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CONFORMAL WELDING OF JORDAN CURVES USING WEIGHTED DIRICHLET SPACES

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Abstract. This paper gives an alternative description of Lehto's results on homeomorphisms which admit conformal welding [6]. A normal families argument is controlled by the action of the homeomorphism as a composition operator on weighted Dirichlet spaces. This point of view admits a discussion of uniqueness.

1. Introduction

Let J be a Jordan curve on the Riemann sphere S^2. Then $S^2 \setminus J = D_1 \cup D_2$, where D_1 and D_2 are disjoint simply connected subdomains of the sphere. The conformal maps $f\colon U \to D_1$, $g\colon U^* \to D_2$, where U is the unit disc, and U^* the complement of the closed unit disc, extend as homeomorphisms from S^1, the unit circle, and the welding $h = g^{-1} \circ f$ is an element of $\mathrm{Homeo}_+(S^1)$. It is an open question to establish the relationship between J and h. The case that is well understood is where J is a quasi-circle which corresponds to h being quasi-symmetric [2] . One may rephrase this as

Theorem 1.1 [2]. *The homeomorphism $h\colon S^1 \longrightarrow S^1$ is a welding of a quasi-circle if and only if $V_h\colon f \longrightarrow f \circ h$ is a bounded operator on $\mathscr{D}$ the Dirichlet space.*

The Dirichlet space $\mathscr{D}$ consists of $f \in L^2(S^1)$ with Poisson extensions to the unit disc f with $\int |\nabla f|^2 \, dA < \infty$. The contribution of this paper is to weaken the hypothesis to obtain for certain spaces D_w given by $\int |\nabla f(z)|^2 w(z) \, dA(z) < \infty$ and certain weights w (defined in Section 3).

Theorem 1.2. *The homeomorphism $h\colon S^1 \longrightarrow S^1$ is a welding of a Jordan curve if h is a uniform limit of quasi-symmetric mappings, h_n, so that $V_{h_n}\colon f \longrightarrow f \circ h_n$ are uniformly bounded operators from D_w to $\mathscr{D}$, where D_w is a weighted Dirichlet space.*

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The aim is to obtain some information on the welding problem without being too dependent on the Beltrami equation cf. [6], [3]. The paper falls into two parts, in the first we give a simple argument for welding to exist cf. [4] and in the second we give an example where we can explicitly calculate with the weight for Theorem 1.2. Sadly the elementary calculations only give the same conclusions as [6], and are weaker than [4] but they do offer a different framework to discuss uniqueness.

2. Outline

The argument is simple. Consider a homeomorphism h, fixing three points, $1, i, -1$, say. Then we may approximate h by h_n uniformly, where the h_n are nice normalized homeomorphisms for which we can solve the welding. These give Jordan curves J_n with Riemann maps f_n to the interior, and we may assume the f_n fix $1, i, -1$. So the $\{f_n\}$ form a normal family and on a subsequence converge to f a univalent map of the unit disc, U. If $f(S^1)$ is a Jordan curve J then all of the convergence is uniform on S^1 and h is the welding of J. Further f takes the unit disc to the interior of a Jordan curve if and only if f extends continuously and injectively to the boundary. This holds if the family $\{f_n\}$ is uniformly continuous and uniformly injective on S^1, i.e. every distinct pair of points on S^1 have images uniformly separated under the f_n. We verify the uniform conditions in terms of module calculations for a set of annuli. Recall that any doubly connected region of the Riemann sphere, A, is conformally equivalent to one of $\{z : 1 < |z| < r\}$, or the punctured plane. The modulus of A is then $\operatorname{Mod} A = \log r/2\pi$, or ∞ for the punctured plane.

We consider two families of annuli $\mathscr{A}$ and $\mathscr{B}$ defined as follows. Given any pair of points x, y on the unit circle with $0 < d(x,y) < \frac{1}{5}$, not both in $\{1, i, -1\}$, form the annulus $A_{x,y}$. This is given by deleting the shorter arc $(x,y) \subset S^1$ and one of the arcs of the unit circle $I \in \{(1,i), (i,-1), (-1,1)\}$ so that $d\big(I, (x,y)\big) > \frac{1}{5}$ from the Riemann sphere. The annuli $A_{x,y}$ form the family $\mathscr{A}$ and we will use them to test the uniform continuity. To test injectivity consider the family $\mathscr{B}$ consisting of the $B_{x,y}$ defined next. Form arcs I_1, I_2 where I_1 has endpoint x and one of $1, i, -1$ and I_2 with endpoints y and one of $1, i, -1$ so that $0 < d(I_1, I_2) \le d(x,y)$, then $B_{x,y}$ is the annulus formed by deleting I_1, I_2 from the Riemann sphere. There are corresponding versions of these annuli under the weldings h_n. If E, F are arcs in the unit circle then we can consider the annulus $A_{E,F} = S^2 \setminus (E \cup F)$, we denote the annulus in the sphere formed by welding h_n as $A^n_{E,F} = S^2 \setminus \big(f_n(E) \cup f_n(F)\big)$. Then following Lehto [6], [7] we have the following lemma.

Lemma 2.1. *Let h be an orientation preserving homeomorphism of the unit circle, fixing $\{1, i, -1\}$, and let h_n be a normalized family of quasi-symmetric homeomorphisms converging uniformly to h. If for all annuli in $\mathscr{A}$ $\operatorname{Mod} A^n \longrightarrow \infty$*

as Mod $A \longrightarrow \infty$ *uniformly in* Mod A*, and if for all annuli in* $\mathscr{B}$, Mod $B^n \longrightarrow 0$ *implies* Mod $B \longrightarrow 0$ *uniformly then* h *welds a Jordan curve.*

Proof. The proof is [7] where euclidean bounds on separation of boundary components of annuli are obtained from modulus inequalities and normalization of annuli. Then as outlined h welds a Jordan curve. To obtain estimates on the moduli of annuli we use an alternative definition. From [1] $\operatorname{Mod} A_{E,F} = L_{E,F}^{-1/2}$ where $L_{E,F}$ is the square of the Dirichlet norm of the harmonic function u on $A_{E,F} = S^2 \setminus (E \cup F)$ with boundary values 0 on E, 1 on F. In particular we have

$$L_{E,F} \le \inf\{|u|_D^2 : u|_E = 0,\ u|_F = 1,\ u \in C^1(A_{E,F})\}.$$

Therefore we may rewrite Lemma 2.1 in terms of the extremal length $L^{1/2}$.

Corollary 2.1. *Let* h *be an orientation preserving homeomorphism of the unit circle, fixing* $\{1, i, -1\}$*, and let* h_n *be a normalized family of quasi-symmetric homeomorphisms converging uniformly to* h*. If for all annuli in* $\mathscr{A}$, $L^n \longrightarrow 0$ *as* $L \longrightarrow 0$ *uniformly, and if for all annuli in* $\mathscr{B}$, $L^n \longrightarrow \infty$ *implies* $L \longrightarrow \infty$ *uniformly, then* h *welds a Jordan curve* J*.*

It remains to determine inequalities for this to hold for a given homeomorphism.

3. The welding as operator

Given arcs E, F on the unit circle we have formed the annulus $A_{E,F}$ and the welded annulus $A_{E,F}^n = S^2 \setminus \big(f_n(E) \cup f_n(F)\big)$. We wish to compute the extremal length of A^n by considering functions which take values 0 on $f_n(E)$ and 1 on $f_n(F)$ and evaluating the Dirichlet norm. So given a function $u \in C^0(J_n)$ say with $u|_{f_n(E)} = 0$, $u|_{f_n(F)} = 1$ we may form the harmonic extensions u_+, u_- to Ω_n, Ω_n^* the interior and exterior of J_n. Since J_n is a quasi-circle the Dirichlet norm may be calculated by integrating off J_n and we have

$$|u|_D^2 = \int_{\Omega_n} |\nabla u|^2 \, dA + \int_{\Omega_n^*} |\nabla u|^2 \, dA.$$

Denoting the family of such u as $U^n(E,F)$ we have

$$L_{E,F}^n = \inf\{|u|_D^2 : u \in U^n(E,F)\}.$$

But as Dirchlet norms are conformally invariant we pull back by the conformal maps f_n and g_n to the round sphere. Then if we pick $u \in U^n(E,F)$ we consider $v = u \circ f_n$, $r = u \circ g_n$ defined on U, U^* respectively. In which case

$$L_{E,F}^n = \inf\{|v|_D^2 + |r|_D^2\}$$

where the infimum is over v harmonic on U, continuous on S^1, $v|_E = 0$, $v|_F = 1$ and over r harmonic on U^* continuous on S^1, $r|_{h_n(E)} = 0$, $r|_{h_n(F)} = 1$. That is

$$L^n_{E,F} \leq \inf\{|v|^2_D + |v \circ h_n|^2_D : v \in U(E,F)\}.$$

We require that if L tends to 0, L^n tends to 0 and as L^n tends to infinity L does, uniformly. So it is clear if V_{h_n} are uniformly bounded on $\mathscr{D}$ the above argument produces a Jordan curve i.e. welding for quasi-symmetric maps. We give a slight weakening.

Theorem 3.1. *Let D_w be a weighted Dirichlet space, so that the weight has properties* (P) *outlined below. If h is a homeomorphism of the circle which is the uniform limit of quasi-symmetric homeomorphisms h_n, for which the operator norm $V_{h_n} : D_w \longrightarrow \mathscr{D}$ is uniformly bounded, then h welds a Jordan curve.*

Given an annulus $A_{E,F}$ determined by a pair of arcs E, F on the unit circle, we have

$$L_{E,F} = 2\inf\{|u|^2_D : u \in U(E,F)\}.$$

We define

$$L^w_{E,F} = 2\inf\{|u|^2_w : u \in U(E,F)\}$$

where

$$|u|^2_w = \int_U |\nabla u(z)|^2 w(z)\, dA(z).$$

Property (P) of the weight w is now the three conditions:
(1) $w(z) > c > 0$.
(2) For $A_{E,F} \in \mathscr{A}$, $L_{E,F} \longrightarrow 0$ implies $L^w_{E,F} \longrightarrow 0$ uniformly.
(3) For $B_{E,F} \in \mathscr{B}$ $L^w_{E,F} \longrightarrow \infty$ implies $L_{E,F} \longrightarrow \infty$, uniformly.

Proof of 3.1. We have

$$\begin{aligned} L^n_{E,F} &\leq \inf\{|v|^2_D + |v \circ h_n|^2_D : v \in U(E,F)\} \\ &\leq \inf\{|v|^2_w + |v \circ h_n|^2_D : v \in U(E,F)\} \leq \inf\{C|v|^2_w : v \in U(E,F)\} \end{aligned}$$

where C is the bound on the operator norms of $V_{h_n}\colon D_w \longrightarrow \mathscr{D}$. But parts (2) and (3) of (P) now imply the appropriate modulus estimates for the annuli $A^n(E,F)$, and h welds by Corollary 2.1.

4. Examples

We look at examples of radially symmetric weights for which the computation is simple. Let $\alpha < 1$ and let $w(z) = O(|\log(1-|z|)^{-1}|^{\alpha})$ as $|z| \longrightarrow 1$. We show for some h, homeomorphisms with $V_h\colon D_w \longrightarrow \mathscr{D}$ bounded, that h welds.

Step 1. First we check that annuli are behaved appropriately in D_w. For this step by conformal invariance we may as well work on the upper-half plane. Let $f_{0,a,1,\infty}$ denote the unique conformal map of the upper-half plane with distinguished points $\{0,a,1,\infty\}$ to the rectangle with corners at $0,1$ and $ib, 1+ib$. Then $\operatorname{Re} f$ is a candidate for $U(E,F)$ where $F=(0,a)$ and $E=(1,\infty)$. By radial symmetry to check properties (2) and (3) of (P) we need that $|f|_w$ tends to zero as a tends to zero and $|f|_w$ is uniformly bounded as a stays away from 1. But the derivative is given (up to a constant $O(1)$) as

$$f'(z) = \frac{1}{\log|a|\sqrt{z(z-a)(z-1)}}.$$

So with $w(z) = O(|\log \operatorname{Im} z|^{\alpha})$ in a neighbourhood of the origin an integration verifies D_w has properties (P).

Step 2. To construct the approximations we need to control the homeomorphisms, we work on the real line (by Möbius conjugation). First consider h a quasi-symmetric homeomorphism, we estimate $|V_h|_{D_w,\mathscr{D}}$ the norm of the composition operator $V_h\colon D_w \longrightarrow \mathscr{D}$. Given H an extension of h to the upper half-plane W, we apply the observation [1]

$$(*) \qquad |f\circ h^{-1}|_D^2 \le 2\int_W |\partial f(z)|^2 \frac{1}{1-|\mu_H(z)|}\,dA(z).$$

Thus we obtain $|V_{h^{-1}}|_{D_w,\mathscr{D}} \le [(1-\mu_H(z))w(z)]^{-1}$. Writing

$$\sup_x \frac{h(x+t)-h(x)}{h(x)-h(x-t)} = \rho_t$$

the Beurling–Ahlfors extension H of h satisfies $(1-|\mu_H(z)|)^{-1} \le c\rho_{\operatorname{Im} z}$. So for h a homeomorphism of the reals we have

$$|V_{h^{-1}}|_{D_w,\mathscr{D}} \le c\sup \rho_t/w(t).$$

Therefore given any homeomorphism h satisfying

$$O(|\log t|^{-\alpha}) \le \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \le O(|\log t|^{\alpha})$$

we may approximate h by piecewise linear quasi-symmetric maps h_n with

$$\sup_n |V_{h_n^{-1}}|_{D_w,\mathscr{D}} \le C \sup \rho_t(h)/w(t) < \infty$$

and applying Theorem 3.1 h^{-1} welds. This implies that h welds.

To prove Lehto's result [6] with weight $w(z) = O(|\log(1-|z|)|)$ we apply the above argument modifying Step 1. Consider $f_{0,a,1,2}$ mapping the upper half-plane to a rectangle with vertices $0, 1, 1+ib, ib$. Let ψ be a conformal map of the upper half-plane to a smooth region W in the upper half-plane, where $\overline{W}$ meets the real line in $[0,2]$, so that $\psi(0) = 0$, $\psi(2) = 2$. Then we consider up to a constant $O(1)$

$$F_a(z) = \frac{1}{\log|\log a| \log\bigl(\psi(z)/4\bigr)\sqrt{z(z-a)(z-1)(z-2)}}.$$

The additional logarithmic singularity at the origin is sufficient to force $|F_a|_w$ to zero as a tends to zero, and h welds.

5. Uniqueness

The operator formulation allows a different approach to the uniqueness of the welding following [5]. A welding h is said to be unique if the Jordan curve J with welding h forces any homeomorphism of the sphere, conformal off J to be Möbius. It is well known [5] that if functions continuous on the sphere with $\int_{S^2\setminus J} |\nabla f|^2\, dA < \infty$ automatically satisfy $\int_{S^2} |\nabla f|^2\, dA < \infty$ ('J Dirichlet removable') then J is unique ('conformally removable'). We outline this for quasi-circles and then point out the extension to curves with weldings of the previous section (for which uniqueness follows from [3]).

Let J be a quasi-circle. Pick F continuous, $\int_{S^2\setminus J} |\nabla F|^2\, dA < \infty$, and harmonic off J. We need to show that $\int_{S^2} |\nabla F|^2\, dA < \infty$. As [5] we show F is the uniform limit of smooth F_n with $\int_{S^2} |\nabla F_n|^2 < C$. Think of F as $F_+ = F|_{\mathrm{int} J} \circ f$ defined on the unit disc and the harmonic extension to the exterior of $F_- = F_+ \circ h|_{S^1}$. Then $\int_{S^2\setminus J} |\nabla F|^2\, dA$ is $\int_U |\nabla F_+|^2\, dA + \int_{U^*} |\nabla F_-|^2\, dA$ by conformal invariance. Approximate F_+ by smooth harmonic G_n on U and h by smooth h_n. The functions G_n on the unit disc and the harmonic extensions of $G_n \circ h_n$ on the exterior represent, after welding, smooth functions F_n on the sphere which converge uniformly to F. Further the Dirichlet norms of F_n are controlled by $|F_+|_D < \infty$ and $|F_n \circ h_n|_D \le K |F_n|_D$ with K the operator norm on the composition V_{h_n}. So F is uniformly approximable by smooth Dirichlet bounded functions and J is Dirichlet removable, hence unique. We repeat this argument with a different norm to obtain

Theorem 5.1. *Let $\alpha < 1$ and h be a homeomorphism with*

$$\frac{h(x+t) - h(x)}{h(x) - h(x-t)} \le O(|\log t|^\alpha).$$

The Jordan curve J with welding h is unique.

Proof. Consider functions F continuous on the sphere harmonic off J. Let f, g be the conformal maps from U, U^* to the interior and exterior of J, respectively. We consider the norm

$$|F|_{\alpha,1} = \int_U |\nabla(F \circ f)|^2 w_\alpha(z)\, dA + \int_{U^*} |\nabla(F \circ g)|^2\, dA.$$

We show if $|F|_{\alpha,1} < \infty$ then

$$|F|_W^2 = \int_{S^2} |\nabla F|^2 W(z)\, dA < \infty$$

where $W(z) = 1$ for $z \in \overline{\operatorname{ext} J}$ and $W(z) = w_\alpha\big(f^{-1}(z)\big)$ for $z \in \operatorname{int} J$. This follows as above by writing F as two distinct functions $F_+ = F|_{\operatorname{int} J} \circ f$ on the unit disc and $F_- = F|_{\operatorname{ext} J} \circ g$ on the exterior—i.e. $F_+, F_+ \circ h$. Now approximate F_+ by smooth harmonic functions Φ_n, and approximate h by smooth h_n with uniformly bounded operator norm $D_w \longrightarrow \mathscr{D}$. We regard the harmonic functions Φ_n on the unit disc and the harmonic extension of $\Phi_n \circ h_n$ as representing, after welding, a smooth function on the sphere F_n, harmonic off J_n. Now $|F_n|_{w_\alpha} \longrightarrow |F|_{w_\alpha}$ and the smoothness forces $|F_n|_W = |F_n|_{\alpha,1}$ which are uniformly bounded, hence $|F|_W$ is finite.

To show that J is conformally removable we wish to show that for Ψ a homeomorphism of the sphere conformal off J, $|\Psi(z) - z|_W < \infty$, hence finite Dirichlet norm and by Weyl's lemma is thus Möbius. To do this we just need to check that $f \in D_{w_\alpha}$, and that J has zero area. For this we have recourse to the powerful estimates of [3], which give $|J| = 0$ immediately. We require

$$\int_U |f'(z)|^2 |\log(1 - |z|)|^\alpha\, dA(z) < \infty$$

that is

$$\sum_{n>0} n^\alpha \int_{A_n} |f'(z)|^2\, dA(z) < \infty,$$

where $A_n = \{z : 1 - 2^{-n} > |z| \geq 1 - 2^{-n-1}\}$. But the bounds from [3] show

$$\int_{A_n} |f'(z)|^2\, dA(z) < O(n^{-\gamma})$$

for some $\gamma > 2$ so $f \in D_w$ and the welding is unique.

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