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GROWTH PROPERTIES OF SPHERICAL MEANS FOR MONOTONE BLD FUNCTIONS IN THE UNIT BALL

Yoshihiro Mizuta and Tetsu Shimomura

Hiroshima University, The Division of Mathematical and Information Sciences
Faculty of Integrated Arts and Sciences, Higashi-Hiroshima 739-8521, Japan
mizuta@mis.hiroshima-u.ac.jp

Akashi National College of Technology, General Studies
Nishioka Uozumi 674-8501, Japan; shimo@akashi.ac.jp

Abstract. Let u be a monotone (in the sense of Lebesgue) function on the unit ball $\mathbf{B}$ of $\mathbf{R}^n$ satisfying

$$\int_{\mathbf{B}} |\nabla u(x)|^p (1-|x|)^{\alpha}\, dx < \infty,$$

where ∇ denotes the gradient, $1 < p < \infty$ and $-1 < \alpha < p-1$. Then u has a boundary limit $f(\xi)$ for almost every $\xi \in \partial\mathbf{B}$, and u may be considered as a Dirichlet solution for the boundary function f. Our aim in this paper is to deal with growth properties of the spherical means

$$S_q(u_r - f) \equiv \left(\int_{\partial\mathbf{B}} |u(r\xi) - f(\xi)|^q \, dS(\xi) \right)^{1/q}.$$

In fact, we prove that

$$\lim_{r\to 1-0} (1-r)^{-\omega} S_q(u_r - f) = 0$$

when $p \leqq q \leqq \infty$ and $\omega = (n-1)/q - (n-p+\alpha)/p > 0$.

1. Introduction

If f is a pth summable function on the boundary $\mathbf{S}$ of the unit ball $\mathbf{B}$ in $\mathbf{R}^n$, then we can find a (weak) Dirichlet solution u which is harmonic in $\mathbf{B}$ and

$$\lim_{r\to 1-0} S_p(u_r - f) = 0, \tag{1}$$

where $1 \leqq p < \infty$, $u_r(z) = u(rz)$ for $z \in \mathbf{S}$ and

$$S_p(v) = \left(\frac{1}{\sigma_n} \int_{\mathbf{S}} |v(z)|^p \, dS(z) \right)^{1/p}$$

with σ_n denoting the surface area of $\mathbf{S}$. Furthermore, for functions f in some Lipschitz spaces $\Lambda_\beta^{p,p}(\mathbf{S})$ (see Stein [16]), we can find Dirichlet solutions u in BLD

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(Beppo Levi and Deny) spaces for which (1) is satisfied. Conversely, if u is a harmonic function on $\mathbf{B}$ satisfying

$$\int_{\mathbf{B}} |\nabla u(x)|^p (1-|x|)^{\alpha}\, dx < \infty, \tag{2}$$

where $1 < p < \infty$ and $-1 < \alpha < p-1$, then it is well known that u has nontangential limits $f(\xi)$ at many boundary points $\xi \in \mathbf{S}$ and f belongs to a certain Lipschitz space (see Stein [16] and the first author [9], [10]). Of course, u is a Dirichlet solution for f.

In this paper, we are concerned with weighted limits of $S_q(u_r - f)$ for monotone BLD functions u on $\mathbf{B}$ with boundary values f. We say that a continuous function u is monotone in an open set G, in the sense of Lebesgue, if both

$$\max_{\overline{D}} u(x) = \max_{\partial D} u(x) \qquad \text{and} \qquad \min_{\overline{D}} u(x) = \min_{\partial D} u(x)$$

hold for every relatively compact open set D with the closure $\overline{D} \subset G$ (see [5]). Clearly, harmonic functions are monotone, and more generally, solutions of elliptic partial differential equations of second order and weak solutions for variational problems may be monotone. For these facts, see Gilbarg–Trudinger [2], Heinonen–Kilpeläinen–Martio [3], Reshetnyak [13], Serrin [14] and Vuorinen [17] and [18].

Our starting point is a result of Gardiner [1, Theorem 2] which states that if u is a Green potential in the unit ball $\mathbf{B}$, then

$$\liminf_{r\to 1-0} (1-r)^{(n-1)(1-1/q)} S_q(u_r) = 0$$

when $(n-3)/(n-1) < 1/q \leqq (n-2)/(n-1)$ and $q > 0$.

Our first aim in this paper is to show the following result.

Theorem 1. *Let u be a monotone function on $\mathbf{B}$ satisfying (2) with $n-1 < p \leqq n+\alpha$. If $p \leqq q < \infty$ and*

$$\omega = \frac{n-1}{q} - \frac{n-p+\alpha}{p} > 0,$$

then

$$\lim_{r\to 1-0} (1-r)^{-\omega} S_q(U_r) = 0,$$

where $U_r(\xi) = u(r\xi) - u(\xi)$ for $\xi \in \mathbf{S}$.

The sharpness of the exponent will be discussed in the final section. We also find a BLD function u satisfying (2) and

$$\limsup_{r\to 1-0} (1-r)^{-\omega} S_q(U_r) = \infty,$$

when $\alpha < 0$.

Corollary 1. *Let u be a coordinate function of a quasiregular mapping on* $\mathbf{B}$ *satisfying* (2). *If* $n-1 < p \leqq n+\alpha$, $p \leqq q < \infty$ *and* $\omega = (n-1)/q-(n-p+\alpha)/p > 0$, *then*

$$\lim_{r\to 1-0}(1-r)^{-\omega}S_q(U_r) = 0.$$

For the definition and basic properties of quasiregular mappings, we refer to [3], [13], and [17]. In particular, a coordinate function $u = f_i$ of a quasiregular mapping $f = (f_1, \dots, f_n)\colon \mathbf{B} \to \mathbf{R}^n$ is $\mathscr{A}$-harmonic (see [3, Theorem 14.39] and [13]) and monotone in $\mathbf{B}$, so that Theorem 1 gives the present corollary.

It is well known that the coordinate functions of a bounded quasiconformal mapping on $\mathbf{B}$ have finite n-Dirichlet integral (see Vuorinen [18]), so that Corollary 1 gives the following result.

Corollary 2. *Let u be a coordinate function of a bounded quasiconformal mapping on* $\mathbf{B}$. *If* $n \leqq q < \infty$, *then*

$$\lim_{r\to 1-0}(1-r)^{-(n-1)/q}S_q(U_r) = 0.$$

Next we are concerned with the case $q = \infty$. In order to give a general result, we consider a nondecreasing positive function φ on the interval $[0,\infty)$ such that φ is of log-type, that is, there exists a positive constant M satisfying

$$\varphi(r^2) \leqq M\varphi(r) \qquad \text{for all } r \geqq 0.$$

Set $\Phi_p(r) = r^p\varphi(r)$ for $p > 1$. Our next aim is to study the boundary behavior of monotone BLD functions u on $\mathbf{B}$, which satisfy the weighted condition

$$\int_{\mathbf{B}} \Phi_p(|\nabla u(x)|)(1-|x|)^\alpha\, dx < \infty. \tag{4}$$

Consider the function

$$\kappa(r) = \left[\int_0^r \left(t^{n-p+\alpha}\varphi(t^{-1})\right)^{-1/(p-1)} \frac{dt}{t}\right]^{1-1/p}$$

for $r > 0$. We see (cf. [15, Lemma 2.4]) that if $n-p+\alpha < 0$, then

$$\kappa(r) \sim \left[r^{n-p+\alpha}\varphi(r^{-1})\right]^{-1/p} \qquad \text{as } r \to 0$$

and if $n-p+\alpha = 0$ and $\varphi(r) = \left(\log(e+r)\right)^\sigma$ with $\sigma > p-1$, then

$$\kappa(r) \sim \left[\log(1/r)\right]^{(p-1-\sigma)/p} \qquad \text{as } r \to 0.$$

Theorem 2. *Let u be a monotone function on* $\mathbf{B}$ *satisfying* (4) *with* $-1 < \alpha \leqq p-n$. *If* $\kappa(1) < \infty$, *then*

$$\lim_{r\to 1-0}\left[\kappa(1-r)\right]^{-1}S_\infty(U_r) = 0,$$

where $U_r(\xi) = u(r\xi) - u(\xi)$ *for* $\xi \in \mathbf{S}$.

Corollary 3. *Let u be a coordinate function of a quasiregular mapping on* $\mathbf{B}$ *satisfying* (4) *with* $-1 < \alpha \leqq p-n$. *If* $\kappa(1) < \infty$, *then*

$$\lim_{r\to 1-0} \bigl[\kappa(1-r)\bigr]^{-1} S_\infty(U_r) = 0.$$

For related results, we also refer to Herron–Koskela [4], the first author [8], [11] and the authors [12].

Finally we wish to express our deepest appreciation to the referee for his useful suggestions.

2. Proof of Theorem 1

Throughout this paper, let M denote various constants independent of the variables in question.

For a proof of Theorem 1, we need the following result, which gives an essential tool in treating monotone functions.

Lemma 1 (cf. [4], [6], [10]). *Let $p > n-1$. If u is a monotone BLD function on $B(x_0, 2r)$, then*

$$(5)\quad |u(x)-u(y)|^p \leqq M r^{p-n} \int_{B(x_0,2r)} |\nabla u(z)|^p\,dz \qquad \text{whenever } x,y \in B(x_0,r).$$

Lemma 1 is a consequence of Sobolev's theorem, so that the restriction $p > n-1$ is needed; for a proof of Lemma 1, see for example [4, Lemma 7.1] or [10, Theorem 5.2, Chapter 8].

Now we are ready to prove Theorem 1, along the same lines as in the proof of [12, Theorem 2].

Proof of Theorem 1. Let u be a monotone function on $\mathbf{B}$ satisfying (2) with $n-1 < p \leqq n+\alpha$. If $|s-t| \leqq r < \frac{1}{2}(1-t)$, then Lemma 1 gives

$$\begin{aligned} |S_q(u_s - u_t)| &= \left(\frac{1}{\sigma_n} \int_{\mathbf{S}} |u(s\xi) - u(t\xi)|^q\,dS(\xi) \right)^{1/q} \\ &\leqq M r^{(p-n)/p} \left(\int_{\mathbf{S}} \left(\int_{B(t\xi,2r)} |\nabla u(z)|^p\,dz \right)^{q/p} dS(\xi) \right)^{1/q}, \end{aligned}$$

so that Minkowski's inequality for integrals yields

$$\begin{aligned} |S_q(u_s - u_t)| &\leqq M r^{(p-n)/p} \Biggl(\int_{B(0,t+2r)-B(0,t-2r)} |\nabla u(z)|^p \\ &\qquad \times \left(\int_{\{\xi\in\mathbf{S}: |\xi - z/t| < 2r/t\}} dS(\xi) \right)^{p/q} dz \Biggr)^{1/p} \\ &\leqq M r^{(p-n)/p} (2r/t)^{(n-1)/q} \left(\int_{B(0,t+2r)-B(0,t-2r)} |\nabla u(z)|^p\,dz \right)^{1/p}. \end{aligned}$$

Let $r_j = 2^{-j-1}$, $t_j = 1-r_{j-1}$ and $A_j = B(0,1-r_j)-B(0,1-3r_j)$ for $j = 1,2,\dots$. For simplicity, set

$$\omega = -(n-p+\alpha)/p + (n-1)/q > 0.$$

Then we find

$$|S_q(u_{t_j} - u_r)| \leqq Mr_{j+1}^{\omega}\left(\int_{A_j} |\nabla u(z)|^p(1-|z|)^\alpha\,dz\right)^{1/p}$$

for $t_j \leqq r < t_j + r_{j+1}$,

$$|S_q(u_r - u_s)| \leqq Mr_{j+2}^{\omega}\left(\int_{A_j} |\nabla u(z)|^p(1-|z|)^\alpha\,dz\right)^{1/p}$$

for $t_j + r_{j+1} \leqq r < s < t_j + r_{j+1} + r_{j+2}$, and

$$|S_q(u_s - u_{t_{j+1}})| \leqq Mr_{j+2}^{\omega}\left(\int_{A_{j+1}} |\nabla u(z)|^p(1-|z|)^\alpha\,dz\right)^{1/p}$$

for $t_j + r_{j+1} + r_{j+2} \leqq s < t_{j+1}$. Collecting these results, we have

$$\begin{aligned}|S_q(u_{t_j} - u_r)| &\leqq Mr_j^{\omega}\left(\int_{A_j} |\nabla u(z)|^p(1-|z|)^\alpha\,dz\right)^{1/p}\\ &\quad + Mr_{j+1}^{\omega}\left(\int_{A_{j+1}} |\nabla u(z)|^p(1-|z|)^\alpha\,dz\right)^{1/p}\end{aligned}$$

for $t_j \leqq r < t_{j+1}$. Hence it follows that

$$|S_q(u_r - u_{t_{j+m}})| \leqq M\sum_{l=j}^{j+m} r_l^{\omega}\left(\int_{A_l} |\nabla u(z)|^p(1-|z|)^\alpha\,dz\right)^{1/p}$$

for $t_j \leqq r < t_{j+m}$. Since $A_l \cap A_k = \emptyset$ for $l \geqq k+2$, Hölder's inequality gives

$$\begin{aligned}|S_q(u_r - u_{t_{j+m}})| &\leqq M\left(\sum_{l=j}^{j+m} r_l^{p'\omega}\right)^{1/p'}\left(\sum_{l=j}^{j+m}\int_{A_l} |\nabla u(z)|^p(1-|z|)^\alpha\,dz\right)^{1/p}\\ &\leqq Mr_j^{\omega}\left(\int_{B(0,1-r_{j+m})-B(0,1-3r_j)} |\nabla u(z)|^p(1-|z|)^\alpha\,dz\right)^{1/p}\end{aligned}$$

for $t_j \leqq r < t_{j+m}$, where $1/p + 1/p' = 1$. Now, letting $m \to \infty$, we establish

$$|S_q(U_r)| \leqq M(1-r)^{\omega}\left(\int_{\mathbf{B}-B(0,1-3r_j)} |\nabla u(z)|^p(1-|z|)^\alpha\,dz\right)^{1/p}$$

for $t_j \leqq r < t_{j+1}$, which implies that

$$\lim_{r\to 1}(1-r)^{-\omega}S_q(U_r) = 0,$$

as required.

3. Proof of Theorem 2

If $B(x,2r) \subset \mathbf{B}$, then, applying Lemma 1 and dividing the domain of integration into two parts

$$E_1 = \{z \in B(x,2r) : |\nabla u(z)| > r^{-\delta}\},$$
$$E_2 = B(x,2r) - E_1,$$

we have

$$(6) \qquad |u(x)-u(y)|^p \leqq Mr^{(1-\delta)p} + Mr^{p-n}\bigl[\varphi(r^{-1})\bigr]^{-1}\int_{B(x,2r)} \Phi_p(|\nabla u(z)|)\,dz$$

for $y \in B(x,r)$, where M may depend on δ, $0<\delta<1$.

Let $r_j = 2^{-j-1}$, $j=0,1,\dots$. For $x \in \mathbf{B}$, let $x_j = (1-2r_j)\xi$ with $\xi = x/|x|$. Then we find

$$|u(x_j)-u(x)| \leqq Mr_j^{1-\delta} + Mr_j^{(p-n)/p}\bigl[\varphi(r_j^{-1})\bigr]^{-1/p}\left(\int_{B(x_j,r_j)} \Phi_p(|\nabla u(z)|)\,dz\right)^{1/p}$$

when $\frac{3}{2}r_j < \varrho(x) < \frac{5}{2}r_j$, where $\varrho(x) = \bigl|1-|x|\bigr|$. Hölder's inequality gives

$$\begin{aligned}
|u(x_{j+m})-u(x_j)| &\leqq M\sum_{l=j}^{j+m} r_l^{1-\delta} + \sum_{l=j}^{j+m} r_l^{\omega}\bigl[\varphi(r_l^{-1})\bigr]^{-1/p} \\
&\quad \times \left(\int_{B(x_l,r_l)} \Phi_p(|\nabla u(z)|)\varrho(z)^{\alpha}\,dz\right)^{1/p} \\
&\leqq Mr_j^{1-\delta} + M\left(\sum_{l=j}^{j+m} r_l^{p'\omega}\bigl[\varphi(r_l^{-1})\bigr]^{-p'/p}\right)^{1/p'} \\
&\quad \times \left(\sum_{l=j}^{j+m}\int_{B(x_l,r_l)} \Phi_p(|\nabla u(z)|)\varrho(z)^{\alpha}\,dz\right)^{1/p} \\
&\leqq Mr_j^{1-\delta} + M\kappa(r_j)\left(\int_{\mathbf{B}-B(0,1-3r_j)} \Phi_p(|\nabla u(z)|)\varrho(z)^{\alpha}\,dz\right)^{1/p},
\end{aligned}$$

where $\omega = (p-n-\alpha)/p$. Since $\lim_{r\to 0}\kappa(r) = 0$, we see that $\{u(x_j)\}$ is a Cauchy sequence. Thus (6) implies that $\lim_{r\to 1} u(r\xi)$ exists and is finite for every $\xi \in \mathbf{S}$. Now, letting $m \to \infty$, we have

$$\begin{aligned}
|U(x_j)| &= |u(x_j)-u(\xi)| \\
&\leqq Mr_j^{1-\delta} + M\kappa(r_j)\left(\int_{\mathbf{B}-B(0,1-3r_j)} \Phi_p(|\nabla u(z)|)\varrho(z)^{\alpha}\,dz\right)^{1/p}.
\end{aligned}$$

Hence we establish

$$|U(x)| \leqq M\varrho(x)^{1-\delta} + M\kappa\big(\varrho(x)\big)\left(\int_{\mathbf{B}-B(0,1-3\varrho(x))} \Phi_p(|\nabla u(z)|)\varrho(z)^{\alpha}\,dz\right)^{1/p}.$$

Note that

$$\kappa(r) \geqq \left[\int_{r/2}^{r} \big(t^{n-p+\alpha}\varphi(t^{-1})\big)^{-1/(p-1)}\,\frac{dt}{t}\right]^{1-1/p} \geqq \big[r^{n-p+\alpha}\varphi(r^{-1})\big]^{-1/p}.$$

If we take δ so that $(n-p+\alpha)/p + 1 > \delta > 0$, then

$$\lim_{r\to 0}\big[\kappa(r)\big]^{-1} r^{1-\delta} = 0.$$

Now it follows that

$$\limsup_{|x|\to 1}\big[\kappa(\varrho(x))\big]^{-1}|U(x)| = 0,$$

as required.

The above proof also shows the following (see [7] and [9]).

Proposition 1. *Let* u *be a monotone function on* $\mathbf{B}$ *satisfying* (4) *with* $-1 < \alpha \leqq p-n$. *If* $\kappa(1) < \infty$, *then* u *has a finite nontangential limit at every* $\xi \in \mathbf{S}$.

Proposition 2. *Let* u *be a monotone function on* $\mathbf{B}$ *satisfying* (4) *with* $n-1 < p \leqq n+\alpha$. *If* $\kappa(1) = \infty$, *then*

$$\lim_{r\to 1}(1-r)^{-\omega}\big[\varphi\big((1-r)^{-1}\big)\big]^{1/p} S_q(U_r) = 0$$

whenever $p \leqq q < \infty$ *and* $\omega = (n-1)/q - (n-p+\alpha)/p > 0$.

4. Sharpness

1. *The sharpness of the exponent* $-\omega$. Let $-1 < \alpha < p-1$. For $\delta > 0$, consider the function u on $\mathbf{B}$ defined by

$$u(x) = (1-|x|)^{1+a}|x-e|^{-b},$$

where $a = \delta - (\alpha+1)/p$, $b = (n-1)/p$ and $e = (1,0,\dots,0)$. Then the function u is monotone on $\mathbf{B}$. To show this, let D be a relatively compact open set with the closure $\overline{D} \subset \mathbf{B}$, and suppose u attains a maximum on $\overline{D}$ at an interior point $c \in D$, and set

$$E = \{x \in \overline{\mathbf{B}} : (1-|x|)^{1+a} = u(c)|x-e|^{b}\}.$$

Since $e \in E$,

$$\max_{\overline{D}} u(x) = \max_{\partial D} u(x)$$

holds. We also see that u attains a minimum on $\overline{D}$ at a boundary point. Hence u is monotone on $\mathbf{B}$. Further we have

$$\int_{\mathbf{B}} |\nabla u(x)|^p \varrho(x)^\alpha \, dx < \infty.$$

If $k(x) = |x-e|^{-b}$, then

$$S_q(u_r) = M(1-r)^{1+a} S_q(k_r) \geqq M(1-r)^{1+a+(1/q-1/p)(n-1)} = M(1-r)^{\omega+\delta}.$$

This implies that the exponent $-\omega$ is sharp in Theorem 1.

2. *The limits for BLD functions.* Theorem 1 fails to hold for BLD functions, when $\alpha < 0$ and $(n-p+\alpha)/p < (n-1)/q < (n-p)/p$.

For $0 < r_j < 1$ and $\gamma > 1$, set

$$C_j = \{x : |x - r_j e| \leqq (1-r_j)^\gamma\},$$

where $|e| = 1$. We take $\{r_j\}$ such that $\{C_j\}$ are mutually disjoint. For $\varphi \in C_0^\infty\big(B(0,1)\big)$ such that $\varphi = 1$ on $B\big(0,\frac{1}{2}\big)$, define

$$u_j(x) = \varphi(|x - r_j e|/(1-r_j)^\gamma)|x - r_j e|^{-a},$$

where

$$(n-p+\alpha)/p < a < (n-p+\alpha/\gamma)/p. \tag{7}$$

Then

$$\int_{\mathbf{B}} |\nabla u_j(x)|^p (1-|x|)^\alpha \, dx \leqq M(1-r_j)^{\alpha+\gamma(n-(a+1)p)}.$$

Note further that

$$S_q\big((u_j)_{r_j}\big) \geqq c(1-r_j)^{\gamma(-a+(n-1)/q)}$$

with a positive constant $c > 0$, when $a < (n-1)/q$. If we set $u = \sum_j u_j$, then

$$\lim_{j\to\infty} (1-r_j)^{-\omega} S_q(u_{r_j}) = \infty,$$

when a and γ are chosen so that

$$(n-p+\alpha/\gamma)/p + (1-\gamma^{-1})\{(n-1)/q - (n-p)/p\} < a < (n-1)/q. \tag{8}$$

Now we suppose that a satisfies (7) and (8). Then, since $\alpha+\gamma\big(n-(a+1)p\big)>0$, we can take $\{r_j\}$ so that

$$\int_{\mathbf{B}} |\nabla u(x)|^p(1-|x|)^\alpha\,dx<\infty.$$

Thus u satisfies (2) and

$$\limsup_{r\to 1}(1-r)^{-\omega}S_q(U_r)=\infty.$$

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