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A REGULARITY PROPERTY OF p-HARMONIC FUNCTIONS

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Abstract. The paper is concerned with the $\mathscr{A}$-harmonic equation

$$\operatorname{div}[\langle G(x)\nabla u, \nabla u\rangle^{(p-2)/2} G(x)\nabla u] = 0$$

where $1 < p < \infty$ and G is a positive definite matrix whose entries are in $L^\infty \cap \mathrm{VMO}$. We show that for every $r > 1$, very weak solutions of class $W^{1,r}_{\mathrm{loc}}$ actually belong to $W^{1,p}_{\mathrm{loc}}$ and are solutions in the distributional sense, provided p is sufficiently close to 2.

0. Introduction

We consider the familiar $\mathscr{A}$-harmonic equation

$$\operatorname{div} \mathscr{A}(x, \nabla u) = 0 \tag{0.1}$$

in an open set $\Omega \subset \mathbf{R}^n$, with $\mathscr{A} = \mathscr{A}(x,\xi)$ verifying usual conditions, in particular the growth condition $|\mathscr{A}(x,\xi)| \approx |\xi|^{p-1}$, with $p > 1$. A distributional solution to (0.1) is a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega \langle \mathscr{A}(x, \nabla u), \nabla \varphi\rangle \, dx = 0 \tag{0.2}$$

for all $\varphi \in C_0^\infty(\Omega)$. Hereafter, $\langle\ ,\ \rangle$ denotes the scalar product of vectors in $\mathbf{R}^n$. Of course, (0.2) extends to arbitrary $\varphi \in W^{1,p}(\Omega)$ with compact support. The p-integrability of ∇u is not required for (0.2) to be meaningful, but it is a *natural* assumption because it is used in studying regularity of solutions. Actually, properties of solutions are often deduced by a suitable choice of test functions in (0.2), typically $\varphi = \lambda u$, with λ a cut-off function. Here, we mention only the

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well-known higher integrability of ∇u proved first in [ME], see [G] and references therein. This result is achieved by the technique of reverse Hölder inequalities which are obtained by testing (0.2) with appropriate φ.

In the paper [IS], the notion of *very weak* solution is considered, relaxing the natural integrability assumption. A function $u \in W^{1,r}_{\mathrm{loc}}(\Omega)$, with $r \geq \max\{1, p-1\}$, is called a very weak solution to (0.1) if it satisfies (0.2), for all $\varphi \in C_0^\infty(\Omega)$.

It is interesting to know how far below the natural exponent can r be and still allow for a comprehensive theory of very weak solutions. The first result in this context was obtained in [IS]. They found exponents r_1, r_2 verifying $1 < r_1 < p < r_2$ and such that any solution $u \in W^{1,r_1}_{\mathrm{loc}}$ actually belongs to W^{1,r_2}_{loc}. Thus u is a solution in the distributional sense. Obviously, for very weak solutions (0.2) holds for all $\phi \in W^{1,r/(r-p+1)}(\Omega)$ and we no longer can use λu as a test function. In [IS] a test function for (0.2) is produced using the Hodge decomposition. We mention that a different method for constructing test functions is proposed in [L].

In [KZ], the authors considered a particular case of (0.1), namely the p-Laplacian

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0.$$

Essentially, they observed that the exponent r_1 in the result of [IS] may be taken arbitrarily close to 1, provided p is sufficiently close to 2. Here, we obtain this kind of result for the more general equation

$$\operatorname{div}\big[\langle G(x)\nabla u, \nabla u\rangle^{(p-2)/2} G(x)\nabla u\big] = 0, \tag{0.3}$$

where $G(x)$ is a definite positive symmetric matrix whose entries are bounded VMO functions. We use the Hodge decomposition which reduces our problem to linear ($p = 2$) elliptic equations with VMO coefficients, see [D], [IS2].

Remark. Equations of type (0.3) arise in quasiconformal geometry and non-linear elasticity, see for instance [I]. These are variational equations for the energy functionals

$$\mathscr{E}[u] = \int_\Omega \langle G(x)\nabla u, \nabla u\rangle^{p/2}\, dx.$$

Here $G\colon \Omega \to \mathbf{R}^{n\times n}$ is viewed as a metric tensor on Ω. In this sense our equation $\operatorname{div} \mathscr{A}(x, \nabla u) = 0$ is no other than a p-harmonic equation with respect to this metric. In a recent paper [IKM] the authors study mappings with BMO-distortion. It becomes clear that mappings whose distortion tensor $G(x)$ is in VMO will play a central role in further developments.

During the publication of the paper, we learned that S. Zhou in a very recent preprint [Z] has given independently a result similar to our Theorem 1. We thank the referee for pointing this out.

1. Hodge decomposition with VMO coefficients

For the definitions and basic properties of the classical spaces BMO and VMO we refer to [JN] and [Sa], where those spaces were introduced. Here we want only to recall that a BMO or VMO function a, defined, for example, on a cube Q, has an extension $\tilde{a}$ which is BMO or VMO, respectively, on the entire space $\mathbf{R}^n$. Also, the extension can be made so that, if a is bounded, $\tilde{a}$ is bounded and $\|\tilde{a}\|_\infty \le C\|a\|_{\infty,Q}$. Therefore, it is not a restriction to assume that the functions in question are defined on $\mathbf{R}^n$. For these questions, see e.g. [A].

Now, let Ω be a smooth open subset of $\mathbf{R}^n$. In the familiar Hodge decomposition, one expresses a vector field $\omega \in L^p(\Omega, \mathbf{R}^n)$, $1 < p < \infty$, as

$$\omega = \nabla\phi + H, \tag{1.1}$$

where $\phi \in W_0^{1,p}(\Omega)$ and $H \in L^p(\Omega, \mathbf{R}^n)$ is a divergence free vector field. Moreover, the following estimate holds

$$\|\nabla\phi\|_p + \|H\|_p \le C_p(\Omega)\|\omega\|_p. \tag{1.2}$$

This is done by solving the Poisson equation

$$\Delta\phi = \operatorname{div}\omega.$$

Next, in [I], [IS], a stability property for the Hodge decomposition is proved, which amounts to saying that if ω is a small perturbation of a gradient field, then the term H in (1.1) is small. More precisely, if $\omega = |\nabla u|^\varepsilon \nabla u$, with $u \in W_0^{1,r}(\Omega)$, $1 < r < \infty$ and ε is sufficiently small, say $-1 < 2\varepsilon < r - 1$, we have

$$\|H\|_{r/(1+\varepsilon)} \le C_r(\Omega)|\varepsilon| \|\nabla u\|_r^{1+\varepsilon}. \tag{1.3}$$

In this paper, we shall use a somewhat more general version of the Hodge decomposition, which is more suitable for our purposes. Let $A = A(x)$ and $B = B(x)$ be $n \times n$ matrices such that

$$|\xi|^2 \le \langle A\xi, \xi\rangle \le \Lambda|\xi|^2, \qquad |\xi|^2 \le \langle B\xi, \xi\rangle \le \Lambda|\xi|^2.$$

Also, we assume that the entries of the product matrix $G = BA$ are in VMO. Then we can decompose a vector field $\omega \in L^p(\Omega, \mathbf{R}^n)$ as

$$\omega = A\nabla\phi + H, \tag{1.4}$$

where $\phi \in W_0^{1,p}(\Omega)$, and $H \in L^p(\Omega, \mathbf{R}^n)$ verifies

$$\operatorname{div} BH = 0. \tag{1.5}$$

Note that estimate (1.2) still holds. Simply we solve the equation

$$\operatorname{div}(G\nabla\phi) = \operatorname{div}(B\omega)$$

and use the L^p-estimate for variational equations with VMO coefficients, see [D], [IS2]. Moreover, we have the following stability property in our decomposition.

Lemma 1.1. *Under the above assumptions, if* $u \in W_0^{1,r}(\Omega)$, $1 < r < \infty$, $-1 < 2\varepsilon < r-1$, *we have*

$$|A\nabla u|^\varepsilon A\nabla u = A\nabla\phi + H$$

with $\phi \in W_0^{1,r/(1+\varepsilon)}(\Omega)$ *and* $H \in L^{r/(1+\varepsilon)}(\Omega, \mathbf{R}^n)$ *verifying both* (1.5) *and* (1.3).

Proof. We mimic the proof of stability in the case of the classical Hodge decomposition.

By the uniqueness of the decomposition (1.4) we can define a linear operator $\mathscr{T}\colon L^p(\Omega,\mathbf{R}^n) \to L^p(\Omega,\mathbf{R}^n)$, for all $1 < p < \infty$, by the rule $\mathscr{T}\omega = H$. Note that $\mathscr{T}$ vanishes on vector fields of the form $A\nabla u$. Therefore, $\mathscr{T}(|A\nabla u|^\varepsilon A\nabla u)$ can be written as a commutator

$$H = \mathscr{T}(|A\nabla u|^\varepsilon A\nabla u) - |\mathscr{T}A\nabla u|^\varepsilon \mathscr{T}A\nabla u.$$

Then we conclude the lemma using a result for commutators in [IS].

Remark. If A and B are equal to the identity matrix, we get the classical Hodge decomposition.

Remark. The regularity of Ω is required in order to apply L^p theory, compare with [D].

Remark. The following Hodge decomposition is also worth noting. Let $a \in L^\infty$ and $u \in W^{1,r}(\mathbf{R}^n)$; then

$$a|\nabla u|^\varepsilon \nabla u = \nabla\phi + H$$

with $H \in L^{r/(1+\varepsilon)}(\mathbf{R},\mathbf{R}^n)$ verifying $\operatorname{div} H = 0$ and

$$\|H\|_{r/(1+\varepsilon)} \le C_r\big(\|a\|_{\mathrm{BMO}} + \|a\|_\infty\,|\varepsilon|\big)\|\nabla u\|_r^{1+\varepsilon}.$$

In fact the operator $\mathscr{T}$ defined above is a singular integral operator and we have

$$\begin{aligned} H = \mathscr{T}(a|\nabla u|^\varepsilon\nabla u) &= \mathscr{T}(a\,|\nabla u|^\varepsilon\nabla u) - a\mathscr{T}(|\nabla u|^\varepsilon\nabla u)\\ &\quad + a\mathscr{T}(|\nabla u|^\varepsilon\nabla u) - a|\mathscr{T}\nabla u|^\varepsilon\mathscr{T}\nabla u. \end{aligned}$$

Therefore, we are in a position to use the result of [CRW] on the commutators of $\mathscr{T}$ with multiplication by a BMO function.

2. The main results

Here we need to consider the nonhomogeneous counterpart of equation (0.3), that is,

$$\operatorname{div}\big[\langle G(\nabla u+g),\nabla u+g\rangle^{(p-2)/2}G(\nabla u+g)\big]=\operatorname{div}h, \tag{2.1}$$

on Ω, with $g\in L^r(\Omega,\mathbf{R}^n)$ and $h\in L^{r/(p-1)}(\Omega,\mathbf{R}^n)$ for some $r>\max\{1,p-1\}$. The matrix $G=G(x)$ is assumed to be symmetric and verifying

$$\langle G\xi,\xi\rangle\ge|\xi|^2\qquad\text{for all } \xi\in\mathbf{R}^n.$$

The entries of G are in $L^\infty\cap\mathrm{VMO}$.

Theorem 1. *Under these assumptions, for each $1<r<2$, there exist $\delta>0$ and $C>0$ such that, if $|p-2|<\delta$ then for all solutions $u\in W_0^{1,r}(\Omega)$ to (2.1) the following estimate holds*:

$$\int_\Omega|\nabla u|^r\,dx\le C\int_\Omega\big(|g|^r+|h|^{r/(p-1)}\big)\,dx. \tag{2.2}$$

Here, δ and C depend on n, r, Λ, the VMO-modulus of A and Ω. In the case that Ω is a ball or a cube, δ and C can be found independent of Ω.

Proof. Let A be the positive square root of G, i.e. $G=A^2$. Then $\langle A\xi,\xi\rangle\ge|\xi|^2$, for all $\xi\in\mathbf{R}^n$ and equation (2.1) can be rewritten as

$$\operatorname{div}\big[|A(\nabla u+g)|^{p-2}G(\nabla u+g)\big]=\operatorname{div}h. \tag{2.3}$$

We have

$$\int_\Omega|\nabla u|^r\le\int_\Omega\langle|\nabla u|^{r-p}\nabla u,|A\nabla u|^{p-2}G\nabla u\rangle. \tag{2.4}$$

Using Hodge decomposition, we write

$$|\nabla u|^{r-p}\nabla u=\nabla\phi+H \tag{2.5}$$

where $\phi\in W_0^{1,r/(r-p+1)}(\Omega)$. Here $H\in L^{r/(r-p+1)}(\Omega,\mathbf{R}^n)$ with $\|H\|_{r/(r-p+1)}\le C\|\nabla u\|_r^{r-p+1}$ and satisfies

$$\operatorname{div}(GH)=0.$$

Inserting (2.5) in (2.4) and using equation (2.3) yields

$$\begin{aligned}\int_\Omega|\nabla u|^r&\le\int_\Omega\langle\nabla\phi+H,|A\nabla u|^{p-2}G\nabla u\rangle\\&=\int_\Omega\langle H,|A\nabla u|^{p-2}G\nabla u\rangle+\int_\Omega\langle\nabla\phi,h\rangle\\&\quad+\int_\Omega\langle\nabla\phi,|A\nabla u|^{p-2}G\nabla u-|A(\nabla u+g)|^{p-2}G(\nabla u+g)\rangle\\&=I_1+I_2+I_3.\end{aligned}$$

To estimate I_1, we write

$$I_1 = \int_\Omega \langle H, |A\nabla u|^{p-2} G\nabla u\rangle = \int_\Omega \langle AH, |A\nabla u|^{p-2} A\nabla u\rangle$$

and then decompose

$$|A\nabla w|^{p-2} A\nabla w = A\nabla\psi + K,$$

where $\psi \in W_0^{1,r/(p-1)}(\Omega)$, and $K \in L^{r/(p-1)}(\Omega, \mathbf{R}^n)$ verifies

$$\operatorname{div} AK = 0$$
$$\|K\|_{r/(p-1)} \le C|p-2|\, \|\nabla u\|_r^{p-1}.$$

Hence

$$\begin{aligned} I_1 &= \int_\Omega \langle AH, A\nabla\psi + K\rangle = \int_\Omega \langle AH, A\nabla\psi\rangle + \int_\Omega \langle AH, K\rangle \\ &= \int_\Omega \langle \nabla\psi, GH\rangle + \int_\Omega \langle AH, K\rangle = \int_\Omega \langle H, AK\rangle \\ &= \int_\Omega \langle |\nabla u|^{r-p}\nabla u - \nabla\phi, AK\rangle = \int_\Omega \langle |\nabla u|^{r-p}\nabla u, AK\rangle \\ &\le \|A\|_\infty \|K\|_{r/(p-1)} \|\,|\nabla u|^{r-p+1}\|_{r/(r-p+1)} \le C|p-2|\, \|\nabla u\|_r^r. \end{aligned}$$

The terms I_2 and I_3 can be estimated like in [IS]–[KZ]. In this way, we end up with the following estimate

$$\{1 - C|p-2| - \theta\} \|\nabla u\|_r^r \le C_\theta \big(\|g\|_r^r + \|h\|_{r/(p-1)}^{r/(p-1)}\big)$$

where $0 < \theta < 1$ is arbitrary. This clearly concludes the proof.

Finally, we consider local solutions to (0.3). The technique of reverse Hölder inequalities as used in [IS] can be easily applied to conclude the following

Theorem 2. *Let Ω be an open subset of $\mathbf{R}^n$ and $G \in \mathrm{VMO}(\Omega, \mathbf{R}^{n\times n}) \cap L^\infty(\Omega, \mathbf{R}^{n\times n})$. For any $1 < r_1 < 2$ there exists $\delta > 0$ such that, if $|p-2| < \delta$ and $u \in W_{\mathrm{loc}}^{1,r_1}(\Omega)$ is a solution to (0.3), then $u \in W_{\mathrm{loc}}^{1,r_2}(\Omega)$ for all exponents $r_2 < \infty$.*

Remark. That we need G to be in VMO is clear. Examples in [S] show that, even in the linear case, corresponding to $p = 2$, boundedness of G is not sufficient to obtain r_1 close to 1 and r_2 close to ∞.

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