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INFINITESIMAL GEOMETRY OF QUASIREGULAR MAPPINGS

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Abstract. We introduce the notion of the infinitesimal space for a quasiregular mapping at a point. This can be used to study local topological and geometric properties of the mapping also at those points where the mapping is not differentiable.

1. Introduction

The local behavior of an analytic function f at z_0 can be read from its Taylor expansion at z_0. In particular, f behaves like $z \mapsto z^k$ for some $k = 0, 1, \ldots,$ up to a dilation, rotation and translation. For quasiregular mappings f in the plane or in space such a simple characterization is impossible. Moreover, no characterization for the local topological behavior of f is known in space.

The main purpose of this paper is to propose an approach to study the local behavior of a quasiregular mapping f at the points where f need not be differentiable. This approach is based on the convergence and compactness theory, cf., e.g., [Be], [BI], [BIK], [Fer], [GMRV$_1$], [GMRV$_2$], [GMRV$_3$], [Iw$_1$], [Iw$_2$], [IK], [LV], [MRV$_1$], [MRV$_2$], [MRV$_3$], [MRVu], [MSa], [MS$_1$], [MS$_2$], [Re$_1$], [Re$_2$], [Sr], [Vä], [Vu]. In the case of analytic functions our approach gives the aforementioned representation.

Let $f\colon D \to \mathbf{R}^n$, $n \geq 2$, be a nonconstant quasiregular mapping. It is known that f has a total differential almost everywhere in D and, moreover, the Jacobian matrix $f'(x)$ is also nondegenerate almost everywhere. At every point x_0 of differentiability, under the assumption that $f'(x_0) \neq 0$, the local behavior of the mapping f is well-described by the linear transformation $L(z) = f'(x_0)z$, $z \in \mathbf{R}^n$. The local behavior of the mapping f at points where f either fails to be differentiable or has a degenerate Jacobian, for instance at branch points, is much more complicated.

Our method is based on the concept of the infinitesimal space $T(x_0, f)$ of f at x_0. If x_0 is a regular point of f, then $T(x_0, f)$ consists of the unique linear

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transformation $L(z) = f'(x_0)z$, up to a constant positive factor. At an irregular point the infinitesimal space provides a useful extension of the tangent space notion for quasiregular mappings.

We prove that the infinitesimal space is always nonempty and consists of entire quasiregular mappings $g\colon \mathbf{R}^n \to \mathbf{R}^n$, $g(0) = 0$, of polynomial type, that is $g(z) \to \infty$ as $z \to \infty$. The local behavior of a quasiregular mapping f at a point x_0 can be translated into the corresponding global behavior of the elements g of the infinitesimal space $T(x_0, f)$. In particular, we prove that the local injectivity of f at x_0 is equivalent to the fact that the corresponding infinitesimal space consists of quasiconformal mappings only. Moreover, we establish that f at x_0 is weakly conformal if and only if $T(x_0, f)$ consists of orthogonal mappings and that f at x_0 is asymptotically linear if and only if $T(x_0, f)$ consists of linear mappings. As a consequence of the last statement, we obtain that asymptotic linearity of f at x_0 implies injectivity of f in a neighborhood of x_0.

A new kind of local regularity emerges from the structure of the infinitesimal space. This is the case when $T(x_0, f)$ consists of one mapping $g\colon \mathbf{R}^n \to \mathbf{R}^n$ only. We show that now B_g has a ray structure and its dilatation tensor and matrix dilatation have ray symmetry. Furthermore, in this case g is a homogeneous function of a positive order. The asymptotic behavior of the mapping f at the point x_0 has now a simple description in terms of g.

We recall the analytic definition of a quasiregular mapping.

Let $D \subset \mathbf{R}^n$ be a domain. A continuous mapping $f\colon D \to \mathbf{R}^n$ of class $W^1_{n,\mathrm{loc}}$ with the Jacobian $J_f(x) > 0$ a.e. is called Q-quasiregular, $Q \geq 1$, if its local dilatation $K_f(x)$ satisfies the inequality

$$K_f(x) = \|f'(x)\|^n / J_f(x) \leq Q \qquad \text{a.e.} \tag{1.1}$$

As usual, we denote by $f'(x)$ and $J_f(x)$ the Jacobian matrix of the mapping f and its determinant, $J_f(x) = \det f'(x)$, respectively. This definition can be applied to all nonconstant quasiregular mappings [Re$_1$].

Here and later on, we use the ordinary matrix norm

$$\|M\| = \sup_{y \in \mathbf{R}^n \setminus \{0\}} \frac{|My|}{|y|}.$$

The normalized Jacobian matrix of a nonconstant quasiregular mapping f is

$$M_f(x) = f'(x) / J_f(x)^{1/n} \qquad \text{a.e.} \tag{1.2}$$

and the symmetrized normalized Jacobian matrix is

$$G_f(x) = M_f^*(x) M_f(x)$$

where M^* is the transpose of M. Note that (1.2) is, first, given at every regular point $x \in D$, i.e. at points where f is differentiable and $J_f(x) \neq 0$. For the other points $x \in D$ we set $M_f(x) = I$ to complete the definition. We also call $M_f(x)$ and $G_f(x)$ the *matrix dilatation* and the *dilatation tensor* of the mapping f at x, respectively, see, e.g., [A_1], [A_2].

In what follows, we also use the so-called *inner dilatation* $K_I(f)$ of the mapping f, that is the smallest number $K \geq 1$ for which the inequality

$$J_f(x) \leq K\|f'(x)\|^n$$

holds a.e. The smallest number $Q \geq 1$ of (1.1) is called the *outer dilatation* of f and denoted by $K_O(f)$, see [MRV_1].

2. Infinitesimal space

Let $f\colon D \to \mathbf{R}^n$, $n \geq 2$, be a nonconstant Q-quasiregular mapping, $x_0 \in D$, $\varrho_0 = \operatorname{dist}(x_0, \partial D)$, $R(\varrho) = \varrho_0/\varrho$, $\varrho > 0$. For $z \in B\bigl(0, R(\varrho)\bigr)$, let

$$F_\varrho(z) = \frac{f(x_0 + \varrho z) - f(x_0)}{r(x_0, f, \varrho)}, \tag{2.1}$$

where $r(x_0, f, \varrho)$ is the mean radius of the image of the infinitesimal ball $B(x_0, \varrho)$ at the point x_0 under the mapping $f\colon D \to \mathbf{R}^n$,

$$r(x_0, f, \varrho) = \left(\frac{\operatorname{meas} f\bigl(B(x_0, \varrho)\bigr)}{\Omega_n}\right)^{1/n} > 0. \tag{2.2}$$

Here Ω_n denotes the volume of the unit ball $\mathbf{B}^n = B(0,1)$ in $\mathbf{R}^n$. Note that for each $x_0 \in D$, $r(x_0, f, \varrho)$ is defined for small radius of ϱ.

Denote by $T(x_0, f)$ the class of all the limit functions for the family of the mappings F_ϱ as $\varrho \to 0$. The limit is taken in terms of the locally uniform convergence.

We call the set $T(x_0, f)$ *the infinitesimal space* for the mapping f at the point x_0. The elements of $T(x_0, f)$ are called *infinitesimal mappings* for f at x_0. Finally, we call the family (2.1) *an approximating family* for f at x_0.

Note that $R(\varrho) \to \infty$ as $\varrho \to 0$ and that all the mappings $F_\varrho(z)$, $z \in B\bigl(0, R(\varrho)\bigr)$, $F_\varrho(0) = 0$, $\varrho > 0$, are Q-quasiregular mappings. Hence, by the Reshetnyak theorem applied in every fixed ball $B(0, m)$, $m = 1, 2, \ldots$, in $\mathbf{R}^n$, see [Re_1, p. 180], $T(x_0, f)$ consists only of Q-quasiregular mappings $g\colon \mathbf{R}^n \to \mathbf{R}^n$, $g(0) = 0$. We will soon see that they are all nonconstant.

At every regular point x_0, the space $T(x_0, f)$ consists of a single linear mapping

$$\mathscr{A}(z) = M_f(x_0)z, \qquad z \in \mathbf{R}^n, \tag{2.3}$$

where $M_f(x)$ is the matrix dilatation of the mapping f.

In what follows, we will use the term *entire* for mappings which are quasiregular in the whole space $\mathbf{R}^n$, $n \geq 2$. Quasiregular mappings $g\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $g(z) \to \infty$ as $z \to \infty$ are called *mappings of polynomial type.*

2.4. Remark. It is well known, see [MRV$_2$], [MS$_1$], [Sr], that the normalization $g(z) \to \infty$ as $z \to \infty$ holds if and only if

$$\sup\{\operatorname{card} g^{-1}(w) : w \in \mathbf{R}^n\} < \infty, \tag{2.5}$$

that is, g has a bounded valency in the whole space $\mathbf{R}^n$. Moreover, g can be extended to a quasiregular mapping $g\colon \overline{\mathbf{R}^n} \to \overline{\mathbf{R}^n}$ with $g(\infty) = \infty$ and then the degree of g can be defined by the formula

$$\sum_{z \in g^{-1}(w)} i_g(z) = \deg g < \infty \tag{2.6}$$

for all $w \in \mathbf{R}^n$. Here $i_g(z)$ denotes the local topological index of g at z, see [MRV$_1$].

2.7. Theorem. *Let $f\colon D \to \mathbf{R}^n$, $n \geq 2$, be a nonconstant Q-quasiregular mapping and let $x_0 \in D$. Then the infinitesimal space $T(x_0, f)$ is not empty and each $g \in T(x_0, f)$ is a nonconstant Q-quasiregular mapping of $\mathbf{R}^n$ onto $\mathbf{R}^n$ such that* (i) $g(0) = 0$, (ii) $\operatorname{meas} g(\mathbf{B}^n) = \operatorname{meas} \mathbf{B}^n$, (iii) $i_g(0) = i_f(x_0)$, (iv) $\deg g = i_f(x_0)$, *and* (v) $g(z) \to \infty$ *as* $z \to \infty$.

2.8. Corollary. *Let $f\colon D \to \mathbf{R}^n$, $n \geq 2$, be a quasiregular mapping. If f is locally injective at $x_0 \in D$, then each $g \in T(x_0, f)$ is a quasiconformal mapping of $\mathbf{R}^n$ onto $\mathbf{R}^n$. The converse conclusion is also true.*

Thus, $T(x_0, f)$ contains quasiregular mappings $g\colon \overline{\mathbf{R}^n} \to \overline{\mathbf{R}^n}$ which not only inherit the aforementioned local topological properties of f at the origin but also transform them in the corresponding global properties. Section 3 contains some detailed analysis of this phenomenon.

Recall, see [MRV$_1$], [Re$_3$] and [Vu, Theorem 10.22], that for nonconstant Q-quasiregular mappings $f\colon D \to \mathbf{R}^n$, $n \geq 2$, at every point $x_0 \in D$

$$H(x_0, f) = \limsup_{\varrho \to 0} \frac{L(x_0, f, \varrho)}{l(x_0, f, \varrho)} \leq C < \infty, \tag{2.9}$$

$$C = 1 + \tau_n^{-1}\left(\frac{\tau_n(1)}{K_O(f) i_f(x_0)}\right) \tag{2.10}$$

where $\tau_n(s) = \operatorname{cap}\big([-e_1, 0], [se_1, \infty)\big)$, $s > 0$, is the capacity of the Teichmüller condenser. Here we denote as usual

$$l(x_0, f, \varrho) = \inf_{|x - x_0| = \varrho} |f(x) - f(x_0)| \tag{2.11}$$

and

$$L(x_0, f, \varrho) = \sup_{|x-x_0|=\varrho} |f(x) - f(x_0)|, \tag{2.12}$$

for every $0 < \varrho < \varrho_0$ where $\varrho_0 = \operatorname{dist}(x_0, \partial D)$.

The mean radius $r(x_0, f, \varrho)$ of the image of the infinitesimal ball $B(x_0, \varrho)$ under the mapping f has l and L as the natural bounds

$$l(x_0, f, \varrho) \le r(x_0, f, \varrho) \le L(x_0, f, \varrho). \tag{2.13}$$

The proof of Theorem 2.7 is based on convergence and compactness properties and the following distortion estimates of infinitesimal spherical rings for quasiregular mappings, see [GMRV$_3$, p. 252].

2.14. Lemma. *Let $f\colon D \to \mathbf{R}^n$, $n \ge 2$, be a nonconstant Q-quasiregular mapping. Then for all $x_0 \in D$ the inequalities*

$$C^{-2}\Lambda^{\alpha} \le \liminf_{\varrho\to 0} \frac{l(x_0, f, \varrho\Lambda)}{L(x_0, f, \varrho)} \le \Lambda^{\beta} \tag{2.15}$$

and

$$\Lambda^{\alpha} \le \limsup_{\varrho\to 0} \frac{L(x_0, f, \varrho\Lambda)}{l(x_0, f, \varrho)} \le C^2\Lambda^{\beta} \tag{2.16}$$

hold for all $\Lambda \ge 1$ on the left side and for all $\Lambda > \gamma$ on the right side, where

$$\alpha = 1/Q, \tag{2.17}$$

$$\beta = \bigl(Q \cdot i_f(x_0)\bigr)^{1/(n-1)}, \tag{2.18}$$

$$\gamma = C^{2Q} > 1. \tag{2.19}$$

Here C depends only on n and the product $Q \cdot i_f(x_0)$ and is the same as in (2.9).

Using the maximum principle for the open mappings we deduce the following consequence.

2.20. Corollary. *Under the hypothesis of Lemma* 2.14,

$$\limsup_{\varrho\to 0} \frac{\displaystyle\max_{|x-x_0|\le\varrho\Lambda} |f(x) - f(x_0)|}{\displaystyle\min_{|x-x_0|=\varrho} |f(x) - f(x_0)|} \le C^2\Lambda^{\beta} \tag{2.21}$$

for all $\Lambda > \gamma$.

Proof of Theorem 2.7. From the inequality (2.21) we obtain that the approximating family $F_\varrho(z)$ is locally bounded and, consequently, by the Arzela–Ascoli theorem and Theorem 3.17 from [MRV$_2$] is a normal family. Hence, the infinitesimal space $T(x_0,f)$ is not empty and, by the Reshetnyak theorem applied in every fixed ball $B(0,m)$, $m=1,2,\dots$, in $\mathbf{R}^n$, see [Re$_1$, p. 180], consists only of Q-quasiregular mappings $g\colon \mathbf{R}^n\to\mathbf{R}^n$.

The equality $g(0)=0$ is obvious because the approximating mappings satisfy $F_\varrho(0)=0$, $\varrho>0$.

Next, from (2.1) and (2.13) we have the following obvious estimates of the approximating mappings

$$\frac{l(x_0,f,\varrho)}{L(x_0,f,\varrho)} \le \frac{l(x_0,f,\varrho)}{r(x_0,f,\varrho)} \le |F_\varrho(z)| \le \frac{L(x_0,f,\varrho)}{r(x_0,f,\varrho)} \le \frac{L(x_0,f,\varrho)}{l(x_0,f,\varrho)}$$

and by (2.9) we obtain the inequalities $C^{-1}\le |g(z)|\le C$ on the unit sphere for all $g\in T(x_0,f)$. Since $g(0)=0$, it implies that $T(x_0,f)$ does not include constant mappings.

Further, the equality $\operatorname{meas} g(\mathbf{B}^n)=\operatorname{meas}\mathbf{B}^n$ easily follows from the second normalization of approximating mappings $\operatorname{meas} F_\varrho(\mathbf{B}^n)=\operatorname{meas}\mathbf{B}^n$, $\varrho>0$. Indeed, let $\varrho_j\to 0$ and $F_{\varrho_j}\to g$ locally uniformly as $j\to\infty$. Then for every $\varepsilon>0$ there is $N=N(\varepsilon)$ such that for all $j>N$

$$F_{\varrho_j}(S^{n-1})\subset g(S^{n-1}_\varepsilon)$$

where S^{n-1}_ε denotes the ε-neighborhood of the unit sphere $S^{n-1}=\partial\mathbf{B}^n$. Note that $g(S^{n-1}_\varepsilon)$ is an open neighborhood of $g(S^{n-1})$. Thus, for $j>N$, the symmetric difference

$$g(\mathbf{B}^n)\Delta F_{\varrho_j}(\mathbf{B}^n)=\bigl[g(\mathbf{B}^n)\setminus F_{\varrho_j}(\mathbf{B}^n)\bigr]\cup\bigl[F_{\varrho_j}(\mathbf{B}^n)\setminus g(\mathbf{B}^n)\bigr]$$

is contained in $g(S^{n-1}_\varepsilon)$ and since $g(\mathbf{B}^n)$ and $F_{\varrho_j}(\mathbf{B}^n)$ are connected open sets with $0\in g(\mathbf{B}^n)\cap F_{\varrho_j}(\mathbf{B}^n)$, we obtain

$$|\operatorname{meas} g(\mathbf{B}^n)-\operatorname{meas}\mathbf{B}^n|\le \operatorname{meas} g(S^{n-1}_\varepsilon). \tag{2.22}$$

Since g is quasiregular, $\operatorname{meas} B_g=\operatorname{meas} g(B_g)=0$, see, e.g., [MRV$_1$, p. 38–39], and

$$\operatorname{meas} g(S^{n-1}_\varepsilon)\le \int_{S^{n-1}_\varepsilon} J_g(z)\,dm,$$

see, e.g., [Vä, p. 113]. Hence by the absolute continuity of the indefinite integral $\operatorname{meas} g(S^{n-1}_\varepsilon)\to 0$ as $\varepsilon\to 0$ and by (2.22) $\operatorname{meas} g(\mathbf{B}^n)=\operatorname{meas}\mathbf{B}^n$.

Now, there is an $r_0>0$, for all $0<\varrho<r_0$, such that the x_0-component $U(x_0,f,\varrho)$ of $f^{-1}\bigl(B(f(x_0),\varrho)\bigr)$ is a normal neighborhood of x_0. Using this and

the convergence of the topological degree for the approximating sequence we see that $\deg g = i_f(x_0)$ and it easily follows that $i_g(0) = i_f(x_0)$ as well.

Finally, by (2.15) for $|z| \geq 1$

$$|g(z)| \geq C^{-2}|z|^{1/Q}$$

for all $g \in T(x_0, f)$. Thus, $g(z) \to \infty$ as $z \to \infty$ and we have completed the proof.

2.23. Remark. The following properties of the elements g of the infinitesimal space $T(x_0, f)$ follow immediately either by the definition, by Theorem 2.7 or by its proof:

(1) The origin is the only zero of the mapping g,

$$g(z) \neq 0, \qquad z \in \mathbf{R}^n \setminus \{0\}.$$

(2) For all $z \in \mathbf{R}^n$,

$$i_g(z) \leq i_f(x_0).$$

(3) For every $r > 0$, the only component of $g^{-1}\big(B(0,r)\big)$ is a normal neighborhood of 0, see [MRV$_1$, p. 9–10]. The same is true for every $z \in \mathbf{R}^n$ such that

$$i_g(z) = i_f(x_0).$$

(4) For every $z, y \in \mathbf{R}^n$ and $|z| = |y| > 0$,

$$|g(z)|/|g(y)| \leq H(0, g) = H(x_0, f) \leq C.$$

(5) For z on the unit sphere $\partial B(0,1)$ in $\mathbf{R}^n$ it holds

$$C^{-1} \leq |g(z)| \leq C.$$

Here C, as in (2.9), depends only on n and the product $Q \cdot i_f(x_0)$.

Later on, the following lemma on the infinitesimal space will be useful, too.

2.24. Lemma. *Let $f\colon D \to \mathbf{R}^n$, $n \geq 2$, be a nonconstant quasiregular mapping and let $x_0 \in B_f$. Then for every $g \in T(x_0, f)$:*

$$\liminf_{\substack{x \to x_0, \\ x \in B_f}} i_f(x) \leq \liminf_{\substack{z \to 0, \\ z \in B_g}} i_g(z). \tag{2.25}$$

To prove the lemma we need the following simple consequence of the upper semicontinuity of the local topological index, see [MRV$_3$, p. 24]:

2.26. Proposition. *Let $f_j\colon D \to \mathbf{R}^n$, $n \geq 2$, $j = 0,1,2,\ldots$, be a sequence of discrete open mappings such that $f_j \to f_0$ locally uniformly as $j \to \infty$ and let $x_j \in D$, $j = 0,1,2,\ldots$, be a sequence of points in D such that $x_j \to x_0$ as $j \to \infty$. Then:*

$$\limsup_{j\to\infty} i_{f_j}(x_j) \leq i_{f_0}(x_0). \tag{2.27}$$

Indeed, consider the sequence of the mappings

$$h_j(x) = f_j(x + x_j - x_0), \qquad j = 0,1,2,\ldots .$$

It is clear that $h_j \to f_0$ locally uniformly as $j \to \infty$ and by the semicontinuity of the topological index applied to the sequence h_j at the point x_0 we obtain (2.27).

Proof of Lemma 2.24. By the definition of the limes interior, there is a sequence $z_j \to 0$, $z_j \in B_g \setminus \{0\}$, $j = 1,2,\ldots$, such that

$$\lim_{j\to\infty} i_g(z_j) = \liminf_{\substack{z\to 0,\\ z\in B_g\setminus\{0\}}} i_g(z). \tag{2.28}$$

Further, by the definition of $T(x_0, f)$ there is a sequence of approximating mappings, see (2.1), F_{ϱ_k}, $k = 1,2,\ldots$, for some $\varrho_k > 0$, $\varrho_k \to 0$, such that $F_{\varrho_k} \to g$ locally uniformly as $k \to \infty$. Denote by $r_k(z)$, $k = 0,1,2,\ldots$, the injectivity radius at the point $z \in \mathbf{R}^n$ of the mappings g and F_{ϱ_k}, $k = 1,2,\ldots$, respectively. In view of the continuity of the radius of injectivity, see [GMRV$_3$, p. 268], $r_k(z) \to r_0(z)$ as $k \to \infty$ for each fixed $z \in \mathbf{R}^n$.

However, $r_0(z_j) = 0$ because $z_j \in B_g$ for every $j = 1,2,\ldots$. Hence by Proposition 2.26 for every $j = 1,2,\ldots$ there is a branch point $\overline{z_j}$ of the mapping $h_j = F_{\varrho_{k_j}}$ such that

$$|\overline{z_j} - z_j| < \frac{1}{j} \tag{2.29}$$

and

$$i_{h_j}(\overline{z_j}) \leq i_g(z_j). \tag{2.30}$$

By (2.29) and (2.30)

$$\liminf_{j\to\infty} i_f(x_j) \leq \lim_{j\to\infty} i_g(z_j) \tag{2.31}$$

where $x_j \to x_0$ as $j \to \infty$,

$$x_j = x_0 + \varrho_j \overline{z_j} \in B_f \tag{2.32}$$

by the construction.

Finally, (2.28), (2.31) and (2.32) imply the inequality (2.25).

3. The local behavior in terms of $T(x_0, f)$

In this section we study, using the infinitesimal space, asymptotic linearity, weak conformality and local injectivity.

We first recall some terminology from [GMRV$_1$]. Let D be a domain in $\mathbf{R}^n$ containing the origin 0 and let $v, w\colon D \to \mathbf{R}^m$ be mappings, not necessarily continuous. We say that $v(x) = o\big(w(x)\big)$ if for each $\varepsilon > 0$ there is a neighborhood V of 0 such that $\|v(x)\| \le \varepsilon \|w(x)\|$ for all $x \in V \setminus \{0\}$. Here the norm $\|\cdot\|$ need not be the usual Euclidean norm.

The functions v and w are said to be equivalent as $x \to 0$, denoted as

$$v(x) \sim w(x), \tag{3.1}$$

if

$$\|v(x) - w(x)\| = o\big(\|w(x)\| + \|v(x)\|\big). \tag{3.2}$$

It is easy to see that the relation $v(x) \sim w(x)$ is an equivalence relation and it is equivalent to either one of the relations

$$v(x) - w(x) = o\big(v(x)\big), \tag{3.3}$$

or

$$v(x) - w(x) = o\big(w(x)\big). \tag{3.4}$$

Moreover, if $m = 1$, then we have the usual equivalence of the real quantities.

Later on, the usual Euclidean norm and the usual inner product of $\mathbf{R}^m$ are denoted by $|\ |$ and $(\ ,\ )$, respectively. It is shown in [GMRV$_1$] that the equivalence $v(x) \sim w(x)$ with respect to the usual Euclidean metric is equivalent to the following two geometric conditions:

$$|v(x)| \sim |w(x)|. \tag{3.5}$$

and

$$\big(v(x), w(x)\big) \sim |v(x)|\,|w(x)|. \tag{3.6}$$

The first condition means the equivalence of the lengths of the vectors $v(x)$ and $w(x)$ and the second one means that the angle between $v(x)$ and $w(x)$ converges to zero as $x \to 0$.

We write $v(x) \approx w(x)$ and say that v and w have the same order of smallness at the origin if

$$c^{-1}|v(x)| \le |w(x)| \le c|v(x)| \tag{3.7}$$

for some $c \ge 1$ as $x \to 0$.

The equivalence relations (3.1) and (3.7) are quasiconformal invariants, see [GMRV$_1$]. Now we are ready to give the main definition.

A mapping $f\colon D \to \mathbf{R}^m$, $f(0)=0$, is said to be *asymptotically linear* at 0 if for each $\varrho \in \mathbf{R} \setminus \{0\}$

$$f(\varrho x) \sim \varrho f(x) \tag{3.8}$$

as $x \to 0$ and

$$f(x+z) \sim f(x) + f(z) \tag{3.9}$$

as $x \to 0$ and $x+z$, x, z have the same order of smallness.

If (3.8) holds uniformly with respect to ϱ, $c^{-1} \le |\varrho| \le c$, for each $1 \le c < \infty$, then we say that f is *uniformly asymptotically linear* at 0.

We have proved in [GMRV$_1$] that, for quasiconformal mappings, asymptotic linearity always implies uniform asymptotic linearity.

3.10. Theorem. *Let $f\colon D \to \mathbf{R}^n$, $n \ge 2$, be a nonconstant quasiregular mapping and $x_0 \in D$. Then the following assertions are equivalent*:

(1) *$T(x_0, f)$ consists of linear mappings.*

(2) *f is asymptotically linear at the point x_0.*

Proof. Without loss of generality we may assume that $f(x_0) = x_0 = 0 \in D$.

(1) $\Rightarrow$ (2). Assume that (3.8) does not hold for f. Then there exist $\varrho \in \mathbf{R}\setminus\{0\}$ and a sequence $x_j \to 0$, $x_j \in \mathbf{R}^n \setminus \{0\}$, such that

$$\frac{|f(\varrho x_j) - \varrho f(x_j)|}{|f(\varrho x_j)| + |\varrho|\,|f(x_j)|} \ge \varepsilon \tag{3.11}$$

for some $\varepsilon > 0$. Without loss of generality we may assume that $\eta_j = x_j/|x_j| \to \eta_0 \in \mathbf{R}^n$, $|\eta_0| = 1$. Moreover, setting $t_j = |x_j|$ and arguing as in the proof of Theorem 2.7 we may also assume that the corresponding approximating sequence F_{t_j} converges to $F_0 \in T(x_0, f)$ locally uniformly as $j \to \infty$ where $|F_0| \ge C^{-1} > 0$ on the unit sphere. Thus, we obtain

$$\begin{aligned}\lim_{j\to\infty} \frac{|f(\varrho x_j) - \varrho f(x_j)|}{|f(\varrho x_j)| + |\varrho f(x_j)|} &= \lim_{j\to\infty} \frac{|F_{t_j}(\varrho \eta_j) - \varrho F_{t_j}(\eta_j)|}{|F_{t_j}(\varrho\eta_j)| + |\varrho F_{t_j}(\eta_j)|}\\ &= \frac{|F_0(\varrho\eta_0) - \varrho F_0(\eta_0)|}{|F_0(\varrho\eta_0)| + |\varrho F_0(\eta_0)|} = 0\end{aligned}$$

and this contradicts (3.11).

Next, let us suppose that (3.9) does not hold for f. Then there exist sequences $x_j, z_j \in \mathbf{R}^n \setminus \{0\}$, $x_j, z_j \to 0$ as $j \to \infty$ such that $x_j + z_j \approx x_j \approx z_j$ and

$$\alpha_j = \frac{|f(x_j + z_j) - f(x_j) - f(z_j)|}{|f(x_j+z_j)| + |f(x_j) + f(z_j)|} \ge \varepsilon \tag{3.12}$$

for some $\varepsilon > 0$.

Let $y_j = (x_j + z_j)/|x_j + z_j|$, $\eta_j = x_j/|x_j + z_j|$, $\zeta_j = z_j/|x_j + z_j|$ and assume that $y_j \to y_0$, $\eta_j \to \eta_0$, $\zeta_j \to \zeta_0$, $y_0, \eta_0, \zeta_0 \in \mathbf{R}^n$ as $j \to \infty$ and $|y_0| = 1$. Then, setting $t_j = |x_j + z_j|$ and arguing once more as above we deduce that for the corresponding approximating sequence $F_{t_j} \to F_0$:

$$\lim_{j\to\infty} \alpha_j = \lim_{j\to\infty} \frac{|F_{t_j}(y_j) - F_{t_j}(\eta_j) - F_{t_j}(\zeta_j)|}{|F_{t_j}(y_j)| + |F_{t_j}(\eta_j) + F_{t_j}(\zeta_j)|} = \frac{|F_0(y_0) - F_0(\eta_0) - F_0(\zeta_0)|}{|F_0(y_0)| + |F_0(\eta_0) + F_0(\zeta_0)|} = 0$$

and this contradicts (3.12).

So the mapping f is asymptotically linear.

(2) $\Rightarrow$ (1). Let $F_0 \in T(x_0, f)$. Then $F_{t_j} \to F_0$ locally uniformly for some approximating sequence F_{t_j}, $t_j > 0$, $t_j \to 0$ as $j \to \infty$. For each fixed $x \in \mathbf{R}$, $\varrho \in \mathbf{R} \setminus \{0\}$ we have that $F_{t_j}(x) \to F_0(x)$ and $F_{t_j}(\varrho x) \to F_0(\varrho x)$ and, since $f(\varrho t_j x) \sim \varrho f(t_j x)$, also $F_{t_j}(\varrho x) \sim \varrho F_{t_j}(x)$ as $j \to \infty$. Consequently, $F_0(\varrho x) = \varrho F_0(x)$.

Similarly, for each fixed $x, z, x+z \in \mathbf{R}^n \setminus \{0\}$ we have that $F_{t_j}(x) \to F_0(x)$ and $F_{t_j}(z) \to F_0(z)$ and, since $f\bigl(t_j(x+z)\bigr) \sim f(t_j x) + f(t_j z)$, also $F_{t_j}(x+z) \sim F_{t_j}(x) + F_{t_j}(z)$ as $j \to \infty$. Consequently, $F_0(x+z) = F_0(x) + F_0(z)$ and we have proved that F_0 is linear.

3.13. Corollary. *Let $f\colon D \to \mathbf{R}^n$, $n \geq 2$, be a nonconstant quasiregular mapping that is asymptotically linear at $x_0 \in D$. Then f is homeomorphic in a neighborhood of x_0.*

Indeed, by Theorem 3.10, $T(x_0, f)$ consists of linear mappings $g\colon \mathbf{R}^n \to \mathbf{R}^n$. On the other hand, by Theorem 2.7, $T(x_0, f)$ contains only nonconstant quasiregular mappings. Since every nonconstant quasiregular mapping is an open mapping, see, e.g., [Re_1] and [MRV_1], we deduce that $g(\mathbf{R}^n)$ is an open subset of $\mathbf{R}^n$. Thus, the linear mappings $g \in T(x_0, f)$ are nondegenerate because otherwise the image of the whole space under the mapping g should be contained in some hyperplane. The corollary now follows from Theorem 2.7.

3.14. Corollary. *Let $f\colon D \to \mathbf{R}^n$, $n \geq 2$, be a nonconstant quasiregular mapping that is asymptotically linear at $x_0 \in D$. Then f is uniformly asymptotically linear at x_0.*

Indeed, Corollary 3.13 implies that the mapping f is quasiconformal in a neighborhood of x_0 and the statement follows by Lemma 4.14 from [GMRV_1].

We denote by $H_f(r)$ the least upper bound of the numbers $t \geq 1$ such that

$$\sup_{|x|=r} |f(x)| \leq t \inf_{|x|=r} |f(x)|, \tag{3.15}$$

where $0 < r < d(0, \partial D)$. The origin is called a *point* of *spherical analyticity* for f if

$$\lim_{r \to 0} H_f(r) = 1. \tag{3.16}$$

Later on we say for brevity that a mapping $f\colon D \to \mathbf{R}^m$, $f(0) = 0 \in D$, is *weakly conformal at the origin* if it is, simultaneously, asymptotically linear and spherically analytic there. In this case the mapping f preserves infinitesimal spheres and spherical rings and angles, in some generalized sense, between rays emanating from the origin. In other words, it has properties typical of conformal mappings. For some examples of mappings in these classes see [GMRV$_1$].

3.17. Theorem. *Let $f\colon D \to \mathbf{R}^n$, $n \geq 2$, be a nonconstant quasiregular mapping and $x_0 \in D$. Then the following assertions are equivalent*:

(1) *$T(x_0, f)$ consists of orthogonal mappings.*

(2) *f is weakly conformal at the point x_0.*

Proof. We may assume that $f(x_0) = x_0 = 0 \in D$.

(1) $\Rightarrow$ (2). By Theorem 3.10 f is asymptotically linear at the origin. To prove that the origin is a point of spherical analyticity for f let us suppose the converse. Then there exist sequences x_j and $z_j \in \mathbf{R}^n \setminus \{0\}$, $|x_j| = |z_j| \to 0$ as $j \to \infty$ such that

$$\lim_{j \to \infty} \frac{|f(z_j)|}{|f(x_j)|} = 1 + \varepsilon \tag{3.18}$$

for some $\varepsilon > 0$. Without loss of generality we may assume that $y_j = z_j/|z_j| \to y_0 \in \mathbf{R}^n$, $\eta_j = x_j/|x_j| \to \eta_0 \in \mathbf{R}^n$, $|\eta_0| = |y_0| = 1$, as $j \to \infty$. Moreover, arguing as in the proof of Theorem 2.7 we may assume also that the corresponding approximating sequence converges, $F_{t_j} \to F_0 \in S_f(0)$ locally uniformly as $j \to \infty$, where we set $t_j = |x_j|$. Consequently, $F_{t_j}(y_j) \to F_0(y_0)$, $F_{t_j}(\eta_j) \to F_0(\eta_0)$ as $j \to \infty$.

Now, the mapping F_0 is an orthogonal mapping. Hence

$$\lim_{j \to \infty} \frac{|F_{t_j}(y_j)|}{|F_{t_j}(\eta_j)|} = 1$$

and this contradicts (3.18).

(2) $\Rightarrow$ (1). Again, by Theorem 3.10, $T(x_0, f)$ consists of linear mappings. To prove that $T(x_0, f)$ consists of orthogonal mappings let us suppose the converse. Then there is $F_0 \in T(x_0, f)$ and $y, \eta \in \mathbf{R}^n$, $|y| = |\eta| = 1$, such that

$$\frac{|F_0(y)|}{|F_0(\eta)|} = 1 + \varepsilon$$

for some $\varepsilon > 0$. By the definition of the infinitesimal space there is a sequence of t_j of positive numbers tending to zero such that the approximating sequence F_{t_j} converges locally uniformly to F_0. In particular,

$$\lim_{j\to\infty} \frac{|F_{t_j}(y)|}{|F_{t_j}(\eta)|} = 1 + \varepsilon$$

and, consequently,

$$\lim_{j\to\infty} \frac{|f(t_j y)|}{|f(t_j \eta)|} = 1 + \varepsilon.$$

However, this last relation contradicts the weak conformality of the mapping f.

3.19. Theorem. *Let $f\colon D \to \mathbf{R}^n$, $n \geq 3$, be a nonconstant quasiregular mapping and let the dilatation tensor $G_f(x)$ or the matrix dilatation $M_f(x)$ be approximately continuous at $x_0 \in D$. Then the infinitesimal space $T(x_0, f)$ contains only mappings of the form*

$$g(z) = U \circ L(z) \tag{3.20}$$

where U are orthogonal mappings, $U \in \mathscr{O}(n)$, and $L(z) = M_f(x_0)z$, $z \in \mathbf{R}^n$.

Proof. The approximate continuity of $G_f(x)$ and $M_f(x)$ at x_0 means that the dilatation tensors $G(z,t) = G_f(x_0 + tz)$ and the matrix dilatations $M(z,t) = M_f(x_0 + tz)$ of the approximating family $F_t(z)$ converge as $t \to 0$ in measure to $G_f(x_0)$ and $M_f(x_0)$, respectively. If $g \in T(x_0, f)$, then there is an approximating sequence $F_{t_j} \to g$ converging locally uniformly where $t_j \to 0$ as $j \to \infty$ and by the corresponding convergence theorems, see [IK] and [GMRV$_2$], $G_g(z) \equiv G_f(x_0)$ and $M_g(z) \equiv O(z)M_f(x_0)$ where $O(z) \in \mathscr{O}(n)$, respectively. Since the mapping $\varphi = g \circ L^{-1}$ is a 1-quasiregular mapping, by the well-known Liouville theorem φ is a Mbius transformation. In view of the normalization of g, see Theorem 2.7, φ is an orthogonal mapping. The proof is complete.

3.21. Remark. The above theorems have a number of consequences. For example:

(1) Theorems 3.10 and 3.19 imply that f is asymptotically linear at x_0, if G_f or M_f is approximately continuous at x_0, cf. [GMRV$_3$].

(2) By Corollary 3.13 we conclude that f is a homeomorphism in a neighborhood of x_0, if G_f is either continuously differentiable as in [Fer] or continuous as in [BIK], and also if G_f or M_f is approximately continuous [GMRV$_3$].

(3) By the well-known theorem of Zorich [Zo$_1$] we obtain global injectivity of $f\colon \mathbf{R}^n \to \mathbf{R}^n$, $n \geq 3$, under the conditions in Theorems 3.10 and 3.19 and Corollary 3.13.

(4) Theorems 3.17 and 3.19 imply that f is weakly conformal at x_0, if G_f or M_f is approximately continuous at x_0 and $G_f(x_0)$ or $M_f(x_0)$ is an orthogonal matrix.

(5) Extending the approach used for the proof of Theorem 3.19 we may show that $T(x_0, f)$ contains only linear mappings and therefore, by Theorem 3.17, f is asymptotically linear and by Corollary 3.13 f is injective in a neighborhood of x_0, if G_f or M_f has *vanishing mean oscillation* at x_0, see [MRVu].

4. Description of simple infinitesimal mappings

We say that the infinitesimal space $T(x_0, f)$ is *simple* if it consists of one mapping g only. In this case g has interesting additional properties. We call every such g a *simple infinitesimal mapping.*

4.1. Theorem. *Let $f\colon D \to \mathbf{R}^n$, $n \geq 2$, be a nonconstant quasiregular mapping and $x_0 \in D$. If $T(x_0, f)$ consists of one mapping $g\colon \mathbf{R}^n \to \mathbf{R}^n$ only, then g is a positively homogeneous nonconstant quasiregular mapping, i.e. there is $d > 0$ such that for all $t > 0$*

$$g(tz) = t^d g(z). \tag{4.2}$$

It easy to see that, in general, every positively homogeneous nonconstant quasiregular mapping of $\mathbf{R}^n$ is simple infinitesimal for itself at the origin.

Theorem 4.1 has useful consequences.

4.3. Corollary. *Under the conditions of Theorem* 4.1, *the following assertions hold*:

(1) *The branch set B_g of g and its image $g(B_g)$ have a ray structure at the origin, i.e. each of them is a union of rays emanating from the origin.*

(2) *The local topological index $i_g(z)$, $z \in \mathbf{R}^n$, is constant in every such ray, i.e.*

$$i_g(tz) = i_g(z), \qquad t > 0,\ z \in \mathbf{R}^n \setminus \{0\}. \tag{4.4}$$

(3) *The matrix dilatation $M_g(z)$, $z \in \mathbf{R}^n$, also has a ray symmetry, i.e.*

$$M_g(tz) = M_g(z), \qquad t > 0,\ z \in \mathbf{R}^n \setminus \{0\}. \tag{4.5}$$

The same is true for the dilatation tensor of the mapping g.

(4) *The infinitesimal mapping g is absolutely continuous on every ray,*

$$g(R\eta) = \int_0^R \partial g_r(r\eta)\, dr, \qquad R > 0,\ \eta \in S^{n-1}. \tag{4.6}$$

Proof of Theorem 4.1. We may assume that $x_0 = 0$ and $f(0) = 0$.

Fix $\zeta \in \mathbf{R}^n$, $\zeta \neq 0$, and $\tau > 0$. Now, if g is the only mapping in $T(x_0, f)$, then

$$f(t\tau\zeta) \sim r(t\tau)g(\zeta) \sim r(t)g(\tau\zeta)$$

as $t \to 0$, $t > 0$. Here, as above, we use the abbreviation $r(t) = r(x_0, f, t)$ for the mean radius (2.2) and the symbol $\sim$ for the equivalence relation (3.1) of vector-valued functions.

Thus, the vectors $g(\zeta)$ and $g(\tau\zeta)$ are collinear, i.e.

$$g(\tau\zeta) = \omega(\tau) g(\zeta)$$

where

$$\omega(\tau) = \lim_{t\to 0} r(t\tau)/r(t) > 0$$

is independent of $\zeta \in \mathbf{R}^n \setminus \{0\}$.

Hence we have for each fixed $\tau > 0$ that

$$r(t\tau) \sim \omega(\tau) r(t)$$

as $t \to 0$. So for any fixed $\tau, s > 0$ we have

$$r(t\tau s) \sim \omega(\tau s) r(t) \sim r(t\tau)\omega(s) \sim \omega(\tau)\omega(s) r(t)$$

as $t \to 0$ and, consequently,

$$\omega(\tau s) = \omega(\tau) \cdot \omega(s).$$

Write

$$\varphi(T) = \ln \omega(e^T).$$

Then

$$\varphi(T_1 + T_2) = \varphi(T_1) + \varphi(T_2)$$

for all T_1 and $T_2 \in \mathbf{R}$ and hence

$$\varphi(m) = m \cdot \varphi(1)$$

where $m = 0, \pm 1, \pm 2, \ldots$. In general, for each rational number we obtain the same

$$\varphi\left(\frac{m}{k}\right) = \frac{m}{k} \cdot \varphi(1).$$

Since the set of all rational numbers is everywhere dense in $\mathbf{R}$, we conclude that

$$\varphi(\gamma) = \gamma \cdot \varphi(1)$$

for all $\gamma \in \mathbf{R}$, provided that the function φ is continuous. But the continuity of φ holds because, by Theorem 2.7, $g(\zeta) \neq 0$, $\zeta \in \mathbf{R}^n \setminus \{0\}$, and ω has the representation

$$\omega(\tau) = \frac{|g(\tau\zeta)|}{|g(\zeta)|}$$

for each fixed $\zeta \in \mathbf{R}^n \setminus \{0\}$ and the function g is clearly continuous.

Consequently,

$$\omega(\tau) = e^{\varphi(\ln \tau)} = e^{\varphi(1)\ln \tau} = \tau^d$$

where $d = \varphi(1)$. Since $g(0) = 0$ and $g(\tau\zeta) \to 0$ as $\tau \to 0$, we obtain that $d > 0$.

From Theorem 4.1 we also have the following statement on the asymptotic behavior of the mapping f.

4.7. Proposition. *Under the conditions of Theorem* 4.1, *the mapping* $f\colon D \to \mathbf{R}^n$, $f(0) = 0 \in D$, *has the asymptotic representation*

$$f(x) \sim \varrho(|x|) \cdot g(x) = r(|x|) \cdot g(x/|x|) \tag{4.8}$$

as $x \to 0$ *where the function*

$$\varrho(t) = \frac{r(t)}{t^d} \tag{4.9}$$

is such that for all $s > 0$

$$\lim_{t\to 0} \frac{\varrho(st)}{\varrho(t)} = 1. \tag{4.10}$$

Here $r(t) = r(0, f, t)$ *denotes the mean radius* (2.2) *of the image of the infinitesimal ball* $B(0, t)$ *under the mapping* f.

Proof. Fix $\zeta \in \mathbf{R}^n$, $\zeta \neq 0$, and $s > 0$. If $T(0, f)$ contains one mapping g only, then by (4.2)

$$r(st)g(z) \sim f(stz) \sim r(t)g(sz) = s^d r(t)g(z)$$

as $t \to 0$, $t > 0$. Thus, for all $s > 0$

$$r(st) \sim s^d r(t)$$

as $t \to 0$, $t > 0$, and (4.10) holds.

Finally, setting $t = |x|$, $z = x/|x|$, $x \in D \setminus \{0\}$, $x \to 0$, we see, again from (4.2) and from the definition of the infinitesimal mapping g, that

$$f(x) \sim r(|x|)g\big(x/|x|\big) = g(x)\frac{r(|x|)}{|x|^d}$$

and (4.8) follows.

Next we shall give a complete description of the infinitesimal mappings g in the simple case. In view of the positive homogeneity (4.2), it is enough to describe the corresponding mappings $\varphi = g|_{S^{n-1}}\colon S^{n-1} \to \mathbf{R}^n$ of the unit sphere $S^{n-1} = \partial\mathbf{B}^n \subset \mathbf{R}^n$ into $\mathbf{R}^n$.

Recall to this end a class of mappings, see [MV]. Let $L \geq 1$ and let D be a domain in $\mathbf{R}^n$, $n \geq 2$. A continuous mapping $h\colon D \to \mathbf{R}^n$ is said to be of *L-bounded length distortion*, abbreviated L-BLD, if h is discrete, open, sense-preserving and

$$\frac{l(\alpha)}{L} \leq l(h\alpha) \leq Ll(\alpha) \tag{4.11}$$

for every path α in D. Here $l(\alpha)$ denotes the length of the path α.

We also say that $h\colon D \to \mathbf{R}^n$ is of *bounded length distortion* and write $h \in \mathrm{BLD}$, if h is of L-BLD for some $L \geq 1$. Note that $h \in \mathrm{BLD}$ if and only if h is quasiregular and $|h'(x)|$ is essentially bounded away from 0 and ∞.

Similarly, on the unit sphere $S^{n-1} = \partial \mathbf{B}^n$, $n \geq 2$, we say that a continuous mapping $\varphi\colon S^{n-1} \to S^{n-1}$ is of *bounded length distortion in* S^{n-1} and write $\varphi \in \mathrm{BLD}(S^{n-1})$, if φ is discrete, open, sense-preserving and, for some $L \geq 1$ and every path γ in S^{n-1}

$$\frac{l(\gamma)}{L} \leq l(\varphi\gamma) \leq Ll(\gamma) \tag{4.12}$$

where $l(\gamma)$ denotes the length of the path γ in the metric of the unit sphere S^{n-1} induced by the Euclidean metric in $\mathbf{R}^n$.

A number of criteria for BLD are known, see [MV, p. 429].

4.13. Theorem. *Let $L \geq 1$ and D be a domain in $\mathbf{R}^n$, $n \geq 2$, and let $h\colon D \to \mathbf{R}^n$ be a continuous mapping. Then each of the following conditions is equivalent to the property of h to be of L-BLD:*

(1) *h is ACL,*

$$\frac{|y|}{L} \leq |h'(x)y| \leq L|y| \qquad \text{for all } y \in \mathbf{R}^n, \tag{4.14}$$

and $J_h(x) \geq 0$ a.e.

(2) *For each $x \in D$, there is $r > 0$ such that*

$$\frac{|z-x|}{L} \leq |h(z) - h(x)| \leq L|z-x| \qquad \text{for all } z \in B(x,r), \tag{4.15}$$

and $J_h(x) \geq 0$ a.e.

(3) *For each $x \in D$, $L(x,h) \leq L$, $l(x,h) \geq 1/L$ and $J_h(x) \geq 0$ a.e.*

Here we use the standard notations

$$L(x,h) = \limsup_{z \to x} \frac{|h(z)-h(x)|}{|z-x|}, \qquad l(x,h) = \liminf_{z \to x} \frac{|h(z)-h(x)|}{|z-x|}. \tag{4.16}$$

4.17. Remark. Note also the following facts related to BLD-mappings:

(a) Although a BLD mapping satisfies (4.15), it need not be a local homeomorphism. As typical examples we can consider the winding mappings $f_k\colon \mathbf{R}^n \to \mathbf{R}^n$, defined by $f_k(r,\vartheta,y) = (r,k\vartheta,y)$, $k = 2,3,\dots$, in cylinder coordinates. These mappings are k-BLD.

(b) If $h\colon D \to \mathbf{R}^n$ is a local homeomorphism, then h is L-BLD if and only if h is locally L-bilipschitz. Such mappings are called also *local quasi-isometries* or *quasi-isometries*, see, e.g., [Ge$_2$], [Jo$_1$], [Jo$_2$], [MSa].

(c) If a BLD-mapping $h\colon D \to \mathbf{R}^n$ is of the class C^1, then its Jacobian $J_f(x)$ never vanishes. Hence h is a local homeomorphism. For nonconstant C^1 quasiregular mappings in dimensions $n \geq 3$, this is an open question.

(d) Let $n = 2$ and a mapping $h\colon D \to \mathbf{R}^2$ be complex analytic. If h has a branch point $z \in D$, then $h'(z) = 0$, and so h cannot be of BLD.

(e) BLD-mappings, being locally lipschitz, are absolutely continuous on every line segment.

4.18. Proposition. *Let $f\colon D \to \mathbf{R}^n$, $n \geq 2$, be a nonconstant quasiregular mapping and $x_0 \in D$. If $T(x_0,f)$ consists of one mapping $g\colon \mathbf{R}^n \to \mathbf{R}^n$ only, then g has the representation*:

$$g(z) = h\big(s(z)\big) \tag{4.19}$$

where the mapping $s\colon \mathbf{R}^n \to \mathbf{R}^n$,

$$s(z) = z|z|^{d-1}, \qquad d > 0, \tag{4.20}$$

is the radial stretching of $\mathbf{R}^n$ and the mapping $h\colon \mathbf{R}^n \to \mathbf{R}^n$, $h(0) = 0$,

$$h(w) = |w| \cdot g(w/|w|), \qquad w \in \mathbf{R}^n \setminus \{0\}, \tag{4.21}$$

is homogeneous of degree 1,

$$h(tw) = th(w), \qquad t > 0, \tag{4.22}$$

and of the class $\mathrm{BLD}(\mathbf{R}^n)$. *The mapping g is of the class* $\mathrm{BLD}(\mathbf{R}^n)$ *if and only if $d = 1$.*

In other words, every simple infinitesimal mapping is a *homogeneous* BLD *map up to a radial stretching of* $\mathbf{R}^n$. The converse conclusion is also true because every BLD map is quasiregular.

Proof. In view of (4.2) $h = g \circ s^{-1}$ where s is a quasiconformal mapping of $\mathbf{R}^n$ in (4.20). Consequently, the mapping h is nonconstant quasiregular and, in particular, discrete, open and sense-preserving.

By Theorem 2.7 for all $z \in \mathbf{R}^n$

$$i_g(z) \le i_f(x_0), \tag{4.23}$$

and, in particular, we have the inequality for the linear dilatation for all $\eta \in S^{n-1}$

$$H(\eta, g) \le C, \tag{4.24}$$

where the constant C as in (2.9) depends only on n and the product $Qi_f(x_0)$.

Moreover, by (4.2) there exists the derivative in the radial direction for all $\eta \in S^{n-1}$

$$\partial_r g(\eta) = d \cdot g(\eta). \tag{4.25}$$

Hence, see Remark 2.23(4), we obtain the following estimates

$$d \cdot C^{-1} \le |\partial_r g(\eta)| \le d \cdot C, \qquad \eta \in S^{n-1}. \tag{4.26}$$

Consequently,

$$L(\eta, g) = \limsup_{\zeta \to \eta} \frac{|g(\zeta) - g(\eta)|}{|\zeta - \eta|} \le d \cdot C^2, \qquad \eta \in S^{n-1}, \tag{4.27}$$

and

$$l(\eta, g) = \liminf_{\zeta \to \eta} \frac{|g(\zeta) - g(\eta)|}{|\zeta - \eta|} \ge d \cdot C^{-2}, \qquad \eta \in S^{n-1}. \tag{4.28}$$

Now, the stretching s deforms the space $\mathbf{R}^n$ only in the radial direction and

$$|\partial_r g(\eta)| = d \cdot |\partial_r h(\eta)|, \qquad \eta \in S^{n-1}. \tag{4.29}$$

Hence, in view of (4.21), for $d \ge 1$,

$$L(z, h) \le d \cdot C^2, \qquad l(z, h) \ge C^{-2}, \qquad z \in \mathbf{R}^n, \tag{4.30}$$

and, for $d \le 1$,

$$L(z, h) \le d^2 \cdot C^2, \qquad l(z, h) \ge d \cdot C^{-2}, \qquad z \in \mathbf{R}^n. \tag{4.31}$$

So by Theorem 4.13(3), we conclude that $h \in \mathrm{BLD}(\mathbf{R}^n)$.

4.32. Remark. As we have seen from the proof, the mapping h of the representation (4.19) is of the class L-BLD$(\mathbf{R}^n)$ with

$$L = C^2 \max\{d, d^{-1}\}. \tag{4.33}$$

Theorems 7.2 and 7.3 of [Re$_1$] give bounds for the degree $d > 0$ in (4.2) in terms of f

$$\big(K_I(f)/i_f(x_0)\big)^{-1/(n-1)} \le d \le \big(K_O(f) \cdot i_f(x_0)\big)^{1/(n-1)} \tag{4.34}$$

because $K_I(g) \le K_I(f)$, $K_O(g) \le K_O(f)$ by the lower semicontinuity of the dilatations, see, e.g., [Re$_1$], [Vä], [Ri], and $i_g(0) = i_f(x_0)$ by Theorem 2.7. Hence

$$L \le C^2 \big[K(f) \cdot i_f(x_0)\big]^{1/(n-1)}, \tag{4.35}$$

where $K(f)$ is the maximal dilatation of the mapping f,

$$K(f) = \max\{K_O(f), K_I(f)\}, \tag{4.36}$$

and C is a constant as in (2.9) depending only on n and the product $K_O(f) i_f(x_0)$.

In particular, if f is locally injective at the point x_0, then h is of the class L-BLD$(\mathbf{R}^n)$ with

$$L = C^2 \big[K(f)\big]^{1/(n-1)} \tag{4.37}$$

where the constant C depends only on n and $K_O(f)$.

Now, it is easy to describe all simple infinitesimal mappings g when $g(S^{n-1}) \subset S^{n-1}$. As observed above this is sufficiently large.

4.38. Corollary. *The formula*

$$g(z) = |z|^d \varphi(z/|z|), \qquad d > 0,\ \varphi \in \mathrm{BLD}(S^{n-1}), \tag{4.39}$$

represents all simple infinitesimal mappings preserving the unit sphere S^{n-1}.

Indeed, if g is a simple infinitesimal mapping preserving S^{n-1}, by Proposition 4.18, $\varphi = g|_{S^{n-1}}\colon S^{n-1} \to S^{n-1}$ satisfies (4.15) and hence φ is differentiable almost everywhere in the unit sphere. In view of (4.2), we have almost everywhere in the unit sphere

$$J_g(\eta) = d \cdot |g(\eta)| \cdot J_\varphi(\eta) > 0, \tag{4.40}$$

i.e. φ is sense-preserving, see, e.g., [FG, p. 115].

Moreover, as the restriction of the quasiregular mapping $g\colon \mathbf{R}^n \to \mathbf{R}^n$, the mapping $\varphi\colon S^{n-1} \to S^{n-1}$ is open in the topology of the unit sphere S^{n-1} induced by the topology of $\mathbf{R}^n$. The discreteness of the mapping φ is clear, of course. Hence φ belongs to the class BLD(S^{n-1}).

Conversely, for any $d > 0$ and $\varphi \in \mathrm{BLD}(S^{n-1})$ the mapping g in (4.39) is quasiregular, see [MRV$_1$, p. 23], and g is the only mapping in $T(0, g)$.

4.41. Remark. Note also that the inclusion $g(S^{n-1}) \subset S^{n-1}$ always implies the equality

$$\varphi(S^{n-1}) = S^{n-1} \tag{4.42}$$

because the image $\varphi(S^{n-1})$ must be simultaneously open and closed in the topology of the unit sphere S^{n-1} induced by the topology of $\mathbf{R}^n$.

4.43. Corollary. *The formula*

$$g(z) = |z|\varphi(z/|z|), \qquad \varphi \in \mathrm{BLD}(S^{n-1}), \tag{4.44}$$

gives the representation of all simple infinitesimal mappings in the class $\mathrm{BLD}(\mathbf{R}^n)$, *preserving the unit sphere* S^{n-1}.

Some examples of such mappings have been given in [MS_2].

5. The low dimensional cases $n = 2, 3$

Corollary 4.43 shows that the set of all simple infinitesimal mappings in the space $\mathbf{R}^n$, $n \geq 2$, is at least as large as the class $\mathrm{BLD}(S^{n-1})$ of all the bounded length distortion mappings of the unit sphere S^{n-1}.

In the case $n = 3$, we may interpret S^{n-1} as the so-called Riemann sphere or, what is the same, as the extended complex plane $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ with the spherical metric. By the well-known Stoilow theorem, see, e.g., [St], [LV, p. 252], every discrete open mapping φ of $\overline{\mathbf{C}}$ may be represented in the form $\varphi = A \circ H$ where A and H are an analytic function and a homeomorphism of $\overline{\mathbf{C}}$, respectively. Since $A' = 0$ in each branch point of A, we obtain by the uniqueness theorem for analytic functions that the branch points of A are isolated and, in view of the compactness of the Riemann sphere $\overline{\mathbf{C}}$, the function A has only a finite number of branch points. So it is easy to see that $A\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ must be a rational function, i.e. a quotient of two polynomials, see Theorem 2.2, [MS_2]. In particular, *every mapping in the class* $\mathrm{BLD}(\overline{\mathbf{C}})$ *is topologically equivalent to a rational function* $A\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$.

Note that a rational function A cannot be in $\mathrm{BLD}(\overline{\mathbf{C}})$, if A has at least one branch point, see Remark 4.17(d). However, every rational function $A\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is topologically equivalent to a function of $\mathrm{BLD}(\overline{\mathbf{C}})$, cf. [$\mathrm{MS}_2$].

For example, consider the function

$$A_k(z) = z^k, \qquad k = 2, 3, \ldots, \tag{5.1}$$

that has branch points at the origin and at ∞ of the same order k and the corresponding radial stretching

$$s_k(\zeta) = \zeta \cdot |\zeta|^{-1+1/k}, \qquad \zeta \in \overline{\mathbf{C}}, \tag{5.2}$$

that is a quasiconformal homeomorphism of $\overline{\mathbf{C}}$. Then the composition $f_k = A_k \circ s_k$ is the winding of the order k, that is of the class k-BLD($\overline{\mathbf{C}}$), see Remark 4.17(a).

Below, as canonical quasiconformal homeomorphisms of $\overline{\mathbf{C}}$, we use the mappings

$$H_k\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}, \qquad k = 2, 3, \dots,$$

that coincide with the corresponding radial stretchings s_k of (5.2) inside of the unit disk $\mathbf{B}^2$ and with the identity mapping outside of $\mathbf{B}^2$.

5.3. Remark. Let $A\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ be a rational function, b_j be its branch points of the orders $k_j > 1$, $j = 1, 2, \dots, l$, respectively. Further, let $B_j \subset \overline{\mathbf{C}}$, $j = 1, 2, \dots, l$, be arbitrary spherical disks with the the centers at b_j such that $\overline{B_j}$ are mutually disjoint and let $M_j\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$, $j = 1, 2, \dots, l$, be Möbius transformations translating b_j into the origin and B_j into $\mathbf{B}^2$. Denote

$$h_j = M_j^{-1} \circ H_{k_j} \circ M_j, \qquad j = 1, 2, \dots, l, \tag{5.4}$$

where H_{k_j} is as above and

$$H = h_1 \circ \cdots \circ h_l. \tag{5.5}$$

Then the composition

$$\varphi = A \circ H \tag{5.6}$$

is of the class BLD($\overline{\mathbf{C}}$).

Note, that $H\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is a quasiconformal homeomorphism of $\overline{\mathbf{C}}$ which is the identity mapping outside the disks B_j and it is similar to the stretchings (5.2) up to the Möbius transformations M_j inside every disk B_j, $j = 1, 2, \dots, l$.

In the plane case simple infinitesimal mappings can be explicitly described, cf. [GR].

5.7. Proposition. *Let $f\colon D \to \mathbf{R}^2$, $f(0) = 0 \in D$, be a locally injective Q-quasiregular mapping for which its infinitesimal space $T(0, f)$ consists of a unique element $g(z)$. Then there exists a real number τ, $0 \le \tau < 2\pi$, and a measurable function $\nu(e^{i\arg z})$, $|\nu(e^{i\arg z})| \le (Q-1)(Q+1)^{-1}$, such that*

$$g(z) = Ce^{i\tau}\omega(z), \qquad C = \left(\frac{\pi}{\operatorname{meas}\omega(\mathbf{B}^2)}\right)^{1/2}, \tag{5.8}$$

where

$$\omega(z) = \left\{ |z| \exp\left(i \int_0^{\arg z} \gamma(\theta)\, d\theta \right) \right\}^{1/a}, \qquad a > 0, \tag{5.9}$$

with

$$\gamma(\theta) = \frac{1-\nu(e^{i\theta})e^{-2i\theta}}{1+\nu(e^{i\theta})e^{-2i\theta}}, \tag{5.10}$$

satisfying the assumption

$$\frac{1}{2\pi}\int_0^{2\pi} \gamma(\theta)\, d\theta = a. \tag{5.11}$$

The function $\omega(z)$ *for each* $\nu(e^{i\arg z})$ *is a* Q*-quasiconformal automorphism of the complex plane with the complex dilatation* $\mu(z) = \nu(z/|z|)$, *keeping the points* 0, 1 *and* ∞ *fixed.*

Proof. By Theorem 2.7 and Theorem 4.1 the infinitesimal space $T(0,f)$ consists of positively homogeneous Q-quasiconformal mappings $g\colon \mathbf{C}\to\mathbf{C}$ normalized by the conditions $g(0)=0$, $g(\infty)=\infty$, and such that $\operatorname{meas} g(\mathbf{B}^2) = \operatorname{meas}\mathbf{B}^2$. Next, the positive homogeneity implies that the complex dilatation μ of the mapping g has the ray symmetry. Since $\mu(sz)=\mu(z)$, $s>0$, μ is a function of the variable $\arg z$ only. Set $\mu = \nu(e^{i\theta})$, where $\theta = \arg z$. Using the well-known Stoilow representation formulae for the open discrete mappings, see, e.g., [St], [LV, p. 252], we find that

$$g(z) = A\circ\omega(z).$$

Here ω is a Q-quasiconformal automorphism of the complex plane with the complex dilatation $\nu(e^{i\arg z})$, keeping the points 0, 1 and ∞ fixed, and $A(w)$ is an entire locally injective analytic function having zeros only at the origin and such that $A(w)\to\infty$ as $w\to\infty$. This last fact implies that $A(w)=cw$ where c is a complex constant.

In order to define $\omega(z)$ we have to find all homeomorphic solutions of the Beltrami equation

$$\omega_{\bar z}(z) = \nu(z/|z|)\omega_z(z) \qquad \text{a.e.} \tag{5.12}$$

keeping the points 0, 1 and ∞ fixed. By Schatz's theorem [Sch] all such solutions can be found explicitly

$$\omega(z) = \left\{ |z|\exp\left(i\int_0^{\arg z} \gamma(\theta)\, d\theta\right)\right\}^{1/(a+ib)}, \qquad a>0, \tag{5.13}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\gamma(\theta)\, d\theta = a+ib \tag{5.14}$$

where the integrand has the form (5.10).

It is easy to see that ω maps radial lines $\arg z = \mathrm{const}$ into infinitely winding spirals if and only if $b \neq 0$. When $b = 0$, all radial lines map to radial lines and therefore this condition is necessary and sufficient for $\omega(z)$ to be positive homogeneous. To complete the proof we need to choose

$$c = e^{i\tau}\left(\frac{\pi}{\operatorname{meas}\omega(\mathbf{B}^2)}\right)^{1/2}, \qquad 0 \le \tau < 2\pi, \tag{5.15}$$

in order to make the corresponding mapping $g(z)$ volume-preserving.

5.16. Proposition. *Let $f\colon D \to \mathbf{R}^2$, $f(0) = 0 \in D$, be a Q-quasiregular mapping for which its infinitesimal space $T(0, f)$ consists of a unique element g. Then there exists a real number τ, $0 \le \tau < 2\pi$, an integer k, $k = 1, 2, \ldots$, and a measurable function $\nu(e^{i\arg z})$, $|\nu(e^{i\arg z})| \le (Q-1)(Q+1)^{-1}$, such that*

$$g(z) = Ce^{i\tau}\omega^k(z), \tag{5.17}$$

where the mapping ω and the constant C are defined by (5.8)–(5.11).

Proof. Arguing as in the proof of Proposition 5.7 we find that

$$g(z) = A \circ \omega(z)$$

where ω is the normalized homeomorphism of the plane with complex dilatation $\mu(z) = \nu(e^{i\arg z})$ and $A(w)$ is an entire analytic function with the following properties: $A(0) = 0$, $A(w) \neq 0$, $w \neq 0$, for the rest of the points and $A(w) \to \infty$ as $w \to \infty$. By the Liouville theorem $A(w) = cw^k$, for some $k = 1, 2, \ldots$. Next, the positive homogeneity of g implies the same property for the mapping $\omega(z)$ and therefore it has the representation (5.8)–(5.11). The condition $\operatorname{meas} g(\mathbf{B}^2) = \pi$ yields the value of c.

5.18. Remark. Each of the functions given by (5.17) is a simple infinitesimal mapping. So, (5.17) is a complete and explicit description of the class of all simple infinitesimal mappings in the plane case.

Below we will identify the (x_1, x_2)-plane with the complex plane $z = x_1 + ix_2$ and, correspondingly, (u_1, u_2)-plane as the complex plane $w = u_1 + iu_2$.

5.19. Proposition. *Let $a > 0$ and $\omega(z)\colon \mathbf{C} \to \mathbf{C}$ be given by* (5.9)–(5.11). *Then the mappings $g\colon \mathbf{R}^n \to \mathbf{R}^n$, $n \ge 2$,*

$$(z, y) \mapsto \big(\omega^k(z), y^{k/a}\big), \tag{5.20}$$

where $y^{k/a} = (x_3^{k/a}, \ldots, x_n^{k/a})$, $k = 1, 2, \ldots$, are infinitesimal mappings satisfying $g \in T(0, g)$.

In particular, if $a = 1$ and $\omega(z) \equiv z$ is the identity mapping, then the mappings (5.20) are topologically equivalent to the k-winding mappings $f_k\colon \mathbf{R}^n \to \mathbf{R}^n$, $k,n \geq 2$

$$(r,\vartheta,y) \mapsto (r,k\vartheta,y) \tag{5.21}$$

written in terms of polar coordinates in the (x_1,x_2)-plane, cf., e.g., [Sr, p. 106].

Let $\mathscr{O}(n)$ be the space of all orthogonal transformations of $\mathbf{R}^n$. Denote by G the class of all entire quasiregular mappings $G\colon \mathbf{R}^n \to \mathbf{R}^n$

$$G(x) = U \circ \Omega \circ V \tag{5.22}$$

where $U,V \in \mathscr{O}(n)$ and let Ω be defined by (5.20).

5.23. Proposition. *Let $G_1,\dots,G_l \in G$. Then*

$$g = G_1 \circ G_2 \circ \cdots \circ G_l \tag{5.24}$$

is a simple infinitesimal mapping for itself at the origin.

The formula (5.24) provides us, in particular, with a wide family of mappings that have a ray structure of the branch set B_f and fB_f.

5.25. Remark. Note that the example g of (4.44) with

$$\varphi(\zeta) = \zeta^2(\zeta-1)^3, \qquad \zeta \in \overline{\mathbf{C}},$$

where we identify the sphere S^2 with the extended complex plane $\overline{\mathbf{C}}$ shows that (5.24) does not describe all simple infinitesimal mappings because the degrees must be multiplied under compositions of mappings (5.22), see Remark 2.4.

6. Bounds for local degree and dilatations

Here we give some consequences of Theorem 4.1 and the following theorem that was proved in [Ma, p. 13]:

6.1. Theorem. *Let $f\colon D \to \mathbf{R}^n$, $n \geq 3$, be a quasiregular mapping and $x_0 \in D$. If $K_I(f) < i_f(x_0)$, then f is differentiable at x_0 and $f'(x_0) = 0$.*

6.2. Corollary. *Let $f\colon D \to \mathbf{R}^n$, $n \geq 3$, be a nonconstant quasiregular mapping and $x_0 \in D$. If $T(x_0,f)$ consists of one mapping $g\colon \mathbf{R}^n \to \mathbf{R}^n$ only, then for all $z \in \mathbf{R}^n \setminus \{0\}$*

$$i_g(z) \leq K_I(g). \tag{6.3}$$

Indeed, if $i_g(z) > K_I(g)$ for a point $z \in \mathbf{R}^n \setminus \{0\}$, then $i_g(tz) > K_I(g)$ for all $t > 0$ by Corollary 4.3(2). Consequently, by Theorem 6.1, $g' \equiv 0$ on the ray $tz, t > 0$. So by Corollary 4.3(4), $g \equiv 0$ on this ray. However, this last conclusion contradicts the fact that g is discrete.

The estimate (6.3) can be used to study the initial mapping f.

6.4. Theorem. *Let $f\colon D \to \mathbf{R}^n$, $n \geq 3$, be a nonconstant quasiregular mapping and $x_0 \in D$. If $T(x_0, f)$ is simple, then*

$$\limsup_{\substack{x \to x_0,\\ x \in D\setminus\{x_0\}}} i_f(x) \leq K_I(f).$$

Proof. Let us assume that there exists a sequence $x_j \in D \setminus \{x_0\}$, $x_j \to x_0$ as $j \to \infty$, such that

$$\lim_{j\to\infty} i_f(x_j) > K_I(f),$$

i.e. in terms of the approximating mappings, see (2.1),

$$\lim_{j\to\infty} i_{F_{\varrho_j}}(\eta_j) > K_I(f) \tag{6.5}$$

where $\varrho_j = |x_j - x_0|$ and

$$\eta_j = (x_j - x_0)/|x_j - x_0| \in S^{n-1}.$$

In view of the sequential compactness of the unit sphere S^{n-1}, we may assume, without loss of generality, that $\eta_j \to \eta_0 \in S^{n-1}$ as $j \to \infty$.

Consider the sequence

$$h_j(\zeta) = F_{\varrho_j}(\zeta + \eta_j - \eta_0).$$

It is clear that $h_j \to g$ locally uniformly as $j \to \infty$. By the upper semicontinuity of the local topological index, see [MRV$_3$, p. 24], we obtain that

$$\limsup_{j\to\infty} i_{h_j}(\eta_0) \leq i_g(\eta_0).$$

Hence, in view of (6.5),

$$i_g(\eta_0) > K_I(f)$$

and by the lower semicontinuity of the inner dilatation, see, e.g., [Vä], [Ri],

$$i_g(\eta_0) > K_I(g).$$

However, the last inequality contradicts Corollary 6.2.

Theorem 6.4 enables us to prove the conjecture [Ma] on $K_I(f)$ in the case of a simple $T(x_0, f)$, cf. [Se].

6.6. Corollary. *Let $f\colon D \to \mathbf{R}^n$, $n \geq 3$, be a nonconstant quasiregular mapping, $x_0 \in D$ and let $T(x_0, f)$ be simple. If*

$$K_I(f) < 2, \tag{6.7}$$

then f is injective in a neighborhood of x_0.

Indeed, by Theorem 6.4, the condition (6.7) implies that

$$\limsup_{\substack{x \to x_0,\\ x \in D \setminus \{x_0\}}} \ i_f(x) = 1.$$

However, space quasiregular mappings cannot have isolated branch points, see, e.g., [AM], [Zo$_2$]. Consequently, x_0 is not a branch point.

Below we also use the following consequence of Theorem 6.1, see [Ma, p. 14]:

6.8. Corollary. *Let $f\colon D \to \mathbf{R}^n$, $n \geq 3$, be a nonconstant quasiregular mapping and and let $\alpha \subset B_f$ be a rectifiable curve. Then*

$$K_I(f) \geq \inf_{x \in \alpha} i_f(x). \tag{6.9}$$

Using Corollary 6.8 we can now obtain upper bounds for $i_f(x_0)$ in terms of $K_I(f)$, if B_f has a very weak tangential structure at x_0. In this case $T(x_0, f)$ need not be simple.

We say that the branch set B_f *has a tangent ray at a point* $x_0 \in B_f$, if there is a ray R emanating from the point x_0 and a positive function $\varepsilon(r)$ such that $\varepsilon(r) \to 0$ as $r \to 0$ and

$$\operatorname{dist}\big(R \cap \partial B(x_0, r), B_f \cap \partial B(x_0, r)\big) \leq r\varepsilon(r) \tag{6.10}$$

for $0 < r < r_0$.

6.11. Theorem. *Let $f\colon D \to \mathbf{R}^n$, $n \geq 3$, be a nonconstant quasiregular mapping such that B_f has a tangent ray R at a point $x_0 \in B_f$. Then*

$$\liminf_{\substack{x \to x_0,\\ x \in B_f}} i_f(x) \leq K_I(f). \tag{6.12}$$

Proof. From (6.10) it easily follows that $R_0 \subset B_g$ for every $g \in T(x_0, f)$ where $R_0 = R - x_0$ is the corresponding ray emanating from the origin.

Next, arguing locally we obtain from Corollary 6.8 that

$$\liminf_{\substack{z \to 0,\\ z \in B_g \setminus \{0\}}} \ i_g(z) \leq K_I(g). \tag{6.13}$$

By the lower semicontinuity of the inner dilatation

$$K_I(g) \leq K_I(f), \tag{6.14}$$

and the inequalities (6.13), (6.14) and from Lemma 2.24 we obtain (6.12).

6.15. Corollary. *Let $f\colon D \to \mathbf{R}^n$, $n \geq 3$, be a nonconstant quasiregular mapping such that B_f has a tangent ray at a point $x_0 \in B_f$. Then $K_I(f) \geq 2$.*

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