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INVOLUTIONS AND SIMPLE CLOSED GEODESICS ON RIEMANN SURFACES

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Abstract. A new geometric characterization of Riemann surfaces with an orientation preserving involution is given. It is proved that a closed Riemann surface M of genus $g \geq 2$ has an involution with exactly $k = 2g + 2 - 4j$ fixed points, $0 \leq j \leq \frac{1}{2}g$, if and only if M has a set F of $2g$ simple closed geodesics which all intersect in the same two points A and B (such that among the elements of F there are no further intersection points). Moreover, A and B are fixed points of the involution and F partitions M into $2g$ hyperbolic quadrilaterals such that exactly $k-2$ of them are symmetric (opposite sides of a symmetric quadrilateral are contained in the same element of F).

1. Introduction

Let M be a closed Riemann surface of genus $g \geq 2$ equipped with a metric of constant curvature -1. Assume that M has an orientation preserving isometric involution $\phi \neq \mathrm{id}$. By the Riemann–Hurwitz relation ϕ has $k = 2g + 2 - 4j$ different fixed points for an integer j with $0 \leq j \leq \frac{1}{2}(g+1)$; we will always exclude the case that ϕ has no fixed points. Let A and B be two fixed points of ϕ, $A \neq B$. Let u be a simple geodesic segment from A to B. Then $u \cup \phi(u)$ is a simple closed geodesic of M. In this manner we can construct a maximal set F of simple closed geodesics of M such that (every) two elements of F intersect only in A and B; we shall see that the order of F is always $2g$. Let $M(F)$ be the surface obtained by cutting M along all elements of F. It will be showed that $M(F)$ has exactly $2g$ connected components which all are (hyperbolic) quadrilaterals. The following questions are then natural.

(1) Let M be a closed Riemann surface of genus g and assume that M has a set F of $2g$ simple closed geodesics such that all elements of F intersect in the same two points A and B and such that there are no further intersection points among the elements of F. Does this imply that M has an orientation preserving involution ϕ such that A and B are among the fixed points of ϕ?

(2) If the answer is yes, in which way the topological properties of F determine the number of fixed points of ϕ?

(3) If the answer to the first question is yes, is the number $2g$ best possible or is a smaller set F already sufficient in order to determine an involution?

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Theorem A. *The answer to the first question is yes.*

Moreover (*this concerns the second question*), *if* $k > 0$ *is the number of fixed points of* ϕ *and* s *the number of connected components of* $M(F)$ *which are* symmetric *quadrilaterals, then* $k - 2 = s$. (*In a* symmetric *quadrilateral the opposite sides are parts of the same element of* F.)

Theorem B. *Concerning the third question, the number* $2g$ *is best possible. Namely, for every integer* $g \geq 2$ *there exists a closed surface* M *of genus* g *with a set* F *of* $2g - 1$ *elements which does not induce an involution.*

The proofs are given in Section 2.

One may ask whether the intersection points of the geodesics in Theorem A and Theorem B are Weierstrass points (for general references on Weierstrass points see [4], [2]). By Lewittes [6], the fixed points of an orientation preserving involution ϕ are ordinary (or 1-fold) Weierstrass points if ϕ has more than four fixed points. If ϕ has four fixed points, then, by Accola [1], the fixed points are at least 2-fold Weierstrass points (but, in general, not ordinary Weierstrass points). Finally, if the involution ϕ has only two fixed points, then these fixed points may miss the dense set of q-fold Weierstrass points, $q = 1, 2, 3, \ldots$, as has been showed by Guerrero [5]. The latter may well be true also for the intersection points of the "counter-examples" in Theorem B.

In the hyperelliptic case we also have the following related result.

Theorem C (Schmutz Schaller). *Let* M *be a closed surface of genus* g. *Then* M *is hyperelliptic if and only if* M *has a set* G *of at least* $2g - 2$ *simple closed geodesics which all intersect in the same point such that among the elements of* G *there are no further intersection points.*

This result has first been proved in [7] (see also the survey paper [8] and [9]). Note that by sets of simple closed geodesics which all intersect in a unique point (as in Theorem C), other involutions than the hyperelliptic one cannot be characterized.

For some results related to those of this paper see Birman and Series [3].

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2. Geometric characterization of involutions

Definition (i) A *surface* is a Riemann surface equipped with a metric of constant curvature -1.

(ii) A (g, n)-*surface* is a surface of genus g with n boundary components which are simple closed geodesics. A *closed* surface is a compact surface without boundary.

(iii) A *simple* geodesic is one without selfintersections.

(iv) An *involution* is an isometry $\phi \neq \mathrm{id}$ with $\phi^2 = \mathrm{id}$ (id is the identity).

(v) Let M be a closed surface. A *geodesic 2-set* F of order $k > 0$ on M is a set of k different simple closed geodesics of M which all intersect in the same two points, the *intersection points of* F, such that among the elements of F, there are no further intersection points. Define by $M(F)$ the surface obtained by cutting M along all elements of F.

(vi) Let M be a closed surface and F a geodesic 2-set. Let A and B be the intersection points of F. Let $u \in F$. Then u is separated by A and B into two parts, the *segments* of u.

Remark and Definition. Let N be (the closure of) a connected component of $M(F)$ (F and M are defined as above). Then the boundary of N consists of a number of simple closed curves which are called *boundary components* of N; they are considered as disjoint, taking different copies of A and B on each boundary component. If N has genus zero and only one boundary component, then I call N a *polygon* and treat the vertices of this polygon as different copies of A and B, respectively. The same convention is used in related cases.

Lemma 1. *Let M be a closed surface of genus g with an orientation preserving involution ϕ with exactly k fixed points. If g is even, then $k \equiv 2 \bmod (4)$. If g is odd, then $k \equiv 0 \bmod (4)$.*

Proof. This is a consequence of the Riemann–Hurwitz relation. □

Remark. Let M be a closed surface of genus g which has an orientation preserving involution ϕ with fixed points. It then follows by Lemma 1 that ϕ has at least two fixed points. This fact will be used throughout without comment.

Lemma 2. *Let M be a closed surface of genus g. Let A and B be two different points on M. Let F be a set of $4g$ simple geodesic segments starting in A and ending in B which are all mutually disjoint in $M \setminus \{A, B\}$. Then the elements of F cut M into exactly $2g$ connected components which all are hyperbolic quadrilaterals.*

Proof. Let $F' \supset F$ such that all elements of F' are simple geodesic segments starting in A and ending in B and such that all elements of F' are mutually disjoint in $M \setminus \{A, B\}$. Assume further that F' is maximal with respect to these conditions. Let $M(F')$ be the surface obtained by cutting M along all elements of F'. Define $M(F)$ analogously.

Let N be a connected component of $M(F')$. Then each boundary component of N contains an even number of geodesic segments. Assume that N has two different boundary components b_1 and b_2. Let A_1 be a copy of A on b_1 and let B_2 be a copy of B on b_2. Since N is convex, N contains a simple geodesic segment v_N from A_1 to B_2. Since $v(N)$ is not in F', this contradicts the maximality of F'.

Assume now that N has only one boundary component b and that N has genus $g(N) > 0$. Then N has a simple geodesic segment $w_N \not\subset b$ from a copy of A on b to a copy of B on b. This again contradicts the maximality of F'.

Assume finally that N has only one boundary component b and that $g(N) = 0$. It follows that N is a polygon. Assume that b has at least six vertices (recall that the number of vertices must be even). Then b contains a copy A_1 of A and a copy B_1 of B such that N has a simple geodesic segment $t(N)$ from A_1 to B_1, $t(N) \not\subset b$. This contradicts the maximality of F'.

We therefore have proved that each connected component of $M(F')$ is a quadrilateral (note that a connected component of $M(F')$ cannot be a polygon with two sides).

Assume that $M(F')$ has q connected components $Q_1, Q_2, \ldots, Q_q$. Let S be the sum of all (inner) angles of the quadrilaterals Q_i, $i = 1, \ldots, q$. Then $S = 4\pi$ since all vertices of Q_i are copies of A and B. We obtain

$$\sum_{i=1}^{q} \mathrm{vol}(Q_i) = 2q\pi - 4\pi = \mathrm{vol}(M) = 4(g-1)\pi$$

where vol is the (hyperbolic) volume. This implies that $q = 2g$. Therefore, there are $8g$ segments as sides of the quadrilaterals Q_i, $i = 1, \ldots, 2g$. It follows that F' has order $4g$. This proves $F = F'$ and hence the lemma. ▫

Corollary 1. *Let M be a closed surface of genus g. Let F be a geodesic 2-set of M of order $2g$. Then $M(F)$ has $2g$ connected components which all are hyperbolic quadrilaterals.*

Proof. Clear by Lemma 2. ▫

Corollary 2. *Let M be a closed surface of genus g which has an orientation preserving involution ϕ with fixed points. Let A and B be fixed points of ϕ, $A \neq B$. Then M has a geodesic 2-set of order $2g$ with fixed points A and B.*

Proof. Let F be a maximal geodesic 2-set on M with intersection points A and B. It is clear that F is not empty since M has a simple geodesic segment u_1 from A to B which implies that $u = u_1 \cup \phi(u_1)$ is a simple closed geodesic passing through A and B. Let N be a connected component of $M(F)$. Assume that N has a simple geodesic segment v starting in a copy of A (on a boundary component of N) and ending in a copy of B (on a boundary component of N) such that v is not a segment of an element of F. Then $v \cup \phi(v)$ is a simple closed geodesic, $v \notin F$, and $F \cup \{v\}$ is a geodesic 2-set. This contradicts the maximality of F. It therefore follows analogously as in the proof of Lemma 2 that N must be a quadrilateral, that the number of connected components of $M(F)$ is $2g$, and that the order of F is $2g$. ▫

Definition. Let F be a geodesic 2-set of order $2g$ in a closed surface M of genus g.

(i) A *quadrilateral of* F is a connected component of $M(F)$.

(ii) Let Q be a quadrilateral of F. Then the sides s_i (in the natural order) of Q are segments of elements u_i of F, $i = 1, 2, 3, 4$. If $u_1 = u_3$ and $u_2 = u_4$, then Q is called *symmetric*.

Lemma 3. *Let F be a geodesic 2-set of order $2g$ in a closed surface M of genus g. Let Q be a quadrilateral of F. Let $u_1, \ldots, u_4$ be the elements of F which form the boundary of Q. Then there is a quadrilateral Q' of F such that u_i, $i = 1, \ldots, 4$, form the boundary of Q' and $Q \neq Q'$ if and only if Q is not symmetric. Moreover, Q' has the same inner angles as Q.*

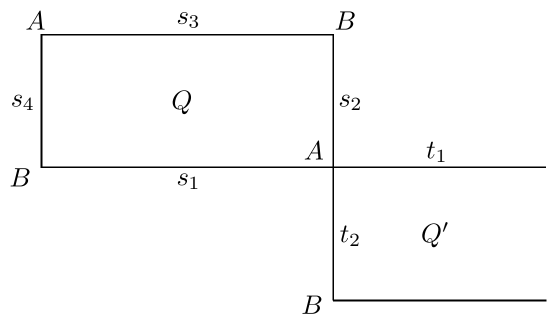

Figure 1. The quadrilaterals Q and Q'.

Proof. (i) We may assume that the notation is such that the segments $s_i \subset u_i$ which form the boundary of Q, appear in the natural order, counter-clockwise say (compare Figure 1). Let α_i be the directed angle from s_i to s_{i+1}, $i = 1, \ldots, 4$ (taking the indices modulo 4), such that α_i is an inner angle of Q. Then all four angles α_i, $i = 1, \ldots, 4$, are measured clockwise. Let A and B be the intersection points of F where the notation is such that A is a vertex in Q between s_1 and s_2 as well as between s_3 and s_4.

(ii) For each $i \in \{1, 2, 3, 4\}$ let $t_i \subset u_i$ be the segment of u_i which is different from s_i. Denote by β_i the directed angle from t_i to t_{i+1}, measured clockwise, $i = 1, \ldots, 4$. Then $\alpha_i = \beta_i$, $i = 1, \ldots, 4$. It follows that there is a quadrilateral Q' of F, containing t_1 and t_2 as sides, such that β_1 is an inner angle of Q'. Since the vertex between t_1 and t_2 in Q' is a copy of A, t_2 will end in a copy B_1 of B. In order to obtain the other side of Q' ending in B_1, we have to turn clockwise around B from u_2 to the next element of F. But this must be u_3 (by the existence of Q), more precisely, this third side of Q' is t_3. The same argument proves that t_4 is the fourth side of Q'. It also follows that $\beta_1, \ldots, \beta_4$ are the inner angles of Q', therefore, Q' and Q have the same inner angles.

(iii) By the existence of Q, u_2 is the next element of F when we turn clockwise around A from u_1, and u_4 is the next element of F when we turn clockwise around A from u_3. It follows that $u_1 = u_3$ if and only if $u_2 = u_4$. Assume that Q is symmetric. It follows by (ii) that t_i and s_{i+2} is the same segment of u_i, $i = 1, \ldots, 4$ (taking the indices modulo 4), which shows that $Q = Q'$.

On the other hand, if $Q = Q'$ then t_i must be a side of Q, $i = 1, \ldots, 4$. Since u_i is simple, it follows that t_i equals s_{i+2}, $i = 1, \ldots, 4$, and hence $u_1 = u_3$ and $u_2 = u_4$. □

Corollary 3. *Let F be a geodesic 2-set of order $2g$ in a closed surface M of genus g. Let Q be a quadrilateral of F, let $u_1, \ldots, u_4$ be the elements of F which form the boundary of Q. Then either all elements u_i, $i = 1, 2, 3, 4$, are different or Q is symmetric. In the latter case, M has an embedded $(1,1)$-surface $S(Q)$ which contains Q.*

Proof. It was already shown during the proof of Lemma 3 that either all four elements u_i, $i = 1, 2, 3, 4$, are different or Q is symmetric. Assume that Q is symmetric. Let the notation be such that $u_1 = u_3$ and $u_2 = u_4$. Let s_2 and s_4 be the segments of u_2. Cut M along u_1 yielding a $(g-1, 2)$-surface M'; denote the boundary geodesics of M' by v_1 and w_1. It then follows that M' contains a unique simple closed geodesic z and an embedded $(0,3)$-surface Y with boundary geodesics z, v_1, w_1 such that $s_2 \subset Y$ (in M, the subsurface Y is an embedded $(1,1)$-surface). Since s_4 is freely homotopic to s_2, it follows that s_4 is contained in Y. □

Corollary 4. *Let M be a closed surface M of genus g which has an orientation preserving involution ϕ with $k > 0$ fixed points. Let A and B be fixed points of ϕ. Let F be a geodesic 2-set of order $2g$ with intersection points A and B. Then among the $2g$ quadrilaterals of F, there are exactly $k-2$ which are symmetric.*

Proof. Let s be the number of symmetric quadrilaterals of F.

If $k > 2$, then ϕ has a fixed point $C \notin \{A, B\}$. C lies in the interior of a quadrilateral Q_C of F (C cannot lie on an element of F since A and B already lie on each element of F). It follows that $\phi(Q_C) = Q_C$ which implies that Q_C is symmetric. This proves $k - 2 \leq s$.

On the other hand, let Q be a quadrilateral of F which is symmetric. By Corollary 3, M has an embedded $(1,1)$-surface $S(Q)$ which contains Q. To Q correspond two elements u_1 and u_2 of F which lie in $S(Q)$. Every $(1,1)$-surface S has a (hyperelliptic) involution ψ with three fixed points, and if v and w are two simple closed geodesics of S which intersect twice, then both intersection points are among the fixed points of ψ. It follows that the hyperelliptic involution ψ of $S(Q)$ is the restriction of ϕ and therefore, ϕ has a third fixed point in $S(Q)$ which lies in the interior of Q. This proves $s \leq k - 2$. □

Definition. Let F be a geodesic 2-set of order $2g$ in a closed surface M of genus g. Let $u \in F$. Then u is called *symmetric* if the two segments of u have equal length.

Corollary 5. *Let F be a geodesic 2-set of order $2g$ in a closed surface M*

of genus g. Let Q be a quadrilateral of F which is symmetric. Let $u \in F$ such that the segments of u are sides of Q. Then u is symmetric.

Proof. By Corollary 3, Q is contained in an embedded $(1,1)$-surface $S(Q)$ of M. As already noted in the proof of Corollary 4, $S(Q)$ has a hyperelliptic involution ψ and the intersection points of F are among the fixed points of ψ. This proves the corollary. ◻

Lemma 4. *Let Q_1 and Q_2 be quadrilaterals with sides a_i, b_i, c_i, d_i, $i = 1, 2$ (in the natural order). Let Q_1 and Q_2 have the same inner angles (the angle between a_1 and b_1 equals the angle between a_2 and b_2, and so on). Then*

(i) $L(a_1) = L(a_2)$ *if and only if Q_1 and Q_2 are isometric, and*

(ii) $L(a_1) > L(a_2) \Longleftrightarrow L(b_1) < L(b_2)$

(where $L(x)$ is the length of x).

Proof. (i) is obvious by hyperbolic trigonometry so assume that $L(a_1) > L(a_2)$. It then follows by (i) that $L(b_1) \neq L(b_2)$. Assume that $L(b_1) > L(b_2)$. Let R_i be the vertex of Q_i between a_i and b_i, $i = 1, 2$. In the hyperbolic plane place Q_2 on Q_1 such that $R_1 = R_2$ and such that $a_2 \subset a_1$ and $b_2 \subset b_1$. Then c_1 and c_2 cannot intersect (since Q_1 and Q_2 have the same angles). By the same argument also d_1 and d_2 cannot intersect. It follows that $Q_2 \subset Q_1$. But since Q_1 and Q_2 have the same volume, this yields a contradiction. ◻

Corollary 6. *Let F be a geodesic 2-set of order $2g$ in a closed surface M of genus g. Let Q be a quadrilateral of F which is not symmetric. Let $u_1, \ldots, u_4$ be the four elements of F which form the boundary of Q. Then either all u_i, $i = 1, \ldots, 4$, are symmetric or none.*

Proof. Let Q' be defined as in Lemma 3. Assume that one of the u_i is symmetric. Since Q and Q' have different segments, it follows by Lemma 4(i) that Q and Q' are isometric and therefore, all u_i are symmetric. ◻

Theorem 1. *Let M be a closed surface of genus g. Then M has a geodesic 2-set F of order $2g$ if and only if M has an orientation preserving involution with fixed points.*

Proof. One direction has already been proved by Corollary 2. Assume now that M has a geodesic 2-set F of order $2g$ with intersection points A and B. Assume that F has a symmetric quadrilateral. It then follows by Corollary 5 and Corollary 6 (and by the fact that all elements of F intersect in A) that every element of F is symmetric.

Let ϕ be the π-rotation around A. It follows that, by ϕ, the quadrilaterals of F are mapped into quadrilaterals of F. More precisely, $\phi(Q) = Q$ if Q is symmetric (by the proof of Corollary 4) and $\phi(Q) = Q'$ if Q is not symmetric where Q' is defined as in Lemma 3. It follows that ϕ is an involution of M.

We therefore can assume that none of the quadrilaterals of F is symmetric. Denote the elements of F by $u_1, \dots, u_{2g}$ such that the u_i lie in the natural order around A. Denote the segments of u_i by v_i and v_{i+2g}, $i = 1, \dots, 2g$, such that the v_j lie in the natural order around A ($j = 1, \dots, 4g$). If an element of F is symmetric, then all elements of F are symmetric by Corollary 6. Assume that the elements of F are not symmetric and that $v_1 > v_{2g+1}$. It then follows by Lemma 4 that $v_2 < v_{2g+2}$. The same argument shows that $v_3 > v_{2g+3}$. By repeating this argument we obtain that $v_{2g} < v_{4g}$ and hence $v_{2g+1} > v_1$, a contradiction. We have therefore proved that all elements of F are symmetric. This implies that the π-rotation around A is an involution of M. □

Theorem 2. *For every integer $g \geq 2$ there exists a closed surface M of genus g which has a geodesic 2-set F of order $2g-1$ with intersection points A and B, but no involution such that A and B are among the fixed points of M.*

Proof. Let $\varepsilon > 0$ be small. For $1 \leq t < 2$, let $T(t)$ be a (hyperbolic) triangle with (inner) angles

$$\alpha(t) = \frac{t\pi}{4(2g-1)} - \varepsilon, \qquad \beta(t) = \frac{t\pi}{4(2g-1)} + \varepsilon, \qquad \gamma = \frac{\pi}{2g-1}.$$

Denote by A, B, C the vertices of $T(t)$ and by a, b, c the sides of $T(t)$ (with the usual convention of notation: a is opposite to A and to $\alpha(t)$, and so on). Take $4g-2$ copies of $T(t)$ and glue them along a or along b such that the vertex C is the same for all $4g-2$ copies. We obtain a $4g-2$-gon $P(t)$ where all sides have the length $L(c)$ and where $2g-1$ angles are $2\alpha(t)$ and $2g-1$ angles are $2\beta(t)$. Denote the sides of $P(t)$ by c_i, $i = 1, \dots, 4g-2$, in the natural order.

(i) Assume now that g is odd. Let $t = 1$. Let S be a triangle of (hyperbolic) area $\frac{1}{2}\pi$ such that two sides x and y of S have the same length while the third side z of S has length $L(c)$. It is clear that S exists and is unique up to isometry. Glue a copy S_1 (with sides x_1, y_1, z_1) of S along z_1 and along c_1 of $P(t)$ such that the interior of $P(t)$ is not intersected by S_1. Glue a copy S_2 (with sides x_2, y_2, z_2) of S along z_2 and along c_{2g} of $P(t)$ such that the interior of $P(t)$ is not intersected by S_2 and such that x_2 is opposite to x_1 and y_2 is opposite to y_1 (the orientation of S_1 and S_2 is the same). Thereby $P(t)$ has been enlarged to a $4g$-gon $R(t)$. By construction, the area of $R(t)$ is $4\pi(g-1)$. $R(t)$ is the fundamental domain of a closed surface $M(t)$ of genus g and we obtain $M(t)$ by the following identifications of the sides of $R(t)$ (the identification is symbolized by a $+$).

$x_1 + x_2$, $y_1 + y_2$,
$c_{4m-2} + c_{4m}$ ($m = 1, \dots, \frac{1}{2}(g-1)$),
$c_{4m} + c_{4m+2}$ ($m = \frac{1}{2}(g+1), \dots, g-1$),
$c_{4m-1} + c_{4m+1}$ ($m = 1, \dots, g-1$).

Let T be one of the copies of $T(t)$ in $R(t)$. By construction, there is a copy T' of $T(t)$ in $R(t)$ such that the side b_1 of T' is the prolongation of the side a_1 of T. Let $u_1 = a_1 \cup b_1$. It is then easy to verify that u_1 is a simple closed geodesic in $M(t)$. Since we have $4g-2$ copies of $T(t)$ we obtain $2g-1$ simple closed geodesics u_i, $i = 1, \dots, 2g-1$, in $M(t)$ which all intersect in C and in V where V corresponds to the $4g$ vertices of $R(t)$ (which all are identified in $M(t)$). Therefore, $\{u_1, \dots, u_{2g-1}\}$ is a geodesic 2-set of order $2g-1$. Since $L(a) < L(b)$, $M(t)$ has not an involution with fixed points C and V.

(ii) Assume now that g is even. Let W be a triangle with three sides of equal length $L(c)$. Denote by δ an (inner) angle of W. Glue a copy W_1 of W along c_1 and glue a copy W_2 of W along c_{2g} (such that the interior of $P(t)$ is not intersected by W_i, $i = 1, 2$). Thereby, $P(t)$ has been enlarged to a $4g$-gon $R(t)$. Denote the new sides of $R(t)$ by x_1, y_1 (coming from W_1) and by x_2, y_2 (coming from W_2) such that x_1 is a neighbour of c_{4g-2} and x_2 is a neighbour of c_{2g-1}.

Let us now assume that t is chosen such that the area of $R(t)$ is

$$(4g-2)\pi - 6\delta - (4g-2)\big(\alpha(t) + \beta(t)\big) = 4\pi(g-1). \tag{1}$$

$R(t)$ is then the fundamental domain of a closed surface $M(t)$ of genus g and we obtain $M(t)$ by the following identifications of the sides of $R(t)$.

$x_1 + c_{4g-3}$, $y_1 + c_3$, $x_2 + c_{2g-2}$, $y_2 + c_{2g+2}$,

and, if $g \neq 2$,

$c_{4m-2} + c_{4m}$ $(m = 1, \dots, \frac{1}{2}(g-2))$,

$c_{4m} + c_{4m+2}$ $(m = \frac{1}{2}(g+2), \dots, g-1)$,

$c_{4m+1} + c_{4m+3}$ $(m = 1, \dots, g-2)$.

It is now easy to see (as above in (i)) that $M(t)$ has a geodesic 2-set of order $2g-1$ with intersection points C and V, but C cannot be a fixed point of an involution since $L(a) < L(b)$.

It remains to show that (1) is possible. Note first that when (1) holds, then

$$\delta = \tfrac{1}{6}\pi(2-t). \tag{2}$$

Let $t = 1$. It then follows (by a calculation) that $\cosh\big(L(c)\big) > 21$ (if ε is small enough) which yields $\cos\delta < 21/22$ and δ is too small (by (2) δ should equal $\pi/6$). Let now $t \longrightarrow 2$. Then $L(c)$ is shorter than in the case $t = 1$ and therefore, δ becomes larger than in the case $t = 1$. But now δ is too large since, by (2), δ should tend to zero. This proves that (1) is possible. □

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