xi_t(x)\big)(X,Y)\,dt$$

for $x\in\Omega$ and matrices X, Y. Then (3.1) takes the form

$$\int_\Omega B_x\big(\Delta_h\nabla^2 u_\delta, \nabla^2(\eta^6\Delta_h[u_\delta-\Phi])\big)\,dx \le 0. \tag{3.2}$$

We have

$$\begin{aligned}\nabla^2(\eta^6\Delta_h u_\delta) &= \eta^6\nabla^2\Delta_h u_\delta + (\partial_\alpha\partial_\beta\eta\Delta_h u_\delta + \partial_\alpha\eta^6\partial_\beta\Delta_h u_\delta + \partial_\beta\eta\partial_\alpha\Delta_h u_\delta)_{1\le\alpha,\beta\le n} \\ &=: \eta^6\nabla^2\Delta_h u_\delta + T_h,\end{aligned}$$

and (3.2) implies

$$\int_\Omega \eta^6 B_x(\Delta_h\nabla^2 u_\delta, \Delta_h\nabla^2 u_\delta)\,dx \le \int_\Omega B_x\big(\Delta_h\nabla^2 u_\delta, \nabla^2(\eta^6\Delta_h\Phi) - T_h\big)\,dx.$$

On the right-hand side we may apply Cauchy–Schwarz's inequality in the form

$$B_x(X,Y) \le B_x(X,X)^{1/2} B_x(Y,Y)^{1/2}.$$

This together with Young's inequality leads to

$$\begin{aligned}&\int_\Omega \eta^6 B_x(\Delta_h\nabla^2 u_\delta, \Delta_h\nabla^2 u_\delta)\,dx \\ &\le c_1(\eta)\int_{\operatorname{spt}\eta}\|B_x\|(|\Delta_h\Phi|^2 + |\nabla\Delta_h\Phi|^2 + |\nabla^2\Delta_h\Phi|^2 + |\Delta_h u_\delta|^2 + |\nabla\Delta_h u_\delta|^2)\,dx\end{aligned} \tag{3.3}$$

for some constant c_1 depending on η.

It is easy to check that the following bounds hold for the parameter dependent bilinear form $D^2 g_\delta(Z)(X,Y)$:

$$\begin{aligned}\|D^2 g_\delta(Z)\| &= \sup_{|X|=1} D^2 g_\delta(Z)(X,X) \\ &\le \delta + c_2(p)(1+|Z|^2)^{(p/2)-1} \le \delta + c_2(p),\end{aligned} \tag{3.4}$$

$$\begin{aligned}D^2 g_\delta(Z)(X,X) &= \delta X : X + p(1+|Z|^2)^{(p/2)-1}|X|^2 \\ &\quad + p(p-2)(1+|Z|^2)^{(p/2)-2}(Z:X)^2 \ge \delta|X|^2.\end{aligned} \tag{3.5}$$

Inserting this into (3.3), we find that

$$\int_\Omega \eta^6 |\Delta_h \nabla^2 u_\delta|^2 \, dx \le c_3(\delta, \eta, p)\big\{\|u_\delta\|^2_{W^2_2(\Omega)} + \|\phi\|^2_{W^3_2(\Omega)}\big\},$$

therefore $u_\delta \in W^3_{2,\mathrm{loc}}(\Omega, \mathbf{R}^M)$. For this reason we can replace Δ_h in (3.1) by the derivative ∂_γ. Then, following the calculations after (3.1), we see that (3.3) has to be replaced by (summation over γ)

$$\text{(3.6)} \qquad \int_\Omega \eta^6 D^2 g_\delta(\nabla^2 u_\delta)(\partial_\gamma \nabla^2 u_\delta, \partial_\gamma \nabla^2 u_\delta)\, dx$$
$$\le c_4(\eta) \int_{\operatorname{spt} \eta} \|D^2 g_\delta(\nabla^2 u_\delta)\| (|\nabla u_\delta|^2 + |\nabla^2 u_\delta|^2 + |\nabla \phi|^2 + |\nabla^2 \phi|^2 + |\nabla^3 \phi|^2)\, dx,$$

c_4 being independent of δ. Recall from Section 2 that

$$\delta \int_\Omega |\nabla^2 u_\delta|^2 \, dx \to 0 \quad \text{as } \delta \downarrow 0.$$

Our assumption concerning p together with $\sup_{\delta>0} \|u_\delta\|_{W^2_p(\Omega)} < \infty$ implies that $\{\nabla u_\delta\}$ is uniformly bounded in $L^2(\Omega, \mathbf{R}^{nM})$. In view of (3.4) it remains to discuss

$$\begin{aligned} \int_\Omega \|D^2 g_\delta(\nabla^2 u_\delta)\| |\nabla^2 u_\delta|^2 \, dx & \\ &\le \delta \int_\Omega |\nabla^2 u_\delta|^2 \, dx + c_2(p) \int_\Omega (1 + |\nabla^2 u_\delta|^2)^{(p/2)-1} |\nabla^2 u_\delta|^2 \, dx \\ &\le \delta \int_\Omega |\nabla^2 u_\delta|^2 \, dx + c_2(p) \int_\Omega (1 + |\nabla^2 u_\delta|^2)^{p/2} \, dx. \end{aligned}$$

We therefore get from (3.6)

$$\text{(3.7)} \qquad \sup_{\delta>0} \int_{\Omega'} D^2 g_\delta(\nabla^2 u_\delta)(\partial_\gamma \nabla^2 u_\delta, \partial_\gamma \nabla^2 u_\delta)\, dx < \infty$$

for each subdomain Ω' of Ω with compact closure.

Let us introduce the auxiliary function $h_\delta := (1 + |\nabla^2 u_\delta|^2)^{p/4}$. It is easy to check that (using the equation in (3.5))

$$\begin{aligned} |\nabla h_\delta|^2 = \partial_\gamma h_\delta \partial_\gamma h_\delta &\le \tfrac14 p^2 (1 + |\nabla^2 u_\delta|^2)^{(p/2)-2} |\nabla^2 u_\delta|^2 \partial_\gamma \nabla^2 u_\delta : \partial_\gamma \nabla^2 u_\delta \\ &\le c_5(p) D^2 g_\delta(\nabla^2 u_\delta)(\partial_\gamma \nabla^2 u_\delta, \partial_\gamma \nabla^2 u_\delta), \end{aligned}$$

hence $\{h_\delta\}_{\delta>0}$ is bounded in $W^1_{2,\mathrm{loc}}(\Omega)$ which is a consequence of (3.7). Let $h \in W^1_{2,\mathrm{loc}}(\Omega)$ denote a weak limit as $\delta \downarrow 0$ of some subsequence. We claim

$$\text{(3.8)} \qquad h = (1 + |\nabla^2 u|^2)^{p/4}.$$

For proving (3.8) let us write

$$\begin{aligned} J_\delta(u_\delta) - J(u) &= \frac{\delta}{2}\int_\Omega |\nabla^2 u_\delta|^2\,dx + J(u_\delta) - J(u) \\ &= \frac{\delta}{2}\int_\Omega |\nabla^2 u_\delta|^2\,dx + \int_\Omega Dg(\nabla^2 u) : (\nabla^2 u_\delta - \nabla^2 u)\,dx \\ &\quad + \int_\Omega \int_0^1 D^2 g\big((1-t)\nabla^2 u + t\nabla^2 u_\delta\big) \\ &\qquad \times (\nabla^2 u_\delta - \nabla^2 u, \nabla^2 u_\delta - \nabla^2 u)(1-t)\,dt\,dx. \end{aligned}$$

From $J_\delta(u_\delta) \to J(u)$, $\delta \downarrow 0$, we infer

$$\text{(3.9)} \quad \lim_{\delta\downarrow 0}\left\{\int_\Omega Dg(\nabla^2 u) : (\nabla^2 u_\delta - \nabla^2 u)\,dx + \int_\Omega\int_0^1 D^2 g(\cdots)(\cdots)(1-t)\,dt\,dx\right\} = 0.$$

On the other hand, minimality of u implies

$$\text{(3.10)} \qquad \int_\Omega Dg(\nabla^2 u) : (\nabla^2 u_\delta - \nabla^2 u)\,dx \geq 0.$$

Next we observe the estimate

$$\begin{aligned} &\int_\Omega\int_0^1 D^2 g(\cdots)(\cdots,\cdots)(1-t)\,dt\,dx \geq \qquad (3.5) \\ &\qquad c_6(p)\int_\Omega (1 + |\nabla^2 u_\delta|^2 + |\nabla^2 u|^2)^{(p/2)-1}|\nabla^2 u_\delta - \nabla^2 u|^2\,dx =: \alpha_\delta. \end{aligned}$$

By (3.9) and (3.10) we must have $\alpha_\delta \to 0$. Hence we can choose a subsequence such that

$$\text{(3.11)} \qquad (1 + |\nabla^2 u_\delta|^2 + |\nabla^2 u|^2)^{(p/2)-1}|\nabla^2 u_\delta - \nabla^2 u|^2 \underset{\delta\downarrow 0}{\longrightarrow} 0 \quad \text{a.e.}$$

Clearly we may assume that also $h_\delta \to h$ a.e. By definition of h_δ this implies

$$|\nabla^2 u_\delta|^2 \to h^{4/p} - 1 \quad \text{a.e.},$$

and the function $h^{4/p} - 1$ is finite a.e. Returning to (3.11) and taking care of our observation that $(1 + |\nabla^2 u_\delta|^2 + |\nabla^2 u|^2)^{(p/2)-1}$ has a pointwise limit a.e. which is not zero, we finally get

$$\nabla^2 u_\delta \to \nabla^2 u \quad \text{a.e. on } \Omega,$$

and in conclusion $h_\delta \to (1+|\nabla^2 u|^2)^{p/4}$ which proves (3.8). The proof of Lemma 3.1 is complete. □

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