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THE ALGEBRA PROPERTY OF THE INTEGRALS OF SOME ANALYTIC FUNCTIONS IN THE UNIT DISK

R. Aulaskari, N. Danikas, and R. Zhao

University of Joensuu, Department of Mathematics
P.O. Box 111, FIN-80101 Joensuu, Finland; Rauno.Aulaskari@joensuu.fi

Aristotelian University of Thessaloniki, Department of Mathematics
GR-54006 Thessaloniki, Greece; danikas@olymp.ccf.auth.gr

University of Joensuu, Department of Mathematics
P.O. Box 111, FIN-80101 Joensuu, Finland rzhao@cc.joensuu.fi

Abstract. We consider the integrals of functions in the α-Bloch space $\mathscr{B}^\alpha$, in the Besov space B_p, in the Möbius invariant subspace Q_p of the weighted Dirichlet space, and we show that they form algebras with respect to the multiplication.

Introduction

Let X be a Banach space of analytic functions in the unit disk $D=\{z\in\mathbf{C}, |z|<1\}$. For $f\in X$ we denote by F the function

$$F(z)=\int_0^z f(\zeta)\,d\zeta,\qquad z\in D.$$

The Banach spaces X we are considering in this paper are the following.

The α-Bloch space $\mathscr{B}^\alpha$, $\alpha>0$, is defined to be the space of all functions f with

$$\|f\|_{\mathscr{B}^\alpha}=|f(0)|+\sup_{z\in D}(1-|z|^2)^\alpha|f'(z)|<\infty.$$

By $\mathscr{B}^\alpha_0$ we denote the space of all functions f with

$$\lim_{|z|\to1}(1-|z|^2)^\alpha|f'(z)|=0$$

([10], [11], [13]). For $\alpha=1$ we get the well-known Bloch space, denoted by $\mathscr{B}$, and the little Bloch space, denoted by $\mathscr{B}_0$ ([1], [2]).

The Besov space B_p, $1<p<\infty$, consists of all functions f with

$$\|f\|_{B_p}=|f(0)|+\left\{\iint_D(1-|z|^2)^{p-2}|f'(z)|^p\,dx\,dy\right\}^{1/p}<\infty,$$

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$z = x + iy$. It is clear that $\mathscr{B} = B_\infty$. The space B_1 is defined as the space of all functions f with

$$\|f\|_{B_1} = |f(0)| + |f'(0)| + \iint_D |f''(z)|\,dx\,dy < \infty.$$

It is known that $B_1 \subset \mathscr{A}$, where $\mathscr{A}$ denotes the disk algebra of D, that is, all functions analytic in D and continuous on $\overline{D}$. Also $B_p \subset B_q \subset \mathscr{B}$ if $1 \le p < q$. Note that B_2 is the classical Dirichlet space $\mathscr{D}$ of the functions f with $\iint_D |f'(z)|^2\,dx\,dy < \infty$ (for the basic theory of B_p spaces see [3] or [12, p. 88–93]).

Finally we denote by Q_p, $0 < p < \infty$, the space of all functions f with

$$\|f\|_{Q_p} = |f(0)| + \left\{\sup_{\zeta\in D} \iint_D |f'(z)|^2 \log^p\left|\frac{1-\overline{\zeta}z}{\zeta - z}\right| dx\,dy\right\}^{1/2} < \infty,$$

and by $Q_{p,0}$, $0 < p < \infty$, the space of all functions f with

$$\lim_{|\zeta|\to 1} \iint_D |f'(z)|^2 \log^p\left|\frac{1-\overline{\zeta}z}{\zeta - z}\right| dx\,dy = 0.$$

It is known ([4]) that for $1 < p < \infty$ the spaces Q_p ($Q_{p,0}$) are all the same and equal to the Bloch space $\mathscr{B}$ ($\mathscr{B}_0$). We note that in [8] a boundary value criterion for functions in Q_p ($Q_{p,0}$) is given, and in [9] a Corona type theorem is proved for Q_p, $0 < p < 1$.

For $p = 1$ we have $Q_1 = \mathrm{BMOA}$, $Q_{1,0} = \mathrm{VMOA}$, where BMOA and VMOA are the classical spaces of analytic functions of bounded mean oscillation and vanishing mean oscillation ([12, p. 179]). For $0 < p < q$ it is known $Q_p \subset Q_q$ ([5]). The space $Q_{1,0} = \mathrm{VMOA}$ contains all B_p functions for $1 \le p < \infty$ ([12, p. 188]).

For the integrals F of Bloch and BMOA functions it is known that they form algebras with respect to the multiplication ([2], [6]).

1. The main result

In this paper we prove that the integrals F of functions in $\mathscr{B}^\alpha$ and in $\mathscr{B}_0^\alpha$ for $0 < \alpha < 2$, in B_p for $1 \le p < \infty$, in Q_p and in $Q_{p,0}$ for $0 < p < \infty$ form algebras with respect to the multiplication.

Our proposition is formulated as follows.

Theorem 1. *Let f_1, f_2 be two functions in X, where X is one of the following Banach spaces of analytic functions in the unit disk D: $\mathscr{B}^\alpha$, $\mathscr{B}_0^\alpha$ for $0 < \alpha < 2$, B_p for $1 \le p < \infty$, and Q_p, $Q_{p,0}$ for $0 < p < \infty$. If*

$$F_j(z) = \int_0^z f_j(\zeta)\,d\zeta, \qquad j = 1, 2,\ z \in D,$$

then

$$F_1(z)F_2(z) = \int_0^z h(\zeta)\,d\zeta, \qquad z \in D,$$

where $h \in X$ *and* $\|h\|_X \le C\|f_1\|_X\,\|f_2\|_X$, *where* C *depends only on* X.

Revising the proof of Theorem 1 we are able to prove a slightly different version.

Theorem 2. *Let* X *be one of the Banach spaces in Theorem* 1. *If*

$$\Phi_j(z) = \frac{1}{z}\int_0^z f_j(\zeta)\,d\zeta, \qquad j = 1, 2,\ z \in D,$$

then

$$\Phi_1(z)\Phi_2(z) = \frac{1}{z}\int_0^z h(\zeta)\,d\zeta, \qquad z \in D,$$

where $h \in X$. *Furthermore,* $\|h\|_X \le C\|f_1\|_X\,\|f_2\|_X$, *where* C *depends only on* X.

2. Proofs of the theorems

Proof of Theorem 1. First we show that the functions $F(z) = \int_0^z f(\zeta)\,d\zeta$, $z \in D$, belong to the disk algebra $\mathscr{A}$ if $f \in X$, where X is one of the Banach spaces in our theorem. If $X \subset \mathscr{B}$, we have (2.3) below which is a sufficient condition for F to be in $\mathscr{A}$ (cf. [7, Theorem 5.5.2]). This covers all cases except $\mathscr{B}^\alpha$, $1 < \alpha < 2$: here we have to use [7, Theorem 5.5.1] and estimate (2.2) below.

We show that if X is one of our Banach spaces, then

$$\|F\|_\infty \le C\|f\|_X. \tag{2.1}$$

For a function $f \in \mathscr{B}^\alpha$ we have $|f'(z)| \le \|f\|_{\mathscr{B}^\alpha}/(1-|z|^2)^\alpha$. By integration we get

$$\begin{aligned} |f(z)| &\le C\|f\|_{\mathscr{B}^\alpha}, && 0 < \alpha < 1,\\ |f(z)| &\le C\frac{\|f\|_{\mathscr{B}^\alpha}}{(1-|z|^2)^{\alpha-1}}, && 1 < \alpha < 2, \end{aligned} \tag{2.2}$$

and

$$|f(z)| \le \|f\|_{\mathscr{B}}\left(1 + \log\frac{1}{1-|z|}\right) \qquad \text{for } \alpha = 1. \tag{2.3}$$

Integrating again, we obtain (2.1) in the case $X = \mathscr{B}^\alpha$.

In the cases $X = B_p$, $1 \le p < \infty$, or $X = Q_p$, $0 < p < \infty$, we have $X \subset \mathscr{B}$ and we know that F must be in $\mathscr{A}$. Furthermore,

$$\|F\|_\infty \le C\|f\|_{\mathscr{B}} \le C\|f\|_X,$$

which is (2.1).

Next, we consider separately the various cases of our Banach spaces X.

(i) $X = \mathscr{B}^\alpha$, $0 < \alpha < 2$. First we note that $(F_1F_2)'(0) = 0$. Further it follows from (2.2) and (2.3) that

$$\begin{aligned} |(F_1F_2)''| &= |F_1''F_2 + F_2''F_1 + 2F_1'F_2'| \\ &\le |f_1'||F_2| + |f_2'||F_1| + 2|f_1||f_2| \\ &\le C|f_1'|\,\|f_2\|_{\mathscr{B}^\alpha} + C|f_2'|\,\|f_1\|_{\mathscr{B}^\alpha} + C\|f_1\|_{\mathscr{B}^\alpha}\|f_2\|_{\mathscr{B}^\alpha} \end{aligned} \tag{2.4}$$

for $0 < \alpha < 1$,

$$|(F_1F_2)''| \le C|f_1'|\,\|f_2\|_{\mathscr{B}^\alpha} + C|f_2'|\,\|f_1\|_{\mathscr{B}^\alpha} + C\|f_1\|_{\mathscr{B}^\alpha}\|f_2\|_{\mathscr{B}^\alpha}\frac{1}{(1-|z|^2)^{2(\alpha-1)}} \tag{2.5}$$

for $1 < \alpha < 2$, and

$$|(F_1F_2)''| \le C|f_1'|\,\|f_2\|_{\mathscr{B}} + C|f_2'|\,\|f_1\|_{\mathscr{B}} + C\|f_1\|_{\mathscr{B}}\|f_2\|_{\mathscr{B}}\left(1 + \log\frac{1}{1-|z|}\right)^2 \tag{2.6}$$

for $\alpha = 1$.

By the above estimates and an easy calculation we get

$$\|(F_1F_2)'\|_{\mathscr{B}^\alpha} \le C\|f_1\|_{\mathscr{B}^\alpha}\|f_2\|_{\mathscr{B}^\alpha}$$

for $0 < \alpha < 2$.

(ii) $X = B_p$, $1 \le p < \infty$. First we assume that $1 < p < \infty$. We have

$$\begin{aligned} |(F_1F_2)''|^p &= |f_1'F_2 + f_2'F_1 + 2f_1f_2|^p \\ &\le C|f_1'|^p\|f_2\|_{B_p}^p + C|f_2'|^p\|f_1\|_{B_p}^p + C|f_1|^p|f_2|^p, \end{aligned} \tag{2.7}$$

where C depends only on p. Since $f_j \in B_p$ implies $f_j \in \mathscr{B}$, we have

$$\begin{aligned} |f_j(z)| &\le \|f_j\|_{\mathscr{B}}\log\left(1 + \frac{1}{1-|z|}\right) \\ &\le C\|f_j\|_{B_p}\log\left(1 + \frac{1}{1-|z|}\right), \qquad z \in D,\ j = 1, 2. \end{aligned}$$

It follows that

$$\begin{aligned} \|(F_1F_2)'\|_{B_p}^p &= \iint_D (1-|z|^2)^{p-2}|(F_1F_2)''|^p\,dx\,dy \\ &\le C\|f_1\|_{B_p}^p\|f_2\|_{B_p}^p \\ &\quad + C\|f_1\|_{B_p}^p\|f_2\|_{B_p}^p\iint_D (1-|z|^2)^{p-2}\left(1 + \log\frac{1}{1-|z|}\right)^{2p}dx\,dy \\ &\le C\|f_1\|_{B_p}^p\|f_2\|_{B_p}^p, \end{aligned}$$

which implies that

$$\|h\|_{B_p} \le C\|f_1\|_{B_p}\,\|f_2\|_{B_p}.$$

Keeping track of the constants, we can let $p \longrightarrow \infty$ to obtain the algebra property of integrals of functions in $\mathscr{B}$ (cf. [2, Section 3.5]).

We consider now the case $X = B_1$. By definition

$$\|(F_1F_2)'\|_{B_1} = |(F_1F_2)'(0)| + |(F_1F_2)''(0)| + \iint_D |(F_1F_2)'''|\,dx\,dy.$$

We observe that

$$(F_1F_2)'(0) = 0$$

and

$$|(F_1F_2)''(0)| = 2|f_1(0)|\,|f_2(0)| \le 2\|f_1\|_{B_1}\|f_2\|_{B_1}.$$

A simple calculation shows

$$|(F_1F_2)'''| \le C|F_1|\,|f_2''| + C|F_2|\,|f_1''| + C|f_1|\,|f_2'| + C|f_2|\,|f_1'|. \tag{2.8}$$

Now the functions f_j, $j = 1,2$, are in the disk algebra $\mathscr{A}$, and

$$\|f_j\|_\infty \le C\|f_j\|_{B_1}, \qquad j = 1,2. \tag{2.9}$$

Further

$$\|F_j\|_\infty \le C\|f_j\|_{B_1}, \tag{2.10}$$

and by [12, p. 58, Remark]

$$\iint_D |f_j'(z)|\,dx\,dy \le C|f_j'(0)| + C\iint_D |f_j''(z)|\,dx\,dy \le C\|f_j\|_{B_1}. \tag{2.11}$$

From (2.8), (2.9), (2.10) and (2.11) it follows immediately that

$$\|(F_1F_2)'\|_{B_1} \le C\|f_1\|_{B_1}\|f_2\|_{B_1}.$$

(iii) $X = Q_p$, $0 < p < \infty$. For a function $f \in Q_p$, $0 < p < \infty$, we have $f \in \mathscr{B}$, so that

$$|f(z)| \le \|f\|_{\mathscr{B}}\left(1 + \log\frac{1}{1-|z|}\right) \le C\|f\|_{Q_p}\left(1 + \log\frac{1}{1-|z|}\right), \qquad z \in D,$$

and $\|F\|_\infty \le C\|f\|_{Q_p}$, where C is a constant depending only on p.

By using the same notation as in (i) and (ii) we get again

$$(F_1F_2)'(0) = 0$$

and

$$|(F_1F_2)''|^2 \le C|f_1'|^2\|f_2\|_{Q_p}^2 + C|f_2'|\,\|f_1\|_{Q_p}^2 + C\|f_1\|_{Q_p}^2\|f_2\|_{Q_p}^2\left(1+\log\frac{1}{1-|z|}\right)^4.$$

We will use here the equivalent norm for functions in Q_p, which involves the Möbius transformation $\varphi_\zeta(z) = (\zeta - z)/(1-\overline{\zeta}z)$, $\zeta, z \in D$, and obtain

$$\|f\|_{Q_p} \sim |||f|||_{Q_p} = |f(0)| + \sup_{\zeta\in D}\left\{\iint_D |f'(z)|^2\bigl(1-|\varphi_\zeta(z)|^2\bigr)^p\,dx\,dy\right\}^{1/2}$$

(see [5, Proposition 1]).

We have now

$$\begin{aligned}\|(F_1F_2)'\|_{Q_p}^2 &\le C\sup_{\zeta\in D}\iint_D |(F_1F_2)''|^2\bigl(1-|\varphi_\zeta(z)|^2\bigr)^p\,dx\,dy\\ &\le C\|f_1\|_{Q_p}^2\|f_2\|_{Q_p}^2 + C\|f_1\|_{Q_p}^2\|f_2\|_{Q_p}^2\iint_D\left(1+\log\frac{1}{1-|z|}\right)^4 dx\,dy\\ &\le C\|f_1\|_{Q_p}^2\|f_2\|_{Q_p}^2.\end{aligned}$$

(iv) $X = \mathscr{B}_0^\alpha$, $0<\alpha<2$ and $X = Q_{p,0}$, $0<p<\infty$. We consider first the case $X = \mathscr{B}_0^\alpha$, $0<\alpha<2$. It suffices to prove that if $f_1, f_2 \in \mathscr{B}_0^\alpha$, then $h \in \mathscr{B}_0^\alpha$.

By (2.4), (2.5) and (2.6) we have

$$\begin{aligned}(1-|z|^2)^\alpha|h'(z)| &= (1-|z|^2)^\alpha|F''(z)|\\ &\le C\|f_2\|_{B_\alpha}(1-|z|^2)^\alpha|f_1'(z)| + C\|f_1\|_{B_\alpha}(1-|z|^2)^\alpha|f_2'(z)|\\ &\quad + C\|f_1\|_{B_\alpha}\|f_2\|_{B_\alpha}(1-|z|^2)^\alpha\end{aligned}$$

for $z\in D$, $0<\alpha<1$,

$$\begin{aligned}(1-|z|^2)^\alpha|h'(z)| &\le C\|f_2\|_{B_\alpha}(1-|z|^2)^\alpha|f_1'(z)| + C\|f_1\|_{B_\alpha}(1-|z|^2)^\alpha|f_2'(z)|\\ &\quad + C\|f_1\|_{B_\alpha}\|f_2\|_{B_\alpha}(1-|z|^2)^{2-\alpha}\end{aligned}$$

for $z\in D$, $0<\alpha<2$, and

$$\begin{aligned}(1-|z|^2)|h'(z)| &\le C\|f_2\|_B(1-|z|^2)|f_1'(z)| + C\|f_1\|_B(1-|z|^2)|f_2'(z)|\\ &\quad + C\|f_1\|_B\|f_2\|_B(1-|z|^2)\left(1+\log\frac{1}{1-|z|}\right)^2\end{aligned}$$

for $z \in D$, $\alpha = 1$.

We see immediately that in all cases

$$(1-|z|^2)^\alpha |h'(z)| \to 0 \qquad \text{as } |z| \to 1.$$

(v) If $X = Q_{p,0}$, $0 < p < \infty$, we have to prove that

$$\iint_D |h'(z)|^2 \big(1-|\varphi_\zeta(z)|^2\big)^p \, dx\, dy \to 0 \qquad \text{as } |\zeta| \to 1.$$

Inequality (2.7) for $p = 2$ and the definition of $Q_{p,0}$ imply that it suffices to show that

$$\iint_D |f_1|^2 |f_2|^2 B(\zeta, z)\, dx\, dy \to 0,$$

where $B(\zeta, z) = \big(1-|\varphi_\zeta(z)|^2\big)^p \le 1$, $\zeta, z \in D$, and $f_1, f_2 \in Q_{p,0}$.

From the estimate

$$|f_j(z)| \le C\|f_j\|_{Q_p}\left(1+\log\frac{1}{1-|z|}\right), \qquad z \in D,$$

(cf. the proof of case (iii)) it follows that

$$\begin{aligned}|f_1(z)|^2|f_2(z)|^2 B(z,\zeta) &\le CM(z)\\ &= C\|f_1\|_{Q_p}^2 \|f_2\|_{Q_p}^2 \left(1+\log\frac{1}{1-|z|}\right)^4, \qquad z, \zeta \in D.\end{aligned}$$

Obviously,

$$\iint_D M(z)\, dx\, dy < \infty$$

and $|f_1(z)|^2|f_2(z)|^2 B(z,\zeta) \to 0$ for every $z \in D$ when $|\zeta| \to 1$.

Lebesgue's dominated convergence theorem implies that

$$\iint_D |f_1(z)|^2|f_2(z)|^2 B(z,\zeta)\, dx\, dy \to 0 \qquad \text{as } |\zeta| \to 1.$$

The proof of our theorem for $X = Q_{p,0}$, $0 < p < \infty$, is now complete.

After this, by minor changes, we are able to prove Theorem 2.

Proof of Theorem 2. We have $\Phi(z) = \int_0^1 f(zt)\, dt$. We start with the case $X \subset \mathscr{B}$. If $L(z) = 1 + \log\big(1/(1-|z|)\big)$, then

$$|\Phi'(z)| \le \left|\int_0^1 t f'(tz)\, dt\right| \le C\|f\|_{\mathscr{B}} L(z) \le C\|f\|_X L(z), \qquad \|\Phi\|_\infty \le C\|f\|_X.$$

As in the proof of Theorem 1, we see that $\Phi \in \mathscr{A}$. It is clear that $h' = (z\Phi_1\Phi_2)''$ is a sum of terms of type $(z\Phi_1)''\Phi_2 = f_1'\Phi_2$, $z\Phi_1'\Phi_2'$ and $\Phi_1(z\Phi_2)'' = \Phi_1 f_2'$. The contributions to the estimate of $\|h\|_X$ from terms of the first and third type are of the form $C\|f_1\|_X\|f_2\|_X$.

A term of the second type is majorized by

$$C\|f_1\|_X\|f_2\|_X L(z)^2.$$

The same computation as in the proof of Theorem 1 will prove Theorem 2. There are some slight differences in the case $X = B_1$, which we leave to the reader.

If $X = \mathscr{B}^\alpha$, $1 < \alpha < 2$, we have

$$|\Phi'(z)| \le \|f\|_{\mathscr{B}^\alpha} \int_0^1 (1 - |z|^2 t^2)^{-\alpha}\, dt \le C\|f\|_X (1 - |z|)^{1-\alpha}, \qquad \|\Phi\|_\infty \le C\|f\|_X,$$

omitting again the details.

Question. If we consider Möbius invariant function spaces, we always have $X \subset \mathscr{B}$. If $\|f\|_X$ is essentially $|||f'|||$ for some norm $|||\cdot|||$ such that $|||L^2||| < \infty$, then the argument above works both for the F- and the Φ-transforms. Does it work for any other transforms?

3. Remark

If we consider the α-Bloch space $\mathscr{B}^\alpha$ with $\alpha \ge 2$, then we do not have the algebra property, since in this case $h \notin \mathscr{B}^\alpha$ in general.

To see this we take for example $f_1(z) = f_2(z) = (1 - z)^{1-\alpha}$, $\alpha \ge 2$, $z \in D$. Then

$$F_1(z) = F_2(z) = \log \frac{1}{1 - z} \qquad \text{if } \alpha = 2$$

and

$$F_1(z) = F_2(z) = \frac{1}{\alpha - 2}\big((1 - z)^{2-\alpha} - 1\big) \qquad \text{if } \alpha > 2.$$

It is easy to check that the function $h(z) = \big(F_1^2(z)\big)'$ does not belong to $\mathscr{B}^\alpha$.

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