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CHARACTERIZATION OF THE ω-HYPOELLIPTIC CONVOLUTION OPERATORS ON ULTRADISTRIBUTIONS

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Abstract. We achieve characterizations of those ultradistributions $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ (resp. $\mathscr{E}'_{\{\omega\}}(\mathbf{R}^N)$) with compact support such that for each ultradifferentiable function f in $\mathscr{E}_{(\omega)}(\mathbf{R}^N)$ (resp. in $\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$) each solution $\nu \in \mathscr{D}'_{(\omega)}(\mathbf{R}^N)$ (resp. in $\mathscr{D}'_{\{\omega\}}(\mathbf{R}^N)$) of the convolution equation $\mu * \nu = f$ belongs to the same class as f. These characterizations extend classical results of Ehrenpreis and Hörmander for distributions and Björck and Chou for ultradistributions.

Introduction

Let $\mathscr{D}'_{(\omega)}(\mathbf{R}^N)$ denote the (ω)-ultradistributions of Beurling type on $\mathbf{R}^N$ in the sense of Beurling–Björck [1] or Braun, Meise and Taylor [5]. Then each (ω)-ultradistribution with compact support $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ induces a convolution operator S_μ on $\mathscr{D}'_{(\omega)}(\mathbf{R}^N)$. S_μ or μ is called (ω)-hypoelliptic if each solution $\nu \in \mathscr{D}'_{(\omega)}(\mathbf{R}^N)$ of the convolution equation

$$S_\mu(\nu) = \mu * \nu = f$$

belongs in fact to $\mathscr{E}_{(\omega)}(\mathbf{R}^N)$, when $f \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$. In the classical case of distributions, i.e. $\omega(t) = \log(1+t)$, the hypoelliptic convolution operators were characterized by Ehrenpreis [8] in terms of a slowly decreasing condition and the location of the zeros in $\mathbf{C}^N$ of the Fourier–Laplace transform $\hat{\mu}$ of μ, while Hörmander [9] showed that the singular support of the solution ν of $\mu * \nu = f$ can be controlled by the singular supports of f and μ and the support of μ. For a complete characterization in this case we refer to Hörmander [10, II, 16.6].

The results of Ehrenpreis and Hörmander were extended in part to ultradistributions by Björck [1] and Chou [7]. Björck characterized the (ω)-hypoelliptic differential operators $P(D)$, while Chou gave necessary as well as sufficient conditions

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for hypoelliptic convolution operators μ acting on $\mathscr{D}'_{\{M_p\}}(\mathbf{R}^N)$ and $\mathscr{D}'_{(M_p)}(\mathbf{R}^N)$. For results concerning pseudo-differential operators on Gevrey classes we refer to Rodino [19, 3.2, 3.3] and the references given there.

In the present paper we show that the classical characterizations of hypoellipticity extend completely to convolution operators acting on $\mathscr{D}'_{(\omega)}(\mathbf{R}^N)$ or on $\mathscr{D}'_{\{\omega\}}(\mathbf{R}^N)$, the space of $\{\omega\}$-ultradistributions of Roumieu type. Using the abbreviation $(*)$ for (ω) or $\{\omega\}$, and denoting the convex hull of a set K by $\mathrm{ch}(K)$, our main result can be stated as follows:

Theorem. *For $\mu \in \mathscr{E}'_*(\mathbf{R}^N)$ the following assertions are equivalent*:

(1) *Whenever $\nu \in \mathscr{D}'_*(\mathbf{R}^N)$ and $\mu * \nu \in \mathscr{E}_*(\mathbf{R}^N)$, then $\nu \in \mathscr{E}_*(\mathbf{R}^N)$.*
(2) *μ is slowly decreasing for $(*)$ and $\lim_{z\in V(\hat{\mu}),\,|z|\to\infty}\big(|\operatorname{Im} z|/\omega(z)\big) = \infty$ if $(*) = (\omega)$ (resp. $\liminf_{z\in V(\hat{\mu}),\,|z|\to\infty}\big(|\operatorname{Im} z|/\omega(z)\big) > 0$ if $(*) = \{\omega\}$).*
(3) $\operatorname{Ker} S_\mu \subset \mathscr{E}_*(\mathbf{R}^N)$ *and μ is slowly decreasing for $(*)$.*
(4) *There exists $F \in \mathscr{E}'_*(\mathbf{R}^N)$ and $\psi \in \mathscr{D}_*(\mathbf{R}^N)$ such that $\mu * F = \delta + \psi$.*
(5) *There exists a fundamental solution E for S_μ satisfying*

$$\mathrm{ch}(\text{sing supp}_*(E)) = -\,\mathrm{ch}\big(\text{sing supp}_*(\mu)\big).$$

(6) *For each open set $\Omega \subset \mathbf{R}^N$ and $u \in \mathscr{D}'_*(\Omega - \mathrm{supp}(\mu) + \mathrm{ch}(\text{sing supp}_*(\mu)))$ satisfying $\mu * u \in \mathscr{E}_*\big(\Omega + \mathrm{ch}\big(\text{sing supp}_*(\mu)\big)\big)$, it follows that $u \in \mathscr{E}_*(\Omega)$.*

For $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ the proof of the theorem is given by appropriate modifications of the classical arguments using a recent characterization of those convolution operators S_μ which are surjective on $\mathscr{D}'_{(\omega)}(\mathbf{R}^N)$, given by Bonet, Galbis and Meise [2]. For $\mu \in \mathscr{E}'_{\{\omega\}}(\mathbf{R}^N)$ the proof can be reduced essentially to the Beurling case, due to results of Braun, Meise and Taylor [5] and Meise, Taylor and Vogt [15]. As we show, they imply that for each open set Ω in $\mathbf{R}^N$, $\mathscr{E}_{\{\omega\}}(\Omega)$ coincides topologically with the intersection of the spaces $\mathscr{E}_{(\sigma)}(\Omega)$, where σ runs through all weight functions satisfying $\sigma(t) = o\big(\omega(t)\big)$ as t tends to infinity. Consequently,

$$\text{sing supp}_{\{\omega\}}(u) = \overline{\cup\{\text{sing supp}_{(\sigma)}(u) : \sigma = o(\omega)\}}$$

holds for each $u \in \mathscr{D}'_{\{\omega\}}(\mathbf{R}^N)$. From this it follows that $\mu \in \mathscr{E}'_{\{\omega\}}(\mathbf{R}^N)$ is $\{\omega\}$-hypoelliptic if and only if there exists a weight function σ_0 satisfying $\sigma_0 = o(\omega)$ such that μ is (σ)-hypoelliptic for each weight function $\sigma \geq \sigma_0$ satisfying $\sigma = o(\omega)$.

Note that our results apply in particular to Gevrey ultradistributions and more generally to ultradistributions μ in $\mathscr{E}'_{\{M_p\}}(\mathbf{R}^N)$ or $\mathscr{E}'_{(M_p)}(\mathbf{R}^N)$, provided that the sequence $(M_p)_{p\in\mathbf{N}_0}$ satisfies the conditions (M1), (M2) and (M3) of Komatsu [11] (see Meise and Taylor [13, 3.11]) or the weaker conditions in Braun, Meise and Taylor [5, 8.9].

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1. Preliminaries

In this preliminary section we introduce the nonquasianalytic classes, the spaces of ultradistributions and most of the notation that will be used in the sequel.

1.1. Definition. Let ω: $[0,\infty[\to[0,\infty[$ be a continuous function which is increasing and satisfies $\omega(0)=0$ and $\omega(1)>0$. ω is called a weight function if it satisfies the following conditions:

(α) $\omega(2t)\le K\big(1+\omega(t)\big)$ for some $K\ge 1$ and for all t.

(β) $\int_{-\infty}^{\infty}\big(\omega(t)/(1+t^2)\big)\,dt<\infty$.

(γ) $\log(1+t^2)=o\big(\omega(t)\big)$ as t tends to ∞.

(δ) $\varphi(t)=\omega(e^t)$ is convex in $\mathbf{R}$.

For a weight function ω we define $\tilde\omega$: $\mathbf{C}^N\to[0,\infty[$ by $\tilde\omega(z)=\omega(|z|)$ and again call this function ω, by abuse of notation. The Young conjugate of φ is defined by $\varphi^*(x)=\sup_{y>0}\{xy-\varphi(y)\}$.

1.2. Remark. (a) Each weight function ω satisfies $\lim_{t\to\infty}\big(\omega(t)/t\big)=0$ by the remark following 1.3 of [14].

(b) For each weight function ω there exists a weight function σ satisfying $\sigma(t)=\omega(t)$ for all large $t>0$ and $\sigma\mid[0,1[=0$. This implies $\varphi_\sigma(y)=\varphi_\omega(y)$ for all large y, $\varphi_\sigma^{**}=\varphi_\sigma$. From this it follows that all subsequent definitions do not change if ω is replaced by σ. On the other hand they also do not change if ω is replaced by $\omega+c$, c some positive number.

(c) For each weight function ω there exists a weight function $\sigma\in\mathscr{C}^\infty$ having bounded derivative on $[0,\infty[$ such that σ and ω are equivalent in the sense that $\omega\le\sigma\le A\omega+C$ for some constants $A,C>0$.

(d) Let ω be as in 1.1. Then, for $x,y\in\mathbf{C}^N$ we have, by [5, 1.2], $\omega(x+y)\le K\big(1+\omega(x)+\omega(y)\big)$. Consequently, $\omega(x-y)\ge\omega(|x|-|y|)\ge\omega(x)/K-1-\omega(y)$, for arbitrary $x,y\in\mathbf{C}^N$.

1.3. Definition. Let ω be a weight function.

(a) For a set $K\subset\mathbf{R}^N$ and $\lambda>0$ let

$$\mathscr{E}_\omega(K,\lambda):=\Big\{f\in C^\infty(K):\|f\|_{K,\lambda}:=\sup_{x\in K}\sup_{\alpha}|f^{(\alpha)}(x)|\exp\Big(-\lambda\varphi^*\Big(\frac{|\alpha|}{\lambda}\Big)\Big)<\infty\Big\}.$$

(b) For an open set $\Omega\subset\mathbf{R}^N$ define

$$\begin{aligned}\mathscr{E}_{(\omega)}(\Omega)&:=\mathrm{proj}_{\leftarrow K\subset\subset\Omega}\mathrm{proj}_{m\in\mathbf{N}}\mathscr{E}_\omega(K,m)\\&=\{f\in C^\infty(\Omega):\|f\|_{K,m}<\infty\text{ for each }K\subset\subset\Omega\text{ and each }m\in\mathbf{N}\},\end{aligned}$$

and

$$\begin{aligned}\mathscr{E}_{\{\omega\}}(\Omega)&:=\mathrm{proj}_{\leftarrow K\subset\subset\Omega}\mathrm{ind}_{\to m\in\mathbf{N}}\mathscr{E}_\omega\Big(K,\frac{1}{m}\Big)\\&=\{f\in C^\infty(\Omega):\text{for each }K\subset\subset\Omega\text{ there is }m\in\mathbf{N}\text{ with }\|f\|_{K,1/m}<\infty\}.\end{aligned}$$

The elements of $\mathscr{E}_{(\omega)}(\Omega)$ (resp. $\mathscr{E}_{\{\omega\}}(\Omega)$) are called *$\omega$-ultradifferentiable functions of Beurling* (resp. *Roumieu*) *type* on Ω. We write $\mathscr{E}_*(\Omega)$, where $*$ can be either (ω) or $\{\omega\}$ at all occuring places.

(c) For a compact set K in $\mathbf{R}^N$ we let

$$\mathscr{D}_*(K) := \{f \in \mathscr{E}_*(\mathbf{R}^N) : \operatorname{supp}(f) \subset K\},$$

endowed with the induced topology. For an open set $\Omega \subset \mathbf{R}^N$ and a fundamental sequence $(K_j)_{j\in\mathbf{N}}$ of compact subsets of Ω we let

$$\mathscr{D}_*(\Omega) := \operatorname{ind}_{j\to}\mathscr{D}_*(K_j).$$

The dual $\mathscr{D}'_*(\Omega)$ of $\mathscr{D}_*(\Omega)$ is endowed with its strong topology. The elements of $\mathscr{D}'_{(\omega)}(\Omega)$ (resp. $\mathscr{D}'_{\{\omega\}}(\Omega)$) are called *$\omega$-ultradistributions of Beurling* (resp. *Roumieu*) *type* on Ω.

1.4. Remark. (a) By Meise, Taylor and Vogt [15, 3.3], for each open set Ω in $\mathbf{R}^N$, the seminorms

$$\|\ \|_{K,\sigma}\colon f \to \sup_{x\in K}\sup_{\alpha\in\mathbf{N}_0^n} |f^{(\alpha)}(x)|\exp\bigl(-\varphi^*_\sigma(|\alpha|)\bigr),$$

where K is any compact set in Ω and σ is a weight function satisfying $\sigma = o(\omega)$, form a fundamental sequence of semi-norms for $\mathscr{E}_{\{\omega\}}(\Omega)$.

(b) For each compact set K in $\mathbf{R}^N$, $\mathscr{D}_{\{\omega\}}(K)$ is a (DFN)-space by Braun, Meise and Taylor [5, 3.6]. A fundamental system of bounded sets is given by

$$B_m := \left\{\varphi \in \mathscr{D}_{\{\omega\}}(K) : |\varphi|_{K,m} := \int_{\mathbf{R}^N} |\hat\varphi(t)| e^{\omega(t)/m}\,dt \le 1\right\},$$

where the Fourier transform $\hat\varphi\colon \mathbf{R}^N \to \mathbf{C}$ is defined as

$$\hat\varphi(t) = \int \varphi(x) e^{-i\langle x,t\rangle}\,dx.$$

(c) For each compact set K in $\mathbf{R}^N$, $\mathscr{D}_{(\omega)}(K)$ is a nuclear Fréchet space, by Braun, Meise and Taylor [5, 3.6]. A fundamental system of seminorms on $\mathscr{D}_{(\omega)}(K)$ is given by $(\|\cdot\|_{K,m})_{m\in\mathbf{N}}$ defined in 1.3 but also by

$$|\!|\!|\varphi|\!|\!|_m := \int_{\mathbf{R}^N} |\hat\varphi(t)| e^{m\omega(t)}\,dt, \qquad \varphi \in \mathscr{D}_{(\omega)}(K).$$

1.5. Example. The following functions $\omega\colon [0,\infty[\to [0,\infty[$ are examples of weight functions:

(1) $\omega(t) = t^\alpha$, $0 < \alpha < 1$.

(2) $\omega(t) = \big(\log(1+t)\big)^\beta$, $\beta > 1$.

(3) $\omega(t) = t\big(\log(1+t)\big)^{-\beta}$, $\beta > 1$.

Note that for $\omega(t) = t^\alpha$, the classes $\mathscr{E}_{(\omega)}$ (resp. $\mathscr{E}_{\{\omega\}}$) coincide with the Gevrey classes $\Gamma^{(d)}$ (resp. $\Gamma^{\{d\}}$) for $d := 1/\alpha$.

(4) Let $(M_p)_{p\in\mathbf{N}_0}$ be a sequence of positive numbers which has the following properties:

(M1) $M_j^2 \le M_{j-1}M_{j+1}$ for all $j \in \mathbf{N}$;

(M2) there exists $A, H > 1$ with $M_n \le AH^n \min_{0\le j\le n} M_j M_{n-j}$ for all $n \in \mathbf{N}$;

(M3) there exists $A > 0$ with $\sum_{q=j+1}^\infty M_{q-1}/M_q \le AjM_j/M_{j+1}$;

and define $\omega_M\colon \mathbf{R} \to [0, \infty[$ by

$$\omega_M(t) = \begin{cases} \sup_{j\in\mathbf{N}_0} \log \dfrac{|t|^j M_0}{M_j} & \text{for } |t| > 0, \\ 0 & \text{for } t = 0. \end{cases}$$

Then ω_M is a continuous even function and by Meise and Taylor [13, 3.11] there exists a concave weight function κ with $\omega_M(t) \le \kappa(t) \le C\omega_M(t) + C$ for some $C > 0$ and all $t > 0$ and such that for each open set Ω in $\mathbf{R}^N$ we have

$$\mathscr{E}_{(M_j)}(\Omega) = \left\{ f \in \mathscr{C}^\infty(\Omega) : \sup_{\alpha\in\mathbf{N}_0^N} \sup_{x\in K} \frac{|f^{(\alpha)}(x)|}{h^{|\alpha|}M_{|\alpha|}} < \infty \text{ for each } h > 0 \right.$$
$$\left. \text{and each } K \subset \Omega \text{ compact} \right\} = \mathscr{E}_{(\kappa)}(\Omega).$$

We have an analogous identity for the Roumieu spaces. Note that by [5, 8.9], these identities hold even under weaker hypotheses on $(M_p)_{p\in\mathbf{N}_0}$.

1.6. Convolution operators. Let $\mu \in \mathscr{E}'_*(\mathbf{R}^N)$, $\mu \ne 0$, and open sets Ω_1, Ω_2 in $\mathbf{R}^n$ be given. If $\Omega_1 - \operatorname{supp}(\mu) \subset \Omega_2$ then we define:

(1) $$S_\mu^t\colon \mathscr{D}_*(\Omega_1) \to \mathscr{D}_*(\Omega_2), \qquad S_\mu^t(\varphi) := \check{\mu} * \varphi|_{\Omega_2},$$

where $\check{\mu} * \varphi\colon x \to \check{\mu}\big(\varphi(x - \cdot)\big)$, $x \in \Omega_2$, and where $\check{\mu}(\psi) := \mu(\check{\psi})$ and $\check{\psi}(x) := \psi(-x)$, $x \in \mathbf{R}^N$. Since S_μ^t is continuous and linear, so is its adjoint operator

$$S_\mu := (S_\mu^t)^t\colon \mathscr{D}'_*(\Omega_2) \to \mathscr{D}'_*(\Omega_1).$$

(2) $$T_\mu^t\colon \mathscr{E}'_*(\Omega_1) \to \mathscr{E}'_*(\Omega_2), \qquad T_\mu^t(\nu) := \check{\mu} * \nu|_{\Omega_2},$$

where $\check{\mu} * \nu(\varphi) := (\check{\mu} * (\check{\nu} * \varphi))(0)$. Again, T_μ^t is continuous and linear, so that its adjoint

$$T_\mu := (T_\mu^t)^t \colon \mathscr{E}_*(\Omega_2) \to \mathscr{E}_*(\Omega_1)$$

is continuous and linear.

Note that $S_\mu(\nu) = \mu * \nu$ and $T_\mu(f) = \mu * f$, so that it is reasonable to call the operators S_μ and T_μ *convolution operators*. Note further that $T_\mu^t|_{\mathscr{D}_*(\Omega_1)} = S_\mu^t$ and $S_\mu|_{\mathscr{E}_*(\Omega_2)} = T_\mu$ and that T_μ^t and S_μ^t are injective.

1.7. Definition. For $u \in \mathscr{D}_*'(\mathbf{R}^N)$ the $*$-singular support, denoted by $\operatorname{sing\,supp}_*(u)$, is the set of points in $\mathbf{R}^N$ having no open neighbourhood U to which the restriction $u|_U$ is in $\mathscr{E}_*(U)$.

1.8. Remark. Similar as Hörmander [10, 4.2.5] one proves that

$$\operatorname{sing\,supp}_*(u * v) \subset \operatorname{sing\,supp}_*(u) + \operatorname{sing\,supp}_*(v)$$

whenever $u, v \in \mathscr{D}_*'(\mathbf{R}^N)$ and one has compact support.

1.9. Spaces of entire functions. Let $A(\mathbf{C}^N)$ denote the space of all entire functions on $\mathbf{C}^N$, endowed with the Fréchet space topology of uniform convergence on all compact subsets of $\mathbf{C}^N$. For an upper semi-continuous function $v \colon \mathbf{C}^N \to]0, \infty[$ we define

$$A(v, \mathbf{C}^N) := \left\{ f \in A(\mathbf{C}^N) : \|f\|_v := \sup_{z \in \mathbf{C}^N} |f(z)| v(z) < \infty \right\}$$

and note that $A(v, \mathbf{C}^N)$ is a Banach space.

1.10. Definition. Let K be a convex compact subset of $\mathbf{R}^N$. Then we define the support functional H_K of K by

$$H_K(y) := \sup_{x \in K} \langle x, y \rangle.$$

1.11. Fourier–Laplace transform. For $\mu \in \mathscr{E}_*'(\mathbf{R}^N)$ its *Fourier–Laplace transform* $\hat{\mu} \in A(\mathbf{C}^N)$ is defined as

$$\hat{\mu}(z) := \mu\big(\exp(-i\langle \cdot\,, z\rangle)\big), \qquad z \in \mathbf{C}^N.$$

To characterize its growth behaviour, fix a weight function ω and define the functions w_j, $w_{j,k}$, v_j and $v_{j,k}$ by

$$\begin{aligned} w_j(z) &:= \exp\big(-j\big(|\operatorname{Im} z| + \omega(z)\big)\big), \\ w_{j,k}(z) &:= \exp\left(-j|\operatorname{Im} z| - \frac{1}{k}\omega(z)\right), \\ v_j(z) &:= \exp\left(-j|\operatorname{Im} z| + \frac{1}{j}\omega(z)\right), \\ v_{j,k}(z) &:= \exp\big(-j|\operatorname{Im} z| + k\omega(z)\big). \end{aligned}$$

Then the Fourier–Laplace transform $\mathscr{F}\colon \mu \to \hat{\mu}$ is an isomorphism between the following spaces (see Braun, Meise and Taylor [5, 3.5, 7.4]:

$$\begin{aligned}
\mathscr{E}'_{(\omega)}(\mathbf{R}^N) &\longrightarrow \operatorname{ind}_{j\to} A(w_j, \mathbf{C}^N),\\
\mathscr{E}'_{\{\omega\}}(\mathbf{R}^N) &\longrightarrow \operatorname{ind}_{j\to}\operatorname{proj}_{\leftarrow k} A(w_{j,k}, \mathbf{C}^N),\\
\mathscr{D}_{(\omega)}(\mathbf{R}^N) &\longrightarrow \operatorname{ind}_{j\to}\operatorname{proj}_{\leftarrow k} A(v_{j,k}, \mathbf{C}^N),\\
\mathscr{D}_{\{\omega\}}(\mathbf{R}^N) &\longrightarrow \operatorname{ind}_{j\to} A(v_j, \mathbf{C}^N).
\end{aligned}$$

Moreover, for a given $\mu \in \mathscr{E}'_*(\mathbf{R}^N)$ and a convex compact set K in $\mathbf{R}^N$, $\operatorname{supp}(\mu) \subset K$ if and only if there exists $j \in \mathbf{N}$ such that for each $\varepsilon > 0$ we find $C > 0$ so that $|\hat{\mu}(z)| \le C \exp\bigl(H_K(\operatorname{Im} z) + \varepsilon|\operatorname{Im} z| + j\omega(z)\bigr)$ for every $z \in \mathbf{C}^N$.

Furthermore, for $\mu, \nu \in \mathscr{E}'_*(\mathbf{R}^N)$ and $\varphi \in \mathscr{D}_*(\mathbf{R}^N)$ we have

$$\mathscr{F}\bigl(S^t_\mu(\nu)\bigr) = \mathscr{F}(\check{\mu})\mathscr{F}(\nu) \qquad \text{and} \qquad \mathscr{F}\bigl(T^t_\mu(\varphi)\bigr) = \mathscr{F}(\check{\mu})\mathscr{F}(\varphi),$$

hence $\mathscr{F} \circ S^t_\mu \circ \mathscr{F}^{-1}$ (resp. $\mathscr{F} \circ T^t_\mu \circ \mathscr{F}^{-1}$) is the operator of multiplication by $\mathscr{F}(\check{\mu})$.

1.12. Definition. Given $\mu \in \mathscr{E}'_*(\mathbf{R}^N)$ we let

$$V(\hat{\mu}) := \{z \in \mathbf{C}^N \mid \hat{\mu}(z) = 0\}.$$

1.13. Definition. (a) An ultradistribution $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ is called *slowly decreasing for* (ω) if there exists $C > 0$ such that for each $x \in \mathbf{R}^N$ with $|x| > C$ there is $\xi \in \mathbf{R}^N$ with $|x - \xi| \le C\omega(x)$ and $|\hat{\mu}(\xi)| \ge \exp\bigl(-C\omega(\xi)\bigr)$.

(b) An ultradistribution $\mu \in \mathscr{E}'_{\{\omega\}}(\mathbf{R}^N)$ is called *slowly decreasing for* $\{\omega\}$ if for each $m \in \mathbf{N}$ there exists $R > 0$ such that for each $x \in \mathbf{R}^N$ with $|x| > R$ there exists $\xi \in \mathbf{R}^N$ satisfying $|x - \xi| \le \omega(x)/m$ such that $|\hat{\mu}(\xi)| \ge \exp\bigl(-\omega(\xi)/m\bigr)$.

1.14. Remark. If $\mu \in \mathscr{E}'_*(\mathbf{R}^N)$ is slowly decreasing for $*$ in the sense of Definition 1.13 then μ is slowly decreasing in the sense of Bonet, Galbis and Meise [2, 2.3, 3.1]. Hence it follows from [2, 2.9, 3.4] that the convolution operator $S_{\check{\mu}}$ and consequently S_μ is surjective on $\mathscr{D}'_*(\mathbf{R}^N)$. Note that by Bonet, Galbis and Momm [3] the two slowly decreasing definitions are in fact equivalent. For $N = 1$ and $* = (\omega)$ this follows easily from Meise, Taylor and Vogt [14, 2.7].

1.15. Notation. For a subset A of $\mathbf{R}^N$, we denote the convex hull of A by $\operatorname{ch}(A)$.

2. (ω)-hypoelliptic convolution operators

In this section we define and characterize when the convolution operator S_μ which is induced by $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ is (ω)-hypoelliptic. To do this we use essentially arguments which go back to Ehrenpreis [8], and Hörmander [9], [10, Section 16.6] in the classical case of convolution operators on distributions. Throughout this section, ω denotes a fixed weight function.

2.1. Theorem. *For $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ the following assertions are equivalent*:

(1) *Whenever $\nu \in \mathscr{D}'_{(\omega)}(\mathbf{R}^N)$ and $\mu * \nu \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$, then $\nu \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$,*

(2) $\lim_{z\in V(\hat\mu),\,|z|\to\infty}\big(|\operatorname{Im} z|/\omega(z)\big) = \infty$ *and there exist $A, R > 0$ such that*

$$|\hat\mu(x)| \geq \exp\big(-A\omega(x)\big) \qquad \text{for each } x \in \mathbf{R}^N,\ |x| \geq R, \tag{2.1}$$

(3) $\lim_{z\in V(\hat\mu),|z|\to\infty}\big(|\operatorname{Im} z|/\omega(z)\big) = \infty$ *and μ is slowly decreasing for (ω),*

(4) $\operatorname{Ker} S_\mu \subset \mathscr{E}_{(\omega)}(\mathbf{R}^N)$ *and μ is slowly decreasing for (ω),*

(5) *there exist $F \in \mathscr{D}'_{(\omega)}(\mathbf{R}^N)$ and $\psi \in \mathscr{D}_{(\omega)}(\mathbf{R}^N)$ such that $\mu * F = \delta + \psi$,*

(6) *there exists $F \in \mathscr{D}'_{(\omega)}(\mathbf{R}^N)$ satisfying $\mu * F = \delta + \varphi$ for some $\varphi \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$ and* $\operatorname{ch}\big(\operatorname{sing\,supp}_{(\omega)}(F)\big) = -\operatorname{ch}\big(\operatorname{sing\,supp}_{(\omega)}(\mu)\big)$.

(7) *there exists a fundamental solution E for S_μ satisfying*

$$\operatorname{ch}\big(\operatorname{sing\,supp}_{(\omega)}(E)\big) = -\operatorname{ch}\big(\operatorname{sing\,supp}_{(\omega)}(\mu)\big),$$

(8) *for each open set $\Omega \subset \mathbf{R}^N$ and $u \in \mathscr{D}'_{(\omega)}\big(\Omega - \operatorname{supp}(\mu) + \operatorname{ch}(\operatorname{sing\,supp}_{(\omega)}(\mu))\big)$ satisfying $\mu * u \in \mathscr{E}_{(\omega)}\big(\Omega + \operatorname{ch}(\operatorname{sing\,supp}_{(\omega)}(\mu))\big)$, it follows that $u \in \mathscr{E}_{(\omega)}(\Omega)$.*

2.2. Definition. (a) If $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ satisfies condition 2.1(1), then μ and also S_μ are called (ω)-*hypoelliptic.*

(b) $F \in \mathscr{D}'_{(\omega)}(\mathbf{R}^N)$ is called a *parametrix* for $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ if $\mu * F = \delta + \varphi$ for some $\varphi \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$.

The proof of Theorem 2.1 is prepared by several intermediate results and is given at the end of the section.

2.3. Proposition. *If $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ satisfies* $\operatorname{Ker} S_\mu \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$, *then*

$$\lim_{z\in V(\hat\mu),\,|z|\to\infty} \frac{|\operatorname{Im} z|}{\omega(z)} = \infty.$$

Proof. Proceeding by contradiction we assume there is a sequence $(\zeta_j)_{j\in\mathbf{N}} \subset V(\hat\mu)$ such that $\lim_{j\to\infty}|\zeta_j| = \infty$ and $|\operatorname{Im}\zeta_j| \leq M\omega(\zeta_j)$ holds for every $j \in \mathbf{N}$

and some $M > 0$. If $a = (a_j)_{j\in\mathbf{N}} \in l_1$ and $\varphi \in \mathscr{D}_{(\omega)}(\mathbf{R}^N)$, by the Paley–Wiener–Komatsu theorem [5, 3.5] there is $s \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ there is $C_n > 0$ with

$$|\hat{\varphi}(z)| \le C_n \exp\bigl(s|\operatorname{Im} z| - n\omega(z)\bigr)$$

for all $z \in \mathbf{C}^N$. If $n > sM + 1$ we have, for each $k \in \mathbf{N}$,

$$\sum_{j=1}^{k} |a_j|\,|\langle \mathrm{e}^{i\langle x,\zeta_j\rangle}, \varphi\rangle| = \sum_{j=1}^{k} |a_j|\,|\hat{\varphi}(-\zeta_j)| \le C_n \sum_{j=1}^{k} |a_j|.$$

This implies that the map $\eta\colon l_1 \longrightarrow \bigl(\mathscr{D}'_{(\omega)}(\mathbf{R}^N), \sigma\bigl(\mathscr{D}'_{(\omega)}(\mathbf{R}^N), \mathscr{D}_{(\omega)}(\mathbf{R}^N)\bigr)\bigr)$ defined by $\eta(a) := \sum_{j=1}^{\infty} a_j \exp(i\langle x, \zeta_j\rangle)$ is well-defined, linear and continuous. By the closed graph theorem [17, 24.31], $\eta\colon l_1 \longrightarrow \mathscr{D}'_{(\omega)}(\mathbf{R}^N)$ is continuous and the range of η is contained in $\operatorname{Ker} S_\mu$, since $\zeta_j \in V(\hat{\mu})$ for each $j \in \mathbf{N}$. By assumption $\eta(l_1) \subset \mathscr{E}_{(\omega)}(\mathbf{R}^N)$ and we can apply again the closed graph theorem to conclude that $\eta\colon l_1 \longrightarrow \mathscr{C}^\infty(\mathbf{R}^N)$ is also continuous. This yields $C > 0$ such that, for all $a = (a_j)_{j\in\mathbf{N}}$ in l_1 and $x \in \mathbf{R}^N$ with $|x| \le 1$, we have

$$\sum_{|\alpha|=1} \left| \left(\sum_{j=1}^{\infty} a_j \exp(i\langle x, \zeta_j\rangle) \right)^{\alpha} \right| \le C \sum_{j=1}^{\infty} |a_j|.$$

If $\zeta_j = (\zeta_j^1, \ldots, \zeta_j^N)$ for each $j \in \mathbf{N}$, we get $|\zeta_j^k| \le C$ for each $j \in \mathbf{N}$ and $K = 1, \ldots, N$. This implies that $(\zeta_j)_{j\in\mathbf{N}}$ is bounded, in contradiction with the choice of $(\zeta_j)_{j\in\mathbf{N}}$. □

We need the following lemma which can be seen in Sampson and Zielezny [20, p. 141].

2.4. Lemma. *Let $(\xi_j)_{j\in\mathbf{N}}$ be a sequence in $\mathbf{R}^N$ satisfying $|\xi_j| > 2|\xi_{j-1}| > 2^j$ for every $j \in \mathbf{N}$ and let $(a_j)_{j\in\mathbf{N}}$ be a bounded sequence in $\mathbf{C}$. Then the series $\sum_{j=1}^{\infty} a_j \exp(i\langle\,\cdot\,, \xi_j\rangle)$ converges in $\mathscr{D}'(\mathbf{R}^N)$ to some $\eta \in \mathscr{D}'(\mathbf{R}^N)$. If $\eta \in \mathscr{C}^3$, then $\lim_{j\to\infty} |a_j| = 0$.*

2.5. Proposition. *If $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ is (ω)-hypoelliptic, then there exist $A, R > 0$ such that* (2.1) *holds, i.e.,*

$$|\hat{\mu}(x)| \ge \exp\bigl(-A\omega(x)\bigr) \qquad \textit{for each } x \in \mathbf{R}^N,\ |x| \ge R.$$

Proof. Arguing by contradiction we construct a sequence $(\xi_j)_{j\in\mathbf{N}} \subset \mathbf{R}^N$ such that $|\xi_j| > 2|\xi_{j-1}| > 2^j$ for every $j \ge 2$ and $|\hat{\mu}(\xi_j)| < \exp\bigl(-j\omega(\xi_j)\bigr)$, for each $j \in \mathbf{N}$. By Lemma 2.4, $\eta := \sum_{j=1}^{\infty} \exp(i\langle\,\cdot\,, \xi_j\rangle)$ belongs to $\mathscr{D}'(\mathbf{R}^N)$ and hence to

$\mathscr{D}'_{(\omega)}(\mathbf{R}^N)$ but it is not a $\mathscr{C}^\infty$ function on $\mathbf{R}^N$. We show that $\mu * \eta \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$ to reach a contradiction. First observe that, for $\varphi \in \mathscr{D}_{(\omega)}(\mathbf{R}^N)$, we have

$$\langle \mu * \eta, \varphi \rangle = \langle \eta, \check{\mu} * \varphi \rangle = \sum_{j=1}^{\infty} \hat{\mu}(\xi_j)\hat{\varphi}(-\xi_j) = \sum_{j=1}^{\infty} \hat{\mu}(\xi_j)\big\langle \exp(i\langle\,\cdot\,,\xi_j\rangle), \varphi \big\rangle.$$

Next note that the definition of φ^* implies for $j, m \in \mathbf{N}$, $j > m$ large enough, $\alpha \in \mathbf{N}_0$:

$$\begin{aligned}\big|\big(\hat{\mu}(\xi_j)\exp(i\langle\,\cdot\,,\xi_j\rangle)\big)^{(\alpha)}\big| \exp\Big(-m\varphi^*\Big(\frac{|\alpha|}{m}\Big)\Big) &= |\hat{\mu}(\xi_j)|\,|\xi_j|^{|\alpha|}\exp\Big(-m\varphi^*\Big(\frac{|\alpha|}{m}\Big)\Big)\\ &\le \exp\Big(-j\omega(|\xi_j|) + |\alpha|\log|\xi_j| - m\varphi^*\Big(\frac{|\alpha|}{m}\Big)\Big)\\ &\le \exp(-j+m)\omega(\xi_j) \le \exp(-j+m).\end{aligned}$$

From this it follows easily that $\mu * \eta \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$. □

2.6. Proposition. *If $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ satisfies the following two conditions*:
(a) $\lim_{z\in V(\hat{\mu}),\,|z|\to\infty}\big(|\operatorname{Im} z|/\omega(z)\big) = \infty$,
(b) *$\hat{\mu}$ is slowly-decreasing for (ω).*
Then there exist $A, R > 0$ such that (2.1) *holds.*

Proof. Since ω satisfies condition (α) and (γ) we can find $k > 0$ such that $\omega(x) \le k\omega(\frac{1}{2}x)$ if $|x| > k$. Because of (b), there exists $B > 0$ such that for all $x \in \mathbf{R}^N$ there is $y \in \mathbf{R}^N$ with $|x-y| < B\omega(x)$ and $|\hat{\mu}(y)| > B^{-1}\exp\big(-B\omega(x)\big)$. Since $\omega(t) = o(t)$ as t goes to ∞, we can enlarge k to get also $\omega(x) < |x|/2B$ if $|x| > k$.

By condition (a), for each $m \in \mathbf{N}$ there exists $C_m > 0$ such that

$$|\operatorname{Im}\zeta| \ge (m+1)\omega(|\zeta|) \quad \text{if} \quad \hat{\mu}(\zeta) = 0 \quad \text{and} \quad |\zeta| \ge C_m.$$

We select $m_0 \in \mathbf{N}$, $m_0 > kB$. For $x \in \mathbf{R}^N$ with $|x| > 2\max(B, C_{m_0}, k)$ we define $g(\lambda) := \hat{\mu}\big(x + \lambda(y-x)\big)$, $\lambda \in \mathbf{C}$, where $y \in \mathbf{R}^N$ is the element associated with x in condition (b) (see above). Clearly $g \in \mathscr{H}(\mathbf{C})$ and, if $|\lambda| \le \frac{1}{2}$, we have

$$|x + \lambda(y-x)| \ge |x| - (\tfrac{1}{2}B)\omega(x) \ge \tfrac{1}{2}|x| > C_{m_0}$$

and

$$\begin{aligned}\big|\operatorname{Im}\big(x + \lambda(y-x)\big)\big| &= |\operatorname{Im}\lambda|\,|y-x| \le \tfrac{1}{2}B\omega(x) \le \tfrac{1}{2}kB\omega(\tfrac{1}{2}x)\\ &\le m_0\omega\big(x+\lambda(y-x)\big),\end{aligned}$$

which implies $g(\lambda) \ne 0$ for each $\lambda \in \mathbf{C}$ with $|\lambda| \le \frac{1}{2}$.

Now we apply the minimum-modulus theorem of Chou [7, II.2.1] with $R = 1$, $r = \frac{1}{3}$, $\lambda_0 = 1$, $\eta < \frac{1}{32}$, $H = 2 + \log(3e/2\eta)$, to get

$$|g(0)| \geq \frac{|g(1)|^{3(H+1)}}{(\sup_{|\lambda| \leq 3\mathrm{e}} |g(\lambda)|^{3H})(\sup_{|\lambda| \leq 3/2} |g(\lambda)|^2)}.$$

If $|\lambda| \leq 3\mathrm{e}$, we have $|x+\lambda(y-x)| \leq |x| + 3\mathrm{e}B\omega(x) \leq 4e|x|$ and $\left|\mathrm{Im}\big(x+\lambda(y-x)\big)\right| \leq 3eB\omega(x)$. By the theorem of Paley–Wiener–Schwartz–Komatsu [5, 7.2], there is $D > 0$ such that for all $|\lambda| \leq 3\mathrm{e}$ we have $\left|\hat{\mu}\big(x + \lambda(x - y)\big)\right| \leq D \exp\big(D\omega(x)\big)$ for all $|\lambda| \leq 3e$. This yields

$$|\hat{\mu}(x)| \geq \frac{\exp\big(-A(3H+3)\omega(x)\big)}{\big(D\exp\big(D\omega(x)\big)\big)^{3H+2}},$$

and the proof is complete. □

For the convenience of the reader we recall the following result from Hörmander [9, Lemma 1].

2.7. Lemma. *Given positive constants S, T and $\varepsilon > 0$ we can find $L \in \mathbf{N}$ such that for all $0 < s < S$ and $0 < r < R/L$ if u is harmonic when $x^2 + y^2 < R^2$ and satisfies the inequalities*

(i) $u(x, 0) \leq 0$,

(ii) $u(x, y) \geq -S|y| - \mathrm{Tr}$ *if* $x^2 + y^2 < R^2$,

it follows that

(iii) $u(x, y) \leq s|y| + \varepsilon r$ *if* $x^2 + y^2 < r^2$.

2.8. Proposition. *Let $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ satisfy the hypothesis of 2.6. There exists $D > 0$ such that for each $m \in \mathbf{N}$ and each convex compact neighbourhood K of $\mathrm{ch}\big(\mathrm{supp}(\mu)\big)$ there exists $R_m > 0$ such that*

$$(*) \qquad \frac{1}{|\hat{\mu}(z)|} \leq \exp\big(H_K(-\operatorname{Im} z) + D\omega(\operatorname{Re} z)\big)$$

for $z \in \mathbf{C}^N$, $|z| \geq R_m$ and $|\operatorname{Im} z| \leq m\omega(\operatorname{Re} z)$.

Proof. We first observe that, if $z \in \mathbf{C}^N$ satisfies $|\operatorname{Im} z| < m\omega(z)$ for some $m \in \mathbf{N}$, then $|\operatorname{Re} z|/|z|$ goes to 1 as $|z|$ goes to ∞. This follows from the estimate

$$1 = \frac{|z|}{|z|} \leq \frac{|\operatorname{Re} z|}{|z|} + \frac{|\operatorname{Im} z|}{|z|} < \frac{|\operatorname{Re} z|}{|z|} + m\frac{\omega(z)}{|z|},$$

since $\omega(t) = o(t)$ as $t \to \infty$.

Next fix a compact convex neighbourhood K of $\operatorname{supp}(\mu)$ and $m \in \mathbf{N}$ and consider $z = x + iy \in \mathbf{C}^N$ with $0 < |y| < m\omega(z)$. Define the entire function

$$g_z(\lambda) := \hat{\mu}(x + \lambda y/|y|), \qquad \lambda \in \mathbf{C}.$$

We study g_z for $|\lambda| < M\omega(x)$ for some $M \in \mathbf{N}$, depending on m but not on z, to be selected later.

By condition (a) in Proposition 2.6, for each $s \in \mathbf{N}$ there exists $\widetilde{D}_s$ such that

$$\hat{\mu}(z) \neq 0 \qquad \text{if } |z| \geq \widetilde{D}_s,\ |\operatorname{Im} z| \leq s\omega(z).$$

We claim there is $D_M > 0$ such that if $z = x + iy \in \mathbf{C}^N$ satisfies $|z| > D_M$, $0 < |y| < m\omega(z)$ and $|\lambda| < M\omega(x)$, then $g_z(\lambda) \neq 0$ and

$$|g_z(t)| \geq \exp\bigl(-A\omega(x + ty/|y|)\bigr)$$

for all $t \in \mathbf{R}$, $|t| < M\omega(x)$, where A is the constant in (2.1) (which holds by Proposition 2.6).

To prove the claim, recall there is $k \in \mathbf{N}$ such that $\omega(x) \leq k\omega(\frac{1}{2}x)$ if $|x| \geq k$ and, given M, there is $K_M > 0$ such that $t - M\omega(t) > \frac{1}{2}t$ if $t > K_M$. For $|x| > \max(K_M, 2\widetilde{D}_{kM}, 2k)$, we have

$$\bigl|x + \lambda y/|y|\bigr| \geq |x| - M\omega(x) \geq \tfrac{1}{2}|x| \qquad (\text{and} \geq k).$$

In particular $|x| \leq 2\bigl|x + \lambda y/|y|\bigr|$, hence

$$\begin{aligned}\bigl|\operatorname{Im}(x + \lambda y/|y|)\bigr| = |\operatorname{Im}\lambda| < M\omega(x) &\leq M\omega\bigl(2\bigl|x + \lambda y/|y|\bigr|\bigr)\\ &\leq Mk\omega\bigl(\bigl|x + \lambda y/|y|\bigr|\bigr).\end{aligned}$$

Thus $g_z(\lambda) = \hat{\mu}(x + \lambda y/|y|) \neq 0$. On the other hand, if $|x| > \max(2R, K_M)$ (R as in (2.1)) and $t \in \mathbf{R}$, $|t| < M\omega(x)$, we get

$$\bigl|x + ty/|y|\bigr| \geq |x| - |t| \geq |x| - M\omega(x) \geq \tfrac{1}{2}|x| > R,$$

hence

$$|g_z(t)| = \bigl|\hat{\mu}(x + ty/|y|)\bigr| \geq \exp\bigl(-A\omega(x + ty/|y|)\bigr).$$

By the very first observation of the proof, there is $D_M > 0$ such that

$$|z| \geq D_M, \qquad z = x + iy \in \mathbf{C}^N,\ 0 < |y| < m\omega(z)$$

implies

$$|x| \geq \max(K_M, 2\widetilde{D}_{kM}, 2k, 2R).$$

This completes the proof of the claim.

We write $H_K(y/|y|) = \alpha + \beta$, $H_K(-y/|y|) = \alpha - \beta$. Then $H_K(\operatorname{Im} y\lambda/|y|) = \alpha|\lambda| + \beta\lambda$. By the theorem of Paley–Wiener–Schwartz–Komatsu [5, 7.2], there is $b > 0$ such that

$$|\hat{\mu}(z)| \leq \exp\bigl(H_K(\operatorname{Im} z) + b\omega(z)\bigr), \qquad \text{if } |z| \geq b.$$

Now, if $z = x + iy \in \mathbf{C}^N$ satisfies $|z| > \max(b, K_M, k)$ and $|\lambda| < M\omega(x)$, then

$$|g_z(\lambda)| \leq \exp\bigl(\alpha|\operatorname{Im}\lambda| + \beta\operatorname{Im}\lambda + b\omega(x + \lambda y/|y|)\bigr).$$

Since $\bigl|x + \lambda y/|y|\bigr| \leq |x| + |\lambda| \leq |x| + M\omega(x) \leq 2|x|$, we get

$$|g_z(\lambda)| \leq \exp\bigl(\alpha|\operatorname{Im}\lambda| + \beta\operatorname{Im}\lambda + bk\omega(x)\bigr).$$

We define

$$u_z(\lambda) := \log\left(\frac{\mathrm{e}^{\beta\operatorname{Im}\lambda}\mathrm{e}^{-Ak\omega(x)}}{|g_z(\lambda)|}\right) = \beta\operatorname{Im}\lambda - Ak\omega(x) - \log|g_z(\lambda)|,$$

which is harmonic if $|z| > R_m := \max(D_M, b, K_M, k)$ and $|\lambda| < M\omega(x)$, since $g_z(\lambda)$ does not vanish. For $t \in \mathbf{R}$, $|t| < M\omega(x)$, we get

$$\omega(x + ty/|y|) \leq \omega(2|x|) \leq k\omega(x),$$

hence

$$\exp\bigl(-A\bigl(\omega(x + ty/|y|)\bigr)\bigr) \geq \exp\bigl(-Ak\omega(x)\bigr).$$

By our claim above, if $t \in \mathbf{R}$, $|t| < M\omega(x)$ and $\lambda \in \mathbf{C}$, $|\lambda| < M\omega(x)$ we have

$$u_z(t) \leq -Ak\omega(x) - \bigl(-Ak\omega(x)\bigr) = 0,$$

and

$$u_z(\lambda) \geq \beta\operatorname{Im}\lambda - Ak\omega(x) - \alpha|\operatorname{Im}\lambda| - \beta\operatorname{Im}\lambda - bk\omega(x) = -\alpha|\operatorname{Im}\lambda| - (Ak + bk)\omega(x).$$

Now let

$$\varepsilon := \frac{1}{2(m+1)}, \qquad T := \frac{(Ak + bk)}{(m+1)}, \qquad S := \sup_{|\eta|=1}\bigl(\tfrac{1}{2}\bigl(\bigl(H_K(\eta) + H_K(-\eta)\bigr)\bigr)\bigr)$$

and choose $L \in \mathbf{N}$ according to Lemma 2.7. Observe that the constants depend on m but not on z. Next let $M := L(m+1)+1$, $R := M\omega(x)$ and $r = (m+1)\omega(x)$. Then we conclude from Lemma 2.7. that

$$|u_z(\lambda)| \leq \alpha|\operatorname{Im}\lambda| + \varepsilon(m+1)\omega(x), \qquad \text{if } |\lambda| \leq r.$$

Setting $\lambda = i|y|$, we get

$$u_z(i|y|) = \beta|y| - Ak\omega(x) - \log|\hat{\mu}(z)| \leq \alpha|y| + \omega(x).$$

This yields

$$-\log|\hat{\mu}(z)| \leq (\alpha - \beta)|y| + (Ak+1)\omega(x).$$

Thus $1/|\hat{\mu}(z)| \leq \exp\bigl(H_K(-y) + (Ak+1)\omega(x)\bigr)$ if $z = x + iy \in \mathbf{C}^N$, $|z| \geq R_m$ and $|y| \leq m\omega(x)$. □

2.9. Lemma. *If $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ satisfies the hypotheses of Proposition* 2.6 *and $\hat{\mu}(x)$ does not vanish if $|x| \ge R_0$, then*

$$\langle F, \varphi\rangle := \Big(\frac{1}{2\pi}\Big)^N \int_{|x|\ge R_0} \frac{\hat{\varphi}(-x)}{\hat{\mu}(x)}\,dx, \qquad \varphi \in \mathscr{D}_{(\omega)}(\mathbf{R}^N)$$

defines an element $F \in \mathscr{D}'_{(\omega)}(\mathbf{R}^N)$ satisfying $\operatorname{sing\,supp}_{(\omega)}(F) \subset -\operatorname{ch}\big(\operatorname{supp}(\mu)\big)$ *and $\mu * F = \delta + G$ for some G which is real analytic on $\mathbf{R}^N$.*

Proof. By the theorem of Paley–Wiener–Schwartz–Komatsu [5, 7.2] and (2.1) (which follows from Proposition 2.6), it is easy to see that F defines an element of $\mathscr{D}'_{(\omega)}(\mathbf{R}^N)$.

Moreover, for $\varphi \in \mathscr{D}_{(\omega)}(\mathbf{R}^N)$, we have

$$\begin{aligned}\langle \mu * F, \varphi\rangle &:= \Big(\frac{1}{2\pi}\Big)^N \int_{|x|\ge R_0} \frac{\hat{\varphi}(-x)\hat{\mu}(x)}{\hat{\mu}(x)}\,dx = \int_{|x|\ge R_0} \hat{\varphi}(x)\,dx\\ &= \varphi(0) - \Big(\frac{1}{2\pi}\Big)^N \int_{|x|\le R_0} \hat{\varphi}(x)\,dx = \varphi(0) + \int_{\mathbf{R}^N} \varphi(t)G(t)\,dt,\end{aligned}$$

where $G(t) = -(2\pi)^{-N} \int_{|x|\le R_0} \mathrm{e}^{i\langle x,t\rangle}\,dx$ defines a real analytic function on $\mathbf{R}^N$. Thus $\mu * F = \delta + G$ as desired.

The main step in the proof is to show that $\operatorname{sing\,supp}_{(\omega)}(F) \subset -\operatorname{ch}\big(\operatorname{supp}(\mu)\big)$. To see this we fix $x_0 \notin -\operatorname{ch}\big(\operatorname{supp}(\mu)\big)$. We select a compact convex neighbourhood K of $-\operatorname{ch}\big(\operatorname{supp}(\mu)\big)$ such that $x_0 \notin K$. By Proposition 2.8, there exists $D > 0$ such that for each $m \in \mathbf{N}$ there exists $R_m \ge R_0$ such that

$$\frac{1}{|\hat{\mu}(z)|} \le \exp\big(H_K(\operatorname{Im} z) + D\omega(\operatorname{Re} z)\big), \quad \text{if } z \in \mathbf{C}^N,\ |z| \ge R_m,\ |\operatorname{Im} z| \le m\omega(\operatorname{Re} z).$$

Since $x_0 \notin K$, there are $\eta \in \mathbf{R}^N$, $|\eta| = 1$, $\sigma > 0$ and a compact convex ball V around x_0 of radius $\delta > 0$ such that $H_K(\eta) + H_V(-\eta) \le -\sigma$. Applying an orthogonal and real change of variables, we may assume $\eta = (1, 0, \dots, 0)$. We choose $\varphi \in \mathscr{D}_{(\omega)}(V')$ such that φ is identically 1 near x_0, with $V' = B_{\delta/2}(x_0)$. If we prove that for each $k \in \mathbf{N}$ there exists $C_k > 0$ such that

$$(*) \qquad |\widehat{\varphi F}(\xi)| \le C_k \exp\big(-k\omega(\xi)\big) \qquad \text{for all } \xi \in \mathbf{R}^N,$$

it follows that $\varphi F \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$ and F is of class $\mathscr{E}_{(\omega)}$ near x_0 as desired.

To prove $(*)$, we first observe that, for $\xi \in \mathbf{R}^N$,

$$\widehat{\varphi F}(\xi) = (2\pi)^{-N} \int_{|x|\ge R_0} \frac{\hat{\varphi}(\xi - x)}{\hat{\mu}(x)}\,dx.$$

By 1.2(d) there is $\beta \in \mathbf{N}$ (depending only on ω) such that $\omega(\xi - x) \geq \beta^{-1}\omega(\xi) - \omega(x) - 1$ for all $x, \xi \in \mathbf{R}^N$. We may assume that ω is $\mathscr{C}^1$ on $\mathbf{R}$ and has a bounded derivative (see 1.2(c)). Now we fix $k \in \mathbf{N}$ and select $t \in \mathbf{N}$ such that $\sigma t > D + \beta k$. We denote by $\|x\| := \max(|x_i|,\ i = 1, \ldots, N)$ the sup-norm on $\mathbf{R}^N$. We have

$$(2\pi)^N |\widehat{\varphi F}(\xi)| \leq \left| \int_{\{x \mid |x| \geq R_0,\, \|x\| \leq R_t\}} \frac{\hat{\varphi}(\xi - x)}{\hat{\mu}(x)}\, dx \right| + \left| \int_{\|x\| \geq R_t} \frac{\hat{\varphi}(\xi - x)}{\hat{\mu}(x)}\, dx \right|$$
$$=: A(\xi) + B(\xi).$$

Since $\varphi \in \mathscr{D}_{(\omega)}(\mathbf{R}^N)$ and V is a compact convex neighbourhood of $\operatorname{supp}(\varphi)$, for each $m \in \mathbf{N}$ there is $L_m > 0$ such that

$$|\hat{\varphi}(z)| \leq L_m \exp\bigl(H_V(\operatorname{Im} z) - m\omega(\operatorname{Re} z)\bigr)$$

for each $z \in \mathbf{C}^N$.

Consequently, there is $C_k > 0$ with

$$A(\xi) \leq C_k \exp\bigl(-k\omega(\xi)\bigr) \int_{\{x \mid |x| \geq R_0,\, \|x\| \leq R_t\}} \frac{\exp\bigl(\beta k \omega(x)\bigr)}{|\hat{\mu}(x)|}\, dx \leq C_k' \exp\bigl(-k\omega(\xi)\bigr);$$

since the integral is finite.

On the other hand, writing $x = (x_1, x') \in \mathbf{R}^N$,

$$B(\xi) \leq \left| \int_{\|x'\| \geq R_t} \int_{\mathbf{R}} \frac{\hat{\varphi}(\xi - x)}{\hat{\mu}(x)}\, dx \right| + \left| \int_{\|x'\| \leq R_t} \int_{|x_1| \geq R_t} \frac{\hat{\varphi}(\xi - x)}{\hat{\mu}(x)}\, dx \right| =: I_1(\xi) + I_2(\xi).$$

We treat the two integrals separately. In $I_1(\xi)$ we change the path of integration in the second integral by Cauchy's theorem. We define $z = \gamma(x_1) = x_1 + it\omega(x_1, x')$, $x_1 \in \mathbf{R}$, $x' \in \mathbf{R}^{N-1}$. Then, if $g_t(x_1)$ is the bounded derivative of γ, we have

$$\begin{aligned}
I_1(\xi) &= \left| \int_{\|x'\| \geq R_t} \int_{\mathbf{R}} \frac{\hat{\varphi}\bigl((-\gamma(x_1), -x') + \xi\bigr)}{\hat{\mu}\bigl(\gamma(x_1), x'\bigr)} g_t(x_1)\, dx \right| \\
&\leq \int_{\|x'\| \geq R_t} \int_{\mathbf{R}} \frac{\bigl|\hat{\varphi}\bigl((-\gamma(x_1), -x') + \xi\bigr)\bigr|}{\bigl|\hat{\mu}\bigl(\gamma(x_1), x'\bigr)\bigr|} |g_t(x_1)|\, dx \\
&\leq CL_{\beta k} \int_{\|x'\| \geq R_t} \int_{\mathbf{R}} \exp\bigl(H_K\bigl(t\omega(x_1, x')\eta\bigr) + D\omega(x_1, x') \\
&\quad + H_V\bigl(-t\omega(x_1, x')\eta\bigr) - k\beta\omega(-x + \xi)\bigr)\, dx_1\, dx' \\
&\leq L_k' \int_{\mathbf{R}^{N-1}} \int_{\mathbf{R}} \exp\bigl(\bigl(\bigl(H_K(\eta) + H_V(-\eta)\bigr)t + D + k\beta\bigr)\omega(x) - k\omega(\xi)\bigr)\, dx_1\, dx' \\
&\leq L_k' \exp\bigl(-k\omega(\xi)\bigr) \int_{\mathbf{R}^N} \exp\bigl(-\varepsilon\omega(x)\bigr)\, dx
\end{aligned}$$

for some $\varepsilon > 0$, and the integral is finite.

Since the integral $I_2(\xi)$ can be estimated in the same way, the proof is complete. □

2.10. Remark. Note that arguments similar to the ones given in the proof of 2.9 permit to show the following: For $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ and a compact convex set K in $\mathbf{R}^N$, the inclusion $\operatorname{sing\,supp}_{(\omega)}(\mu) \subset K$ is equivalent to the existence of $b > 0$ such that for each $m \in \mathbf{N}$ there exists $C_m > 0$ such that

$$|\hat{\mu}(z)| \leq C_m \exp\bigl(H_K(\operatorname{Im} z) + b\omega(z)\bigr)$$

for all $z \in \mathbf{C}^N$ with $|\operatorname{Im} z| \leq m\omega(z)$ and $|z| \geq C_m$. Compare with [10, Volume I, 7.3.8].

2.11. Remark. Let $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ be (ω)-hypoelliptic and let F and G be parametrices for μ. Then $\operatorname{sing\,supp}_{(\omega)}(F) = \operatorname{sing\,supp}_{(\omega)}(G)$. Indeed, $\mu * F = \delta + f$, $\mu * G = \delta + g$ and $f, g \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$. Then $\mu * (F - G) = f - g \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$. Since μ is (ω)-hypoelliptic, it follows that $F - G \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$, which yields the conclusion.

As a consequence of Theorem 2.1, every parametrix F of $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ which is (ω)-hypoelliptic satisfies $\operatorname{ch}\bigl(\operatorname{sing\,supp}_{(\omega)}(F)\bigr) = -\operatorname{ch}\bigl(\operatorname{sing\,supp}_{(\omega)}(\mu)\bigr)$.

Proof of Theorem 2.1. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): The implication (1)$\Rightarrow$(2) is a direct consequence of Propositions 2.3 and 2.5. By the very definition of slowly decreasing for (ω), (3) follows from (2).

(3) $\Rightarrow$ (5): By Propositions 2.6 and 2.8 and Lemma 2.9, (3) implies the existence of $F \in \mathscr{D}'_{(\omega)}(\mathbf{R}^N)$ such that $\mu * F = \delta + G$, G is real analytic on $\mathbf{R}^N$ and $\operatorname{sing\,supp}_{(\omega)}(F) \subset -\operatorname{ch}\bigl(\operatorname{supp}(\mu)\bigr)$. Select $\psi \in \mathscr{D}_{(\omega)}(\mathbf{R}^N)$ which is identically 1 on a neighbourhood of $-\operatorname{ch}\bigl(\operatorname{supp}(\mu)\bigr)$ and set $F_1 = \psi F \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$. We have

$$\mu * F_1 = \delta + G - \mu * \bigl((1-\psi)F\bigr).$$

Clearly $\varphi = G - \mu * \bigl((1-\psi)F\bigr) \in \mathscr{D}_{(\omega)}(\mathbf{R}^N)$, and (3) implies (5).

(5) $\Rightarrow$ (1): Let $h \in \mathscr{D}'_{(\omega)}(\mathbf{R}^N)$ satisfy $\mu * h \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$ and choose $F \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ as in (5). Then $F * (\mu * h) = (\delta + \psi) * h = h + \psi * h$ implies $h = -\psi * h + F * (\mu * h) \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$. Hence (1) holds.

(1) $\Rightarrow$ (6): By Lemma 2.9 and the implications already proved, there is a parametrix F of μ with $\operatorname{sing\,supp}_{(\omega)}(F) \subset -\operatorname{ch}\bigl(\operatorname{supp}(\mu)\bigr)$. Next fix $x_0 \notin -\operatorname{ch}\bigl(\operatorname{sing\,supp}_{(\omega)}(\mu)\bigr)$ and choose $\chi \in \mathscr{D}_{(\omega)}(\mathbf{R}^N)$ identically 1 on a neighbourhood of $\operatorname{ch}\bigl(\operatorname{sing\,supp}_{(\omega)}\mu\bigr)$ and such that $-x_0 \notin \operatorname{ch}\bigl(\operatorname{supp}(\chi)\bigr)$. Then let $\nu := \chi\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$. Clearly $x_0 \notin -\operatorname{ch}(\operatorname{supp}\nu)$ and $\mu = \nu + \varphi$ for some $\varphi \in \mathscr{D}_{(\omega)}(\mathbf{R}^N)$. We note that ν is (ω)-hypoelliptic. Indeed, if $\nu * h \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$ for some $h \in \mathscr{D}'_{(\omega)}(\mathbf{R}^N)$, we have $\mu * h = \nu * h + \varphi * h \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$, hence $h \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$. By Lemma 2.9, there is a parametrix F_0 for ν such that $\operatorname{sing\,supp}_{(\omega)}(F_0) \subset -\operatorname{ch}\bigl(\operatorname{supp}(\nu)\bigr)$. In particular F_0 is of class $\mathscr{E}_{(\omega)}$ on a neighbourhood of x_0. It is

easy to see that F_0 is also a parametrix for μ. By Remark 2.11, $\operatorname{sing\,supp}_{(\omega)}(F_0) = \operatorname{sing\,supp}_{(\omega)}(F)$, which implies $x_0 \notin \operatorname{sing\,supp}_{(\omega)}(F)$.

(6) $\Rightarrow$ (7): The same argument used above to show (5)$\Rightarrow$(1), permits to conclude (1) from (6). By what is already proved, we get (3). In particular μ is slowly decreasing for (ω) and by [2, Corollary 2.9] there is a fundamental solution $E \in \mathscr{D}'_{(\omega)}(\mathbf{R}^N)$ with $\mu * E = \delta$. In particular E is a parametrix for μ, which is (ω)-hypoelliptic. By Remark 2.11, $\operatorname{ch}\big(\operatorname{sing\,supp}_{(\omega)}(E)\big) = \operatorname{ch}\big(\operatorname{sing\,supp}_{(\omega)}(F)\big) = -\operatorname{ch}(\operatorname{sing\,supp}_{(\omega)}\mu)$.

(7) $\Rightarrow$ (8): Let Ω be an open subset of $\mathbf{R}^N$ and let $u \in \mathscr{D}'_{(\omega)}(\Omega - \operatorname{supp}(\mu) + \operatorname{ch}\big(\operatorname{sing\,supp}_{(\omega)}\mu)\big)$ satisfy $\mu * u \in \mathscr{E}_{(\omega)}\big(\Omega + \operatorname{ch}\big(\operatorname{sing\,supp}_{(\omega)}(\mu)\big)\big)$. We fix $x_0 \in \Omega$ and we show that the restriction of u to $B_R(x_0)$ belongs to $\mathscr{E}_{(\omega)}\big(B_R(x_0)\big)$ for some $R > 0$. To do so, take $R > 0$ such that $B_{2R}(x_0) \subset \Omega$, choose $\varphi \in \mathscr{D}_{(\omega)}\big(\Omega - \operatorname{supp}(\mu) + \operatorname{ch}\big(\operatorname{sing\,supp}_{(\omega)}(\mu)\big)\big)$ which is identically 1 on a neighbourhood of $B_R(x_0) - \operatorname{supp}(\mu) + \operatorname{ch}(\operatorname{sing\,supp}_{(\omega)}\mu)$ and let $v := \varphi u \in \mathscr{E}'_{(\omega)}(\mathbf{R}^n)$. Since u and v coincide on a neighbourhood of $B_R(x_0) - \operatorname{supp}(\mu) + \operatorname{ch}(\operatorname{sing\,supp}_{(\omega)}\mu)$, we get that $\mu * u$ and $\mu * v$ coincide on $B_R(x_0) + \operatorname{ch}\big(\operatorname{sing\,supp}_{(\omega)}(\mu)\big)$, hence $\mu * v \in \mathscr{E}_{(\omega)}\big(B_R(x_0) + \operatorname{ch}\big(\operatorname{sing\,supp}_{(\omega)}(\mu)\big)\big)$. Therefore,

$$\begin{aligned}\operatorname{sing\,supp}_{(\omega)}(v) &= \operatorname{sing\,supp}_{(\omega)}(E * \mu * v) \subset \operatorname{sing\,supp}_{(\omega)}(E) + \operatorname{sing\,supp}_{(\omega)}(\mu * v)\\ &\subset -\operatorname{ch}\big(\operatorname{sing\,supp}_{(\omega)}(\mu)\big) + \big(\mathbf{R}^n \backslash \big(B_R(x_0) + \operatorname{ch}\big(\operatorname{sing\,supp}_{(\omega)}(\mu)\big)\big)\\ &\subset \mathbf{R}^n \backslash B_R(x_0).\end{aligned}$$

(8) $\Rightarrow$ (1): This is obvious.

(1) $\Rightarrow$ (4): First observe that (1) clearly implies $\operatorname{Ker} S_\mu \subset \mathscr{E}_{(\omega)}(\mathbf{R}^N)$ and, by what is already proved it also implies that μ is slowly decreasing for (ω).

(4) $\Rightarrow$ (1): Conversely, assume that (4) holds and take $\nu \in \mathscr{D}'_{(\omega)}(\mathbf{R}^N)$ such that $\nu * \mu = g \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$. Since μ is slowly decreasing for (ω), we can apply [2, Corollary 2.9] to find $f \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$ such that $\mu * f = g$. Accordingly $\nu - f \in \operatorname{Ker} S_\mu \subset \mathscr{E}_{(\omega)}(\mathbf{R}^N)$. This yields $\nu \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$ and (1) is satisfied. ▫

2.12. Remark. Theorem 2.1 should be compared with the characterization of the (ω)-hypoelliptic differential operators given by [1, 4.1.1] and the results stated in Chou [7, IV.11, Remarque 4]. In both articles classes of ultradifferentiable functions are used which are defined in a different way than those considered here. However, in many cases these different definitions decribed the same classes. Note that there are more equivalences and more precise statements in Theorem 2.1 than in Chou [7].

2.13. Proposition. *Let σ and ω be two weights such that $\omega(t) = o\big(\sigma(t)\big)$ as $t \to \infty$. Then the following assertions hold.*

(a) *There exists $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ which is (σ)-hypoelliptic but not (ω)-hypoelliptic.*

(b) *There exists $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R})$ which is (ω)-hypoelliptic but not (σ)-hypoelliptic.*

Proof. (a) By Braun [4, Theorem 7], there exist an elliptic ultradifferential operator $G(D)$ of class (σ) and $f \in \mathscr{E}_{\{\sigma\}}(\mathbf{R}^N)$ which is real analytic on $\mathbf{R}^N \backslash \{0\}$ such that $G(D)f = \delta$. We select $\varphi \in \mathscr{D}_{\{\sigma\}}(\mathbf{R}^N)$ identically 1 on a neighbourhood of $\{0\}$ and define $\mu := \varphi f \in \mathscr{D}_{\{\sigma\}}(\mathbf{R}^N) \subset \mathscr{D}_{(\omega)}(\mathbf{R}^N)$. Clearly μ is not (ω)-slowly decreasing, hence it is not (ω)-hypoelliptic. On the other hand $G(D)\mu = \delta + \chi$ with $\chi \in \mathscr{D}_{(\sigma)}(\mathbf{R}^N)$, and μ has a parametrix in $\mathscr{D}'_{(\sigma)}(\mathbf{R}^N)$, hence it is (σ)-hypoelliptic.

(b) By [5, 1.7], there is $m\colon [0,\infty[\to [0,\infty[$ such that $\omega(t) = o\big(m(t)\big)$ and $m(t) = o\big(\sigma(t)\big)$ as $t \to \infty$. Proceeding by recurrence, with $r_1 \geq 2$, select a sequence $(r_j)_{j\in\mathbf{N}}$ with $4r_j \leq r_{j+1}$, $j \in \mathbf{N}$, such that, for all $j \in \mathbf{N}$,

(i) $1 + \sigma(r_j) \geq m(r_j)$,

(ii) $j\big(\omega(r_j) + 1\big) \leq m(r_j)$,

(iii) $(j+1)^2 \leq \inf_{t\geq r_j}\big(\omega(t)/\log(t)\big)$.

By (iii), if $n(t) = \operatorname{card}\{j \in \mathbf{N} \mid r_j \leq t\}$, $t \geq 0$, we have $n(t)\log t = o\big(\omega(t)\big)$ as $t \to \infty$. For each $j \in \mathbf{N}$, select $z_j \in \mathbf{C}$ with $|z_j| = r_j$, $\operatorname{Im} z_j = m(r_j)$ and let $f(z) := \prod_{j\in\mathbf{N}}\big(1 - (z/z_j)\big)$, $z \in \mathbf{C}$. The function f is entire, its zeros are the z_j's and

$$\log|f(z)| \leq n(|z|)\log|z| + \log 2 + \tfrac{4}{9},$$

for each $z \in \mathbf{C}$ by Braun, Meise, Vogt [6, 3.11]. Hence there is $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R})$ with $\hat{\mu} = f$, $\operatorname{supp}(\mu) = \{o\}$ and T_μ is surjective on $\mathscr{E}_{(\omega)}(\mathbf{R})$, therefore on $\mathscr{E}_{(\sigma)}(\mathbf{R})$. By our construction and Theorem 2.1, μ is (ω)-hypoelliptic, but not (σ)-hypoelliptic, since the first part of condition (2) in Theorem 2.1 is not satisfied for (σ). □

2.14. Remark. Observe that the example $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ constructed in Proposition 2.11(a) satisfies $\operatorname{Ker} S_\mu \subset \mathscr{E}_{(\sigma)}(\mathbf{R}^N) \subset \mathscr{E}_{(\omega)}(\mathbf{R}^N)$ but it is not (ω)-slowly decreasing. Accordingly the condition $\operatorname{Ker} S_\mu \subset \mathscr{E}_{(\omega)}(\mathbf{R}^N)$ alone does not imply that μ is (ω)-hypoelliptic.

We refer the reader Ehrenpreis [8, pp. 574–579] for interesting examples on hypoelliptic convolution operators in the case of classical distributions.

3. $\{\omega\}$-hypoelliptic convolution operators

In this section we define and characterize when the convolution operator S_μ which is induced by $\mu \in \mathscr{E}'_{\{\omega\}}(\mathbf{R}^N)$ is $\{\omega\}$-hypoelliptic. The statements look very similar, but the proofs require new ingredients. Some of them, as for example Proposition 3.5 and Corollary 3.7, might be of independent interest. Again in this section ω denotes a fixed weight function.

3.1. Theorem. *For $\mu \in \mathscr{E}'_{\{\omega\}}(\mathbf{R}^N)$ the following assertions are equivalent:*

(1) *Whenever $\nu \in \mathscr{D}'_{\{\omega\}}(\mathbf{R}^N)$ and $\mu * \nu \in \mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$, then $\nu \in \mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$,*

(2) $\liminf_{z\in V(\hat{\mu}),\,|z|\to\infty}\big(|\operatorname{Im} z|/\omega(z)\big) > 0$ *and for each $m \in \mathbf{N}$ there is $R_m > 0$ such that $|\hat{\mu}(x)| \geq \exp\big(-\omega(x)/m\big)$ for each $x \in \mathbf{R}^N$, $|x| \geq R_m$,*

(3) $\liminf_{z\in V(\hat{\mu}),\,|z|\to\infty}\big(|\operatorname{Im} z|/\omega(z)\big) > 0$ *and* μ *is slowly decreasing for* $\{\omega\}$,
(4) $\operatorname{Ker} S_\mu \subset \mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$ *and* μ *is slowly decreasing for* $\{\omega\}$,
(5) $\operatorname{Ker} S_\mu \subset \mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$ *and* $T_\mu\colon \mathscr{E}_{\{\omega\}}(\mathbf{R}^N) \to \mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$ *is surjective,*
(6) *there exist* $F \in \mathscr{E}'_{\{\omega\}}(\mathbf{R}^N)$ *and* $\psi \in \mathscr{D}_{\{\omega\}}(\mathbf{R}^N)$ *such that* $\mu * F = \delta + \psi$,
(7) *there exists* $F \in \mathscr{D}'_{\{\omega\}}(\mathbf{R}^N)$ *satisfying* $\mu * F = \delta + \varphi$ *for some* $\varphi \in \mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$ *and the convex hull of* $\operatorname{sing\,supp}_{\{\omega\}}(F)$ *and* $-\operatorname{sing\,supp}_{\{\omega\}}(\mu)$ *coincide,*
(8) *there exists a fundamental solution* E *for* S_μ *satisfying*

$$\operatorname{ch}\big(\operatorname{sing\,supp}_{\{\omega\}}(E)\big) = -\operatorname{ch}\big(\operatorname{sing\,supp}_{\{\omega\}}(\mu)\big),$$

(9) *for each open set* $\Omega \subset \mathbf{R}^N$ *and* $u \in \mathscr{D}'_{\{\omega\}}\big(\Omega - \operatorname{supp}(\mu) + \operatorname{ch}\big(\operatorname{sing\,supp}_{\{\omega\}}(\mu)\big)\big)$ *satisfying* $\mu * u \in \mathscr{E}_{\{\omega\}}\big(\Omega + \operatorname{ch}\big(\operatorname{sing\,supp}_{\{\omega\}}(\mu)\big)\big)$, *it follows that* $u \in \mathscr{E}_{\{\omega\}}(\Omega)$.

3.2. Definition. (a) If $\mu \in \mathscr{E}'_{\{\omega\}}(\mathbf{R}^N)$ satisfies condition 3.1(1), then μ and also S_μ are called $\{\omega\}$-*hypoelliptic.*

(b) $F \in \mathscr{D}'_{\{\omega\}}(\mathbf{R}^N)$ is called a *parametrix* for $\mu \in \mathscr{E}'_{\{\omega\}}(\mathbf{R}^N)$ if $\mu * F = \delta + \varphi$ for some $\varphi \in \mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$.

3.3. Proposition. *If* $\mu \in \mathscr{E}'_{\{\omega\}}(\mathbf{R}^N)$ *satisfies* $\operatorname{Ker} S_\mu \subset \mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$, *then*

$$\lim_{z\in V(\hat{\mu})}\ \inf_{|z|\to\infty} \frac{|\operatorname{Im} z|}{\omega(z)} > 0.$$

Proof. Arguing by contradiction we assume that there exists a sequence $(z_j)_{j\in\mathbf{N}} \subset \mathbf{C}^N$, $|z_j| \to \infty$, $\hat{\mu}(z_j) = 0$ and $|\operatorname{Im} z_j| < \omega(z_j)/j$ for each $j \in \mathbf{N}$. If $a = (a_j)_{j\in\mathbf{N}} \in l_1$ and $\varphi \in \mathscr{D}_{\{\omega\}}(\mathbf{R}^N)$, we can apply the Paley–Wiener–Komatsu theorem [5, 3.5] to conclude

$$\sum_{j=1}^{k} |a_j|\,|\langle \mathrm{e}^{i\langle x, z_j\rangle}, \varphi\rangle| \le C \sum_{j=1}^{k} |a_j| \exp\Big(\Big(\frac{D}{j} - \varepsilon\Big)\omega(z_j)\Big),$$

for some $\varepsilon > 0$, $D, C > 0$ depending only on φ. Now an application of the closed graph theorem (see Meise and Vogt [17, 24.31]) and an obvious modification of the proof of 2.3 leads to a contradiction. ▫

3.4. Proposition. *Let* $\mu \in \mathscr{E}'_{\{\omega\}}(\mathbf{R}^N)$ *is* $\{\omega\}$-*hypoelliptic, then for each* $m \in \mathbf{N}$ *there is* $R_m > 0$ *such that* $|\hat{\mu}(x)| \ge \exp\big(-\omega(x)/m\big)$ *for each* $x \in \mathbf{R}^N$, $|x| \ge R_m$.

Proof. Proceeding by contradiction, we find $m \in \mathbf{N}$ and a sequence $(\xi_j)_{j\in\mathbf{N}}$ in $\mathbf{R}^N$ such that $|\xi_j| > 2|\xi_{j-1}| > 2^j$ for each $j \ge 2$ such that $|\hat{\mu}(\xi_j)| \le \exp\big(-\omega(\xi_j)/m\big)$ for each $j \in \mathbf{N}$. By Lemma 2.4, $\eta := \sum_{j=1}^\infty \exp(i\langle\,\cdot\,, \xi_j\rangle)$ belongs to $\mathscr{D}'_{\{\omega\}}(\mathbf{R}^N)$ but it is not a $\mathscr{C}^\infty$ function on $\mathbf{R}^N$. Proceeding similarly as in the proof of 2.5 we show $\mu * \eta \in \mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$. This is a contradiction. ▫

3.5. Proposition. *Let ω and σ_0 be two weight functions with $\sigma_0 = o(\omega)$. Let $S := \{\sigma : \sigma$ is a weight function, $\sigma_0 \le \sigma,\ \sigma = o(\omega)\}$. Then, for each open set Ω in $\mathbf{R}^N$, the following topological equality holds*

$$\mathscr{E}_{\{\omega\}}(\Omega) = \bigcap_{\sigma\in S} \mathscr{E}_{(\sigma)}(\Omega),$$

when the space on the right hand side is endowed with the corresponding projective limit topology.

Proof. To prove the equality as vector spaces, it suffices to show

$$\mathscr{D}_{\{\omega\}}\big(B_\delta(0)\big) = \bigcap_{\sigma\in S} \mathscr{D}_{(\sigma)}\big(B_\delta(0)\big),$$

for each $\delta > 0$, since the equality holds if it holds locally. By Meise, Taylor and Vogt [5, 3.9] the inclusion "$\subset$" is valid. To prove the converse inclusion, fix $f \in \bigcap_{\sigma\in S} \mathscr{D}_{(\sigma)}\big(B_\delta(0)\big)$ and choose $(\rho_\varepsilon)_{\varepsilon>0}$ in $\mathscr{D}_{\{\omega\}}(\mathbf{R}^N)$ such that $\rho_\varepsilon \to \delta$ in $\mathscr{D}'_{\{\omega\}}(\mathbf{R}^N)$ as $\varepsilon \downarrow 0$. Fix $\sigma \in S$ and note that $f * \rho_\varepsilon \to f$ in $\mathscr{E}_{(\sigma)}\big(B_\delta(0)\big)$ (see the proof of [5, 3.8]). Since this holds for all $\sigma \in S$, it follows from [15, 3.3] that $(f * \rho_\varepsilon)_{\varepsilon>0}$ is a Cauchy net in $\mathscr{E}_{\{\omega\}}\big(B_\delta(0)\big)$, which is a complete space by [5, 4.9]. Accordingly, there exists $g \in \mathscr{E}_{\{\omega\}}\big(B_\delta(0)\big)$ such that $f * \rho_\varepsilon \to g$ in $\mathscr{E}_{\{\omega\}}\big(B_\delta(0)\big)$. Since $f * \rho_\varepsilon \to f$ pointwise, it follows that $f = g \in \mathscr{D}_{\{\omega\}}\big(B_\delta(0)\big)$.

Note that the topological equality in the proposition follows from the algebraic one and [15, 3.3]. □

3.6. Remark. Under certain conditions on the sequence $(M_p)_{p\ge 0}$, see [5, 8.9] and 1.5(4), the spaces $\mathscr{E}^{\{M_p\}}(\Omega)$ coincide with $\mathscr{E}_{\{\omega_M\}}(\Omega)$ for a suitably defined weight function ω_M. In this situation the proposition above shows that in Chou [7, I.2.6] even the topological equality holds.

3.7. Corollary. *For each $u \in \mathscr{D}'_{\{\omega\}}(\mathbf{R}^N)$ we have*

$$\text{sing supp}_{\{\omega\}}(u) = \overline{\bigcup_{\sigma\in S} \text{sing supp}_{(\sigma)}(u)}.$$

Proof. Let $A := \text{sing supp}_{\{\omega\}}(u)$. Then $u|_{\mathbf{R}^N\setminus A} \in \mathscr{E}_{\{\omega\}}(\mathbf{R}^N\setminus A)$. By Proposition 3.5, $u|_{\mathbf{R}^N\setminus A} \in \mathscr{E}_{(\sigma)}(\mathbf{R}^N\setminus A)$ for each $\sigma \in S$. Hence $\text{sing supp}_{(\sigma)}(u) \subset A$ for each $\sigma \in S$. Since A is closed, this implies

$$B := \overline{\bigcup_{\sigma\in S} \text{sing supp}_{(\sigma)}(u)} \subset A.$$

To prove the converse implication, fix $x_0 \in \mathbf{R}^N\setminus B$. If no such x_0 exists, we have $B = \mathbf{R}^N$, hence $A = \mathbf{R}^N$. Choose a neighbourhood V of x_o such that $V \cap B = \emptyset$. Then $u|_V \in \mathscr{E}_{(\sigma)}(V)$ for every $\sigma \in S$ and hence, by Proposition 3.5, $u \in \mathscr{E}_{\{\omega\}}(V)$. This implies $x_0 \notin \text{sing supp}_{\{\omega\}}(u)$. Therefore $\mathbf{R}^N\setminus B \subset \mathbf{R}^N\setminus A$ and consequently $A \subset B$. □

3.8. Lemma. *If* $\mu \in \mathscr{E}'_{\{\omega\}}(\mathbf{R}^N)$ *satisfies the hypothesis of* 3.1(3), *then there is a weight function* σ_0 *with* $\sigma_0 = o(\omega)$ *such that for each weight function* σ *satisfying* $\sigma_0 \le \sigma$ *and* $\sigma = o(\omega)$, μ *belongs to* $\mathscr{E}'_{(\sigma)}(\mathbf{R}^N)$ *and* μ *is* (σ)*-hypoelliptic.*

Proof. By [5, 7.6] there is a weight function σ_1 with $\sigma_1 = o(\omega)$ such that $\mu \in \mathscr{E}'_{(\sigma_1)}(\mathbf{R}^N)$. By [2, 3.2], there is a weight $\sigma_0 \ge \sigma_1$ with $\sigma_0 = o(\omega)$ and such that μ is (σ_0)-slowly decreasing. By the very definition $\mu \in \mathscr{E}'_{(\sigma)}(\mathbf{R}^N)$ and μ is (σ)-slowly decreasing for every weight function σ such that $\sigma_0 \le \sigma$ and $\sigma = o(\omega)$. By Theorem 2.1 it remains to show that $\operatorname{Im} z/\sigma(z)$ tends to ∞ as $|z| \to \infty$, $z \in V(\hat{\mu})$, for every weight function σ. To see this observe that

$$\frac{\operatorname{Im} z}{\sigma(z)} = \frac{\operatorname{Im} z}{\omega(z)} \frac{\omega(z)}{\sigma(z)}, \qquad z \in \mathbf{C}^N$$

and apply $\sigma = o(\omega)$ and the first part of the assumption 3.1(3). □

3.9. Remark. If F and G are two parametrices of the $\{\omega\}$-hypoelliptic operator $\mu \in \mathscr{E}'_{\{\omega\}}(\mathbf{R}^N)$, then $\operatorname{sing\,supp}_{\{\omega\}}(F)$ and $\operatorname{sing\,supp}_{\{\omega\}}(G)$ coincide. The proof is the same as the one of 2.11.

Proof of Theorem 3.1. (1) ⇒ (2): This follows directly from Propositions 3.3 and 3.4.

(2) ⇒ (3): This holds obviously.

(3) ⇒ (6): By 3.8, there is $R_0 > 0$ such that $\hat{\mu}(x) \ne 0$ for every $x \in \mathbf{R}^N$ with $|x| \ge R_0$, since $\mu \in \mathscr{E}'_{(\sigma_0)}(\mathbf{R}^N)$ and is (σ_0)-hypoelliptic. We define, for $\varphi \in \mathscr{D}_{\{\omega\}}(\mathbf{R}^N)$,

$$\langle F_0, \varphi \rangle := \Big(\frac{1}{2\pi}\Big)^N \int_{|x| \ge R_0} \frac{\hat{\varphi}(-x)}{\hat{\mu}(x)}\, dx.$$

Since $\mu \in \mathscr{E}'_{(\sigma)}(\mathbf{R}^N)$ and is (σ)-hypoelliptic for all weight functions σ with $\sigma_0 \le \sigma$ and $\sigma = o(\omega)$, the same definition of F_0 for each $\varphi \in \mathscr{D}_{(\sigma)}(\mathbf{R}^N)$ yields, by Lemma 2.9, that $F_0 \in \mathscr{D}'_{(\sigma)}(\mathbf{R}^N) \subset \mathscr{D}'_{\{\omega\}}(\mathbf{R}^N)$, and F_0 is a parametrix for μ such that $\mu * F_0 = \delta + G$ for a real analytic function G on $\mathbf{R}^N$ and

$$\operatorname{sing\,supp}_{(\sigma)}(F_0) \subset -\operatorname{ch}\big(\operatorname{supp}(\mu)\big).$$

As $\operatorname{supp}(\mu)$ is compact, using Corollary 3.7 we obtain

$$\operatorname{sing\,supp}_{\{\omega\}}(F_0) \subset -\operatorname{ch}\big(\operatorname{supp}(\mu)\big).$$

We fix $\varphi \in \mathscr{D}_{\{\omega\}}(\mathbf{R}^N)$ identically 1 on a neighbourhood of $\operatorname{ch}\big(\operatorname{sing\,supp}_{\{\omega\}}(\mu)\big)$ and we set $F = \varphi F_0 \in \mathscr{E}'_{\{\omega\}}(\mathbf{R}^N)$. Since $(1-\varphi)F_0 \in \mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$, we have

$$\mu * F = \delta + G - \mu * \big((1-\varphi)F_0\big);$$

since $G-\mu*\big((1-\varphi)F_0\big)\in\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$, we reach the conclusion.

(6) $\Rightarrow$ (1): This follows with an argument similar to the one we used to check (5) $\Rightarrow$ (1) in the proof of Theorem 2.1.

(1) $\Rightarrow$ (7): Since (1) implies (2), we can proceed as in the proof of (3) $\Rightarrow$ (6) to get that, for some $R_0>0$,

$$\langle F,\varphi\rangle := \Big(\frac{1}{2\pi}\Big)^N\int_{|x|\geq R_0}\frac{\hat\varphi(-x)}{\hat\mu(x)}\,dx$$

defines $F\in\mathscr{D}'_{(\sigma)}(\mathbf{R}^N)$ with $\operatorname{sing\,supp}_{(\sigma)}(F)\subset-\operatorname{ch}\big(\operatorname{supp}(\mu)\big)$ for every weight function $\sigma\geq\sigma_0$ and $\sigma=o(\omega)$, and $F*\mu=\delta+g$, g real analytic in $\mathbf{R}^N$. By the proof of (1)$\Rightarrow$(6) in Theorem 2.1,

$$\operatorname{sing\,supp}_{(\sigma)}(F)\subset-\operatorname{ch}\big(\operatorname{sing\,supp}_{(\sigma)}(\mu)\big)$$

for every σ and we can apply Corollary 3.7 to conclude

$$\operatorname{sing\,supp}_{\{\omega\}}(F)\subset-\operatorname{ch}\big(\operatorname{sing\,supp}_{\{\omega\}}(\mu)\big).$$

If we apply (6), which follows from (1), we obtain $G\in\mathscr{E}'_{\{\omega\}}(\mathbf{R}^N)$ with $\mu*G=\delta+\psi$ and with $\psi\in\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$, that is, G is $\{\omega\}$-hypoelliptic and μ is a parametrix for G. Since all the parametrices have the same $\{\omega\}$-singular support,

$$\operatorname{sing\,supp}_{\{\omega\}}(\mu)\subset-\operatorname{ch}\big(\operatorname{sing\,supp}_{\{\omega\}}(F)\big).$$

(7) $\Rightarrow$ (8): Multiplying by a cut-off function on $\mathscr{D}_{\{\omega\}}(\mathbf{R}^N)$ which is identically 1 on a neighbourhood of $-\operatorname{ch}\big(\operatorname{sing\,supp}_{\{\omega\}}(\mu)\big)$, we conclude (6), which is already equivalent to (1) and (3). Consequently μ is $\{\omega\}$-slowly decreasing, hence there is a fundamental solution $E\in\mathscr{D}'_{\{\omega\}}(\mathbf{R}^N)$ of μ by [2, 3.4]. Since E is a parametrix for μ, we can apply 3.9 to conclude (8).

(8) $\Rightarrow$ (9): The proof is the same as the proof of (7) $\Rightarrow$ (8) in Theorem 2.1.

(9) $\Rightarrow$ (1) $\Rightarrow$ (4): The first implication is obvious, the second one follows from Proposition 3.4.

(4) $\Rightarrow$ (5): First observe that $\operatorname{Ker}S_\mu$ is a Fréchet space and $\operatorname{Ker}S_\mu=\operatorname{Ker}T_\mu$. Since μ is slowly decreasing for $\{\omega\}$, the operator T_μ is locally surjective by [2, 3.4]. Since $\operatorname{Ker}T_\mu$ is a Fréchet space, a Mittag-Leffler argument shows that T_μ is surjective on $\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$. Alternatively one can use the theory of the vanishing of the derived functor Proj^1 as it is developed in Palamodov [18] and Vogt [21]: since $\operatorname{Ker}S_\mu$ is a Fréchet space and $\operatorname{Ker}S_\mu=\operatorname{Ker}T_\mu$, we have $\operatorname{Proj}^1\operatorname{Ker}T_\mu=0$. Since $\operatorname{Proj}^1\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)=0$, this implies that T_μ is surjective on $\mathscr{E}_{\{\omega)}(\mathbf{R}^N)$.

(5) $\Rightarrow$ (1): We fix $\nu\in\mathscr{D}'_{\{\omega\}}(\mathbf{R}^N)$ such that $g=\nu*\mu\in\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$. Since T_μ is surjective, we find $f\in\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$ with $f*\mu=g$. This implies $f-\nu\in\operatorname{Ker}S_\mu\subset\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$, from where $\nu\in\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$ follows. ◻

3.10. Remark. Theorem 3.1 extends the results of Chou [7, IV.2-1] in case that the classes of ultradifferentiable functions considered in Chou coincide with those considered in the present paper. In this case our results are more precise and give more equivalences.

Note that the proof of (3) ⇒ (6) in Theorem 3.1 shows that the converse of Lemma 3.8 holds, too.

3.11. Corollary. *Let* $\mu \in \mathscr{E}'_{\{\omega\}}(\mathbf{R}^N)$ *be given. Then* μ *is* $\{\omega\}$*-hypoelliptic if and only if there is a weight function* σ_0 *with* $\sigma_0 = o(\omega)$ *such that for each weight function* σ *satisfying* $\sigma_0 \le \sigma$ *and* $\sigma = o(\omega)$, μ *belongs to* $\mathscr{E}'_{(\sigma)}(\mathbf{R}^N)$ *and* μ *is* (σ)*-hypoelliptic.*

3.12. Remark. Note that by [5, 8.8] the intersection of all spaces $\mathscr{E}_{(\omega)}(\mathbf{R}^N)$ is equal to the intersection of all spaces $\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$ and hence equal to the space of all real analytic functions on $\mathbf{R}^N$, by the theorem of Bang–Mandelbrojt (see Chou [7, I.2.2]). Thus, each $\mu \in \mathscr{E}'_{(\omega)}(\mathbf{R}^N)$ (resp. $\mu \in \mathscr{E}'_{\{\omega\}}(\mathbf{R}^N)$) which is (σ)-hypoelliptic (resp. $\{\sigma\}$-hypoelliptic) for every non-quasianalytic weight function σ with $\sigma \ge \omega$, is elliptic in the sense of [7, IV.3].

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