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JØRGENSEN'S INEQUALITY FOR DISCRETE CONVERGENCE GROUPS

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Abstract. We explore in this paper whether certain fundamental properties of the action of Kleinian groups on the Riemann sphere extend to the action of discrete convergence groups on $\overline{\mathbf{R}^2}$. A Jørgensen inequality for discrete K-quasiconformal groups is developed, and it is shown that such an inequality depends naturally on the quasiconformal dilatation K. Furthermore, it is established that no such inequality can hold for general discrete convergence groups. In the discontinuous case a universal constraint on discreteness is formulated for both quasiconformal and general convergence groups.

1. Basic definitions and notation

The group of all orientation-preserving Möbius transformations in $\overline{\mathbf{R}^2}$ is denoted by Möb. All maps in this article are assumed to be orientation-preserving.

A group G of homeomorphisms of $\overline{\mathbf{R}^2}$ is a K-*quasiconformal group* if each of its elements is K-quasiconformal, and we call the group simply *quasiconformal* if it is K-quasiconformal for some K. Recall that every Möbius transformation is 1-quasiconformal, and that the converse also holds; i.e. every 1-quasiconformal homeomorphism of $\overline{\mathbf{R}^2}$ is a Möbius transformation (see [TV2] for a nice geometric proof). One natural way to construct a quasiconformal group is to conjugate a conformal group by a quasiconformal mapping. In $\overline{\mathbf{R}^2}$ one obtains every quasiconformal group in this way ([Sul], [Tuk2]), whereas in higher dimensions there exist quasiconformal groups which are not quasiconformally conjugate to Möbius groups ([Tuk2], [Mar], [McK], [FrSk]).

A group G of homeomorphisms of $\overline{\mathbf{R}^2}$ is *discrete* if no sequence $\{f_n\} \subset G$ of distinct elements converges to the identity uniformly in $\overline{\mathbf{R}^2}$. A discrete subgroup of Möb is called a *Kleinian group.*

A (not necessarily discrete) group G of homeomorphisms of $\overline{\mathbf{R}^2}$ is said to be a *convergence group* if each infinite subfamily of G contains a sequence $\{f_n\}$, such that one of the following is true:

(i) There exists a homeomorphism f of $\overline{\mathbf{R}^2}$ such that

$$\lim_{n\to\infty} f_n = f \qquad \text{and} \qquad \lim_{n\to\infty} f_n^{-1} = f^{-1}$$

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uniformly in $\overline{\mathbf{R}^2}$.

(ii) There exist points $x_0, y_0 \in \overline{\mathbf{R}^2}$ such that

$$\lim_{n\to\infty} f_n = x_0 \qquad \text{and} \qquad \lim_{n\to\infty} f_n^{-1} = y_0$$

locally uniformly in $\overline{\mathbf{R}^2}\setminus\{y_0\}$ and $\overline{\mathbf{R}^2}\setminus\{x_0\}$, respectively. Here we allow $x_0 = y_0$.

In (ii) we call x_0 and y_0 the attracting and repelling limit point of the sequence $\{f_n\}$, respectively, if these two points are distinct. Möbius groups and quasiconformal groups are examples of convergence groups, see [GM1]. Homeomorphic conjugates of quasiconformal groups are also convergence groups, so that the class of convergence groups of $\overline{\mathbf{R}^2}$ is strictly larger than the class of quasiconformal groups.

Convergence groups in many essential ways resemble their conformal counterparts. As with Möbius groups, we define the *limit set* $L(G)$ of the convergence group G to be the set of all limit points of those sequences $\{f_n\}$ converging in the sense of (ii). Likewise, we define the *regular set* $\Omega(G)$ to be the set of points where G acts discontinuously; i.e. the set of all x that have a neighborhood U satisfying $g(U)\cap U = \varnothing$ for all but finitely many $g \in G$. The regular set is an open set, and the limit set is closed; both sets are G-invariant. If $L(G)$ contains more than two points then $L(G)$ is an infinite perfect set. If $\Omega(G) \neq \varnothing$, then G is necessarily discrete. For discrete G, the limit set $L(G)$ is the complement of the regular set $\Omega(G)$. (See e.g. [GM1], [Tuk3] for proofs.)

There exists a classification of the elements of a convergence group that is topologically analogous to the classification of Möbius maps. If G is a convergence group and $g \in G$, then we say that g is *elliptic* if $\langle g\rangle$, the group generated by g, is pre-compact, i.e. if every sequence in $\langle g\rangle$ contains a subsequence converging uniformly to a homeomorphism. If g is not elliptic, then g is *loxodromic* if it has exactly two fixed points, or g is *parabolic* if g fixes exactly one point. It is not hard to see that every element in a convergence group is either elliptic, parabolic or loxodromic ([Tuk3]). In a discrete convergence group the elliptic elements are those $g \in G$ that satisfy $g^n = \mathrm{id}$ for some $n \in \mathbf{N}$. The sequence $\{g^n\}$ of iterates of a loxodromic element of a discrete convergence group converges to a fixed point a of g locally uniformly in the exterior of the other fixed point b; we call a the *attracting* and b the *repelling* fixed point of g. For parabolic g, the sequence of iterates converges to the fixed point of g locally uniformly in the exterior of that fixed point (see [GM1]).

As is customary, we define a discrete convergence group G to be *non-elementary* if $L(G)$ contains more than two points. We can extend this definition to non-discrete G but then in addition we must require that no $x \in L(G)$ is fixed by the entire group G. In both cases one can show that a convergence group G is elementary if and only if either $L(G) = \varnothing$ or there is a one or two-point set which is fixed setwise by G. Furthermore, G is non-elementary if and only if there are two loxodromic $g, h \in G$ without common fixed points (see [Tuk3]).

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2. Measuring discreteness

Let $q(x,y)$ denote the *chordal distance* of the points $x,y \in \overline{\mathbf{R}^2}$; it is the Euclidean distance of their stereographic projections onto $S^2 \subset \mathbf{R}^3$ and is given by

$$q(x,y) = \frac{2|x-y|}{\sqrt{1+|x|^2}\,\sqrt{1+|y|^2}}.$$

For two homeomorphisms f, g of $\overline{\mathbf{R}^2}$, define their chordal distance to be

$$d(f,g) := \sup_{x\in\overline{\mathbf{R}^2}} q\bigl(f(x),g(x)\bigr).$$

Likewise, let $d(f)$ denote the chordal distance of f to the identity:

$$d(f) := d(f,\mathrm{id}) = \sup_{x\in\overline{\mathbf{R}^2}} q\bigl(f(x),x\bigr). \tag{1}$$

Suppose that G is a fixed convergence group acting on $\overline{\mathbf{R}^2}$. If G is discrete, then all $f \in G \setminus \{\mathrm{id}\}$ are uniformly bounded away from the identity in the metric given by (1), i.e. there is a constant $c > 0$ (depending on G) such that

$$d(f) \ge c$$

for all $f \in G \setminus \{\mathrm{id}\}$.

The following theorems of Gehring and Martin [GM2, Theorems 4.19, 4.26, 6.14] make this observation more precise in the case of Kleinian groups; the authors find a uniform estimate in the chordal metric, independent of the group G. The first result is a consequence of Jørgensen's inequality [Jør] and we shall refer to it as *chordal Jørgensen inequality.*

Theorem 2.1 (Chordal Jørgensen inequality). *There is a constant $c_1 > 0$ so that if f and g generate a discrete non-elementary subgroup of* Möb *then f and g satisfy*:

$$\max\{d(f),d(g)\} \ge c_1 \qquad \text{and} \qquad d(f)+d(g) \ge 2c_1.$$

Furthermore,

$$2(\sqrt{2}-1) = 0.828\ldots \le c_1 \le 0.911\ldots = 2\sqrt{\frac{\cos(2\pi/7)+\cos(\pi/7)-1}{\cos(2\pi/7)+\cos(\pi/7)+1}}.$$

Recall that Jørgensen's inequality [Jør] in its original form was stated as follows:

Theorem 2.2 (Jørgensen's inequality). *If the two Möbius transformations* f, g *generate a discrete non-elementary group then*

$$|\operatorname{tr}^2(f) - 4| + |\operatorname{tr}(f \circ g \circ f^{-1} \circ g^{-1}) - 2| \geq 1,$$

where tr *denotes the trace function.*

The next result by Gehring and Martin says that we can conjugate any Kleinian group, so that the resulting group has the property that its non-identity elements are bounded away from zero in the chordal distance given in (1).

Theorem 2.3. *If* G *is a discrete subgroup of* Möb, *then there exists an* $h \in$ Möb *such that*

$$d(f) \geq c_1$$

for all $f \in h \circ G \circ h^{-1} \setminus \{\mathrm{id}\}$. *Here* c_1 *is the same constant as in Theorem* 2.1.

3. Statement of results

Our first results show that there are analogs to Theorem 2.1 and Theorem 2.3 for discrete quasiconformal groups. In particular, there is a chordal Jørgensen inequality for discrete K-quasiconformal groups:

Theorem 3.1. *For each* $K \geq 1$, *there is a constant* $c_K > 0$ *so that if* f *and* g *generate a discrete non-elementary* K*-quasiconformal group on* $\overline{\mathbf{R}^2}$, *then* f *and* g *satisfy*:

$$\max\{d(f), d(g)\} \geq c_K \qquad \text{and} \qquad \big(d(f)\big)^{1/K} + \big(d(g)\big)^{1/K} \geq 2\big(c_K\big)^{1/K}.$$

One (non-sharp) choice for the constant c_K is

$$c_K = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}\,c_1}{128}\right)^K,$$

where c_1 is the constant from Theorem 2.1.

A relative version of Theorem 2.3 holds for discrete K-quasiconformal groups:

Theorem 3.2. *If* G *is a discrete, torsion-free* K*-quasiconformal group on* $\overline{\mathbf{R}^2}$, *then there exists an* $h \in$ Möb *such that*

$$d(f) \geq c_K$$

for all $f \in h \circ G \circ h^{-1} \setminus \{\mathrm{id}\}$. *Here* c_K *is the same constant as in Theorem* 3.1.

Our main result is that Theorem 3.1 does not hold for general discrete convergence groups:

Theorem 3.3 (Main theorem). *There are sequences $\{f_n\}$, $\{g_n\}$ of homeomorphisms on $\overline{\mathbf{R}^2}$ so that the group generated by f_n and g_n is a free, discrete non-elementary convergence group with non-empty regular set for each $n \in \mathbf{N}$, but*

$$\max\{d(f_n), d(g_n)\} \to 0 \qquad \text{as } n \to \infty.$$

We give a brief outline of the proof of Theorem 3.3. First we construct a sequence $\{G_n\}$ of topological Schottky groups, where G_n is freely generated by f_n and g_n. We shall denote this by $G_n = \langle f_n, g_n \rangle$. Suppressing the index n in the notation, the idea of the construction of the groups G is as follows:

We shall find four mutually disjoint, simply connected regions S_0, S_1, S_2, S_3. The homeomorphism f will map the exterior of S_2 onto the interior of S_0 and S_2 onto the exterior of S_0. The boundary of S_2 will be mapped onto the boundary of S_0. The homeomorphism g will be constructed in a similar way, with S_1 and S_3 replacing S_0 and S_2. By appropriately shrinking S_0, S_1 and S_3 under the map f, shrinking S_1, S_2, S_3 under f^{-1}, shrinking S_0, S_1, S_2 under g and shrinking S_0, S_2, S_3 under g^{-1}, the group $G = \langle f, g \rangle$ will have the convergence property. Since G is the topological version of a Schottky group, it is discontinuous by construction.

The main difficulty consists in making $d(f)$ and $d(g)$ as small as desired. To this end it is necessary to have any point of the exterior of S_0 be close to S_2 and any point of the exterior of S_2 be close to S_0. The same must be true with S_1 and S_3 replacing S_0 and S_2. One way to satisfy these assumptions is to give the regions $S_0, \dots, S_3$ the shape of spirals, see Figure 1. The hardest part now is to tune the action of f and g in such a way that $d(f)$ and $d(g)$ are small, while ensuring on the other hand that $\langle f, g \rangle$ has the convergence property.

Once the main theorem has been proved, using the density of diffeomorphisms in the homeomorphisms of $\overline{\mathbf{R}^2}$ it is not hard to see the following:

Corollary 3.4. *There are sequences $\{\hat{f}_n\}$, $\{\hat{g}_n\}$ of quasiconformal mappings such that $\hat{f}_n$ and $\hat{g}_n$ generate a discrete, non-elementary, K_n-quasiconformal group where $K_n \to \infty$ and $d(\hat{f}_n) \to 0$ and $d(\hat{g}_n) \to 0$ as $n \to \infty$.*

Note that by Theorem 3.1 K_n must necessarily become unbounded as $n \to \infty$.

Even though there is no chordal Jørgensen inequality on the full class of discrete convergence groups, there is an analog of Theorem 3.2 for convergence groups with non-empty regular set [Geh1]:

Theorem 3.5. *If G is a convergence group with non-empty regular set, then for each constant c with $0 < c < 2$ there exists an $h \in$ Möb such that*

$$d(f) \geq c$$

for all $f \in h \circ G \circ h^{-1} \setminus \{\mathrm{id}\}$.

4. Proofs

We first show that normalized K-quasiconformal maps satisfy a chordal Hölder inequality. The result is probably known, though for the reader's convenience we include a proof.

Lemma 4.1. *Any K-quasiconformal homeomorphism φ of $\overline{\mathbf{R}^2}$ fixing 0, 1 and ∞ satisfies*

$$q\big(\varphi(x),\varphi(y)\big) \le 128 \cdot 2^{(1-K)/(2K)} q(x,y)^{1/K}$$

for all $x,y \in \overline{\mathbf{R}^2}$.

Proof. The proof uses the following fact [Geh3, Theorem 4.1]: If $\varphi\colon D \to D'$ is a K-quasiconformal map of the domain $D \subset \overline{\mathbf{R}^2}$ onto $D' \subset \overline{\mathbf{R}^2}$ where $\overline{\mathbf{R}^2} \setminus D \neq \varnothing$, then

$$q\big(\varphi(x),\varphi(y)\big) \cdot q(\overline{\mathbf{R}^2} \setminus D') \le 128 \cdot \left(\frac{q(x,y)}{q(x,\partial D)}\right)^{1/K} \tag{2}$$

for all $x,y \in \overline{\mathbf{R}^2}$ with $x \neq y$. Here for a set E the expression $q(E)$ denotes the chordal diameter of E and $q(x,\partial E)$ is the chordal distance of x to the boundary of E.

Note that $q(0,1) = q(1,\infty) = \sqrt{2}$ and $q(0,\infty) = 2$. Let $x,y \in \overline{\mathbf{R}^2}$.

Case 1. $x,y \in \{0,1,\infty\}$. Then

$$\begin{aligned} q\big(\varphi(x),\varphi(y)\big) &= q(x,y) \le 2 \le 128 \cdot 2^{(1-K)/(2K)} \cdot \sqrt{2}^{\,1/K} \\ &\le 128 \cdot 2^{(1-K)/(2K)} \cdot q(x,y)^{1/K}. \end{aligned}$$

Case 2. $y \notin \{0,1,\infty\}$. If we are not in case 1 then we can always assume we are in case 2 by relabeling x and y.

(a) Assume first that $q(x,0) < \sqrt{2}/2$. In this case we have $q(x,1) \ge \sqrt{2}/2$ and $q(x,\infty) \ge \sqrt{2}/2$. Choosing $D = D' = \overline{\mathbf{R}^2} \setminus \{1,\infty\}$ we obtain $q(\overline{\mathbf{R}^2} \setminus D') = q(1,\infty) = \sqrt{2}$ and $q(x,\partial D) \ge 1/\sqrt{2}$. Hence by (2) we have

$$q\big(\varphi(x),\varphi(y)\big) \cdot \sqrt{2} \le 128 \cdot q(x,y)^{1/K} \cdot \sqrt{2}^{1/K},$$

and from this the claim follows.

(b) Assume next that $q(x,1) < \sqrt{2}/2$. Choosing $D = D' = \overline{\mathbf{R}^2} \setminus \{0,\infty\}$ and again applying (2) we obtain the desired result in this case.

(c) Assume finally that $q(x,0) \ge \sqrt{2}/2$ and $q(x,1) \ge \sqrt{2}/2$. Choose $D = D' = \overline{\mathbf{R}^2} \setminus \{0,1\}$ and again use (2) to complete the proof of the lemma. □

We can now prove Theorem 3.1.

Proof of Theorem 3.1. Let G be a K-quasiconformal, discrete, non-elementary convergence group, generated by the quasiconformal mappings $f, g\colon \overline{\mathbf{R}^2} \to \overline{\mathbf{R}^2}$. Then G is K-quasiconformally conjugate to a Möbius group [Sul], [Tuk1], i.e. there is a K-quasiconformal map φ and a Möbius group Γ such that $G = \varphi^{-1} \circ \Gamma \circ \varphi$. By conjugating Γ with a Möbius map we may assume that φ fixes 0, 1, and ∞. Setting $\tilde{f} = \varphi \circ f \circ \varphi^{-1}$ and $\tilde{g} = \varphi \circ g \circ \varphi^{-1}$ we obtain that Γ is generated by $\tilde{f}$ and $\tilde{g}$, and furthermore Γ is a discrete, non-elementary group of Möbius transformations. Hence, by the chordal Jørgensen inequality (Theorem 2.1) we have

$$\max\{d(\tilde{f}), d(\tilde{g})\} \geq c_1, \tag{3}$$

where $c_1 > 0$ is as in Theorem 2.1. Note that φ satisfies the inequality of Lemma 4.1. Let

$$c_K := \frac{1}{\sqrt{2}} \left(\frac{\sqrt{2}\, c_1}{128} \right)^K .$$

We argue by contradiction: Assume that $\max\{d(f), d(g)\} < c_K$. Observe that

$$d(\tilde{f}) = d(\varphi \circ f \circ \varphi^{-1}) = d(\varphi \circ f, \varphi) = \sup_{x \in \overline{\mathbf{R}^2}} q\bigl(\varphi\bigl(f(x)\bigr), \varphi(x)\bigr).$$

Since $\overline{\mathbf{R}^2}$ is compact, the supremum is obtained at some point $x_0 \in \overline{\mathbf{R}^2}$. For this x_0 we have

$$\begin{aligned} q\bigl(\varphi\bigl(f(x_0)\bigr), \varphi(x_0)\bigr) &\leq 128 \cdot 2^{(1-K)/(2K)} q\bigl(f(x_0), x_0\bigr)^{1/K} \\ &\leq 128 \cdot 2^{(1-K)/(2K)} d(f)^{1/K} < 128 \cdot 2^{(1-K)/(2K)} (c_K)^{1/K} = c_1, \end{aligned}$$

hence we have shown that $d(\tilde{f}) < c_1$. In the same way we obtain $d(\tilde{g}) < c_1$, contradicting (3).

For the proof of the second part of the theorem, we again argue by contradiction and assume that $\bigl(d(f)\bigr)^{1/K} + \bigl(d(g)\bigr)^{1/K} < 2(c_K)^{1/K}$. As before, we obtain

$$\begin{aligned} d(\tilde{f}) + d(\tilde{g}) &\leq 128 \cdot 2^{(1-K)/(2K)} \cdot \bigl(d(f)\bigr)^{1/K} + 128 \cdot 2^{(1-K)/(2K)} \cdot \bigl(d(g)\bigr)^{1/K} \\ &< 2 \cdot 128 \cdot 2^{(1-K)/(2K)} \cdot (c_K)^{1/K} = 2c_1. \end{aligned}$$

This contradicts Theorem 2.1. □

We can now prove Theorem 3.2, following an argument given by Waterman [Wat].

Proof of Theorem 3.2. By conjugation with a Möbius transformation we may assume that 0 and ∞ are not fixed by any $g \in G \setminus \{\mathrm{id}\}$. Define $h_t(x) := t \cdot x$ for $0 < t \le 1$. Suppose that there is no $t \in (0,1]$ such that each element $f \in h_t \circ G \circ h_t^{-1} \setminus \{\mathrm{id}\}$ satisfies $d(f) \ge c_K$. We seek a contradiction.

Observe that for $g \in G \setminus \{\mathrm{id}\}$ we have

$$d(h_t \circ g \circ h_t^{-1}) \ge q\big(t \cdot g(\infty), \infty\big) \to 2 \qquad \text{as } t \to 0. \tag{4}$$

Observe also that, for fixed $g \in G$, the distance of $h_t \circ g \circ h_t^{-1}$ to the identity (i.e. $d(h_t \circ g \circ h_t^{-1})$) varies continuously in t.

Let $\tilde t_0 \le 1$. Then by assumption there exists $g_0 \in G \setminus \{\mathrm{id}\}$ so that $d(h_{\tilde t_0} \circ g_0 \circ h_{\tilde t_0}^{-1}) < c_K$. By continuity and using (4) we can find a largest $t_0^* < \tilde t_0$ so that $d(h_t \circ g_0 \circ h_t^{-1}) \ge c_K$ for all $t \le t_0^*$. Hence, by assumption, we find another element $g_1 \in G \setminus \{\mathrm{id}\}$, $g_1 \ne g_0$, so that $d(h_{t_0^*} \circ g_1 \circ h_{t_0^*}^{-1}) < c_K$.

By continuity we can now find $t_0 \in (t_0^*, \tilde t_0)$, so that

$$d(h_{t_0} \circ g_0 \circ h_{t_0}^{-1}) < c_K \qquad \text{and} \qquad d(h_{t_0} \circ g_1 \circ h_{t_0}^{-1}) < c_K.$$

Defining $\tilde t_1 := t_0^*$ we have $d(h_{\tilde t_1} \circ g_1 \circ h_{\tilde t_1}^{-1}) < c_K$ and can restart our construction with $\tilde t_0$ being replaced by $\tilde t_1$.

In general we find $t_{n+1} < t_n$ and mutually distinct elements $g_n \in G \setminus \{\mathrm{id}\}$ so that

$$d(h_{t_n} \circ g_n \circ h_{t_n}^{-1}) < c_K \qquad \text{and} \qquad d(h_{t_n} \circ g_{n+1} \circ h_{t_n}^{-1}) < c_K$$

for all $n \in \mathbf{N}$. By Theorem 3.1 we conclude that the groups $\langle g_n, g_{n+1} \rangle$ are elementary for each $n \in \mathbf{N}$. The discrete elementary torsion-free convergence groups have been studied in [GM1, Theorems 5.7, 5.10, 5.11]. We conclude that either all g_n are loxodromic and have common fixed points a, b; or all g_n are parabolic and fix a common point a. By choosing a subsequence (and relabeling a, b if necessary) and using the convergence property, we may assume that

$$\begin{aligned} g_n &\to a \qquad \text{locally uniformly in } \overline{\mathbf{R}^2} \setminus \{b\} \quad \text{as } n \to \infty \qquad \text{and} \\ g_n^{-1} &\to b \qquad \text{locally uniformly in } \overline{\mathbf{R}^2} \setminus \{a\} \quad \text{as } n \to \infty; \end{aligned}$$

where $a = b$ in the parabolic case. Note that $\{a, b\} \cap \{0, \infty\} = \varnothing$ by assumption.

Since each h_t fixes 0 and ∞, we obtain

$$(h_{t_n} \circ g_n \circ h_{t_n}^{-1})(\infty) = t_n \cdot g_n(\infty) \qquad \text{and} \qquad (h_{t_n} \circ g_n \circ h_{t_n}^{-1})(0) = t_n \cdot g_n(0),$$

where

$$g_n(0) \to a \qquad \text{and} \qquad g_n(\infty) \to a \quad \text{as } n \to \infty.$$

Note that the decreasing sequence $\{t_n\}$ converges to some $t^* \geq 0$ as $n \to \infty$, hence

$$(h_{t_n} \circ g_n \circ h_{t_n}^{-1})(\infty) \to t^* a \qquad \text{and} \qquad (h_{t_n} \circ g_n \circ h_{t_n}^{-1})(0) \to t^* a \quad \text{as } n \to \infty.$$

But $q(0, t^* a) + q(\infty, t^* a) \geq q(0, \infty) = 2$, so that we conclude

$$\liminf_{n\to\infty} d(h_{t_n} \circ g_n \circ h_{t_n}^{-1}) \geq 1,$$

contradicting the fact that $d(h_{t_n} \circ g_n \circ h_{t_n}^{-1}) < c_K \leq c_1 < 1$ for all n. ◻

In the proof of the fact that there is no chordal Jørgensen inequality for general discrete convergence groups we construct a sequence of free two generator discrete convergence groups, where both generators are arbitrarily close to the identity in the chordal distance.

Proof of Theorem 3.3. We construct a sequence of discrete non-elementary convergence groups $\langle f_n, g_n \rangle$, where f_n, g_n are homeomorphisms of $\overline{\mathbf{R}^2}$, and $\max\{d(f_n), d(g_n)\} \to 0$ as $n \to \infty$. In the following we will suppress the index n in the notation; we write f, g instead of f_n, g_n.

Part I: *A topological analog of Schottky groups.* The group generated by f and g that we construct is a topological analog of a two generator Schottky group. That is, there are four disjoint regions S_0, S_1, S_2, S_3; the homeomorphism f will map the exterior of S_2 onto the interior of S_0, its inverse will map the exterior of S_0 onto the interior of S_2, boundary will be mapped onto boundary. The homeomorphism g shall be constructed in the same way, using S_1 and S_3 instead of S_0 and S_2.

The special shape of these regions will make it possible for $d(f)$ and $d(g)$ to be small and at the same time ensure the group generated by f and g has the convergence property. As already mentioned in the outline of the proof, in order to make $d(f)$ as small as desired, it is necessary to have any point of the exterior of S_0 be close to S_2 and any point of the exterior of S_2 be close to S_0. The same must hold for the regions S_1 and S_3 in order to make $d(g)$ as small as desired. We can satisfy these assumptions by giving the regions $S_0, \dots, S_3$ the shapes of spirals: Define

$$\gamma_k(t) := e^{2\pi i t} e^{t/n + k/(8n)}, \qquad k = 0, \dots, 7, \qquad -n^2 \leq t \leq n^2.$$

Let S_0 be the region bounded by γ_0, γ_1, the line segment

$$\{e^{-2\pi i n^2} e^{-n + s/(8n)} \mid 0 \leq s \leq 1\}$$

(which connects $\gamma_0(-n^2)$ and $\gamma_1(-n^2)$) and the line segment

$$\{e^{2\pi i n^2} e^{n + s/(8n)} \mid 0 \leq s \leq 1\}$$

(which connects $\gamma_0(n^2)$ and $\gamma_1(n^2)$). Define S_1 to be the region bounded by γ_2, γ_3 and similar radial line pieces. Define S_2 and S_3 analogously (see Figure 1), that is

$$S_j = \{e^{2\pi it}e^{t/n+(2j+s)/(8n)} \mid -n^2 \le t \le n^2,\ 0 \le s \le 1\}, \qquad j = 0,1,2,3.$$

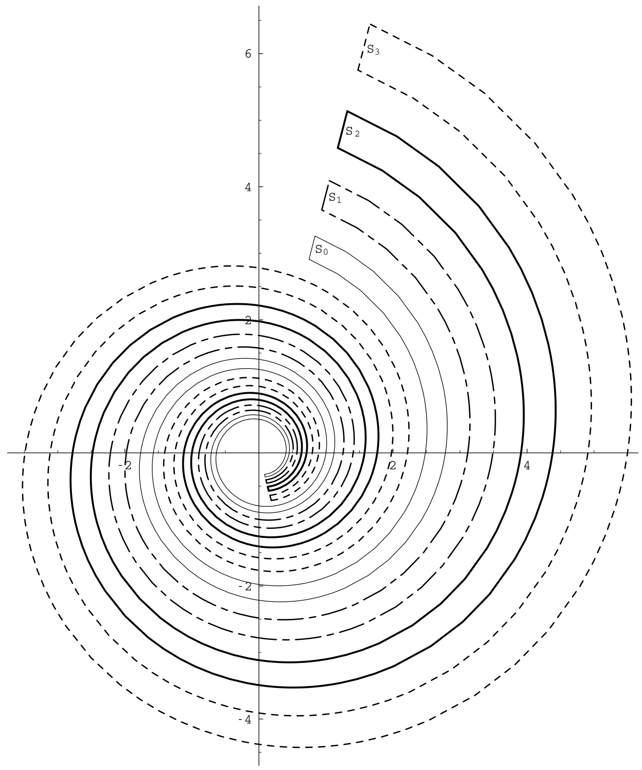

Figure 1. The spiral regions

A point $z \in S_j$, $j = 0,1,2,3$, can be described by its argument parameter t and its radius parameter s:

$$z = z(j,s,t) = e^{2\pi it}e^{t/n+(2j+s)/(8n)}, \qquad -n^2 \le t \le n^2,\ 0 \le s \le 1. \tag{5}$$

Note that for example on the boundary of S_0 we have

$$\gamma_0(t) = z(0,0,t), \qquad \gamma_1(t) = z(0,1,t).$$

Part II: *The construction of the maps* f *and* g. The following seven steps describe the construction of f. In steps 1–3 the map f is constructed on $S_0 \cup S_1 \cup$

S_3 as a composition $h_3 \circ h_2 \circ h_1$ of shrinking (via h_1, h_2) and shifting (via h_3) processes. In step 4, f is defined on ∂S_2, step 5 extends f to all of the exterior of S_2, so that f becomes a homeomorphism between the exterior of S_2 and the interior of S_0; boundary gets mapped onto boundary. In step 6 we define f^{-1} as a homeomorphism between the exterior of S_0 and the interior of S_2 in the same way as f has been defined before. Finally, step 7 combines both definitions and we obtain a homeomorphism $f\colon \overline{\mathbf{R}^2} \to \overline{\mathbf{R}^2}$.

Step 1. *The map h_1 shrinks in t-direction.* The map h_1 shrinks the spirals S_0, S_1, and S_3 by a factor $(1-2/n^2)$ in "length" (t-direction), keeping points with $t=0$ fixed, i.e. for

$$z = z(j,s,t) = e^{2\pi it} e^{t/n+(2j+s)/(8n)} \in S_j, \quad j=0,1,3,\ 0 \le s \le 1,\ -n^2 \le t \le n^2,$$

define

$$h_1(z) := z\biggl(j,s,t\Bigl(1-\frac{2}{n^2}\Bigr)\biggr) = e^{2\pi it(1-2/n^2)} e^{(t/n)(1-2/n^2)+(2j+s)/(8n)}.$$

Then for $j=0,1,3$ $h_1(S_j)$ is a subspiral of S_j, which is as "thick" as S_j, but "shorter":

$$h_1(S_j) = \bigl\{ e^{2\pi it} e^{t/n+(2j+s)/(8n)} \bigm| -n^2+2 \le t \le n^2-2,\ 0 \le s \le 1 \bigr\} \subset S_j.$$

Note that for $t \ne 0$ the point $z(j,s,t)$ travels towards the point $e^{(2j+s)/(8n)}$ on the median of S_j, but chordally no points get moved far as we shall show in the following:

Let $z = z(j,s,t) \in S_j$ as in (5) for $j \in \{0,1,3\}$. We consider three cases:

(i) $-n\sqrt{n} \le t \le n\sqrt{n}$. In this case

$$\begin{aligned}
q\bigl(z,h_1(z)\bigr) &= q\bigl(e^{2\pi it}|z|, e^{2\pi it(1-2/n^2)}|h_1(z)|\bigr) \\
&\le q\bigl(e^{2\pi it}|z|, e^{2\pi it(1-2/n^2)}|z|\bigr) + q\bigl(|z|,|h_1(z)|\bigr) \\
&\le q\bigl(e^{2\pi it}, e^{2\pi it(1-2/n^2)}\bigr) + \frac{2|1-|h_1(z)|/|z||}{\sqrt{|z|^{-2}+1}\sqrt{1+|h_1(z)|^2}} \\
&\le q\bigl(1, e^{-4\pi it/n^2}\bigr) + 2\left|1-\frac{|h_1(z)|}{|z|}\right| \le q(1, e^{4\pi i/\sqrt{n}}) + 2|1-e^{-2t/n^3}| \\
&\le q\bigl(1, e^{4\pi i/\sqrt{n}}\bigr) + 2|1-e^{2/(n\sqrt{n})}|.
\end{aligned}$$

Hence $q\bigl(z,h_1(z)\bigr)$ is arbitrarily small for large n, uniformly in $-n\sqrt{n} \le t \le n\sqrt{n}$.

(ii) $t > n\sqrt{n}$. In this case

$$\begin{aligned}
|z| &= e^{t/n+(2j+s)/(8n)} \ge e^{t/n} \ge e^{\sqrt{n}} \qquad \text{and} \\
|h_1(z)| &= e^{(t/n)(1-2/n^2)+(2j+s)/(8n)} \ge e^{(t/n)(1-2/n^2)} \ge e^{\sqrt{n}-2/(n\sqrt{n})}.
\end{aligned}$$

Hence $q(z,\infty)$ and $q(h_1(z),\infty)$ are arbitrarily small for large n, and the same holds for $q\bigl(z,h_1(z)\bigr)$ by the triangle inequality.

(iii) $t < -n\sqrt{n}$. In this case

$$|z| \le e^{-\sqrt{n}+7/(8n)} \qquad \text{and} \qquad |h_1(z)| \le e^{-\sqrt{n}+2/(n\sqrt{n})+7/(8n)},$$

so that $q(z,0)$, $q\bigl(h_1(z),0\bigr)$ and hence $q\bigl(z,h_1(z)\bigr)$ are arbitrarily small for large n.

Summing up, we have shown that for given $\varepsilon > 0$ we can find a large enough $n \in \mathbf{N}$ such that any $z \in S_0 \cup S_1 \cup S_3$ satisfies $q\bigl(z,h_1(z)\bigr) < \varepsilon$.

Step 2. *The map* h_2 *shrinks in* s*-direction.* On the new shorter spirals $h_1(S_0 \cup S_1 \cup S_3)$ we define a map h_2 which shrinks in “width” (s-direction) by a factor 8, keeping each of the longitudinal center spiral lines given by $s = \frac{1}{2}$ fixed. That is for $-n^2+2 \le t \le n^2-2$, $0 \le s \le 1$, and

$$w = w(j,s,t) = e^{2\pi it}e^{t/n+(2j+s)/(8n)} \in h_1(S_j), \qquad j = 0,1,3$$

define

$$h_2(w) := w\left(j, \frac{s}{8}+\frac{7}{16}, t\right) = e^{2\pi it}e^{t/n+2j/(8n)+(1/8n)(s/8+7/16)}.$$

Thus for $j = 0,1,3$

$$(h_2\circ h_1)(S_j) = \left\{ e^{2\pi it}e^{t/n+(2j+s)/(8n)} \;\middle|\; -n^2+2 \le t \le n^2-2,\ \frac{7}{16} \le s \le \frac{9}{16} \right\} \subset S_j.$$

As before we see that $q\bigl(w,h_2(w)\bigr)$ is arbitrarily and uniformly small for all $w \in h_1(S_0 \cup S_1 \cup S_3)$ given large enough n:

$$\begin{aligned} q\bigl(w,h_2(w)\bigr) &= q\bigl(|w|,|h_2(w)|\bigr) \le q\bigl(e^{t/n+2j/(8n)}, e^{t/n+(2j+1)/(8n)}\bigr) \\ &= 2\frac{|1-e^{1/(8n)}|}{\sqrt{e^{-2t/n-4j/(8n)}+1}\,\sqrt{1+e^{2t/n+(4j+2)/(8n)}}} \\ &\le 2|1-e^{1/(8n)}| \to 0 \qquad \text{as } n\to\infty. \end{aligned}$$

Step 3. *Shifting* $(h_2\circ h_1)(S_0 \cup S_1 \cup S_3)$ *into* S_0. Next with a map h_3 we move $v \in (h_2\circ h_1)(S_0 \cup S_1 \cup S_3)$ along a radial line into S_0 without meeting S_2, i.e. we keep v's argument $e^{2\pi it}$ fixed. The map keeps points in $(h_2\circ h_1)(S_0)$ fixed, decreases the radius of points in $(h_2\circ h_1)(S_1)$, and increases the radius of points in $(h_2\circ h_1)(S_3)$ so that these three shrunk spirals come to be placed equally spaced in S_0. To be precise: For $-n^2+2 \le t \le n^2-2$, $\frac{7}{16} \le s \le \frac{9}{16}$, and $j = 0,1,3$ the point

$$v = v(j,s,t) = e^{2\pi it}e^{t/n+(2j+s)/(8n)}$$

gets mapped to

$$h_3(v) := \begin{cases} v, & \text{for } j = 0, \\ v\big(0, s + \frac{9}{32}, t\big) = e^{2\pi i t} e^{t/n + (s + (9/32))/(8n)}, & \text{for } j = 1, \\ v\big(0, s - \frac{9}{32}, t + 1\big) = e^{2\pi i t} e^{(t+1)/n + (s - (9/32))/(8n)}, & \text{for } j = 3. \end{cases}$$

Again, it is easy to verify that h_3 moves points arbitrarily little, uniformly for all $v \in (h_2 \circ h_1)(S_0 \cup S_1 \cup S_3)$.

Together, the maps constructed above define $f := h_3 \circ h_2 \circ h_1$ on $S_0 \cup S_1 \cup S_3$. Note that the only fixed point of f on these three spirals is the point

$$z = z\big(0, \tfrac{1}{2}, 0\big) = e^{2\pi i \cdot 0} e^{(1/n)\cdot 0 + (1/2)/(8n)} = e^{1/(16n)}$$

which lies in S_0.

Step 4. *The map f on ∂S_2.* The map f maps ∂S_2 onto ∂S_0 as follows:

$$f\big(\gamma_4(t)\big) := \gamma_1(t), \qquad \text{for } -n^2 \le t \le n^2,$$
$$f\big(\gamma_5(t)\big) := \gamma_0(t+1), \qquad \text{for } -n^2 \le t \le n^2 - 1.$$

Furthermore, define f so that it maps the radial segment

$$\{e^{-2\pi i n^2} e^{-n + (4+s)/(8n)} \mid 0 \le s \le 1\} \subset \partial S_2$$

homeomorphically onto the set

$$\{\gamma_0(t) \mid -n^2 \le t \le -n^2 + 1\} \cup \{e^{-2\pi i n^2} e^{-n + s/(8n)} \mid 0 \le s \le 1\} \subset \partial S_0.$$

Finally, let f map the set

$$\{\gamma_5(t) \mid n^2 - 1 \le t \le n^2\} \cup \{e^{2\pi i n^2} e^{n + (4+s)/(8n)} \mid 0 \le s \le 1\} \subset \partial S_2$$

homeomorphically onto the radial segment

$$\{e^{2\pi i n^2} e^{n + s/(8n)} \mid 0 \le s \le 1\} \subset \partial S_0.$$

As before, no point gets moved far, given large enough n.

Step 5. *Extending f to all of the exterior of S_2.* With steps 1–4, f is defined on $S_0 \cup S_1 \cup S_3 \subset \operatorname{ext}(S_2)$ and on the boundary of S_2. We now extend f to the remaining part of the exterior of S_2 to become a homeomorphism mapping the exterior of S_2 onto the interior of S_0, so that no point gets moved far. This can be done as follows: We first extend f to the four regions between the spirals, and then extend f to the two simply connected domains that are left around the origin and around the point infinity, respectively.

In order to extend f to the regions between the spirals, note that points in these regions can be described as in (5), but with $j = 0.5, 1.5, 2.5, 3.5$, i.e.

$$z(j,s,t) = e^{2\pi it} e^{t/n+(2j+s)/(8n)}, \qquad 0 < s < 1,$$

and $-n^2 \le t \le n^2$ for $j = 0.5, 1.5, 2.5$ and $-n^2 \le t \le (n^2-1)$ for $j = 3.5$.

For the region between the spirals S_0 and S_1, i.e. points of the form $z(0.5,s,t)$, we define f by "interpolating" between the images of the points $\gamma_1(t)$ and $\gamma_2(t)$. Note that $\gamma_1(t)$ and $\gamma_2(t)$ are points on the adjacent boundaries of S_0 and S_1, respectively, that have the same argument t as the point z. Since $f\big(\gamma_1(t)\big) \in S_0$ and $f\big(\gamma_2(t)\big) \in S_0$, there are unique parameters $s_1, s_2 \in [0,1]$ and $t_1, t_2 \in [-n^2, n^2]$ such that

$$f\big(\gamma_1(t)\big) = z(0, s_1, t_1) \qquad \text{and} \qquad f\big(\gamma_2(t)\big) = z(0, s_2, t_2).$$

We now set

$$f\big(z(0.5,s,t)\big) := z\big(0, (1-s)\cdot s_1 + s\cdot s_2, (1-s)\cdot t_1 + s\cdot t_2\big),$$

i.e. we interpolate linearly between the "width" and "length" parameters of the images of $\gamma_1(t)$ and $\gamma_2(t)$. Since $z(0.5,s,t)$ is arbitrarily close to both $\gamma_1(t)$ and $\gamma_2(t)$ for large enough n, and since f moves the points $\gamma_1(t)$ and $\gamma_2(t)$ an arbitrarily small distance, we conclude that the extension of f to the region between S_0 and S_1 moves points an arbitrarily small distance, as well.

For points $z(1.5,s,t)$ between the spirals S_1 and S_2 we define f in exactly the same way as above by using the boundary curves γ_3 and γ_4 instead of γ_1 and γ_2. For points $z(2.5,s,t)$ between the spirals S_2 and S_3 we use the boundary curves γ_5 and γ_6 instead. As before, we see that f moves points as little as desired, given large enough n.

For points $z(3.5,s,t)$ $(-n^2 \le t \le (n^2-1))$ we define f by "interpolating" between the images of the points $\gamma_7(t)$ and $\gamma_0(t+1)$, which are on the adjacent boundaries of S_3 and S_0, respectively. Again, f does not move points far.

The only parts in the exterior of S_2, where f has not been defined yet are two simply connected domains. One of these domains, denoted V_0, is bounded by

$$\{\gamma_0(t) \mid -n^2 \le t \le -n^2+1\} \cup \{e^{-2\pi in^2} e^{-n+s/(8n)} \mid 0 \le s \le 8\},$$

i.e. the first spiral part of γ_0 and the line segment joining $\gamma_0(-n^2)$ and $\gamma_0(-n^2+1)$. The other domain, denoted V_1, is bounded by

$$\{\gamma_7(t) \mid n^2-1 \le t \le n^2\} \cup \{e^{2\pi in^2} e^{n+s/(8n)} \mid -1 \le s \le 7\},$$

i.e. the last spiral part of γ_7 and the line segment joining $\gamma_7(n^2-1)$ and $\gamma_7(n^2)$. Note that f has already been defined on the boundaries of the domains V_0 and V_1. Note furthermore, that any two points in V_0 are chordally close to each other for large enough n, and the same holds for any two points in V_1. Hence we can extend f to all of V_0 and V_1 using the topological Schoenflies theorem, so that f becomes a homeomorphism between V_0 and its image, and between V_1 and its image, where no point gets moved far.

With the above definitions, f becomes a homeomorphism of the exterior of S_2 onto the interior of S_0.

Step 6. *Defining f^{-1} on the exterior of S_0.* In the same way as f was defined on $S_0 \cup S_1 \cup S_3$ in steps 1–3, we now define its inverse f^{-1} on $S_1 \cup S_2 \cup S_3$. I.e. f^{-1} shrinks S_1, S_2, and S_3 in length and width, then moves these spirals into S_2. On ∂S_0, we define $f^{-1}\colon \partial S_0 \to \partial S_2$ as the inverse of the already defined map $f\colon \partial S_2 \to \partial S_0$ (compare step 4). Note that f^{-1} fixes $z\bigl(2, \frac{1}{2}, 0\bigr) = e^{9/(16n)} \in S_2$ and no other point on $S_1 \cup S_2 \cup S_3$. In the same way as it was done for f in step 5, we now extend f^{-1} to all of the exterior of S_0, so that f^{-1} maps the exterior of S_0 homeomorphically onto the interior of S_2.

Step 7. *The map f.* Combining the definitions of f and f^{-1}, we obtain a homeomorphism $f\colon \overline{\mathbf{R}^2} \to \overline{\mathbf{R}^2}$, which has exactly two fixed points, and which is as close to the identity as desired.

With these steps the construction of the map f is complete. In the same way as above, we now construct the map g. That is: g maps the exterior of S_3 onto the interior of S_1 and g^{-1} maps the exterior of S_1 onto the interior of S_3, where boundary gets mapped onto boundary.

Next we show that the free group generated by f and g is a convergence group.

Part III: *$\langle f, g\rangle$ is a convergence group.* By construction, $\langle f, g\rangle$ is a free group, i.e. every element $h \in \langle f, g\rangle$ has a unique (shortest) representation as a word

$$h = h^{(k)} \circ h^{(k-1)} \circ \cdots \circ h^{(1)}, \qquad \text{where } h^{(l)} \in \{f, f^{-1}, g, g^{-1}\}.$$

In the following we shall call the letter $h^{(1)}$ the "first letter" or "beginning" and the letter $h^{(k)}$ the "last letter" or "end" of the word h. Denote by $\ell_l(h)$ the lth letter of the unique representation of h, i.e.

$$\ell_l(h) = h^{(l)}, \qquad l = 1, \dots, k.$$

We shall call the regions S_0, S_1, S_2, and S_3 "spirals of generation 0", whereas images of a spiral S_j under a k-letter word h which does not start with the letter that maps the exterior of S_j onto some other spiral will be called "kth generation spirals". For example, $f(S_0)$, $f(S_1)$ and $f(S_3)$ are spirals of generation 1 (they are all small subspirals of S_0), but $f(S_2)$ is not a spiral anymore. Note that the chordal diameter of all kth generation spirals tends uniformly to 0 as k tends to ∞.

Let $\{h_m\}$ be an infinite sequence of distinct elements in $\langle f,g\rangle$. Then the word length of h_m is unbounded as $m\to\infty$. Choose a subsequence of $\{h_m\}$ so that the word length of the mth element of the new sequence is $\geq 2m$. Denote this new sequence by $\{h_m\}$. We can choose a subsequence $\{h_m^1\}$ of $\{h_m\}$ so that $\ell_1(h_m^1)$ is constant for all m, and also $\ell_1\big((h_m^1)^{-1}\big)$ is constant for all m. That is, all words in this subsequence start with the same letter $w_1\in\{f,f^{-1},g,g^{-1}\}$ and end with the same letter v_1. From this sequence $\{h_m^1\}$ we can choose another subsequence $\{h_m^2\}$ so that $\ell_2(h_m^2)$ is equal to some $w_2\in\{f,f^{-1},g,g^{-1}\}$ for all m and $\ell_2\big((h_m^2)^{-1}\big)=v_2^{-1}$ for all m. Proceed like this, and denote the diagonal sequence $\{h_m^m\}$ by $\{H_m\}$. Then

$$H_m = v_1\circ v_2\circ\cdots\circ v_m\circ r_m\circ w_m\circ w_{m-1}\circ\cdots\circ w_1,$$

where r_m is some word of unknown length, which does not start with w_m^{-1} and does not end with v_m^{-1}.

We now show that there are $a,b\in S_0\cup S_1\cup S_2\cup S_3$, so that $H_m\to a$ uniformly on $U=\overline{\mathbf{R}^2}\setminus(S_0\cup S_1\cup S_2\cup S_3)$, and $H_m^{-1}\to b$ uniformly on U.

By definition, the map v_m (which is one of the maps f,f^{-1},g,g^{-1}) maps the exterior of some 0th generation spiral onto the interior of some other 0th generation spiral, and v_m^{-1} reverses this process. Denote by $S_{E(m)}$ the spiral whose exterior is mapped by v_m onto another spiral, called $S_{I(m)}$. Since the last letter of r_m is not v_m^{-1}, we know that $(r_m\circ w_m\circ w_{m-1}\circ\cdots\circ w_1)(U)$ is contained in a 0th generation spiral different from $S_{E(m)}$ and hence is in the exterior of $S_{E(m)}$. Thus $(v_m\circ r_m\circ w_m\circ w_{m-1}\circ\cdots\circ w_1)(U)$ is contained in $S_{I(m)}$. Furthermore, $(v_1\circ v_2\circ\cdots\circ v_{m-1})(S_{I(m)})$ is an $(m-1)$st generation spiral, whose "length" has been shrunk by a factor $(1-2/n^2)^{m-1}$, where n is the variable but now fixed parameter of the construction of f and g. Hence, $H_m(U)$ is contained in this $(m-1)$st generation spiral. Observe now that $H_{m+1}(U)$ is contained in an mth generation spiral, being a subspiral of the previous $(m-1)$st generation spiral. By construction, the chordal diameter of all mth generation spirals converges uniformly to 0 as $m\to\infty$. Hence there exists a unique point a so that $H_m(x)\to a$ uniformly in $x\in U$.

A similar argument shows $H_m^{-1}\to b$ uniformly in U for some $b\in S_0\cup S_1\cup S_2\cup S_3$. Both points a and b are limit points of descending "Cantor-type" spiral sequences.

Finally, we show that

$$H_m\to a\qquad\text{locally uniformly in } \overline{\mathbf{R}^2}\setminus\{b\},$$

and

$$H_m^{-1} \to b \qquad \text{locally uniformly in } \overline{\mathbf{R}^2} \setminus \{a\}.$$

Let K be a compact connected set in $\overline{\mathbf{R}^2} \setminus \{b\}$. Then there is an open neighborhood of b, which does not intersect K. Thus K intersects only finitely many elements of the Cantor spiral sequence converging to b, say only spirals of generation $\le k$. Then $(w_{k+2} \circ w_{k+1} \circ \cdots \circ w_1)(K)$ is entirely contained in one of the spirals S_0, S_1, S_2 or S_3: Otherwise there would be $x \in U$, so that

$$(w_1^{-1} \circ w_2^{-1} \circ \cdots \circ w_{k+2}^{-1})(x) \in K.$$

But this contradicts the fact that $(w_1^{-1} \circ w_2^{-1} \circ \cdots \circ w_{k+2}^{-1})(U)$ is contained in a $(k+1)$st generation spiral, which is part of the sequence converging to b.

Again, since there are no cancellations between the letters of H_m, we see that $(v_m \circ r_m \circ w_m \circ \cdots \circ w_1)(K)$ is contained in $S_{I(m)}$ for $m \ge k+2$, so that $H_m(K)$ is contained in a $(m-1)$st generation spiral for $m \ge k+2$. This shows that $H_m \to a$ uniformly in K. Similarly we obtain $H_m^{-1} \to b$ locally uniformly in $\overline{\mathbf{R}^2} \setminus \{a\}$.

Hence we have shown that $\langle f, g\rangle$ is a convergence group. Furthermore it is obvious that $\langle f, g\rangle$ acts discontinuously on U, so that $\langle f, g\rangle$ is discrete. Finally $\langle f, g\rangle$ is non-elementary, since both f and g are loxodromic and have disjoint fixed point sets. This completes the proof. ▫

Remark 1. Note that the limit set $L(\langle f, g\rangle)$ of $\langle f, g\rangle$ is a Cantor set with

$$L(\langle f, g\rangle) \subset \{e^{s/(8n)} \mid 0 \le s \le 7\}.$$

Hence the chordal diameter of the limit sets converges to 0 as $n \to \infty$.

Remark 2. We can modify step 1 of the above construction, so that for each $\varepsilon > 0$ there exists $n \in \mathbf{N}$ such that $L(\langle f, g\rangle)$ is ε-dense in $\overline{\mathbf{R}^2}$, i.e. for each $x \in \overline{\mathbf{R}^2}$ the chordal ball $B_\varepsilon(x)$ meets $L(\langle f, g\rangle)$. We can do this by only modifying one of the maps, say f. In step 1, instead of giving f an attracting fixed point at $z\big(0, \frac12, 0\big)$ in S_0 and a repelling fixed point at $z\big(2, \frac12, 0\big)$ in S_2, we let the attracting fixed point be $z\big(0, \frac12, -n\sqrt{n}\big)$ in S_0, and the repelling fixed point $z\big(2, \frac12, n\sqrt{n}\big)$ in S_2. This can be done by redefining h_1 on $S_0 \cup S_1 \cup S_3$: Let h_1 map the point

$$z = z(j, s, t) = e^{2\pi i t} e^{t/n + (2j+s)/(8n)} \in S_j$$

onto the point

$$\begin{aligned} h_1(z) &= z\left(j, s, t\left(1 - \frac{2}{n^2}\right) - \frac{2}{\sqrt{n}}\right) \\ &= e^{2\pi i [t(1-2/n^2) - 2/\sqrt{n}]} e^{(1/n)[t(1-(2/n^2)) - (2/\sqrt{n})] + (2j+s)/(8n)}. \end{aligned}$$

With the same modification for the definition of f^{-1} on $S_1 \cup S_2 \cup S_3$ the map f fixes

$$e^{-2\pi i n\sqrt{n}}\, e^{-\sqrt{n}+(1/2)/(8n)} \in S_0$$

and

$$e^{2\pi i n\sqrt{n}}\, e^{\sqrt{n}+(4+1/2)/(8n)} \in S_2.$$

Let g be unchanged. If $\varepsilon > 0$ is given then we can find $n \in \mathbf{N}$ such that $d(f) < \frac{1}{2}\varepsilon$, $d(g) < \frac{1}{2}\varepsilon$, and the attracting fixed point of f is in an $\frac{1}{2}\varepsilon$-neighborhood of 0, the repelling fixed point of f is in an $\frac{1}{2}\varepsilon$-neighborhood of ∞, and the distance from

$$z = z(j,s,t) = e^{2\pi i t} e^{t/n+(2j+s)/(8n)}$$

to

$$z' = z'(j,s,t+1) = e^{2\pi i (t+1)} e^{(t+1)/n+(2j+s)/(8n)}$$

is less than $\frac{1}{2}\varepsilon$ for all j, s, t.

Note that the images of all fixed points of f and g under maps in $\langle f, g\rangle$ are contained in the limit set $L(\langle f, g\rangle)$. Observe now that g moves the fixed points of f in steps of length $\le \frac{1}{2}\varepsilon$ towards its attracting fixed point, i.e. these images will be $\frac{1}{2}\varepsilon$-dense in g's attracting spiral S_3. Similar for g^{-1}. The map f maps this picture into S_0, whereas f^{-1} maps it into S_2. Since by construction the spirals S_j and $S_{(j+1) \bmod 4}$ are within distance $\frac{1}{2}\varepsilon$ of each other, the limit set $L(\langle f, g\rangle)$ is ε-dense in $\overline{\mathbf{R}^2}$.

Proof of Corollary 3.4. Let $G_n = \langle f_n, g_n\rangle$ be the discrete, non-elementary convergence groups constructed in Theorem 3.3. Then by construction G_n does not contain parabolics and its limit set $L(G_n)$ is a Cantor set. Thus by [MS] the group G_n is topologically conjugate to a Möbius group, i.e. there exists a homeomorphism $\Phi_n\colon \overline{\mathbf{R}^2} \to \overline{\mathbf{R}^2}$ and a Möbius group $\Gamma_n = \langle \tilde{f}_n, \tilde{g}_n\rangle$ such that $f_n = \Phi_n \circ \tilde{f}_n \circ \Phi_n^{-1}$, $g_n = \Phi_n \circ \tilde{g}_n \circ \Phi_n^{-1}$, and hence

$$G_n = \Phi_n \circ \Gamma_n \circ \Phi_n^{-1}.$$

Because of the topological conjugacy, Γ_n is a discrete, non-elementary Möbius group. Since the diffeomorphisms of $\overline{\mathbf{R}^2}$ are dense in the homeomorphisms or $\overline{\mathbf{R}^2}$, we can choose a quasiconformal map $\Psi_n\colon \overline{\mathbf{R}^2} \to \overline{\mathbf{R}^2}$ with $d(\Psi_n, \Phi_n) < 1/n$. Define $\hat{f}_n = \Psi_n \circ \tilde{f}_n \circ \Psi_n^{-1}$ and $\hat{g}_n = \Psi_n \circ \tilde{g}_n \circ \Psi_n^{-1}$. Then

$$\begin{aligned} d(\hat{f}_n) &= d(\Psi_n \circ \tilde{f}_n, \Psi_n) \\ &\le d(\Psi_n \circ \tilde{f}_n, \Phi_n \circ \tilde{f}_n) + d(\Phi_n \circ \tilde{f}_n, \Phi_n) + d(\Phi_n, \Psi_n) \\ &= 2d(\Phi_n, \Psi_n) + d(f_n) \to 0 \qquad \text{as } n \to \infty, \end{aligned}$$

and similarly

$$d(\hat{g}_n) \to 0 \qquad \text{as } n \to \infty. \ \square$$

For the proof of Theorem 3.5 we first show a lemma:

Lemma 4.2. *Suppose that G is a convergence group with non-empty regular set. Then there exists an open ball U such that $f(U)\cap U = \varnothing$ for all $f \in G\setminus\{\mathrm{id}\}$.*

Proof. Since by hypothesis the regular set of G is non-empty, there exists a point $y_0 \in \Omega(G)$ and an open neighborhood V of y_0 such that $f(V)\cap V = \varnothing$ for all but a finite number of elements $f \in G$, let $f_1,\dots,f_k \in G \setminus \{\mathrm{id}\}$ be these elements. If E_j denotes the set of fixed points of f_j, then E_j is closed by continuity of f_j. Furthermore, E_j contains no interior points: If E_j is finite then this is clear, otherwise f_j is elliptic and E_j has no interior points by a theorem of Newman's [New] on periodic homeomorphisms of spaces. Thus we find $x_0 \in V$ which is not fixed by any f_j, and hence

$$s := \min_{j=1,\dots,k} q\bigl(f_j(x_0), x_0\bigr) > 0.$$

Let U be a chordal ball about x_0 with radius $r > 0$, where r is chosen so that $r < \frac{1}{2}s$, $U \subset V$, and so that $x \in U$ implies $q\bigl(f_j(x), f_j(x_0)\bigr) < \frac{1}{2}s$ for $j = 1,\dots,k$. Then for $x \in U$ we have

$$q\bigl(f_j(x), x_0\bigr) \ge q\bigl(f_j(x_0), x_0\bigr) - q\bigl(f_j(x), f_j(x_0)\bigr) > s - \tfrac{1}{2}s > r,$$

hence $f_j(x) \notin U$. Thus U satisfies

$$f(U)\cap U = \varnothing \qquad \text{for all } f \in G \setminus \{\mathrm{id}\}. \ \square$$

This lemma enables us to prove our final theorem.

Proof of Theorem 3.5. Let $0 < c < 2$. Choose U as in Lemma 4.2 and let h be a Möbius transformation which maps U onto the chordal ball V with center ∞ and radius c. Then if

$$\tilde f = h\circ f \circ h^{-1}, \qquad f \in G \setminus \{\mathrm{id}\},$$

we have

$$\tilde f(V)\cap V = h\bigl(f(U)\cap U\bigr) = \varnothing,$$

and hence

$$d(\tilde f, \mathrm{id}) \ge q\bigl(\tilde f(\infty), \infty\bigr) \ge c. \ \square$$

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