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THE SIDE-PAIRING ELEMENTS OF MASKIT'S FUNDAMENTAL DOMAIN FOR THE MODULAR GROUP IN GENUS TWO

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Abstract. In this paper we study the hyperbolic geometry on a genus 2 surface. The main object of study is a subset of the set of hyperbolic lengths of closed geodesics on such a surface which arises from an algorithmic choice of shortest loops. Maskit has shown that this data can be used to identify finite sided polyhedral fundamental set for the modular group on the marked hyperbolic surface structures of a given genus. The special nature of genus 2 has made it more accessible than in higher genus and we are able to produce a more detailed picture of the domain and its side-pairing transformations. If the domain can be shown to satisfy certain basic topological criteria, according to a classical theorem of Poincaré, then this would give a set of geometrical generators and relations for the modular group.

0. Introduction

In this paper we study the structural properties of hyperbolic geometry on a genus 2 surface i.e. the crystallographic properties of the Fuchsian groups which uniformise such a surface. Our primary tool, following on from important work of Bernard Maskit ([20]), is a detailed analysis of a type of subset of the set of hyperbolic lengths of closed geodesics on such a surface, which arises from an algorithmic choice of shortest loops in the surface. Maskit shows that this data may be used to identify a finite sided polyhedral fundamental set for the action of the (Teichmüller) modular group on the space of all marked hyperbolic surface structures of a given genus. In genus 2, this action has proved to be more accessible than in higher genus and we are able to produce a more detailed picture of the domain and its side-pairing transformations. If the domain can be shown to satisfy certain basic topological criteria, according to a classical theorem of Poincaré, extended to general discrete group actions, this would then give a set of geometrical generators and relations for the modular group itself.

Maskit's construction in the special case of genus 2 is as follows. Choose a sequence of 4 non-dividing geodesic loops on the surface satisfying the following intersection property: the second loop intersects the first loop in a single point;

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the third loop intersects the second loop in a single point, but does not intersect the first; the fourth loop intersects the third loop in a single point, but does not intersect either the first loop or the second loop. We call such an ordered sequence of loops a standard chain. Cutting the surface open along a standard chain we obtain a topological disc and so a standard chain gives a marking for the surface. So our surface, standard chain pair represents a point in Teichmüller space. Now if each choice of geodesic loop was a shortest possible then we say that the standard chain is minimal. We say a surface, standard chain pair lies in the Maskit domain if the standard chain is minimal.

We wish to consider the intersections of translates of the Maskit domain. Consider an element of the mapping class group. The image of a standard chain under this element is an ordered sequence of loops on the surface. Taking the unique geodesics in the homotopy classes of these loops we obtain another standard chain on the surface. If there exists a surface with both of these standard chains minimal then the Maskit domain and its translate under this mapping class element has non-empty intersection. So solving the problem of which translations have non-empty intersection with the Maskit domain becomes the problem of finding the complete set of allowable minimal standard chain pairs. Due to the special nature of genus 2 surfaces it is known that sequential loops in a standard chain intersect at one of the six Weierstraß points on the surface—the fixed points of the unique hyperelliptic involution that each genus 2 surface exhibits. Theorem 1.1 states that distinct loops in a pair of minimal chains are either disjoint or intersect at Weierstraß points.

Our characterisation of the side-pairing elements of the Maskit domain in genus 2 is as follows: if the Maskit domain has non-empty intersection with a translate under the mapping class group, then this intersection contains a copy of one or other of two special surfaces. One of these special surfaces is the well-known genus 2 surface with maximal symmetry group. The other special surface does not seem to have appeared in the literature before; it is unusual in that it is not defined by its symmetry group alone, it also requires a certain length equality between geodesic loops to be satisfied. From this characterisation it is a combinatorial exercise to obtain a complete list of mapping class elements that are side-pairing elements of the Maskit domain.

We organise the paper as follows. We begin with general preliminaries concerning genus 2 surfaces and the particular model for Teichmüller space that we adopt. With respect to this model we then repeat Maskit's definition for a fundamental domain for the Teichmüller modular group. We then construct a one-parameter family of genus 2 surfaces. Two distinguished members of this family are the two special surfaces that feature in our main result. We then show how the main result can be used to give a full list of side-pairing elements of the Maskit domain. We then have the two main technical parts of the paper. In the first we prove the main result under the assumption of Theorem 1.1. In the second we prove Theorem 1.1. We have chosen this order so as to centre the paper on the geometry of the two special surfaces. Moreover we apply results from the first part in the second.

The history of defining a fundamental domain for Mod_g for $g \geq 2$ goes back to the rough domains of Keen [11]. Maskit covers certain low signature surfaces in his papers [17], [18]. In his doctoral thesis Semmler defined a fundamental domain for closed genus 2 surfaces, based upon locating the shortest dividing geodesic. Recently McCarthy and Papadopoulos [21] have defined a fundamental domain based on the classical Dirichlet construction. For surfaces with one or more punctures there are known triangulations of Teichmüller space. Associated to these are combinatorial fundamental domains—see Harer [14] for an overview of this work. An eventual goal of this work is to give geometrical presentation of the mapping class group in genus 2. The first presentation of the mapping class group in genus 2 was obtained by Birman and Hilden [4] completing the program begun by Bergau and Mennicke [3]. For higher genus surfaces see Hatcher and Thurston [15]. Part of the author's inspiration for this work came from reading Thurston's note [25].

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1. Preliminaries

Throughout our model for the hyperbolic plane $\mathbf{H}^2$ will be the interior of the unit circle of the complex plane with a metric of constant curvature -1. Likewise $\mathscr{S}$ will always denote an oriented closed surface of genus 2. The Teichmüller space of genus 2 surfaces $\mathscr{T}$ is the space of hyperbolic metrics on $\mathscr{S}$ up to isometries that are isomorphic to the identity. Without further mention, all genus 2 surfaces $\mathscr{S}$ will be oriented and endowed with a hyperbolic metric.

Let γ denote a simple closed geodesic on $\mathscr{S}$. We say that γ is dividing if $\mathscr{S}\setminus\gamma$ has two components and non-dividing if $\mathscr{S} \setminus \gamma$ has one component. Throughout the paper '$\setminus$' denotes 'set minus' and by 'non-dividing geodesic' we shall always mean 'simple closed non-dividing geodesic'.

We define a *chain* to be an ordered set of n non-dividing geodesics $\mathscr{A}_n = \alpha_1, \ldots, \alpha_n$ on $\mathscr{S}$ such that: $|\alpha_i \cap \alpha_j| = 1$ for $|i-j| = 1$ and $\alpha_i \cap \alpha_j = \emptyset$ for $|i-j| \geq 2$, where $1 \leq n \leq 5$ and $1 \leq i, j \leq n$. A *necklace* is an ordered set of 6 non-dividing geodesics $\mathscr{A}_6 = \alpha_1, \ldots, \alpha_6$ on $\mathscr{S}$ such that: $|\alpha_i \cap \alpha_j| = 1$ for $|i-j| \bmod 6 = 1$ and $\alpha_i \cap \alpha_j = \emptyset$ for $|i-j| \bmod 6 \geq 2$, where $1 \leq i, j \leq 6$. We call the geodesics in a chain or necklace the *links* and we call n, the number of links in a chain, the *length* of the chain. We note that any length 4 chain extends uniquely to a chain of length 5 and that any chain of length 5 extends uniquely to a necklace, so chains of length 4 and 5 and necklaces can be considered equivalent. We call a chain of length 4 *standard* and will denote it by $\mathscr{A}$.

To a surface, standard chain pair $\mathscr{S}, \mathscr{A}$ Maskit associates discrete faithful representation of $\pi_1(\mathscr{S})$ into $\mathrm{PSL}(2, \mathbf{R})$; see [20, p. 376]. It is well known that there is a real-analytic diffeomorphism between $\mathscr{DF}\big(\mathscr{S}, \mathrm{PSL}(2,\mathbf{R})\big)$ and the Teichmüller space $\mathscr{T}$ (see Abikoff [1]); this diffeomorphism was given explicitly by Maskit in [19]. So there is a one-to-one correspondence between pairs $\mathscr{S}, \mathscr{A}$ and points in $\mathscr{T}$.

We define a chain $\mathscr{A}_n = \alpha_1, \ldots, \alpha_n$ to be *minimal* if α_1 is a shortest non-dividing geodesic and if, for any α'_m such that $\mathscr{A}'_m = \alpha_1, \ldots, \alpha_{m-1}, \alpha'_m$ is a chain, we have that $l(\alpha_m) \leq l(\alpha'_m)$ for $2 \leq m \leq n$.

Firstly, minimal standard chains exist. To see this we use the fact that given any $L > 0$ there are only finitely many closed geodesics on $\mathscr{S}$ that have length $\leq L$ (see Buser [2, p. 27]). An elementary consequence of this fact is that there are only finitely many shortest non-dividing geodesics; we choose one of them and label it by α_1. Choose a non-dividing geodesic that intersects α_1 exactly once. There are only finitely many shorter non-dividing geodesics with the same intersection property. Choose a shortest and label it by α_2. And so on, until we have a minimal standard chain.

Following Maskit we then define $\mathscr{D} \subset \mathscr{T}$, the *Maskit domain*, to be the set of surface, standard chain pairs $\mathscr{S}, \mathscr{A}$ with $\mathscr{A}$ minimal. By the above construction a generic genus 2 surface has exactly one minimal standard chain and so a unique representative on the interior of $\mathscr{D}$. Maskit also shows that the set of surfaces with more than one minimal standard chain has measure zero in $\mathscr{T}$ and hence that the boundary of $\mathscr{D}$ has measure zero. Maskit also gives a proof that the tesselation of $\mathscr{T}$ by $\mathscr{D}$ is locally finite. Maskit then observes that $\mathscr{D}$ satisfies the classical prerequisites to be a fundamental domain for the action of the *Teichmüller modular group*, or *mapping class group*, Mod on $\mathscr{T}$.

The main question addressed in Maskit's paper [20] and the author's paper [13] is the following: given a standard chain, what set of length inequalities must it satisfy in order to be minimal? Maskit, for any genus g, shows that this set is finite and, for genus 2, shows that its cardinality is at most 45. In [13] the author improved this number to 27. The author is confident that this set of inequalities is optimal.

In this paper we examine the tesselation of $\mathscr{T}$ by $\mathscr{D}$. More precisely we consider the elements $\phi \in \mathrm{Mod}$ that have the property $\phi(\mathscr{D}) \cap \mathscr{D} \neq \emptyset$, what we call *side-pairing elements* of $\mathscr{D}$. Let $\phi \in \mathrm{Mod}$ be a side-pairing element and choose some point $\mathscr{S} \in \phi(\mathscr{D}) \cap \mathscr{D}$. So $\mathscr{S}$ has minimal standard chains $\mathscr{A}, \mathscr{B}$ associated to $\mathscr{D}, \phi(\mathscr{D})$, respectively. Here $\mathscr{B} = \beta_1, \ldots, \beta_4$ where $\beta_i = [\phi(\alpha_i)]$, the geodesic in the free homotopy class of $\phi(\alpha_i)$. That is associated to any side-pairing element of $\mathscr{D}$ there is an ordered pair of minimal standard chains $\mathscr{A}, \mathscr{B}$ on some surface $\mathscr{S}$.

Conversely given an ordered pair of minimal standard chains $\mathscr{A}, \mathscr{B}$ on $\mathscr{S}$ there is an associated side-pairing element of $\mathscr{D}$. It suffices to calculate a representative ϕ of the unique mapping class such that $\beta_i = [\phi(\alpha_i)]$ for $i \in \{1, \ldots, 4\}$. The natural basis for this calculation is $\{\tau_i\}$ for $1 \leq i \leq 6$ where τ_i denotes a left Dehn twist about the link α_i in the necklace $\mathscr{A}_6$.

The main fact that enables us to study minimal standard chain pairs is the following: every genus 2 surface $\mathscr{S}$ exhibits a unique involution, the *hyperelliptic involution* $\mathscr{J}$. This order 2 isometry has six fixed points, the *Weierstraß points.* Moreover $\mathscr{J}$ fixes any simple closed geodesic γ on $\mathscr{S}$, the action of $\mathscr{J}$ on γ being classified by the topological type of γ. The restriction of $\mathscr{J}$ to γ has no fixed points if γ is dividing and two fixed points if γ is non-dividing (see Haas–Susskind [8]). It is a simple consequence that sequential links in a chain intersect at Weierstraß points. We say that two distinct non-dividing geodesics *cross* if they intersect in a point that is not a Weierstraß point, and we say that two chains cross if a link in one chain crosses a link in the other. We have that:

Theorem 1.1. *Minimal standard chains do not cross.*

Corollary 1.2. *There are only finitely many side-pairing elements.*

Proof of Corollary 1.2. Let $\mathscr{A}$ be a standard chain on $\mathscr{S}$. It is enough to show that there are only finitely many other standard chains $\mathscr{B}$ on $\mathscr{S}$ that do not cross $\mathscr{A}$. This follows since there are only finitely many non-dividing geodesics that do not cross $\mathscr{A}$. □

An application of Theorem 1.1 is that $\tau_{i-1} \circ \tau_{i+1}$ is not a side-pairing element for $1 \leq i \leq 4$, subscript addition modulo 6. Let $\mathscr{A}$ be a standard chain and let $\mathscr{B} = \tau_{i-1} \circ \tau_{i+1}(\mathscr{A})$. Now $\beta_i = [\tau_{i-1} \circ \tau_{i+1}(\alpha_i)]$ crosses α_i (see Subsection 2.1 where we perform similar calculations). So, by Theorem 1.1, $\mathscr{A}, \mathscr{B}$ cannot both be minimal.

Given surfaces $\mathscr{S}, \mathscr{S}'$ with pairs of minimal standard chains $\mathscr{A}, \mathscr{B}$ and $\mathscr{A}', \mathscr{B}'$, respectively, we say that $\mathscr{A}, \mathscr{B}$ on $\mathscr{S}$ is *equivalent* to $\mathscr{A}', \mathscr{B}'$ on $\mathscr{S}'$ if there exists a homeomorphism $\Psi\colon \mathscr{S} \to \mathscr{S}'$ such that $[\Psi(\mathscr{A})] = \mathscr{A}'$, $[\Psi(\mathscr{B})] = \mathscr{B}'$. Our main result in this paper is:

Theorem 1.3. *Any minimal standard chain pair is equivalent to a minimal standard chain pair on* $\mathscr{O}ct$ *or* $\mathscr{E}$.

In Subsection 1.2 we construct $\mathscr{O}ct$ and $\mathscr{E}$ as members of a one-parameter family of surfaces—each satisfying a certain length equality. Whilst $\mathscr{E}$ does not seem to have appeared in the literature before, $\mathscr{O}ct$ is the well-known genus 2 surface of maximal symmetry group.

A simple consequence of Theorem 1.3 is that if $\varphi(\mathscr{D}) \cap \mathscr{D} \neq \emptyset$ then $\phi(\mathscr{D}) \cap \mathscr{D} \ni \mathscr{O}ct$ or $\mathscr{E}$. Suppose $\phi(\mathscr{D}) \cap \mathscr{D} \ni \emptyset$. Choose a point $\mathscr{S} \in \phi(\mathscr{D}) \cap \mathscr{D}$. By the construction above, there exist a minimal standard chain pair $\mathscr{A}, \mathscr{B}$ on $\mathscr{S}$ such that $\mathscr{B} = \phi(\mathscr{A})$. By Theorem 1.3, $\mathscr{A}, \mathscr{B}$ on $\mathscr{S}$ is equivalent to $\mathscr{A}', \mathscr{B}'$ on $\mathscr{O}ct$ or $\mathscr{E}$. It follows that $\phi(\mathscr{D}) \cap \mathscr{D} \ni \mathscr{O}ct$ or $\mathscr{E}$.

The main complaint about the proofs of Theorems 1.3 and 1.1 is that they are based on a case-by-case analysis. That is, we consider cases and derive contradictions using length inequality results for systems of non-dividing geodesics. The majority of the paper is devoted to the proofs of these results. Unfortunately the author has yet to derive a more satisfactory approach.

1.1. Some notation and nomenclature. All of the hyperbolic formulae we use can be found in Buser [2, p. 454]. Given a pair of points X, Y in $\mathbf{H}^2$ we shall use $d(X,Y)$ to denote the distance between them. For X, Y distinct we shall use $\perp XY$ to denote the bisector of X, Y—the set of points $Z \in \mathbf{H}^2$ such that $d(Z,X) = d(Z,Y)$. Given a triplet of distinct points X, Y, Z in $\mathbf{H}^2$ we shall use $\angle XYZ$ to denote the angle at the Y vertex of the triangle spanned by X, Y, Z. By a *trirectangle* we shall mean a compact hyperbolic quadrilateral with three right angles. By a *birectangle* we shall mean a compact hyperbolic quadrilateral with two adjacent right angles. We shall use curly brackets $\{*,*,*\}$ to indicate unordered sets and round brackets $(*,*,*)$ to indicate ordered sets.

1.2. Special surfaces. Suppose we have a trirectangle with acute angle $\pi/4$. Label the edges incident upon the $\pi/4$ vertex α, β and the edge opposite α (respectively β) by a (respectively b). We label the diagonal from vertex $\alpha \cap \beta$ to the vertex $a \cap b$ by c. Let θ_a denote the angle between a, c, et cetera—see Figure 1. We shall abuse notation by using the same symbol as an edge or diagonal to denote its length. We denote such a trirectangle by $\mathcal{Q}_\alpha$.

Lemma 1.4. *For any given $a > \cosh^{-1}(\sqrt{2})$ there exists such a trirectangle $\mathcal{Q}_\alpha$. Moreover there exist $\mathcal{Q}_\alpha$ such that $c = 2a$ and $c = 2\alpha$.*

Proof. Firstly a triangle in the hyperbolic plane $\mathbf{H}^2$ with angles $\pi/4$, $\pi/2$, 0 has finite edge (between the $\pi/4$ vertex and the $\pi/2$ vertex) length $\cosh^{-1}\sqrt{2}$.

Consider three geodesics such that the first geodesic intersects the second at an angle $\pi/4$ and the second intersects the third at an angle $\pi/2$. Let α denote the distance between these intersections. By the above calculation if $\cosh\alpha = \sqrt{2}$ the three geodesics bound a $\pi/4, \pi/2, 0$ triangle. So for $\cosh\alpha > \sqrt{2}$ there exists a unique common perpendicular between the first and third geodesics. The three geodesics and this common perpendicular now bound a trirectangle.

We now want to show that there exist trirectangles such that $c = 2a$ and $c = 2\alpha$. By the above we consider the range $2 < \cosh^2\alpha < \infty$. A simple calculation gives

$$\cosh^2 c - \cosh^2 2a = -\frac{(\cosh^2\alpha - 1)(\cosh^4\alpha - 4\cosh^2\alpha + 2)}{\cosh^2\alpha - 2}.$$

This expression has exactly one root in the range, $\cosh^2\alpha = 2 + \sqrt{2}$. Similarly

$$\cosh^2 c - \cosh^2 2\alpha = -\frac{4\cosh^6\alpha - 13\cosh^4\alpha + 10\cosh^2\alpha - 2}{\cosh^2\alpha - 2}.$$

Again this expression has exactly one root in the range. Consider the polynomial in the numerator as a polynomial in $\cosh^2\alpha$. This polynomial has a root between 2 and 3, and its turning points lie at $\frac{1}{2}, \frac{5}{3}$. So there exist unique trirectangles such that $c = 2a$ and $c = 2\alpha$. □

We are going to define a fundamental domain in terms of the tesselation of $\mathbf{H}^2$ by $\mathscr{Q}_\alpha$. In Figure 1 we have pictured part of this tesselation, generated by reflecting in each edge. Consider the copy of $\mathscr{Q}_\alpha$ with its edges and diagonal labelled—i.e. in the negative real, negative imaginary quadrant with its β edge along the real axis. Starting at the $a\cap b$ vertex of this trirectangle, in the direction of the a edge: walk a distance $4a$; turn right through an angle $\pi-\theta_a$; walk a distance c; turn right through an angle $\pi/2$; walk a distance c; and turn right through an angle θ_a. Repeat this sequence 3 more times to close the path.

Let Ω_α denote the domain circumscribed by this path. Label the sides of Ω_α in the order we have walked round them by S_1, S_3', S_4, S_6', S_4', S_2', S_1', S_2, S_5', S_6, S_5, S_3. Define side-pairing elements $g_i\in\mathrm{PSL}(2,\mathbf{R})$ for Ω_α so that $g_i(S_i)=S_i'$ for $1\le i\le 6$. This identification pattern has three length 4 vertex cycles—each with angle sum 2π. It is the same identification pattern as that given by Maskit when constructing a discrete faithful representation to a surface, standard chain pair—see [20, p. 376]. So we obtain a genus 2 surface, with a complete hyperbolic metric, which we shall denote by $\mathscr{S}_\alpha$. We define the *octahedral surface* $\mathscr{O}ct$ (respectively *exceptional surface* $\mathscr{E}$) to be $\mathscr{S}_\alpha$ with $\mathscr{Q}_\alpha$ such that $c=2a$ (respectively $c=2\alpha$).

We need to label a distinguished set of non-dividing geodesics on $\mathscr{S}_\alpha$. Label by ω_0, ω_3, ω_0, ω_4, ω_2, ω_1, ω_2, ω_4, ω_0, ω_3, ω_0, ω_4, ω_2, ω_1, ω_2, ω_4 the orbits of $a\cap b$ and $\alpha\cap\beta$ on the boundary of Ω_α in the order that we walked them and label the origin by ω_5. Using the index sets $k=0,1,2,3$, $l=4,5$ and modulo 4 addition label by $\kappa_{k,k+1}$ (respectively $\kappa_{k,l}$) the union of orbits of a or b (respectively c) in Ω_α passing through ω_k, ω_{k+1} (respectively ω_k,ω_l). Label by λ_k the union of orbits of α or β that intersect $\kappa_{k,k+1}$. Using the generators g_i it is a simple exercise to check that each one of $\kappa_{k,k+1}$, $\kappa_{k,l}$, λ_k projects to a non-dividing geodesic on $\mathscr{S}_\alpha$. Likewise it is easy to check that each set of points ω_i projects to a single point on $\mathscr{S}_\alpha$, a Weierstraß point.

Proposition 1.5. *The set $\bigcup\kappa_{k,k+1}\cup\kappa_{k,l}$ is the set of shortest non-dividing geodesics on $\mathscr{O}ct$. The set $\kappa_{1,2}\cup\kappa_{3,0}$ (respectively $\bigcup\kappa_{k,l}\cup\lambda_0\cup\lambda_2$) is the set of shortest (respectively second shortest) non-dividing geodesics on $\mathscr{E}$.*

It is a simple consequence that minimal chains on $\mathscr{O}ct$ (respectively $\mathscr{E}$) lie in the set of shortest (respectively shortest and second shortest) non-dividing geodesics.

Proof. Consider $\mathscr{E}$. By definition $c=2\alpha$. It follows that $2\theta_b=\theta_\alpha$ and hence $\alpha<b$. By elementary geometry $a<\alpha$, $b<\beta$ and so $a<\alpha<b<\beta$.

Take an open disc D_5 (a circle C_5) of radius $c=2\alpha$ centred on ω_5. No other orbit of a Weierstraß point lies in D_5. Around C_5, since $c=2\alpha$, there are orbits of ω_k and of ω_4 in diametrically opposite pairs. The diameter between the ω_k pair projects to $\kappa_{k,5}$. The diameters between ω_4 pairs project to λ_0 and λ_2. So this is the set of shortest non-dividing geodesics passing through ω_5. Likewise for ω_4.

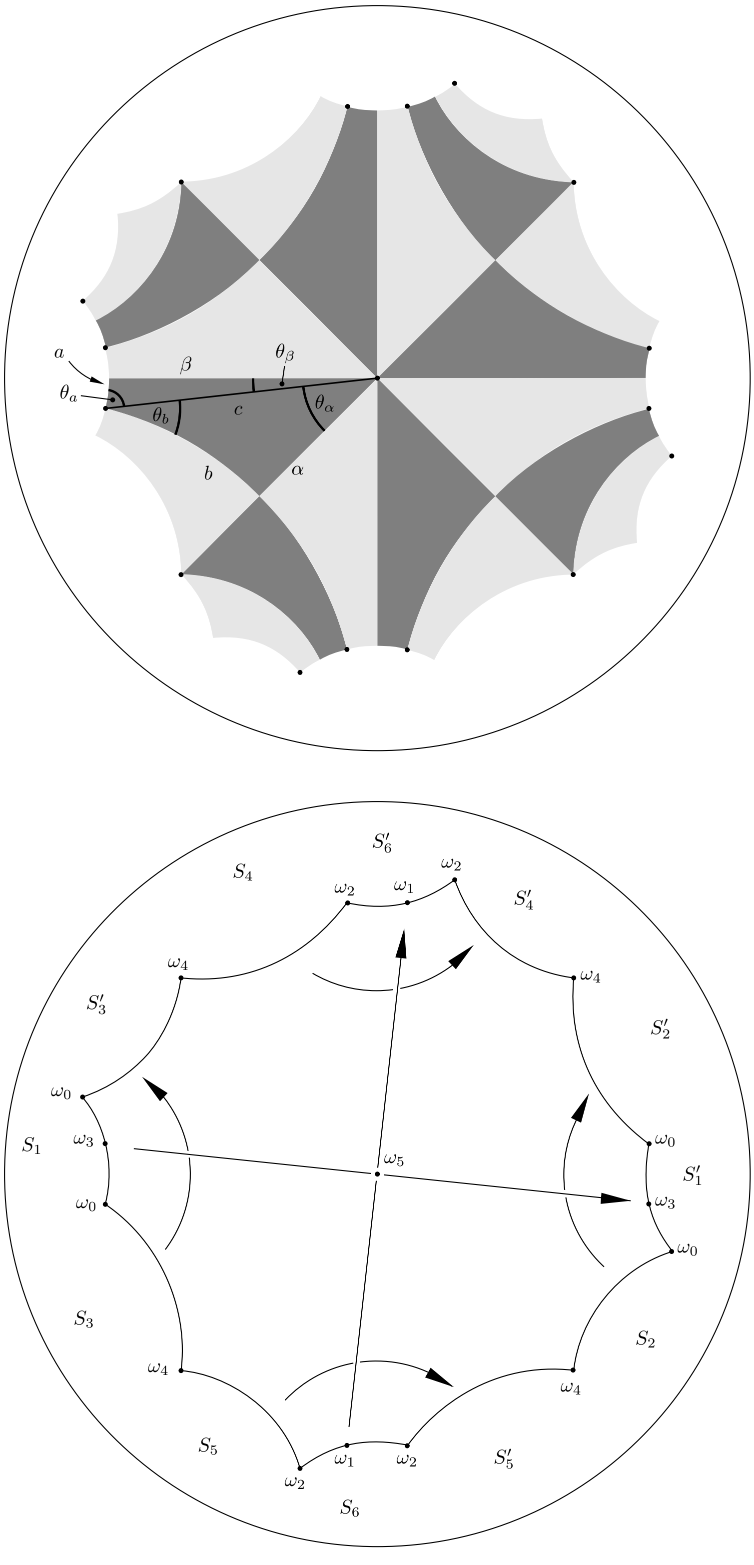

Figure 1. Construction of the one-parameter family of surfaces.

Now consider an open disc D_0 (a circle C_0) of radius $2a$ centred on an orbit of ω_0. No other orbit of a Weierstraß point lies in D_0 and there is a diametrically opposite pair of orbits of ω_3 on C_0. No other orbits of Weierstraß points lie on C_0 since $2a < c = 2\alpha$. So $\kappa_{3,0}$, the image of the diameter between the ω_3 pair, is the shortest non-dividing geodesic passing through ω_0. Let C_0' denote a circle of radius $c = 2\alpha$ about ω_0. There are orbits of ω_l in diametrically opposite pairs, projecting to $\kappa_{0,l}$. There are no orbits of ω_1 or ω_2 on C_0'. The nearest such orbit point is at a distance $2b > c = 2\alpha$. So $\kappa_{0,l}$ is the set of second shortest non-dividing geodesics passing through ω_0. Likewise for ω_1, ω_2, ω_3.

We now consider $\mathscr{O}ct$. By definition: $c = 2a$ and so $\theta_a = 2\theta_\beta$. Suppose that $a < b$. From the formulae we get that $\alpha < \beta$ and hence that $\theta_a > \theta_b$ and $\theta_\beta < \theta_\alpha$. It follows that $\theta_a > \pi/4$ and $\theta_\beta < \pi/8$ giving a contradiction. Likewise for $a > b$. So $a = b$.

Take an open disc D_5 (a circle C_5) of radius $c = 2a = 2b$ centred on ω_5. No other orbit of a Weierstraß point lies in D_5. Since $c = 2a < 2\alpha$ orbits of ω_4 lie outside C_5. Around C_5 there are orbits of ω_k in diametrically opposite pairs. Again the diameter between the ω_k pair projects to $\kappa_{k,5}$ and so this is the set of shortest non-dividing geodesics passing through ω_5. Likewise for ω_4.

Now consider an open disc D_1 (a circle C_1) of radius $c = 2a = 2b$ centred an orbit of ω_1. No other orbit of a Weierstraß point lies in D_1. Around C_1 there are orbits of ω_k for $k = 0, 2$ and ω_l in diametrically opposite pairs. Again the diameter between the ω_k (respectively ω_l) pair projects to $\kappa_{0,1}$ or $\kappa_{1,2}$ (respectively $\kappa_{1,l}$) and so this is the set of shortest non-dividing geodesics passing through ω_1. Likewise for ω_2, ω_3, ω_0. □

2. Listing of side-pairing elements and proof of Theorem 1.3

In this section show how a listing of side-pairing elements can be generated and prove Theorem 1.3 under the assumption of Theorem 1.1. All minimal chain pairs may be assumed to be non-crossing.

We say that minimal standard chain pair $\mathscr{A}, \mathscr{B}$ is of *type* (I) (respectively *type* (II)) if there exist a pair of links Γ (respectively a triplet of links Υ) such that $\mathscr{S} \setminus \Gamma$ (respectively $\mathscr{S} \setminus \Upsilon$) has two components.

The basis of the proof of Theorem 1.3 is to show that a minimal standard chain of type (I) or (II) is equivalent to a standard minimal chain pair on $\mathscr{E}$. To show that a minimal standard chain of neither type (I) nor (II) is equivalent to a standard minimal chain pair on $\mathscr{O}ct$ is a combinatorial exercise.

We label Weierstraß points on $\mathscr{A}_6$ so that $\alpha_i \ni a_i, a_{i+1}$. Likewise for $\mathscr{B}_6$. Consider a permutation element $\sigma \in \mathscr{B}_6$. It is a combinatorial exercise to ennumerate non-equivalent pairs of minimal standard chains on $\mathscr{O}ct, \mathscr{E}$ associated to σ. To each of these pairs it is a simple calculation to write down the corresponding side-pairing element of the Maskit domain. We do these exercises for the identity Id and for $(i\, i+1)$ which exchanges a_i, a_{i+1} for $1 \leq i \leq 6$.

2.1. Listing of side-pairing elements of the Maskit domain. Let τ_i denote a *left Dehn twist* about α_i for $1 \leq i \leq 6$. It is well known that $\{\tau_i\}$ generates the mapping class group—for example Humphries [9] showed that $\{\tau_i\}$ for $1 \leq i \leq 5$ generates it. The action of τ_i on $\mathscr{A}_6$ is also well known. If $j = i$ or $|i-j| > 2$ then $[\tau_i(\alpha_j)] = \alpha_j$. For $j = i-1$ (respectively $j = i+1$) and $[\tau_i(\alpha_j)]$ is a non-dividing geodesic through a_j, a_{i+1} (respectively a_i, a_{j+1}) that does not cross $\mathscr{A}_6$. Moreover $\alpha_j \cup \alpha_i \cup [\tau_i(\alpha_j)]$ bounds a pair of triangles that are exchanged under $\mathscr{J}$. The geodesics α_j, α_i, $[\tau_i(\alpha_j)]$ lie in anticlockwise order around each triangle. Moreover we know that τ_i, τ_j commute if $|i-j| \geq 2$.

Consider $\mathscr{A}, \mathscr{B}$ on $\mathscr{E}$ given by $\alpha_1 = \beta_1 = \kappa_{3,0}$, $\alpha_2 = \beta_2 = \kappa_{0,4}$, $\alpha_3 = \lambda_2$, $\beta_3 = \lambda_0$ and $\alpha_4 = \beta_4 = \kappa_{2,5}$. It is associated to the identity permutation Id since $a_i = b_i$ for $1 \leq i \leq 6$. The corresponding side-pairing element is $\iota = (\tau_2 \circ \tau_1)^3$. We have illustrated the calculation to show that $\mathscr{B} = \iota(\mathscr{A})$ in Figure 2. The first picture shows $\tau_1(\mathscr{A})$; the second $\tau_2 \circ \tau_1(\mathscr{A})$; et cetera. We now note that $\Gamma = \alpha_3 \cup \beta_3$ is a pair of links that divide $\mathscr{S}$ into two components, i.e., $\mathscr{A}, \mathscr{B}$ is of type (I).

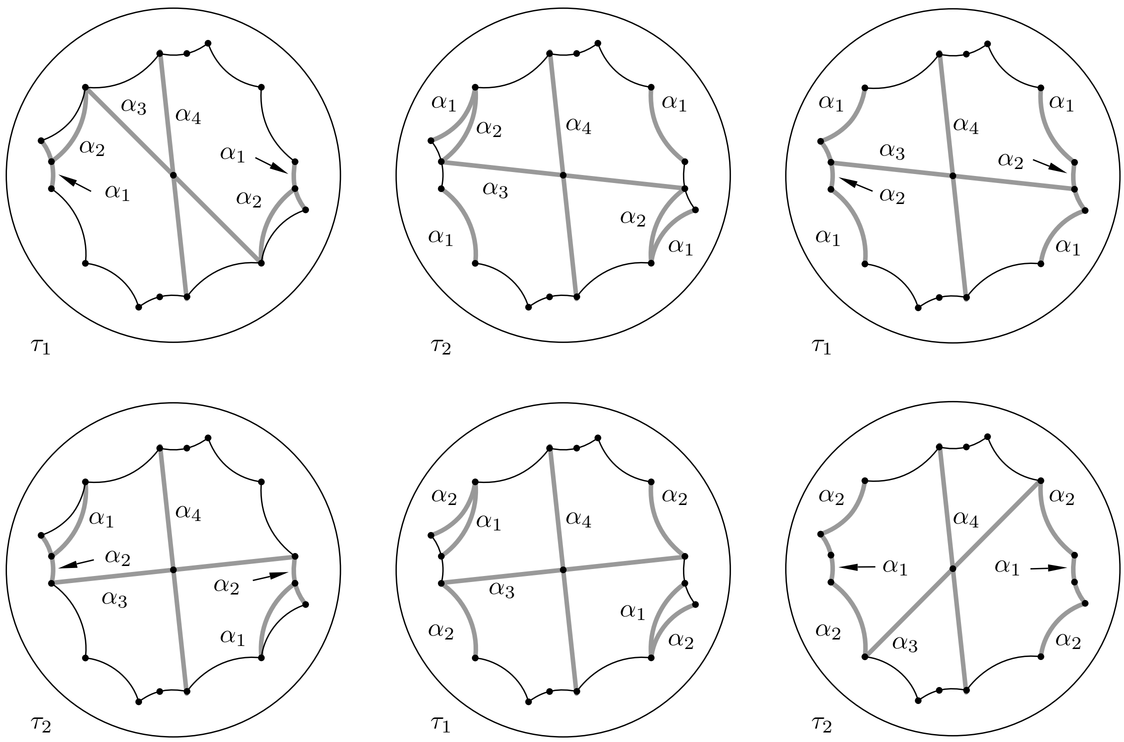

Figure 2. The action of $\iota = (\tau_2 \circ \tau_1)^3$ on the standard chain $\mathscr{A}$

Now consider $\mathscr{A}, \mathscr{B}$ on $\mathscr{E}$ given by $\alpha_1 = \beta_1 = \kappa_{3,0}$, $\alpha_2 = \kappa_{0,4}$, $\beta_2 = \kappa_{3,4}$, $\alpha_3 = \beta_3 = \lambda_2$ and $\alpha_4 = \beta_4 = \kappa_{2,5}$. Here $\mathscr{A}, \mathscr{B}$ is associated to (12) since $a_1 = b_2$, $a_2 = b_1$ and $a_i = b_i$ for $3 \leq i \leq 6$. The corresponding side-pairing element is τ_1.

Let $\Upsilon = \alpha_2 \cup \beta_1 \cup \beta_2$. We note that $\mathscr{S} \setminus \Upsilon$ has *three* components: two triangles and a torus with boundary component. Also associated to (12) is $\tau_1 \circ \iota$.

Next consider $\mathscr{A}, \mathscr{B}$ on $\mathscr{O}ct$ given by $\alpha_1 = \kappa_{3,0}$, $\beta_1 = \kappa_{0,4}$, $\alpha_2 = \beta_2 = \kappa_{3,4}$, $\alpha_3 = \kappa_{2,4}$, $\beta_3 = \kappa_{2,3}$ and $\alpha_4 = \beta_4 = \kappa_{1,2}$, which is associated to (23). The corresponding side-pairing element is τ_2.

Consider $\mathscr{A}, \mathscr{B}$ on $\mathscr{E}$ given by $\alpha_1 = \beta_1 = \kappa_{3,0}$, $\alpha_2 = \kappa_{0,5}$, $\beta_2 = \kappa_{0,4}$, $\alpha_3 = \beta_3 = \lambda_2$, $\alpha_4 = \kappa_{2,4}$ and $\beta_4 = \kappa_{2,5}$, which is associated to (34). The corresponding side-pairing element is $\tau_1^{-2} \circ \tau_3$. Let $\Upsilon = \alpha_2 \cup \beta_2 \cup \beta_3$. We note that $\mathscr{S} \setminus \Upsilon$ has two components: a quadrilateral disc and an annulus. So $\mathscr{A}, \mathscr{B}$ is of type (II). Also associated to (34) is τ_3, $\tau_3 \circ \tau_5^{-2}$, $\tau_1^{-2} \circ \tau_3 \circ \tau_5^{-2}$ and $\tau_3 \circ \iota$, $\tau_1^{-2} \circ \tau_3 \circ \iota$, $\tau_3 \circ \tau_5^{-2} \circ \iota$, $\tau_1^{-2} \circ \tau_3 \circ \tau_5^{-2} \circ \iota$.

Similarly τ_4 is associated to (45); τ_5, $\iota \circ \tau_5$ are associated to (56); and τ_6 is associated to (61). The reader can verify that—up to inverses—we have given each side-pairing element of the mapping class group associated to each of the stated permutation elements.

2.2. Projection to the quotient. The quotient of $\mathscr{S}$ by the hyperelliptic involution $\mathscr{J}$ is a sphere with six order two cone points $\mathscr{S}/\mathscr{J}$. By *orbifold* we shall always mean a sphere with six order two cone points and a fixed hyperbolic metric. We shall use $\mathscr{O}$ to denote an orbifold. For technical and pictorial reasons we shall work on the quotient orbifold for the rest of the paper.

The image of a non-dividing geodesic under projection $\mathscr{J}\colon \mathscr{S} \to \mathscr{O}$ is a simple geodesic between distinct cone points, what we shall call an *arc*. Likewise the image of a Weierstraß point under the projection $\mathscr{J}\colon \mathscr{S} \to \mathscr{O}$ is a cone point. Definitions of chains, necklaces, links and crossing all pass naturally to the quotient. We define a *bracelet* Υ to be a set of arcs that contains no crossing arcs, divides $\mathscr{O}$ and is such that no proper subset of Υ divides $\mathscr{O}$. As with chains, we call the arcs in a bracelet links and call the number of links the length of a bracelet. In particular a necklace is a bracelet of length 6.

A length 3 bracelet Υ always divides the orbifold into two components, dividing either: one cone point (c) from two; or no cone points from three. For the former we say that Υ *cuts off* c. For the latter we say that Υ *bounds a triangle* (the component of $\mathscr{O} \setminus \Upsilon$ containing no interior cone points).

On the double cover $\mathscr{S}$ the lift of Υ divides either: one Weierstraß point c from two; or no Weierstraß points from three. For the former, the single Weierstraß point c lies at the centre of the quadrilateral disc and the two Weierstraß points lie on the interior of the annulus. For the latter, neither triangular disc contains an interior Weierstraß point, whilst the torus with boundary component has three interior Weierstraß points.

We can now restate types (I), (II) on the quotient orbifold. We say that a standard minimal chain pair $\mathscr{A}, \mathscr{B}$ is of *type* (I) if it contains a length 2 bracelet. We say that a standard minimal chain pair $\mathscr{A}, \mathscr{B}$ is of *type* (II) if it contains a length 3 bracelet that cuts off a cone point.

Proof of Theorem 1.3. Consider a minimal standard chain pair $\mathscr{A}, \mathscr{B}$ on $\mathscr{O}$ that is of neither type (I) nor type (II).

We say that an arc set Γ is of *type* (III) if Γ contains no crossing arcs, each vertex of Γ has index at most four, Γ contains no length 2 bracelets and each length 3 bracelet in Γ bounds a triangle. So the arc set $\mathscr{A} \cup \mathscr{B}$ has property (III).

We say that an arc set Γ on $\mathscr{O}$ is *octahedral* if it is graph-isomorphic to a subgraph of the set of shortest arcs on $\mathscr{O}ct$. We will show that all arc sets of type (III) are octahedral. It follows that $\mathscr{A}, \mathscr{B}$ is equivalent to a standard minimal chain pair on $\mathscr{O}ct$.

Let Γ be an arc set of type (III). Suppose Γ has a vertex of index four. It is now a simple combinatorial exercise to show that Γ is octahedral. So each vertex of Γ has index at most three. Suppose Γ contains a bracelet of length 3. Again we can show that Γ is octahedral. So each bracelet is of length at least 4. Likewise for Γ containing bracelets of length 4, 5 and 6. So Γ is a tree and we can again show that it is octahedral. □

2.3. Arc and cone point labelling and pictorial conventions. In this subsection we define an arc system $K \cup \Lambda$ and explain our pictorial conventions. Most length inequality results are given in terms of subsets of this arc system. As its name suggests this arc system is related to the set of non-dividing geodesics we labelled in Subsection 1.2.

Let K be a set of 12 arcs that contains no crossing arcs and has the combinatorial pattern of the edge set of the octahedron. In particular any cone point has four arcs in K incident upon it. Label a pair of cone points having no K arc between them c_l for $l = 4, 5$. We think of c_4 as being at the South Pole and c_5 as being at the North Pole. We think of the other cone points as lying on the equator. We label them c_k for $k = 0, 1, 2, 3$ so that there is a K arc between c_k, c_{k+1}. Throughout the paper subscript addition for k will be modulo 4. Label the arcs in K so that $\kappa_{k,k+1}$ is between c_k, c_{k+1} and $\kappa_{k,l}$ is between c_k, c_l. We define λ_k to be the arc between c_4, c_5 that crosses only $\kappa_{k,k+1} \subset K$. Let $\Lambda = \cup \kappa_k$. We now note that the set of non-dividing geodesics we defined on the one-parameter family of surfaces $\mathscr{S}_\alpha$ projects to an arc set of the form $K \cup \Lambda$ on a one parameter family of orbifolds $\mathscr{O}_\alpha$.

We now explain our pictorial conventions. We always represent the orbifold as a wire-frame figure. Solid (respectively dashed) lines represent arcs in front (respectively behind) the figure. There are three different wire-frames: the octahedral, the exceptional and the triangular prism. The octahedral (respectively exceptional) wire-frame has a wire for each shortest arc (respectively for each shortest and second shortest arc). The triangular prism wire-frame is only used in Section 3. We always represent subsets of $K \cup \Lambda$ on the octahedral wire-frame. Any K (respectively Λ) arc in the subset is drawn in thick black (respectively thick grey). We always orient the figure so that c_4 (respectively c_5) is at the bottom (respectively top). When representing minimal chain pairs, $\mathscr{A}$ arcs are drawn in thick grey, $\mathscr{B}$ arcs are drawn in thick black. We regard α_i as oriented from a_i, a_{i+1} and use an arrow head to indicate this orientation. Similarly for β_j. A single unarrowed thick grey (respectively thick black) line represents the minimal chain $\mathscr{A}_1 = \alpha_1$ (respectively $\mathscr{B}_1 = \beta_1$).

We now note that $\lambda_k \cup \lambda_{k+1}$ is a length 2 bracelet that divides the cone point c_k from c_{k+1}, c_{k+2}, c_{k+3}. Likewise $\Lambda_k = \lambda_{k-1} \cup \lambda_{k+1}$ is a length 2 bracelet that divides c_k, c_{k+1} from c_{k+2}, c_{k+3}. These arc sets feature in the hypotheses of Lemma 2.3, the result we use to prove Propositions 2.1 and 2.2. Similarly $\bigcup_{l=4,5} \kappa_{k,l} \cup \lambda_{k+1}$ is a length 3 bracelet that cuts off c_{k+1}. This arc set features in the hypothesis of Theorem 2.6, an important result in the proof Propositions 2.4 and 2.5.

We will denote the two components of $\mathscr{O} \setminus \Lambda_k$ by $\mathscr{O}_{k,k+1}$, $\mathscr{O}_{k+2,k+3}$ so that $\mathscr{O}_{k,k+1} \supset \kappa_{k,k+1}$. Cutting $\mathscr{O}_{k,k+1}$ open along $\kappa_{k,k+1}$ we obtain an annulus that we will label by $A_{k,k+1}$. Let $P_{l,k}$ denote the perpendicular from c_l to $\kappa_{k,k+1}$ in $A_{k,k+1}$ for $l = 4, 5$. The perpendiculars divide $A_{k,k+1}$ into a pair of birectangles. Denote by $\mathscr{Q}_{k-1,k}$ (respectively $\mathscr{Q}_{k+1,k}$) the birectangle such that λ_{k-1} (respectively λ_{k+1}) lies on its boundary. Similarly for the component $\mathscr{O}_{k+2,k+3}$.

2.4. Proof of Theorem 1.3 under the assumption of Theorem 1.1

Proposition 2.1. *Let $\mathscr{A}_{i_2}, \mathscr{B}_{j_2}$ be a minimal chain pair such that $\Gamma_{i_2,j_2} = \alpha_{i_2} \cup \beta_{j_2}$ is a length 2 bracelet. Then $(i_2, j_2) = (3, 3)$ and $\Gamma_{3,3}$ divides two cone points from two.*

Proposition 2.2. *Any minimal standard chain pair that contains a length 2 bracelet is equivalent to a minimal standard chain pair on $\mathscr{E}$.*

In fact, there is nothing more to prove. To see this, suppose a minimal standard chain $\mathscr{A}, \mathscr{B}$ contains a length 2 bracelet. By Proposition 2.1, $\Gamma_{3,3} = \alpha_3 \cup \beta_3$ is this bracelet, $\Gamma_{3,3}$ divides two cone points from two, and $\mathscr{A}, \mathscr{B}$ contains no other length 2 bracelets. It is now a combinatorial exercise to enumerate standard chain pairs of this kind. Each one of these is equivalent to a minimal standard chain pair on $\mathscr{E}$—see the wire-frames in Figure 3 and two of the wire-frames in Figure 12 for some examples.

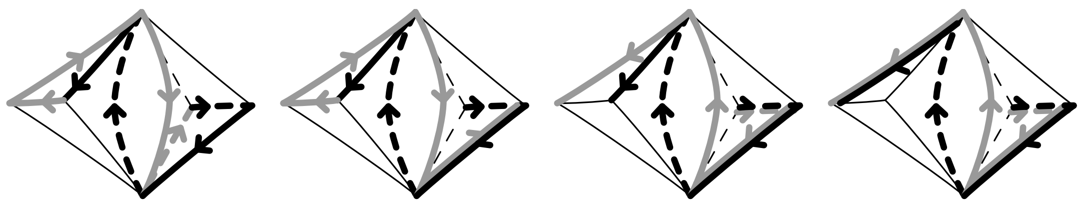

Figure 3. Minimal chain pairs on $\mathscr{E}$ with $(i_2, j_2) = (3, 3)$.

Lemma 2.3. *We have that* (i) $l(\kappa_{k,l}) < \big(l(\lambda_{k-1}) + l(\lambda_k)\big)/2$ *for* $l = 4, 5$ *and* (ii) $\max\{l(\kappa_{k,k+1}), l(\kappa_{k+2,k+3})\} < \big(l(\lambda_{k-1}) + l(\lambda_{k+1})\big)/2$.

We have pictured the arc sets for Lemma 2.3 with $k = 3$ in Figure 4.

Proof. (i) One component of $\mathscr{O}\setminus\lambda_{k-1}\cup\lambda_k$ contains c_k, label it by $\mathscr{O}_k$. Cut $\mathscr{O}_k$ open along $\kappa_{k,l}$ for $l = 4$ *or* 5. The resulting triangular domain has edge lengths $2l(\kappa_{k,l}), l(\lambda_{k-1}), l(\lambda_k)$. By the triangle inequality $2l(\kappa_{k,l}) < l(\lambda_{k-1}) + l(\lambda_k)$.

(ii) Consider the birectangle $\mathscr{Q}_{k-1,k}$. Its $\kappa_{k,k+1}$ edge is strictly shorter than its λ_{k-1} edge. Likewise for the birectangle $\mathscr{Q}_{k+1,k}$. Adding up edge lengths we have $2l(\kappa_{k,k+1}) < l(\lambda_{k-1}) + l(\lambda_{k+1})$. Likewise for the birectangles $\mathscr{Q}_{k-1,k+2}$, $\mathscr{Q}_{k+1,k+2}$. □

Proof of Proposition 2.1. Up to relabelling we may suppose that $i_2 \le j_2$. We have that $\{a_{i_2}, a_{i_2+1}\} = \{b_{j_2}, b_{j_2+1}\}$ and so each one of $a_1, \dots, a_{i_2-1}, b_1, \dots, b_{j_2-1}$ must lie in one or other component of $\mathscr{O}\setminus\Gamma_{i_2,j_2}$.

First: Γ_{i_2,j_2} divides two cone points from two. Suppose $j_2 = 4$. Each one of b_1, b_2, b_3 lies in one or other component of $\mathscr{O}\setminus\Gamma_{i_2,4}$. So b_2 lies in a different component of $\mathscr{O}\setminus\Gamma_{i_2,4}$ to b_1 or b_3 and so β_1 or β_2 crosses $\Gamma_{i_2,4}$—a contradiction.

We need to derive a contradiction for $i_2 \le 2$, otherwise $2 < i_2 \le j_2 < 4$ and $(i_2, j_2) = (3,3)$. Claim: $l(\alpha_{i_2}) = l(\beta_{j_2})$. If $i_2 = 1$, $j_2 \ge 1$ then α_{i_2}, β_{j_2} are both shortest arcs: by definition if $j_2 = 1$ and because $\mathscr{B}'_{j_1} = \beta_1, \dots, \beta_{j_2-1}, \alpha_1$ is a chain for $j_2 > 1$. Similarly, if $i_2 = 2$, $j_2 \ge 2$ then both of $\mathscr{A}'_2 = \alpha_1, \beta_{j_2}$, $\mathscr{B}'_{j_1} = \beta_1, \dots, \beta_{j_2-1}, \alpha_2$ are chains.

By Lemma 2.3(ii) we have a contradiction if $\alpha_{i_2}, \beta_{j_2}$ are both shortest arcs. So $i_2 = 2$. The arc α_1 lies in one component of $\mathscr{O}\setminus\Gamma_{2,j_2}$. Let α'_2 denote the arc disjoint from Γ_{2,j_2} in this component of $\mathscr{O}\setminus\Gamma_{2,j_2}$. By Lemma 2.3(ii) $l(\alpha'_2) < l(\alpha_2)$. Since $\mathscr{A}'_2 = \alpha_1, \alpha'_2$ is a chain, we have a contradiction.

Next: Γ_{i_2,j_2} divides one cone point c from three. Let $\mathscr{O}_c$ denote the component of $\mathscr{O}\setminus\Gamma_{i_2,j_2}$ containing c and let $\mathscr{O}'_c$ denote its complement. As above we can show that $l(\alpha_{i_2}) = l(\beta_{j_2})$. Again, if $i_2 = 1$ then α_1, β_{j_2} are both shortest arcs and Lemma 2.3(i) gives a contradiction.

Suppose $i_2 = 2$. If $c = a_1$ then $\alpha_1 \subset \mathscr{O}_c$. Let α'_2 be the other arc in $\mathscr{O}_c$. Then by Lemma 2.3(i): $l(\alpha'_2) < l(\alpha_2)$ and since $\mathscr{A}'_2 = \alpha_1, \alpha'_2$ is a chain we have a contradiction. Suppose $c \ne a_1$, $\alpha_1 \subset \mathscr{O}'_c$. Let α'_2 be the arc in $\mathscr{O}_c$ between a_2, c. Again $l(\alpha'_2) < l(\alpha_2)$, $\mathscr{A}'_2 = \alpha_1, \alpha'_2$ is a chain and we have a contradiction.

Finally, consider $i_2 > 2$. Each of $\alpha_1, \dots, \alpha_{i_2-1}$ must lie in $\mathscr{O}'_c$, otherwise one of these arcs would cross Γ_{i_2,j_2}. Let α'_{i_2} be the arc in $\mathscr{O}_c$ between a_{i_2}, c. Again $l(\alpha'_{i_2}) < l(\alpha_{i_2})$, $\mathscr{A}'_{i_2} = \alpha_1, \dots, \alpha_{i_2-1}, \alpha'_{i_2}$ is a chain and we have a contradiction. □

Proposition 2.4. *If $\Upsilon_{i_3,j_3} = \alpha_{i_3} \cup \beta_{j_3-1} \cup \beta_{j_3}$ is a length 3 bracelet that cuts off a cone point c, then we have that $i_3 > 1$, $j_3 > 2$; if $(i_3, j_3) = (2,3)$, then $(a_2, a_3) = (b_2, b_4)$, $c = a_1 = b_1$; if $(i_3, j_3) = (2,4)$, then $(a_2, a_3) = (b_5, b_3)$, $a_1 = c \notin \{b_1, b_2\}$; if $(i_3, j_3) = (3,3)$, then $(a_3, a_4) = (b_4, b_2)$, $b_1 = c \notin \{a_1, a_2\}$; if $(i_3, j_3) = (3,4)$, then $(a_3, a_4) = (b_3, b_5)$, $c \notin \{a_1, a_2\} = \{b_1, b_2\}$; if $(i_3, j_3) = (4,3)$, then $(a_4, a_5) = (b_4, b_2)$, $b_1 = c \notin \{a_1, a_2\}$; and if $(i_3, j_3) = (4,4)$, then $(a_4, a_5) = (b_3, b_5)$, $c \notin \{a_1, a_2\} = \{b_1, b_2\}$.*

Proposition 2.5. *Any minimal standard chain pair that contains a length 3 bracelet that cuts off a cone point is equivalent to a minimal standard chain pair on $\mathscr{E}$.*

Unlike Proposition 2.2 which followed directly from Proposition 2.1, Proposition 2.5 does not follow directly from Proposition 2.4; there is still something to prove. However almost all the arguments reproduce arguments given in the proof of Proposition 2.4. The main result we apply to prove Proposition 2.4 is Theorem 2.6 which appeared in paper [12] as Theorem 1.1. Theorem 2.7 also appeared in paper [12] as Theorem 1.2.

Theorem 2.6. *Suppose, for some k, that $l(\kappa_{k,l}) \leq l(\kappa_{k+1,l})$, $l(\lambda_{k+1}) \leq l(\lambda_{k-1})$ for $l = 4,5$, then $l(\kappa_{k,l}) = l(\kappa_{k+1,l})$, $l(\lambda_{k+1}) = l(\lambda_{k-1})$ for $l = 4,5$.*

Theorem 2.7. *Suppose, for some k, that $\kappa_{k,l}$ is a shortest arc for $l = 4,5$ and that $l(\kappa_{k,k+1}) \leq l(\kappa_{k+2,k+3})$, $l(\lambda_{k+1}) \leq l(\lambda_{k-1})$. Then $\mathscr{O}$ is the octahedral orbifold.*

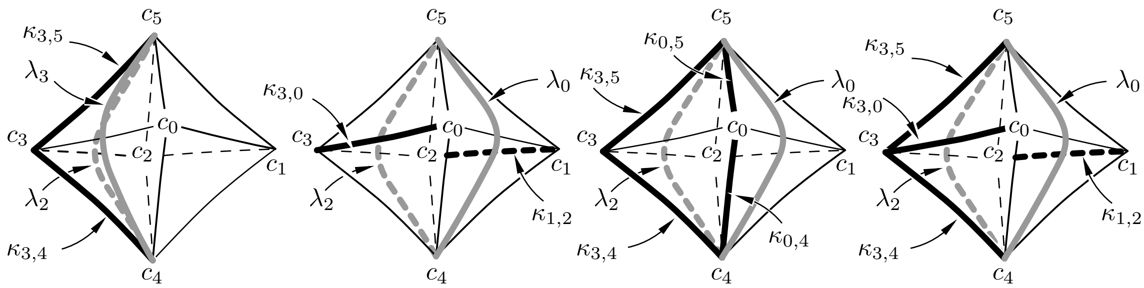

Figure 4. Arc sets for Lemma 2.3 and Theorems 2.6, 2.7 with $k = 3$

Lemma 2.8. *Suppose, for some k, that $l(\kappa_{k,l}) = l(\kappa_{k+1,l})$, $l(\lambda_{k+1}) = l(\lambda_{k-1})$ for $l = 4,5$. Then $l(\kappa_{k,4}) = l(\kappa_{k+1,5})$ if and only if $l(\kappa_{k+2,4}) = l(\kappa_{k+3,5})$.*

Proof. We will show $l(\kappa_{k+2,4}) = l(\kappa_{k+3,5})$ implies $l(\kappa_{k,4}) = l(\kappa_{k+1,5})$. The other direction follows similarly.

Since $l(\lambda_{k+1}) = l(\lambda_{k-1})$ the annulus $\mathscr{A}_{k+2,k+3}$ has mirror symmetry exchanging the birectangles $\mathscr{Q}_{k-1,k+2}$, $\mathscr{Q}_{k+1,k+2}$. That is $P_{4,k+2}$, $P_{5,k+2}$ are equally spaced about the geodesic boundary component $\kappa_{k+2,k+3}$ of $\mathscr{A}_{k+2,k+3}$. We know that c_{k+2}, c_{k+3} are also equally spaced about this boundary component. Since $l(\kappa_{k+2,4}) = l(\kappa_{k+3,5})$ it follows that $l(P_{4,k+2}) = l(P_{5,k+2})$. That is $\mathscr{A}_{k+2,k+3}$ has rotational symmetry exchanging $\mathscr{Q}_{k-1,k+2}$, $\mathscr{Q}_{k+1,k+2}$. Gluing along $\kappa_{k+2,k+3}$ to recover $\mathscr{O}_{k+2,k+3}$ this symmetry is respected. This in turn implies that $\mathscr{O}_{k,k+1}$ has rotational symmetry—c.f. the proof of Theorem 1.2 in [12]—and hence that $l(\kappa_{k,4}) = l(\kappa_{k+1,5})$. ▫

Proof of Proposition 2.4. Suppose Υ is a length 3 bracelet that cuts off a cone point c. Label arcs in Υ by $\kappa_{k,l}, \lambda_{k+1}$ for $l = 4,5$. This labelling then extends uniquely to $K_k \cup \Lambda_k \subset K \cup \Lambda$ where $K_k = \bigcup_{l=4,5} \kappa_{k,l} \cup \kappa_{k+1,l} \cup \kappa_{k,k+1} \cup \kappa_{k+2,k+3}$. To see this we proceed as follows. Label by $\mathscr{O}_c$ the component of $\mathscr{O} \setminus \Upsilon$ containing c. Set $c = c_{k+1}$ and then label the arcs in $\mathscr{O}_c$ between c_{k+1}, c_l by $\kappa_{k+1,l}$ for $l = 4,5$ and between c_k, c_{k+1} by $\kappa_{k,k+1}$. Let $\mathscr{O}'_c$ denote the component of $\mathscr{O} \setminus \Upsilon$ not containing c. Label by λ_{k-1} the arc in $\mathscr{O}'_c$ between c_4, c_5 such that $\bigcup_{l=4,5} \kappa_{k,l} \cup \lambda_{k-1}$ bounds a triangle. Label by $\kappa_{k+2,k+3}$ the arc disjoint from Υ in $\mathscr{O}'_c$. We will use this extension of arc labelling for applications of Theorem 2.6.

Suppose $i_3 = 1$. In Figure 5 the four wire-frames represent all the configurations of $\mathscr{A}_1, \mathscr{B}_{j_3}$ such that Υ_{1,j_3} cuts off a cone point. For all but the third configuration we can use Theorem 2.6 and Lemma 2.3(ii) to derive a contradiction. For the third configuration we can apply Theorem 2.7 to show that $\mathscr{O}$ is the octahedral orbifold. As we have observed before minimal chains on $\mathscr{O}ct$ lie in its set of shortest arcs. Any length 3 bracelet in this set bounds a triangle.

Consider, for example the fourth configuration, with $j_3 = 4$. Set $\alpha_1 = \kappa_{3,4}$, $\beta_3 = \lambda_0$, $\beta_4 = \kappa_{3,5}$. This extends uniquely to $K_3 \cup \Lambda_3$. We note that $\mathscr{B}'_3 = \beta_1, \beta_2, \alpha_1$, $\mathscr{B}'_4 = \beta_1, \beta_2, \beta_3, \kappa_{0,5}$ are both chains and so λ_0 is a shortest arc and $l(\kappa_{3,5}) \leq l(\kappa_{0,5})$. We know that $\kappa_{3,4}$ is a shortest arc, so the hypotheses of Theorem 2.6 are satisfied. So $l(\lambda_0) = l(\lambda_2)$. By Lemma 2.3(ii): $l(\kappa_{3,0}) < l(\lambda_0) = l(\lambda_2)$ which contradicts λ_0 being a shortest arc.

For the third configuration we argue as follows. Set $\alpha_1 = \kappa_{3,4}$, $\beta_2 = \kappa_{3,5}$, $\beta_3 = \lambda_0$ which extends uniquely to $K_3 \cup \Lambda_3$. We note that $\mathscr{B}'_2 = \beta_1, \alpha_1$, $\mathscr{B}'_3 = \beta_1, \beta_2, \lambda_2$ are chains, so $\kappa_{3,5}$ is a shortest arc and $l(\lambda_0) \leq l(\lambda_2)$. So the hypotheses of Theorem 2.7 are satisfied: $\mathscr{O}$ is the octahedral orbifold.

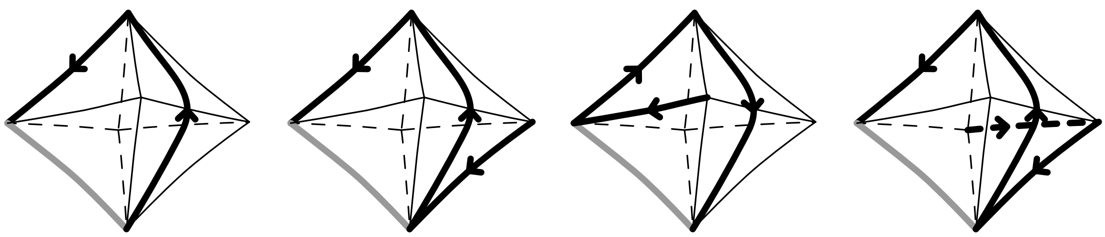

Figure 5. Configurations of $\mathscr{A}_1, \mathscr{B}_{j_3}$

So we have shown that $i_3 > 1$. Next we show that $j_3 > 2$ (Figure 6). We then consider $j_3 = 3$ (Figure 7), then $j_3 = 4$ (Figure 8). By cone point labels on Υ_{i_3,j_3} we know that $\{a_{i_3}, a_{i_3+1}\} = \{b_{j_3-1}, b_{j_3+1}\}$. Suppose $(a_{i_3}, a_{i_3+1}) = (b_{j_3+1}, b_{j_3-1})$. If $a_{i_3-1} = b_{j_3}$ then $\alpha_{i_3-1}, \beta_{j_3}$ share endpoints. Unless $(i_3, j_3) = (4,3)$, by Proposition 2.1, $\alpha_{i_3-1} = \beta_{j_3}$. So $\Upsilon_{i_3,j_3} = \beta_{j_3-1} \cup \alpha_{i_3-1} \cup \alpha_{i_3}$ with $(b_{j_3-1}, b_{j_3}) = (a_{i_3+1}, a_{i_3-1})$ which is covered by the argument we give for $(i'_3, j'_3) = (j_3 - 1, i_3)$ since $b_{j_3-2} \neq a_{i_3} = b_{j_3+1}$. Suppose $(i_3, j_3) = (4,3)$, $(a_4, a_5) = (b_5, b_3)$ and $a_3 = b_4$. Suppose $\alpha_3 = \beta_3$. If $b_1 = c$ then $\mathscr{A}, \mathscr{B}_3$ is equivalent to a minimal chain pair on $\mathscr{E}$; see the third wire-frame, Figure 12. Otherwise, it is covered by an argument we give for $(i_3, j_3) = (2,4)$ since $b_1 \neq a_4 = b_5$. If $\alpha_3 \cup \beta_3$ is a bracelet then $\mathscr{A}, \mathscr{B}_3$ is equivalent to a minimal chain pair on $\mathscr{E}$ by Proposition 2.2; see the fourth wire-frame, in Figure 12. So, for $(a_{i_3}, a_{i_3+1}) = (b_{j_3+1}, b_{j_3-1})$, we may suppose that $a_{i_3-1} \neq b_{j_3}$.

For $j_3 = 2$ we begin by $i_3 = 2$. First we consider $(a_2, a_3) = (b_1, b_3)$. If $a_1 \neq b_2$ then $\mathscr{A}_2' = \alpha_1, \beta_1$ is a chain and α_2 is a shortest arc—this is equivalent to $i_3 = 1$. So $a_1 = b_2$, α_1, β_1 share endpoints and so, by Proposition 2.1, $\alpha_1 = \beta_1$; see the first wire-frame, Figure 6. We can apply Theorem 2.6 and Lemma 2.3(ii) to contradict the shortness of $\alpha_1 = \beta_1$. Next $(a_2, a_3) = (b_3, b_1)$. We may suppose that $a_1 \neq b_2$. So $a_1 = c$ or $a_1 \in \mathscr{O}_c'$; see the second and third wire-frames. We can apply Theorem 2.6 and Lemma 2.3(ii) to contradict the shortness of β_1, β_2, respectively. For the latter , with labelling so that $\beta_2 = \lambda_0$, we show that $l(\kappa_{3,0}) < l(\lambda_0) = l(\lambda_2)$ which gives a contradiction since $\mathscr{B}_2' = \beta_1, \kappa_{3,0}$ is a chain.

Next $i_3 = 3$. Suppose $(a_3, a_4) = (b_1, b_3)$. If $\{a_1, a_2\} \not\ni b_2$ then $\mathscr{A}_3' = \alpha_1, \alpha_2, \beta_1$ is a chain and α_3 is a shortest arc—again equivalent to $i_3 = 1$. So $\{a_1, a_2\} \ni b_2$. If $a_1 = b_2$ then $\{a_1, a_3\} = \{b_1, b_2\}$ and so $\beta_1 \cup \alpha_1 \cup \alpha_2$ is a bracelet. Since $\mathscr{A}_2' = \alpha_1, \beta_1$ is a chain it follows that this is a bracelet of shortest arcs and we can apply Proposition 2.10. So $a_2 = b_2$. Either $a_1 = c$ or $a_1 \in \mathscr{O}_c'$; see the fourth and fifth wire-frames. For the former we can apply Theorem 2.7 to show that $\mathscr{O}$ is the octahedral orbifold. For the latter we can apply Theorem 2.6 and Lemma 2.3(ii) to contradict the shortness of β_1.

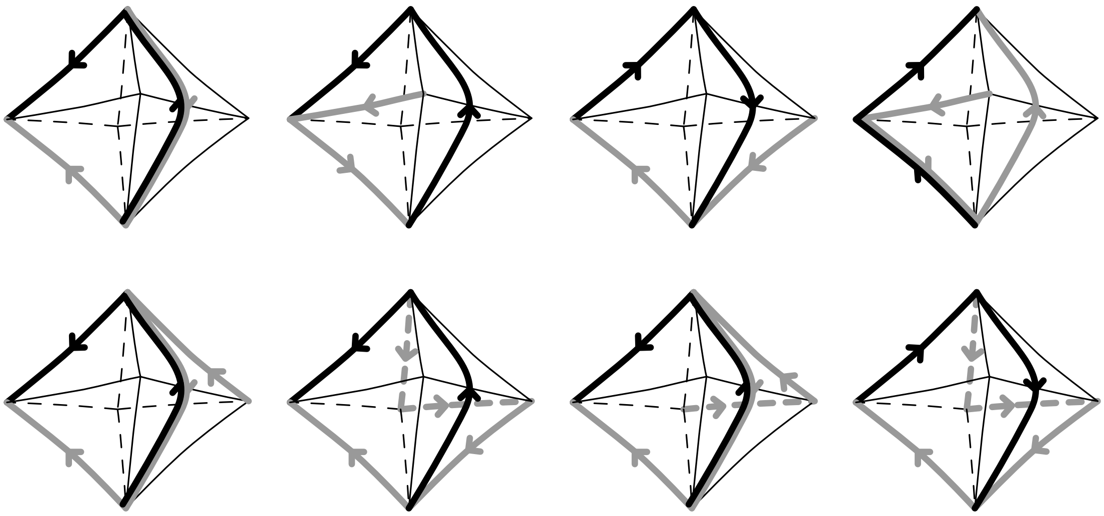

Figure 6. Configurations of $\mathscr{A}_{i_3}, \mathscr{B}_2$

Finally $i_3 = 4$. If $a_2 = b_2$ then the arcs $\alpha_1, \beta_1, \alpha_2, \beta_2$ all have exactly one cone point in common, $\mathscr{A}_2' = \alpha_1, \beta_1$, $\mathscr{B}_2' = \beta_1, \alpha_1$ are both chains and so each arc in this set is shortest. We can now apply Lemma 2.11: $\mathscr{O}$ is the octahedral orbifold. So $a_2 \neq b_2$. Suppose $(a_4, a_5) = (b_1, b_3)$. Either $a_1 = b_2$ or $a_3 = b_2$; see the sixth and seventh wire-frames. We can apply Theorem 2.6 and Lemma 2.3(ii) to contradict the shortness of β_1. For $(a_4, a_5) = (b_3, b_1)$ we may suppose that

$a_3 \neq b_2$. So $a_1 = b_2$ and we can apply Theorem 2.6 and Lemma 2.3(ii) to contradict the shortness of β_2; see the eighth wire-frame.

For $j_3 = 3$ we begin with $i_3 = 2$. Suppose $(a_2, a_3) = (b_2, b_4)$. If $a_1 \neq b_1$ then $\mathscr{A}_2' = \alpha_1, \beta_1$ is a chain and α_2 is a shortest arc, again equivalent to $i_3 = 1$. So $a_1 = b_1$. For $a_1 = b_1 = c$: $\mathscr{A}_2, \mathscr{B}_3$ is equivalent to a minimal chain pair on $\mathscr{E}$; see the first wire-frame, Figure 12. For $a_1 = b_1 \in \mathscr{O}_c'$ we can apply Theorem 2.6 and Lemma 2.3(ii) to contradict the shortness of β_2; see the first wire-frame, Figure 7. Now suppose $(a_2, a_3) = (b_4, b_2)$. We may suppose that $a_1 \neq b_3$. If $a_1 = b_1$ then $\{a_1, a_3\} = \{b_1, b_2\}$ and so $\beta_1 \cup \alpha_1 \cup \alpha_2$ is a bracelet of shortest arcs. So $a_1 \neq b_1$. For $a_1 = c$, $b_1 \in \mathscr{O}_c'$ we can apply Theorem 2.6 and Lemma 2.3(ii) to contradict the shortness of β_2; see the second wire-frame.

Suppose $a_1 \in \mathscr{O}_c'$, $b_1 = c$; see the third wire-frame. Set $\alpha_2 = \kappa_{3,4}$, $\beta_2 = \kappa_{3,5}$, $\beta_3 = \lambda_0$, $\alpha_1 = \kappa_{1,4}$ which extends uniquely to $K_3 \cup \Lambda_3 \cup \kappa_{1,4} \cup \kappa_{2,5}$. The hypotheses of Theorem 2.6 are satisfied: each one of $\mathscr{A}_2' = \alpha_1, \kappa_{0,4}$, $\mathscr{B}_2' = \beta_1, \kappa_{0,5}$, $\mathscr{B}_3' = \beta_1, \beta_2, \lambda_2$ is a chain. So $l(\kappa_{3,l}) = l(\kappa_{0,l})$ for $l = 4, 5$ and $l(\lambda_0) = l(\lambda_2)$. By Lemma 2.3(ii), $\beta_3 = \lambda_0$ is not a shortest arc. Below we show that $\kappa_{2,5}$ is a shortest arc, which gives a contradiction since $\mathscr{B}_3'' = \beta_1, \beta_2, \kappa_{2,5}$ is a chain. First $\mathscr{B}_2'' = \beta_1, \alpha_2$ is a chain and so $l(\beta_2) \leq l(\alpha_2)$. Next, $\kappa_{3,0} = \beta_1$, $\kappa_{1,4} = \alpha_1$ are both shortest arcs and $l(\kappa_{3,5}) = l(\kappa_{0,5})$. By Lemma 2.9 there exists an arc γ such that $l(\gamma) \leq l(\kappa_{3,5}) = l(\kappa_{0,5})$. The arc γ is such that $\mathscr{A}_2'' = \alpha_1, \gamma$ is a chain and so $l(\alpha_2) \leq l(\gamma)$. So, we have that $l(\kappa_{3,5}) = l(\beta_2) \leq l(\alpha_2) = l(\kappa_{3,4}) \leq l(\gamma) \leq l(\kappa_{3,5})$ and hence $l(\kappa_{3,4}) = l(\kappa_{3,5})$. So $l(\kappa_{3,4}) = l(\kappa_{0,5})$ and by Lemma 2.8, $l(\kappa_{1,4}) = l(\kappa_{2,5})$. Since $\alpha_1 = \kappa_{1,4}$ is a shortest arc we are done.

So $a_1 \neq b_1 \in \mathscr{O}_c'$; see the fourth wire-frame. Set $\alpha_2 = \kappa_{3,4}$, $\beta_2 = \kappa_{3,5}$, $\beta_3 = \lambda_0$, $\beta_1 = \kappa_{2,3}$ which extends uniquely to $K \cup \Lambda$. We have that $\mathscr{A}_2' = \alpha_1, \kappa_{0,4}$ is a chain and so $l(\kappa_{3,4}) \leq l(\kappa_{0,4})$. Also each one of $\mathscr{B}_2' = \beta_1, \alpha_2$, $\mathscr{A}_2'' = \alpha_1, \beta_3$, $\mathscr{B}_3' = \beta_1, \beta_2, \kappa_{0,5}$ is a chain and so $l(\kappa_{3,5}) = l(\beta_2) \leq l(\alpha_2) \leq l(\beta_3) \leq l(\kappa_{0,5})$. By Theorem 2.6, $l(\lambda_2) \leq l(\lambda_0)$. Now both $\mathscr{B}_3'' = \beta_1, \beta_2, \lambda_1$, $\mathscr{B}_3''' = \beta_1, \beta_2, \lambda_3$ are chains and so $l(\lambda_0) \leq \min\{l(\lambda_1), l(\lambda_3)\}$. By Theorem 2.12, $\mathscr{O}$ is the octahedral orbifold.

Next $i_3 = 3$. Suppose $(a_3, a_4) = (b_2, b_4)$. If $\{a_0, a_1\} \not\ni b_1$ then $\mathscr{A}_3' = \alpha_1, \alpha_2, \beta_1$ is a chain, α_2 is a shortest arc. So $\{a_1, a_2\} \ni b_1$. If $\{a_1, a_2\} \ni b_3$ then $\alpha_1 \cup \beta_1 \cup \beta_2$ is a bracelet of shortest arcs. So $\{a_1, a_2\} \not\ni b_3$. Since α_1 does not cross $\Upsilon_{3,3}$ it follows that a_1, a_2 lie in the same component of $\mathscr{O} \setminus \Upsilon_{3,3}$ and hence $a_1, a_2 \in \mathscr{O}_c'$. If $a_1 = b_1$ then $\beta_1 \cup \alpha_1 \cup \alpha_2$ is a bracelet of shortest arcs. So $a_2 = b_1$; see the fifth wire-frame; and we can apply Theorem 2.6 and Lemma 2.3(ii) to contradict the shortness of β_2. Now $(a_3, a_4) = (b_4, b_2)$. Again we may suppose that $a_2 \neq b_3$. Suppose $a_1 = b_3$. If $a_2 = b_1$ then we have that $\alpha_1 \cup \beta_1 \cup \beta_2$ is a bracelet of shortest arcs. If $a_2 \neq b_1$ then $\mathscr{B}_3' = \beta_1, \beta_2, \alpha_1$, $\mathscr{A}_2' = \alpha_1, \beta_3$ are both chains and so β_3, α_2 are both shortest arcs. Since $(a_1, a_3) = (b_3, b_4)$ it follows that $\beta_3 \cup \alpha_1 \cup \alpha_2$ is a bracelet of shortest arcs. So $\{a_1, a_2\} \not\ni b_3$. Again this implies that $a_1, a_2 \in \mathscr{O}_c'$. If $b_1 = c$ then $\mathscr{A}_3, \mathscr{B}_3$ is equivalent to a minimal chain pair on $\mathscr{E}$; see the second wire-frame, Figure 12. If $b_1 \in \mathscr{O}_c'$ then $a_1 = b_1$ or $a_2 = b_1$ and we can apply Theorems 2.6 and 2.12 to show that $\mathscr{O}$ is $\mathscr{O}ct$; see the sixth and seventh wire-frames.

Next $i_3 = 4$. First $(a_4, a_5) = (b_2, b_4)$. Suppose $\{a_1, a_2, a_3\} \not\ni b_1$ then $\mathscr{A}'_4 = \alpha_1, \alpha_2, \alpha_3, \beta_1$ is a chain and α_4 is a shortest arc. So $\{a_1, a_2, a_3\} \ni b_1$. Also $\{a_1, a_2, a_3\} \ni b_3$ since neither α_1 nor α_2 crosses $\Upsilon_{4,3}$. So $\{a_1, a_2, a_3\} \supset \{b_1, b_3\}$. If $\{a_1, a_2\} = \{b_1, b_3\}$ then $\alpha_1 \cup \beta_1 \cup \beta_2$ is a bracelet of shortest arcs. If $\{a_2, a_3\} = \{b_1, b_3\}$ then $\alpha_2 \cup \beta_1 \cup \beta_2$ is a bracelet of shortest arcs. For $(a_2, a_3) = (b_1, b_3)$ we have that $\mathscr{A}'_2 = \alpha_1, \beta_1$, $\mathscr{B}'_2 = \beta_1, \alpha_1$ are both chains. For $(a_2, a_3) = (b_3, b_1)$ we have that $\mathscr{B}'_3 = \beta_1, \beta_2, \alpha_1$, $\mathscr{A}'_2 = \alpha_1, \beta_3$, $\mathscr{B}'_2 = \beta_1, \alpha_2$ are all chains. So $\{a_1, a_3\} = \{b_1, b_3\}$ and we can apply Theorem 2.6 and Lemma 2.3(ii) to contradict the shortness of β_2; see the eighth and ninth wire-frames.

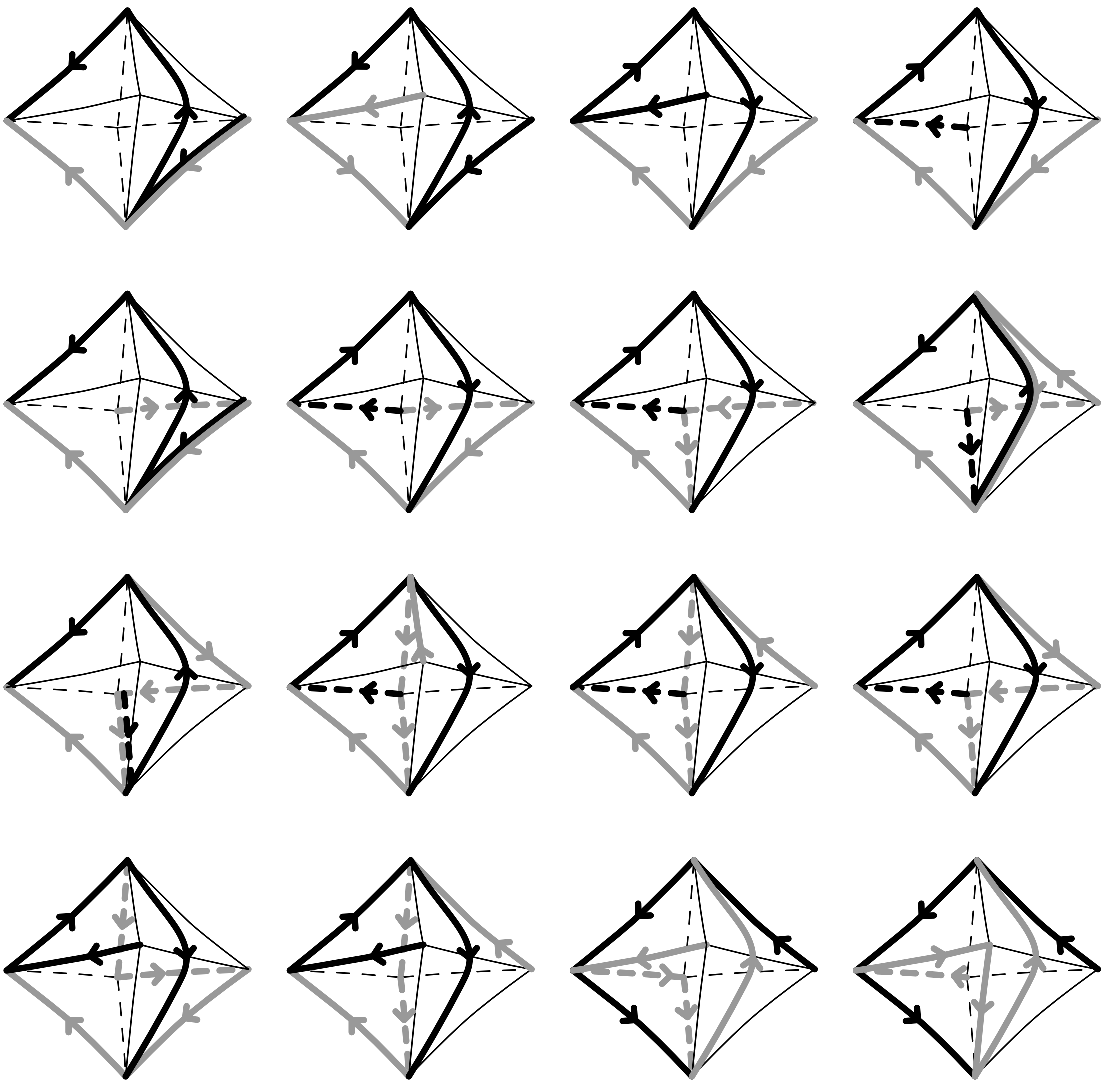

Figure 7. Configurations of $\mathscr{A}_{i_3}, \mathscr{B}_3$

Next $(a_4, a_5) = (b_4, b_2)$. We may suppose that $a_3 \neq b_3$. Again neither α_1

nor α_2 cross $\Upsilon_{4,3}$ so $\{a_1, a_2\} \ni b_3$. If $\{a_1, a_2\} \ni b_1$ then $\alpha_1 \cup \beta_1 \cup \beta_2$ is a bracelet of shortest arcs. So $\{a_1, a_2\} \not\ni b_1$. Suppose $a_3 = b_1$; see the tenth, eleventh and twelfth wire-frames, Figure 7. In each case $\mathscr{B}_3' = \beta_1, \beta_2, \alpha_1$, $\mathscr{A}_2' = \alpha_1, \beta_3$, $\mathscr{B}_2' = \beta_1, \alpha_2$ are chains and so β_2 is a shortest arc. We can now apply Theorems 2.6 and 2.12 to show that $\mathscr{O}$ is $\mathscr{O}ct$. So $\{a_1, a_2, a_3\} \not\ni b_1$ and hence $\mathscr{B}_3' = \beta_1, \beta_2, \alpha_1$ is a chain, β_3 is a shortest arc. If $b_1 = c$; see the 13th and 14th wire-frames. We can apply Theorem 2.6 and Lemma 2.3(ii) to contradict β_3 being a shortest arc. If $a_1 = c$ or $a_3 = c$ (see the 15th and 16th wire-frames) we can again apply Theorems 2.6 and 2.12. The argument differs slightly from that given above. Above we had that $l(\lambda_2) \le l(\lambda_0)$ and $l(\lambda_0) \le \min\{l(\lambda_1), l(\lambda_3)\}$. Again we have that $l(\lambda_2) \le l(\lambda_0)$ but only have $l(\lambda_0) \le l(\lambda_1)$. Here, however, we have that $\kappa_{3,0}$ is a shortest arc. The argument runs as follows; c.f. the proof of Lemma 2.11. Suppose $a_1 = c$. Set $\alpha_4 = \lambda_0$, $\beta_2 = \kappa_{3,5}$, $\beta_3 = \kappa_{3,4}$, $\alpha_2 = \kappa_{2,3}$ which extends uniquely to $K \cup \Lambda$. Since $\mathscr{B}_3' = \beta_1, \beta_2, \alpha_1$, $\mathscr{A}_2' = \alpha_1, \beta_3$ are both chains, $\beta_3 = \kappa_{3,4}$, $\alpha_2 = \kappa_{2,3}$ are both shortest arcs. Since $\mathscr{B}_2' = \beta_1, \kappa_{0,5}$ is also a chain we can apply Theorem 2.6: $l(\lambda_2) \le l(\lambda_0)$. Now $\mathscr{A}_4' = \alpha_1, \alpha_2, \alpha_3, \lambda_1$ is a chain and so $l(\alpha_4) = l(\lambda_0) \le l(\lambda_1)$. If $l(\lambda_2) \le l(\lambda_3)$ we can apply Theorem 2.12 since $\kappa_{2,3}$ is a shortest arc. Otherwise $l(\lambda_3) \le l(\lambda_2) \le l(\lambda_0)$ and we can again apply Theorem 2.12 since $\kappa_{3,0} = \alpha_1$ is a shortest arc.

Now $j_3 = 4$. We know that $b_1, b_2 \in \mathscr{O}_c'$ since β_1 does not cross $\Upsilon_{i_3,4}$. Consider $i_3 = 2$. If $a_1 \in \{b_1, b_2\}$ then $\mathscr{A}_2' = \alpha_1, \beta_1$ is a chain and α_2 is a shortest arc. So $a_1 \notin \{b_1, b_2\}$. Suppose $(a_2, a_3) = (b_3, b_5)$. If $a_1 = b_4$ then we can apply Theorem 2.6 and Lemma 2.3(ii) to contradict $\alpha_1 = \beta_3$ being a shortest arc; see the first wire-frame, Figure 8. So $a_1 = c$ and we have that $\mathscr{B}_3' = \beta_1, \beta_2, \alpha_1$, $\mathscr{A}_2' = \alpha_1, \beta_3$ are both chains and so α_2 is again a shortest arc. Next $(a_2, a_3) = (b_5, b_3)$. We may suppose that $a_1 \ne b_4$. So $a_1 = c$ and $\mathscr{A}_2, \mathscr{B}$ is equivalent to a minimal chain pair on $\mathscr{E}$; see the fifth wire-frame, Figure 12.

Next $i_3 = 3$. First $(a_3, a_4) = (b_3, b_5)$. If $\{a_1, a_2\} \not\ni b_4$ then, as α_1 does not cross $\Upsilon_{3,4}$ it follows that $a_1, a_2 \in \mathscr{O}_c'$, i.e. $\{a_1, a_2\} = \{b_1, b_2\}$ and $\mathscr{A}_3, \mathscr{B}$ is equivalent to a minimal chain pair on $\mathscr{E}$; see the sixth wire-frame, Figure 12. So $\{a_1, a_2\} \ni b_4$. Suppose $a_1 = b_4$. If $a_2 = b_1$ then $(a_2, a_3) = (b_1, b_3)$, $a_1 \ne b_2$ and so $\alpha_2 \cup \beta_1 \cup \beta_2$ is a bracelet of shortest arcs. If $a_2 = b_2$ then we argue as follows; see the second wire-frame, Figure 8. Set $\alpha_3 = \kappa_{3,4}$, $\beta_3 = \lambda_0$, $\beta_4 = \kappa_{3,5}$, $\beta_2 = \alpha_2 = \kappa_{1,4}$, $\alpha_1 = \kappa_{1,5}$ which extends uniquely to $K \cup \Lambda$. Each one of $\mathscr{A}_3' = \alpha_1, \alpha_2, \kappa_{0,4}$, $\mathscr{B}_3' = \beta_1, \beta_2, \lambda_2$, $\mathscr{B}_4' = \beta_1, \beta_2, \beta_3, \kappa_{0,5}$ is a chain so the hypotheses of Theorem 2.6 are satisfied: $l(\lambda_0) = l(\lambda_2)$. Now $\mathscr{A}_2' = \alpha_1, \beta_1$ is a chain, so $\alpha_2 = \kappa_{1,4}$ is a shortest arc. Now $\alpha_1 = \kappa_{1,5}, \beta_1 = \kappa_{1,2}$ are also shortest arcs so, by Theorem 2.7, $\mathscr{O}$ is the octahedral orbifold. So $a_2 \notin \{b_1, b_2\}$ and each one of $\mathscr{B}_4' = \beta_1, \beta_2, \beta_3, \alpha_1$, $\mathscr{A}_2' = \alpha_1, \beta_4$, $\mathscr{B}_3' = \beta_1, \beta_2, \alpha_2$ are chains: $\beta_3 \cup \alpha_1 \cup \alpha_2$ is a bracelet of shortest arcs. Next suppose $a_2 = b_4$. If $a_1 = c$ and we can apply Theorem 2.7; see the third wire-frame. If $a_1 = b_1$ then $\mathscr{B}_2' = \beta_1, \alpha_1$, $\mathscr{A}_3' = \alpha_1, \alpha_2, \beta_2$ are chains and so β_2, α_3 are shortest arcs; we can apply Theorems 2.6 and 2.12 to show $\mathscr{O}$ is $\mathscr{O}ct$;

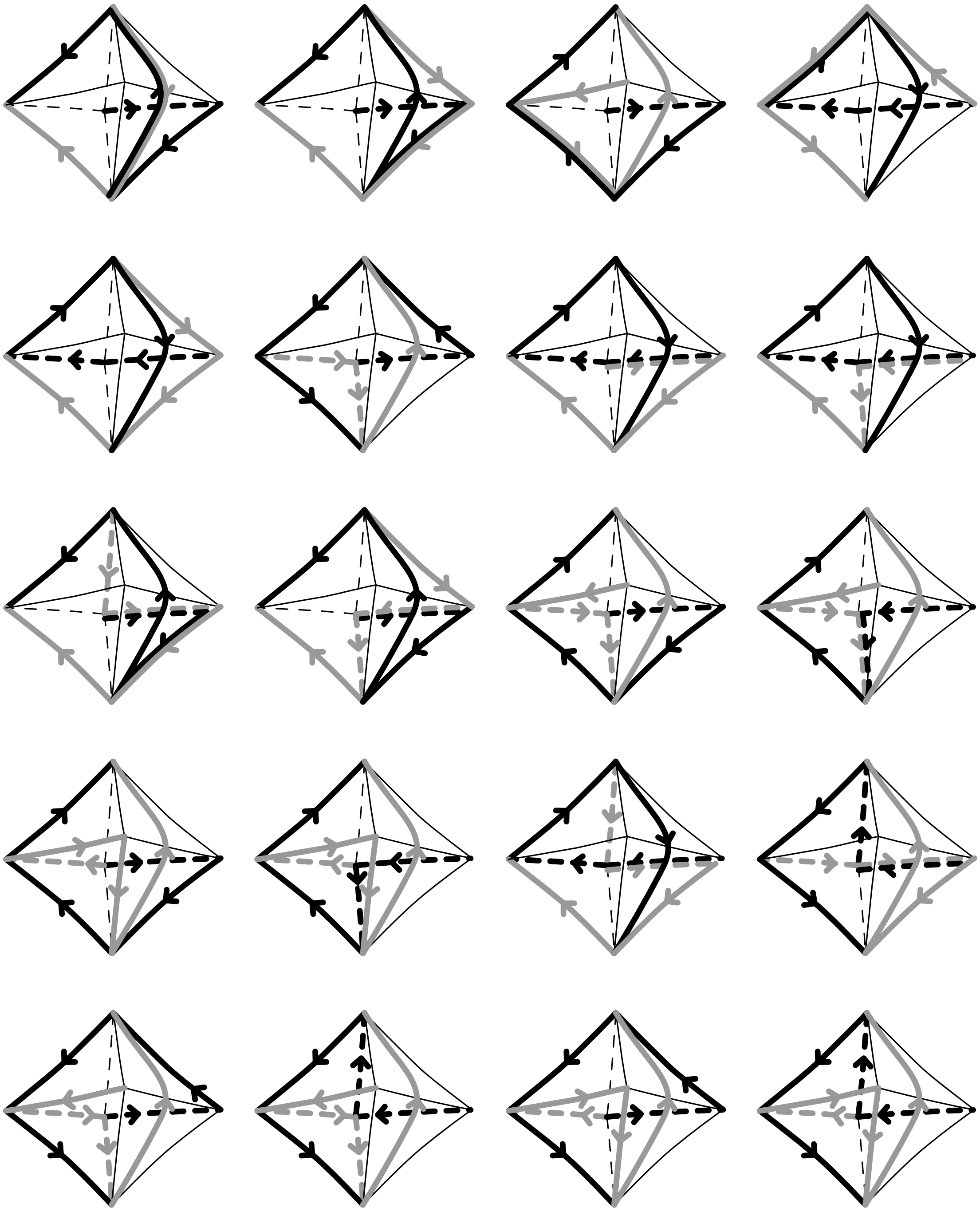

Figure 8. Configurations of $\mathscr{A}_{i_3}, \mathscr{B}_4$

see the fourth wire-frame. If $a_1 = b_2$ then $(a_1, a_3) = (b_2, b_3)$, $a_2 \neq b_1$ and so $\beta_2 \cup \alpha_1 \cup \alpha_2$ is a bracelet of shortest arcs.

Next $(a_3, a_4) = (b_5, b_3)$. We may suppose that $a_2 \neq b_4$. Suppose $a_1 = b_4$. For $a_2 \notin \{b_1, b_2\}$ we have that both of $\mathscr{B}'_4 = \beta_1, \beta_2, \beta_3, \alpha_1$, $\mathscr{A}'_2 = \alpha_1, \beta_4$ are chains and so $\beta_4 \cup \alpha_1 \cup \alpha_2$ is a bracelet of shortest arcs. So $a_2 \in \{b_1, b_2\}$. If

$a_2 = b_2$ then $\alpha_1, \beta_1, \alpha_2, \beta_2$ have exactly one cone point in common and we can apply Lemma 2.11: $\mathscr{O}$ is $\mathscr{O}ct$. If $a_2 = b_1$ we can apply Theorems 2.6 and 2.12—the fifth (respectively sixth) wire-frame illustrates the argument for $l(\alpha_3) \le l(\beta_4)$ (respectively $l(\beta_4) \le l(\alpha_3)$).

So $a_1 \neq b_4$ and hence $\{a_1, a_2\} = \{b_1, b_2\}$; see the seventh and eighth wire-frames. Set $\alpha_3 = \kappa_{3,4}$, $\beta_3 = \kappa_{3,5}$, $\beta_4 = \lambda_0$, $\beta_2 = \kappa_{2,3}$ which extends to $K \cup \Lambda$. First $\mathscr{A}_3' = \alpha_1, \alpha_2, \kappa_{0,4}$ is a chain and so $l(\kappa_{3,4}) \le l(\kappa_{0,4})$. Likewise $\mathscr{B}_3' = \beta_1, \beta_2, \alpha_3$, $\mathscr{A}_3' = \alpha_1, \alpha_2, \beta_4$, $\mathscr{B}_4' = \beta_1, \beta_2, \beta_3, \kappa_{0,5}$ are all chains so $l(\kappa_{3,5}) = l(\beta_3) \le l(\alpha_3) \le l(\beta_4) \le l(\kappa_{0,5})$. Also $\mathscr{B}_2' = \beta_1, \kappa_{2,4}$ is chain and so $l(\kappa_{2,3}) \le l(\kappa_{2,4})$. Again $\mathscr{B}_4' = \beta_1, \beta_2, \beta_3, \kappa_{0,5}$ is a chain and so $l(\lambda_0) \le l(\kappa_{0,5})$. By Theorem 2.13 either $l(\kappa_{0,4}) < l(\kappa_{3,4})$ or $l(\kappa_{0,5}) < l(\kappa_{3,5})$ or $l(\kappa_{2,4}) < l(\kappa_{2,3})$ or $l(\kappa_{0,5}) < l(\lambda_0)$.

Finally $i_3 = 4$. First $(a_4, a_5) = (b_3, b_5)$. Suppose $a_1 = b_4$. Since α_2 does not cross $\Upsilon_{4,4}$ it follows that $a_2, a_3 \in \mathscr{O}_c'$ and hence $\{a_2, a_3\} = \{b_1, b_2\}$. Suppose $(a_2, a_3) = (b_1, b_2)$; see the ninth wire-frame. We argue as follows. Set $\alpha_4 = \kappa_{3,4}$, $\beta_3 = \lambda_0$, $\beta_4 = \kappa_{3,5}$, $\alpha_3 = \beta_2 = \kappa_{1,4}$ which extends to $K_3 \cup \Lambda_3 \cup \kappa_{1,4} \cup \kappa_{2,5}$. We have that $\mathscr{A}_4' = \alpha_1, \alpha_2, \alpha_3, \kappa_{0,4}$, $\mathscr{B}_3' = \beta_1, \beta_2, \lambda_2$, $\mathscr{B}_4' = \beta_1, \beta_2, \beta_3, \kappa_{0,5}$ are all chains and so $l(\kappa_{3,l}) = l(\kappa_{0,l})$, $l(\lambda_0) = l(\lambda_2)$ by Theorem 2.6. Now $\mathscr{B}_2' = \beta_1, \alpha_1$ is a chain and so $\beta_2 = \kappa_{1,4}$ is a shortest arc. Since $\alpha_1 = \kappa_{2,5}$ is also a shortest arc we have $l(\kappa_{1,4}) = l(\kappa_{2,5})$. By Lemma 2.8, $l(\kappa_{3,4}) = l(\kappa_{0,5})$. Now $\mathscr{B}_3' = \beta_1, \beta_2, \kappa_{0,4}$ is a chain and so $l(\lambda_0) \le l(\kappa_{0,4})$. Likewise $\beta_1 = \kappa_{1,2}$ is a shortest arc and so $l(\kappa_{1,2}) \le l(\kappa_{3,0})$. By Theorem 2.14, $l(\kappa_{1,4}) = l(\kappa_{2,5}) > l(\kappa_{1,2})$ which contradicts $\kappa_{2,5} = \alpha_1$, $\kappa_{1,4} = \beta_2$ being shortest arcs. If $(a_2, a_3) = (b_2, b_1)$ we can apply Theorems 2.6 and 2.7; see the tenth wire-frame.

Next $a_2 = b_4$. If $\{a_1, a_3\} = \{b_1, b_2\}$ then $\beta_1 \cup \alpha_1 \cup \alpha_2$ is a bracelet of shortest arcs. For either $a_1 \notin \{b_1, b_2\}$ or $a_2 \notin \{b_1, b_2\}$ and we can apply Theorems 2.6 and 2.12; see the 11–14th wire-frames. For $a_3 = b_4$ we have that $\{a_1, a_2\} = \{b_1, b_2\}$ since α_1 does not cross $\Upsilon_{4,4}$. Here $\mathscr{A}, \mathscr{B}$ is equivalent to a minimal chain pair on $\mathscr{E}$; see the seventh and eighth wire-frames, Figure 12.

To finish $(a_4, a_5) = (b_5, b_3)$. We may suppose that $a_3 \neq b_4$. For $a_1 = b_4$ we again have that $\{a_2, a_3\} = \{b_1, b_2\}$. Suppose $(a_2, a_3) = (b_2, b_1)$. We can apply Theorems 2.6 and 2.12—the 15th (respectively 16th) wire-frame illustrates the argument for $l(\beta_3) \le l(\beta_4)$ (respectively $l(\beta_4) \le l(\beta_3)$). Similarly for $(a_2, a_3) = (b_1, b_2)$. Suppose $a_2 = b_4$. If $\{a_1, a_3\} = \{b_1, b_2\}$ then $\beta_1 \cup \alpha_1 \cup \alpha_2$ is a bracelet of shortest arcs. So $a_1 = c$, $a_3 \in \{b_1, b_2\}$—see the 17th and 18th wire-frames—or $a_1 \in \{b_1, b_2\}$, $a_3 = c$—see the 19th and 2Oth wire-frames—and we can apply Theorems 2.6 and 2.12. □

Lemma 2.9. *If $l(\kappa_{3,0}) = l(\kappa_{1,4})$, $l(\kappa_{0,5}) = l(\kappa_{3,5})$ then there exists an arc γ between either c_2, c_1 or c_2, c_4 such that $l(\gamma) \le l(\kappa_{0,5}) = l(\kappa_{3,5})$.*

Proof. Let $\Gamma = \kappa_{3,0} \cup \kappa_{1,4} \cup \kappa_{2,5}$, a disjoint triple of arcs. As $l(\kappa_{3,0}) = l(\kappa_{1,4})$ the pair of pants $\mathscr{O} \setminus \Gamma$ has rotational symmetry R exchanging $\kappa_{3,0}$, $\kappa_{1,4}$ and fixing $\kappa_{2,5}$ setwise.

The cone points c_0, c_3, c_5 span an isosceles triangle $\mathscr{I}$ bounded by $\kappa_{3,0} \cup$

$\kappa_{3,5} \cup \kappa_{0,5}$. So $R(c_0), R(c_3), R(c_5)$ spans an isometric isosceles triangle $R(\mathscr{I})$. As $R(c_0), R(c_3)$ divides $\kappa_{1,4}$ into equal length subarcs, either c_1 or c_4 lies on the $\kappa_{1,4}$ edge of $R(\mathscr{I})$. If γ denotes the arc between this cone point and $R(c_5) = c_2$ contained in $R(\mathscr{I})$ then γ has the required properties. ▫

Suppose a minimal chain pair $\mathscr{A}, \mathscr{B}$ such that $(a_2, a_3) = (b_1, b_3)$, $a_1 \neq b_2$. Since $a_2 = b_1$ but $a_1 \neq b_2$: $\mathscr{A}_2' = \alpha_1, \beta_1$, $\mathscr{B}_2' = \beta_1, \alpha_1$ are both chains and so α_2, β_2 are shortest arcs. Now $(a_2, a_3) = (b_1, b_3)$ and so $\alpha_2 \cup \beta_1 \cup \beta_2$ is a bracelet of shortest arcs. By Proposition 2.10, $\mathscr{A}, \mathscr{B}$ is equivalent to a minimal standard chain pair on $\mathscr{O}ct$ and no length 3 bracelet of links on $\mathscr{O}ct$ cuts off a cone point. Likewise if $\{a_1, a_2\} = \{b_1, b_3\}$. (For applications to the proof of Theorem 1.1 we note that no minimal chains on $\mathscr{O}$ cross.)

Proposition 2.10. *Suppose an orbifold $\mathscr{O}$ has a length 3 bracelet of shortest arcs. Any minimal chain pair on $\mathscr{O}$ is equivalent to a minimal chain pair on the $\mathscr{O}ct$.*

Proof. Let Υ be a length 3 bracelet of shortest arcs on $\mathscr{O}$. Suppose Υ cuts off a cone point. By an application of Theorem 2.6 and then Lemma 2.3(ii), as in the proof of Proposition 2.4, we derive a contradiction. So Υ bounds a triangle. There exists Υ' another length 3 bracelet of shortest arcs disjoint from Υ that bounds a triangle. Moreover, the conformal symmetry group of $\mathscr{O}$ contains a subgroup isomorphic to S_3. This is a well-known result; see for example Schmutz [23, Lemma 5.1]. We will sketch the following proof of the existence of Υ', along the lines of the proof of Lemma 2.9.

Relabel the arcs in Υ by $\kappa_{3,0}, \kappa_{0,5}, \kappa_{3,5}$; label the shortest arc disjoint from Υ by $\kappa_{1,4}$; and label the unique arc disjoint from $\kappa_{3,0} \cup \kappa_{1,4}$ by $\kappa_{2,5}$. Label the cone points so that $\kappa_{k,l}$ is between c_k, c_l, et cetera. Again, let $\Gamma = \kappa_{3,0} \cup \kappa_{1,4} \cup \kappa_{2,5}$ and so $\mathscr{O} \setminus \Gamma$ is a pair of pants. Denote by $P_{3,0}$ (respectively $P_{1,4}$) the common perpendicular between boundary components $\kappa_{2,5}$ and $\kappa_{3,0}$ (respectively $\kappa_{2,5}$ and $\kappa_{1,4}$); denote by D_5 (respectively D_2) a disc of radius $l(\kappa_{3,0})$ (respectively $l(\kappa_{1,4})$) about c_5 (respectively c_2); and denote by $I_{3,0}$ (respectively $I_{1,4}$) the interval of the boundary component $\kappa_{3,0}$ (respectively $\kappa_{1,4}$) inside D_5 (respectively D_2). So $I_{3,0}$, $I_{1,4}$ are chords to D_5, D_2 respectively. Let θ_5 (respectively θ_2) denote the angle that $I_{3,0}$ (respectively $I_{1,4}$) subtends at c_5 (respectively c_2). Finally, denote by $\theta_{3,0}$ (respectively $\theta_{1,4}$) the angle of an equilateral triangle of edge length $l(\kappa_{3,0})$ (respectively $l(\kappa_{1,4})$).

Since $l(\kappa_{3,0}) \leq l(\kappa_{1,4})$ it follows that $\theta_{3,0} \geq \theta_{1,4}$ and that $l(P_{3,0}) \geq l(P_{1,4})$. Moreover $l(\kappa_{3,0}) \leq l(\kappa_{1,4})$, $l(P_{3,0}) \geq l(P_{1,4})$ implies that $\theta_5 \leq \theta_2$. Since $I_{1,4}$ subtends an angle $\theta_2 \geq \theta_5 = \theta_{3,0} \geq \theta_{1,4}$ it follows that $l(I_{1,4}) \geq l(\kappa_{1,4})$.

Unless $l(I_{1,4}) = l(\kappa_{1,4})$ and the cone points c_1, c_4 lie at the ends of $I_{1,4}$, there exists an arc γ between c_1, c_2 or c_2, c_4 such that $l(\gamma) < l(\kappa_{1,4})$. This would contradict the shortness assumption on $\kappa_{1,4}$. So $l(I_{1,4}) = l(\kappa_{1,4})$ and c_1, c_4 lie at the ends of $I_{1,4}$. So c_2, c_1, c_4 span a bracelet of shortest arcs Υ'.

Let Ξ be the set of shortest arcs between some cone point on Υ and some cone point on Υ'. Using cut-and-paste arguments, arcs in Ξ do not cross either: each other or arcs in $\Upsilon \cup \Upsilon'$. By the $\mathscr{S}_3$ action of the symmetry group: $|\Xi| = 3$ *or* 6. It is not hard to show that any minimal chain on $\mathscr{O}$ must lie in $\Upsilon \cup \Upsilon' \cup \Xi$. Moreover $\Upsilon \cup \Upsilon' \cup \Xi$ is graph isomorphic to a subgraph of the set of shortest arcs on $\mathscr{O}ct$. □

Suppose a minimal chain pair $\mathscr{A}, \mathscr{B}$ is such that $a_2 = b_2$ and a_1, a_3, b_1 and b_3 are all distinct. We have that $\mathscr{A}_2' = \alpha_1, \beta_1$, $\mathscr{B}_2' = \beta_1, \alpha_1$ are both chains and so α_2, β_2 are shortest arcs. It follows that $\alpha_1, \beta_1, \alpha_2, \beta_2$ are four distinct shortest arcs with exactly one cone point in common. By Lemma 2.11, $\mathscr{O}$ is the octahedral orbifold.

Lemma 2.11. *Suppose, for some k, that $\kappa_{k-1,k}$, $\kappa_{k,k+1}$, $\kappa_{k,l}$ are all shortest arcs for $l = 4, 5$. Then $\mathscr{O}$ is the octahedral orbifold.*

Proof. This result appeared as Lemma 5.2 in Schmutz [23]. However we offer the following proof. It illustrates the applications of Theorems 2.6 and 2.12 in the proof of Proposition 2.4. The arc set $\kappa_{k-1,k}$, $\kappa_{k,k+1}$, $\kappa_{k,l}$ for $l = 4, 5$ extends uniquely to the arc set $K \cup \Lambda$.

Suppose $l(\lambda_{k+1}) \leq l(\lambda_{k-2})$. We know that $l(\kappa_{k,l}) \leq l(\kappa_{k+1,l})$ for $l = 4, 5$ and so $l(\lambda_{k-1}) \leq l(\lambda_{k+1})$ by Theorem 2.6. So $l(\lambda_{k-1}) \leq l(\lambda_{k+1}) \leq l(\lambda_{k-2})$. If $l(\lambda_{k-1}) \leq l(\lambda_k)$ then we can apply Theorem 2.12. If $l(\lambda_k) \leq l(\lambda_{k-1})$ then $l(\lambda_k) \leq l(\lambda_{k-1}) \leq l(\lambda_{k+1})$ and we can again apply Theorem 2.12.

Suppose $l(\lambda_{k-2}) \leq l(\lambda_{k+1})$. We know that $l(\kappa_{k,l}) \leq l(\kappa_{k-1,l})$ for $l = 4, 5$ and so $l(\lambda_k) \leq l(\lambda_{k-2})$ by Theorem 2.6. So $l(\lambda_k) \leq l(\lambda_{k-2}) \leq l(\lambda_{k+1})$. If $l(\lambda_k) \leq l(\lambda_{k-1})$ then we can apply Theorem 2.12. If $l(\lambda_{k-1}) \leq l(\lambda_k)$ then $l(\lambda_{k-1}) \leq l(\lambda_k) \leq l(\lambda_{k-2})$ and we can again apply Theorem 2.12. □

Theorem 2.12. *Suppose, for some k, that $\kappa_{k,k+1}$ is a shortest arc and $l(\lambda_k) \leq \min\{l(\lambda_{k-1}), l(\lambda_{k+1})\}$ then $\mathscr{O}$ is the octahedral orbifold.*

Proof. Suppose $k = 3$. Cut $\mathscr{O}_{3,0}$ open along $\kappa_{0,5}$, $\kappa_{3,4}$ so as to obtain a simply connected domain Ω. Take a lift of Ω to the universal cover $\widetilde{\mathscr{O}}$ and, without confusion, give the geodesics having non-trivial intersection with Ω the same labels as on $\mathscr{O}$. So Ω is bounded by $\kappa_{0,5}$, λ_0, $\kappa_{3,4}$, λ_2. In the same cyclic order label the orbits of cone points that lie on the boundary of Ω by c_5, c_0, c_5', c_4', c_3 and c_4. The point c_0 (respectively c_3) lies at the midpoint of the $\kappa_{0,5}$ edge (respectively $\kappa_{3,4}$ edge). By inspection λ_3 is the diagonal of Ω between vertices c_5 and c_4'. As $\kappa_{3,0}$ is a shortest arc c_5, c_4' (and hence c_5', c_4) cannot lie inside $D_0 \cup D_3$, where D_k denotes the closed disc of radius $\kappa_{3,0}$ about c_k.

To prove the result we will show that either: $l(\lambda_3) > l(\lambda_2)$ or $l(\lambda_3) > l(\lambda_0)$ or $l(\lambda_3) = l(\lambda_2) = l(\lambda_0)$ and $\mathscr{O}$ is the octahedral orbifold. Let θ denote the angle in an equilateral triangle of edge length $l(\kappa_{3,0})$. The maximal length of a shortest arc on an orbifold $\mathscr{O}$ occurs exactly for $\mathscr{O}$ the octahedral orbifold; see Näätänen [22] or Schmutz [23]. It is a simple consequence that $\theta \geq \pi/4$ with *equality if and only if* $\mathscr{O}$ is the octahedral orbifold.

Claim. *Either* $\angle c_5 c_0 c_4' \geq 2\theta$ *or* $\angle c_5 c_3 c_4' \geq 2\theta$.

Now $\angle c_5 c_0 c_4' > \pi/2$ implies $\angle c_5' c_0 c_4' < \pi/2$ and hence that $l(\lambda_0) < l(\lambda_3)$. Likewise $\angle c_5 c_3 c_4' > \pi/2$ implies $\angle c_5 c_3 c_4 < \pi/2$ and hence that $l(\lambda_2) < l(\lambda_3)$. So we must have $\angle c_5 c_0 c_4' = \angle c_5 c_3 c_4' = 2\theta = \pi/2$. That is $\mathscr{O}$ is the octahedral orbifold.

Proof of the claim. Since c_5 lies outside $D_0 \cup D_3$ either $\angle c_5 c_0 c_3 \geq \theta$ or $\angle c_5 c_3 c_0 \geq \theta$. Likewise since c_4' lies outside $D_0 \cup D_3$ either $\angle c_4' c_0 c_3 \geq \theta$ or $\angle c_4' c_3 c_0 \geq \theta$. If $\angle c_5 c_0 c_3 \geq \theta$ and $\angle c_4' c_0 c_3 \geq \theta$ then $\angle c_5 c_0 c_4' \geq 2\theta$. Likewise, if $\angle c_5 c_3 c_0 \geq \theta$ and $\angle c_4' c_3 c_0 \geq \theta$ then $\angle c_5 c_3 c_4' \geq 2\theta$. So, up to relabelling, we may suppose that $\angle c_5 c_3 c_0 \leq \theta$ and $\angle c_4' c_0 c_3 \leq \theta$ as in Figure 9. Let $\theta_5 = \angle c_5 c_0 c_3 \geq \theta$ and $\theta_4 = \angle c_4' c_3 c_0 \geq \theta$. If $\theta_5 \geq 2\theta$ or $\theta_4 \geq 2\theta$ we are done. So suppose not.

Consider points x_5, x_4', to the c_5, c_4' side of $\kappa_{3,0}$ respectively, such that $\angle x_5 c_0 c_3 = \theta_5$ and $\angle x_5 c_3 c_0 = 2\theta - \theta_5$; and $\angle x_4' c_3 c_0 = \theta_4$ and $\angle x_4' c_0 c_3 = 2\theta - \theta_4$. Next we show that $x_5 \in D_0$. (Similarly $x_4' \in D_3$).

We have constructed the points A, E, C and C' such that $\angle ABC = \angle BAC = \angle ACB = \theta$ and $\angle BAC' = \theta_5$, $\angle ABC' = 2\theta - \theta_5$. We let O denote the intersection of AC, BC'. Now $\angle OAC' = \angle OBC = \theta_5 - \theta$ and $\angle AOC' = \angle BOC$ and OA is shorter than OB. So comparing the triangles OAC' and OBC we see that AC' is shorter than BC and we are done.

Now $\angle c_5 c_0 c_4' \geq \angle c_5 c_0 x_4' = 2\theta + (\theta_5 - \theta_4)$, $\angle c_5 c_3 c_4' \geq \angle x_5 c_3 c_4' = 2\theta + (\theta_4 - \theta_5)$. □

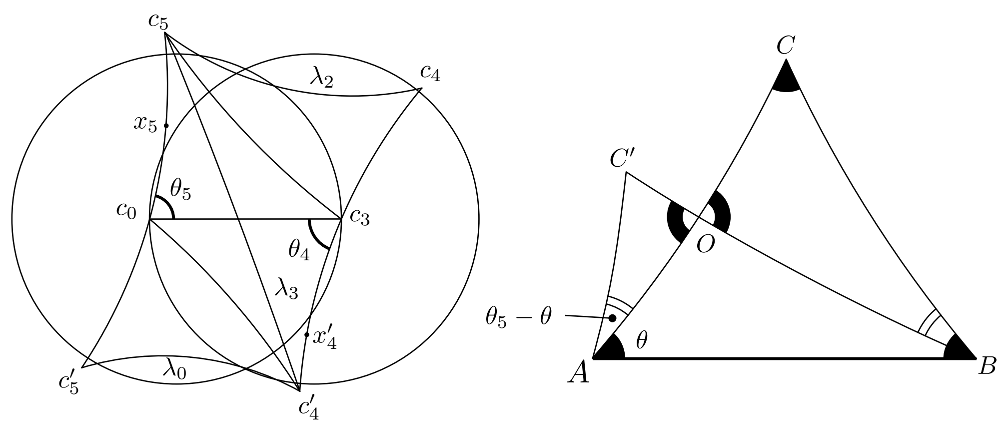

Figure 9. Arcs and lift of Ω in Theorem 2.12

Theorem 2.13. *Either* $l(\kappa_{0,4}) < l(\kappa_{3,4})$ *or* $l(\kappa_{0,5}) < l(\kappa_{3,5})$ *or* $l(\kappa_{2,4}) < l(\kappa_{2,3})$ *or* $l(\kappa_{0,5}) < l(\lambda_0)$.

Proof. We suppose that $l(\kappa_{3,4}) \leq l(\kappa_{0,4})$, $l(\kappa_{3,5}) \leq l(\kappa_{0,5})$, $l(\kappa_{2,3}) \leq l(\kappa_{2,4})$ and show that $l(\kappa_{0,5}) < l(\lambda_0)$. First, by Theorem 2.6, $l(\lambda_2) \leq l(\lambda_0)$. Also

$\bigcup_{l=4,5} \kappa_{2,l} \cup \kappa_{3,l}$ bound the quadrilateral spanned by $\lambda_2 \cup \kappa_{2,3}$. By Lemma 3.3, $l(\kappa_{2,4}) < l(\kappa_{2,3})$ or $l(\kappa_{3,5}) < l(\lambda_2)$. So $l(\kappa_{3,5}) < l(\lambda_2)$.

Cut $A_{3,0}$ open along $\kappa_{3,5}$ to obtain a simply connected domain Ω. Choose a lift of Ω in the universal cover of the annulus $\mathscr{A}_{3,0}$. Label the geodesics around the boundary of Ω by $\kappa_{3,0}$, $\kappa_{3,5}$, λ_2, λ_0 and $\kappa'_{3,5}$, in cyclic order. Without confusion, give the lifts of $\kappa_{0,4}$, $\kappa_{0,5}$ and $\kappa_{3,4}$ having non-trivial intersection with Ω the same labels. In the same cyclic order, label orbits of cone points around the boundary of Ω: c_0, c_3, c_5, c_4, c'_5 and c'_3. Let $P_{5,3}$, $P_{4,3}$ and $P'_{5,3}$ denote the perpendiculars to $\kappa_{3,0}$ from c_5, c_4, c'_5 and let $f_{5,3} = P_{5,3} \cap \kappa_{3,0}$, $f_{4,3} = P_{4,3} \cap \kappa_{3,0}$, $f'_{5,3} = P'_{5,3} \cap \kappa_{3,0}$. By choosing orientation we may suppose that $P_{5,3}$ is to the left and $P'_{5,3}$ to the right of $P_{4,3}$.

We now observe that c_0 must lie strictly between $P_{4,3}$ and $P'_{5,3}$. If c_0 is left of $P_{4,3}$ then $l(\kappa_{0,4}) < l(\kappa_{3,4})$ and if c_0 is to the right of $P'_{5,3}$ then $l(\kappa_{0,5}) < l(\kappa_{3,5})$.

If $c_3 = f_{5,3}$ set $\eta = \pi/2$, if $c_3 = f_{4,3}$ set $\nu = \pi/2$, $\phi = 0$. Otherwise we label angles as follows. Set $\theta = \angle c_5c_3c_4$, $\theta' = \angle c'_5c_0c_4$ and $\psi = \angle c_5c_4c_3$, $\psi' = \angle c'_5c_4c_0$ and $\phi = \angle c_3c_4f_{4,3}$, $\phi' = \angle c_0c_4f_{4,3}$ and $\nu = \angle c_4c_3f_{4,3}$, $\nu' = \angle c_4c_0f_{4,3}$ and $\eta = \angle c_5c_3f_{5,3}$, $\eta' = \angle c'_5c_0f'_{5,3}$; see Figure 10.

Since $l(\kappa_3, l) \leq l(\kappa_{0,l})$ for $l = 4, 5$ we have that $\nu \geq \nu'$, $\eta \geq \eta'$. If c_3 is between $P_{5,3}$ and $P_{4,3}$ then $\theta = \pi - (\nu + \eta)$. If c_3 lies strictly to the left of $P_{5,3}$ then $\theta = \eta - \nu < (\pi - \eta) - \nu$ since $\eta < \pi/2$. If c_3 lies strictly to the right of $P_{4,3}$ then $\theta = \nu - \eta < (\pi - \nu) - \eta$ since $\nu < \pi/2$. In each case $\theta \leq \pi - (\nu + \eta) \leq \pi - (\nu' + \eta') = \theta'$.

Also $l(\kappa_{3,4}) \leq l(\kappa_{0,4})$ implies that $\phi \leq \phi'$. Since $l(\lambda_2) \leq l(\lambda_0)$, by comparing the birectangles $\mathscr{Q}_{2,3}, \mathscr{Q}_{0,3}$, we have that $\phi + \psi \geq \phi' + \psi'$. It follows that $\psi \geq \psi'$. Now $l(\kappa_{3,5}) < l(\lambda_2)$ and so $\theta > \psi$. Therefore $\theta' \geq \theta > \psi \geq \psi'$: $l(\kappa_{0,5}) < l(\lambda_0)$. □

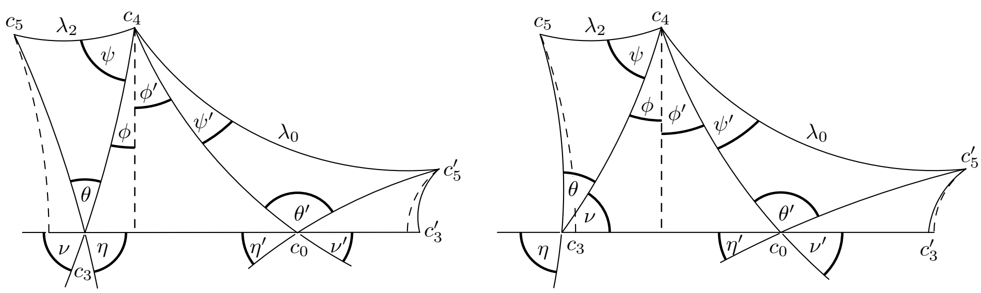

Figure 10. c_3 between $P_{5,3}$, $P_{4,3}$ and c_3 to the left of $P_{5,3}$

Theorem 2.14. *If $l(\kappa_{3,4}) = l(\kappa_{3,5}) = l(\kappa_{0,4}) = l(\kappa_{0,5})$ and $l(\kappa_{1,4}) = l(\kappa_{2,5})$, $l(\lambda_0) = l(\lambda_2) \leq l(\kappa_{0,4})$ and $l(\kappa_{1,2}) \leq l(\kappa_{3,0})$ then $l(\kappa_{1,4}) = l(\kappa_{2,5}) > l(\kappa_{1,2})$.*

Proof. By Lemma 2.8 both annuli $A_{1,2}, A_{3,0}$ have rotational symmetry exchanging $\lambda_0 \leftrightarrow \lambda_2$. It follows that $\mathscr{O} \setminus \kappa_{1,2} \cup \kappa_{3,0}$ has rotational symmetry exchanging $\lambda_0 \leftrightarrow \lambda_2$. So $\kappa_{1,2} \cup P_{4,1} \cup P_{4,3} \cup \kappa_{3,0} \cup P_{5,3} \cup P_{5,1}$ divides $\mathscr{O}$ into a pair of isometric right hexagons, $\mathscr{H}_0 \supset \lambda_0$, $\mathscr{H}_2 \supset \lambda_2$.

Consider $\mathscr{H}_0$. First we note that c_0 lies at the midpoint of the $\kappa_{3,0}$ edge of $\mathscr{H}_0$. The common perpendicular between the $\kappa_{1,2}$, $\kappa_{3,0}$ edges of $\mathscr{H}_0$ divides $\mathscr{H}_0$ into a mirror pair of right pentagons $\mathscr{P}_l$ where $\mathscr{P}_l$ has c_l as a vertex. Relabel the edges of $\mathscr{P}_4$: $A, \ldots, E$ in cyclic order such that $A = l(P_{4,1}), \ldots, E = l(P_{4,3})$. Label by F the diagonal from $A \cap E$ to $D \cap C$; by G the perpendicular from $A \cap E$ to C; and by H the diagonal from $A \cap E$ to $C \cap B$. Label by θ the angle between A, H; ψ the angle between G, H; and ϕ the angle between C, H. We observe that $l(\kappa_{1,4}) \geq l(P_{4,1}) = A$. To complete the proof we show that $A > 2B = l(\kappa_{1,2})$.

Consider the triangle ABH:

$$\sinh H = \frac{\sinh A}{\sin(\pi/2 - \phi)} \qquad \text{or} \qquad \sinh H = \frac{\sinh A}{\cos \phi}.$$

Now consider the double of ABH along A:

$$\frac{\sinh 2B}{\sin 2\theta} = \frac{\sinh H}{\sin(\pi/2 - \phi)} \qquad \text{or} \qquad \frac{\sinh 2B}{\sinh H} = \frac{\sin 2\theta}{\cos \phi}$$

and so we have

$$\frac{\sinh 2B}{\sinh A} = \frac{\sin 2\theta}{\cos^2 \phi}.$$

Now $B = l(\kappa_{1,2})/2 \leq l(\kappa_{3,0})/2 = D$ implies that $0 < \theta + \psi \leq \pi/4$. So $0 < 2\theta \leq \pi/2 - 2\psi < \pi/2$. Therefore $\sin 2\theta \leq \sin(\pi/2 - 2\psi)$.

Likewise $B \leq D$ implies that $H \geq F$. Moreover $F = l(\kappa_{0,4}) \geq l(\lambda_0) = 2G$. Consider CGH doubled along C. As $H \geq 2G$ we have that $2\phi \leq \psi$. Since $\psi < \pi/4$ it follows that $0 < \pi/4 - \psi \leq \pi/4 - 2\phi < \pi/4$. So $0 < \pi/2 - 2\psi \leq \pi/2 - 4\phi < \pi/2$, and hence $\sin(\pi/2 - 2\psi) \leq \sin(\pi/2 - 4\phi)$. Therefore $\sin 2\theta \leq \sin(\pi/2 - 2\psi) \leq \sin(\pi/2 - 4\phi) = \cos 4\phi$ and so

$$\frac{\sinh 2B}{\sinh A} \leq \frac{\cos 4\phi}{\cos^2 \phi}.$$

Now $\cos^2\phi - \cos 4\phi = \cos^2\phi - 2(2\cos^2\phi - 1)^2 - 1 = (8\cos^2\phi - 1)(1 - \cos^2\phi)$. Since $0 < \phi < \pi/8$ it follows that $\cos^2\phi > \cos 4\phi$. So $\sinh A > \sinh 2B$, $A > 2B$. □

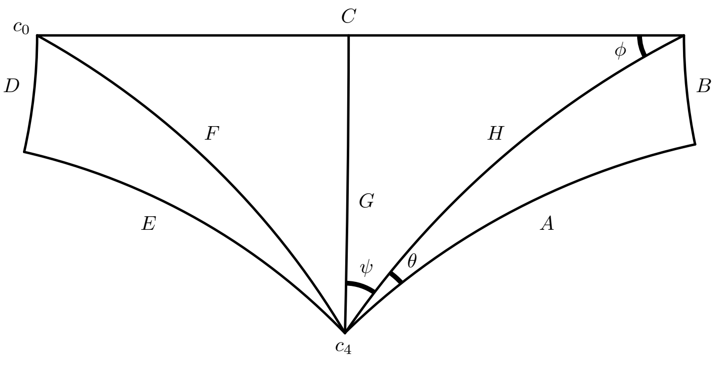

Figure 11. The pentagon $\mathscr{P}_4$

Proof of Proposition 2.5. Suppose that a minimal chain pair $\mathscr{A}_{i_3}, \mathscr{B}_{j_3}$ satisfies the hypotheses of Proposition 2.4. We need to show that any minimal standard chain pair $\mathscr{A}, \mathscr{B}$ containing $\mathscr{A}_{i_3}, \mathscr{B}_{j_3}$ is equivalent to a minimal standard chain pair on $\mathscr{E}$. Throughout, Proposition 2.1 will be used without mention.

Consider $(i_3, j_3) = (2, 3)$. By Proposition 2.4, $(a_2, a_3) = (b_2, b_4)$ and $a_1 = b_1 = c$. Set $\alpha_2 = \kappa_{3,4}$, $\beta_2 = \kappa_{3,5}$, $\beta_3 = \lambda_0$ which extends to $K_3 \cup \Lambda_3$. Each one of $\mathscr{A}_2' = \alpha_1, \kappa_{0,4}$, $\mathscr{B}_2' = \beta_1, \kappa_{0,5}$, $\mathscr{B}_3' = \beta_1, \beta_2, \lambda_2$ is a chain and so $l(\lambda_0) = l(\lambda_2)$ by Theorem 2.6. Consider the possibilities for $\beta_4, \alpha_3, \alpha_4$. Unless $\Upsilon_{3,4} = \beta_3 \cup \alpha_3 \cup \alpha_4$ is a bracelet, $\mathscr{A}, \mathscr{B}$ is equivalent to a minimal standard chain pair on $\mathscr{E}$.

Suppose $\Upsilon_{3,4}$ is a bracelet. Exchange labels $\alpha_* \leftrightarrow \beta_*$ and set $\alpha_3' = \lambda_2$. Suppose $\Upsilon_{3,4} = \alpha_3 \cup \beta_3 \cup \beta_4$ cuts off a cone point. Then $(a_3, a_4) = (b_5, b_3)$ with $\{a_1, a_2\} \neq b_4$ and by applying Lemma 3.3 and Theorem 2.13 we can derive a contradiction. If $\Upsilon_{3,4} = \alpha_3 \cup \beta_3 \cup \beta_4$ bounds a triangle, then $\Upsilon_{3,4}' = \alpha_3' \cup \beta_3 \cup \beta_4$ cuts off a cone point. The same argument can now be applied, where we use the fact that $l(\alpha_3') = l(\alpha_3)$.

Next, consider $(i_3, j_3) = (2, 4)$. Again, by Proposition 2.4, $(a_2, a_3) = (b_5, b_3)$ and $c = \alpha_1 \notin \{b_1, b_2\}$. Set $\alpha_2 = \kappa_{3,4}$, $\beta_4 = \kappa_{3,5}$, $\beta_3 = \lambda_0$ which again extends to $K_3 \cup \Lambda_3$. Again by Theorem 2.6, $l(\lambda_0) = l(\lambda_2)$. Now we consider α_3, α_4. Unless $\Upsilon_{3,4} = \beta_3 \cup \alpha_3 \cup \alpha_4$ is a bracelet, $\mathscr{A}, \mathscr{B}$ is equivalent to a standard minimal chain on $\mathscr{E}$. Suppose $\Upsilon_{3,4}$ is a bracelet. Now both of $\mathscr{A}_4' = \alpha_1, \alpha_2, \alpha_3, \beta_1$, $\mathscr{B}_2' = \beta_1, \alpha_4$ are chains and so α_4, β_2 are shortest arcs. If $\alpha_3 \neq \beta_2$ then $\mathscr{A}_3' = \alpha_1, \alpha_2, \beta_2$ is a chain and so α_3 is a shortest arc. In either case $\beta_1, \alpha_3, \alpha_4$ are all shortest arcs and $l(\lambda_0) = l(\lambda_2)$. By Theorem 2.7, $\mathscr{O}$ is the octahedral orbifold. As in the proof of Proposition 2.4 this gives a contradiction.

For $(i_3, j_3) = (3,3), (4,3), (4,4)$ there is nothing to prove. It remains to consider $(i_3, j_3) = (3, 4)$. By Proposition 2.4 we have that $(a_3, a_4) = (b_3, b_5)$ and $\{a_1, a_2\} = \{b_1, b_2\}$. Set $\alpha_3 = \kappa_{3,4}$, $\beta_4 = \kappa_{3,5}$, $\beta_3 = \lambda_0$ which extends to $K_3 \cup \Lambda_3$. We can apply Theorem 2.6: $l(\lambda_0) = l(\lambda_2)$. Unless $\alpha_4 = \beta_4$, $\mathscr{A}, \mathscr{B}$ is equivalent to a minimal standard chain pair on $\mathscr{E}$. Suppose $\alpha_4 = \beta_4$. Set $\beta_3 = \widehat{\kappa_{3,4}}$, $\alpha_4 = \widehat{\kappa_{3,5}}$, $\alpha_3 = \widehat{\lambda_0}$ which again extends to $\widehat{K_3} \cup \widehat{\Lambda_3}$. Again we can apply Theorem 2.6: $l(\widehat{\lambda_0}) = l(\widehat{\lambda_2})$. The arcs $\lambda_2, \widehat{\lambda_2}$ between b_3, b_4 and a_3, a_4, respectively, cross in a single point and have an endpoint $b_3 = a_3$ in common. By one of the cut-and-paste arguments we use in the proof of Proposition 3.1 we have that $l(\alpha_3) < l(\widehat{\lambda_2})$ or $l(\beta_3) < l(\lambda_2)$ which gives a contradiction. □

3. Proof of Theorem 1.1

As in the previous section we shall work on the quotient orbifold $\mathscr{O} = \mathscr{S} / \mathscr{J}$. We consider pairs of crossing standard minimal chains $\mathscr{A}, \mathscr{B}$ and derive contradictions. We say that $\alpha_{i_1}, \beta_{j_1}$ are the *first crossing links* if $i_1 = \min i \in \{1, \ldots, 4\}$ such that α_i, β_j cross, $j_1 = \min j \in \{1, \ldots, 4\}$ such that α_{i_1}, β_j cross. The crossing minimal chain pair $\mathscr{A}_{i_1}, \mathscr{B}_{j_1}$ has exactly one pair of crossing arcs: $\alpha_{i_1}, \beta_{j_1}$. In Proposition 3.1 we show that $|\alpha_{i_1} \cap \beta_{j_1}| < 2$. It remains to consider $|\alpha_{i_1} \cap \beta_{j_1}| = 1$

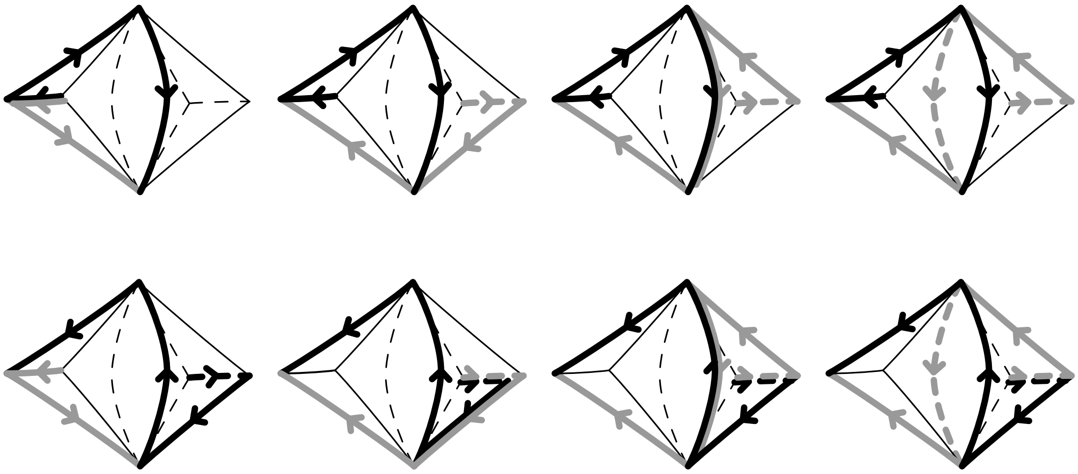

Figure 12. Minimal chain pairs with $j_3 = 3, 4$

i.e., $\alpha_{i_1}, \beta_{j_1}$ have distinct endpoints and a single crossing point $-\alpha_{i_1} \cup \beta_{j_1}$ has the form an 'X' on $\mathscr{O}$.

Proposition 3.1. *We have* $|\alpha_{i_1} \cap \beta_{j_1}| < 2$.

Proof. First we consider $\alpha_{i_1}, \beta_{j_1}$ having more than one crossing point. We then consider $\alpha_{i_1}, \beta_{j_1}$ having one crossing point and one or two endpoints in common.

On α_{i_1} label the crossing point nearest to its tail by a_t and the crossing point nearest to its head by a_h. Likewise for the crossing points on β_{j_1}. The points a_t, a_h, b_t, b_h divide α_{i_1} into either three, four or five subarcs. Label the subarc of α_{i_1} between a_{i_1}, a_t by α_t; between b_t, b_h by α_m; and between a_h, a_{i_1+1} by α_h. Likewise for the subarcs of β_{j_1}.

Let α'_{i_1} denote the geodesic in the endpoint fixed homotopy class of $\alpha_t \cup \beta_m \cup \alpha_h$. By its definition α'_{i_1} is simple. Let $\mathscr{A}_{i_1} = \alpha_1, \ldots, \alpha'_{i_1}$. Suppose α'_{i_1} crosses α_i for some $1 \leq i < i_1$. Choose lifts of $\alpha_t, \beta_m, \alpha_h, \alpha'_{i_1}$ in $\mathbf{H}^2$ so that $\alpha_t \cup \beta_m \cup \alpha_h \cup \alpha'_{i_1}$ bounds a rectangle (crossed or uncrossed). A lift of α_i intersects α'_{i_1} between a_{i_1}, a_{i_1+1}. This lift also intersects either α_t or β_m or α_h. This intersection projects to a crossing point of α_i with α_{i_1} or β_{j_1} on $\mathscr{O}$ which gives a contradiction. So $\mathscr{A}'_{i_1}$ is a chain.

Let β'_{j_1} denote the geodesic in the endpoint fixed homotopy class of $\beta_t \cup \alpha_m \cup \beta_h$. Again $\mathscr{B}'_{j_1} = \beta_1, \ldots, \beta'_{j_1}$ is a chain. We have that $l(\alpha'_{i_1}) < l(\alpha_{i_1})$ or $l(\beta'_{j_1}) < l(\beta_{j_1})$. If $l(\alpha_m) \leq l(\beta_m)$ then $l(\beta'_{j_1}) < l(\beta_t) + l(\alpha_m) + l(\beta_h) \leq l(\beta_t) + l(\beta_m) + l(\beta_h) \leq l(\beta_{j_1})$. If $l(\beta_m) \leq l(\alpha_m)$ then $l(\alpha'_{i_1}) < l(\alpha_t) + l(\beta_m) + l(\alpha_h) \leq l(\alpha_t) + l(\alpha_m) + l(\alpha_h) \leq l(\alpha_{i_1})$.

We now consider $\alpha_{i_1}, \beta_{j_1}$ having one crossing point and one or two endpoints in common. We will suppose that $a_{i_1} = b_{j_1}$ (the other possibilities follow similarly). The crossing point divides α_{i_1} into exactly two subarcs which we label α_t, α_h, so that $\alpha_t \ni \alpha_{i_1}$. Likewise for β_{j_1}. Let $\alpha'_{i_1}, \beta'_{j_1}$ denote the geodesic in the endpoint fixed homotopy class of $\beta_t \cup \alpha_h, \alpha_t \cup \beta_h$ respectively. Again we have that

$\mathscr{A}'_{i_1} = \alpha_1, \ldots, \alpha_{i_1-1}, \alpha'_{i_1}$ and $\mathscr{B}'_{j_1} = \beta_1, \ldots, \beta_{j_1-1}, \beta'_{j_1}$ are both chains and that $l(\alpha'_{i_1}) < l(\alpha_{i_1})$ or $l(\beta'_{j_1}) < l(\beta_{j_1})$. □

We say that $\mathscr{A}_{i_1}, \mathscr{B}_{j_1}$ is of *type* (I) if $\{a_{i_1}, a_{i_1+1}\} = \{b_j, b_{j+1}\}$, for some $j < j_1 - 1$, up to relabelling $\alpha_* \leftrightarrow \beta_*$. We say that $\mathscr{A}_{i_1}, \mathscr{B}_{j_1}$ is of *type* (II) if $\{a_{i_1}, a_{i_1+1}\} = \{b_{j-1}, b_{j+1}\}$ or $\{a_{i_1-1}, a_{i_1+1}\} = \{b_j, b_{j+1}\}$, for some $j < j_1 - 1$, up to relabelling $\alpha_* \leftrightarrow \beta_*$.

Proposition 3.2. *Type* (I), (II) *minimal chain pairs give a contradiction.*

Proof. Suppose $\mathscr{A}_{i_1}, \mathscr{B}_{j_1}$ is such that $\{a_{i_1}, a_{i_1+1}\} = \{b_j, b_{j+1}\}$ for some $j < j_1 - 1$. So arcs α_{i_1}, β_j share endpoints. We know that these arcs do not cross and that β_{j_1} crosses α_{i_1} but does not cross β_j. So $\Gamma_{i_1,j} = \alpha_{i_1} \cup \beta_j$ must be a bracelet. As $j < j_1 - 1 \le 3$ it follows that $j = 1$ or $j = 2$ which by Proposition 2.1 gives a contradiction.

Suppose now that $\mathscr{A}_{i_1}, \mathscr{B}_{j_1}$ is such that $\{a_{i_1-1}, a_{i_1+1}\} = \{b_j, b_{j+1}\}$ for some $j < j_1 - 1$. The other possibility follows similarly. So $\Upsilon_{j,i_1} = \beta_j \cup \alpha_{i_1-1} \cup \alpha_{i_1}$ is a bracelet. The arc β_{j_1} crosses only $\alpha_{i_1} \subset \Upsilon_{j,i_1}$. The endpoints b_{j_1}, b_{j_1+1} of β_{j_1} lie off Υ_{j,i_1}; b_{j_1}, b_{j_1+1} are distinct from $b_j, b_{j+1} \in \beta_j$ since $j < j_1 - 1$ and we know that b_{j_1}, b_{j_1+1} are both distinct from a_{i_1}, the other cone point on Υ_{j,i_1}. So b_{j_1}, b_{j_1+1} must lie in different components of $\mathscr{O} \setminus \Upsilon_{j,i_1}$. As there is only one other cone point lying off Υ_{j,i_1} it follows that Υ_{j,i_1} cuts off b_{j_1} or b_{j_1+1}. By Proposition 2.4, $1 < j$, $2 < i_1$. Since $j < j_1 - 1 \le 3$, $i_1 \le 4$ it follows that $(j, i_1) = (2, 3)$ or $(j, i_1) = (2, 4)$. As $2 = j < j_1 - 1 \le 3$ we have that $j_1 = 4$. So $\Upsilon_{2,3}$ or $\Upsilon_{2,4}$ cuts off b_4 or b_5. By Proposition 2.4 this gives a contradiction. □

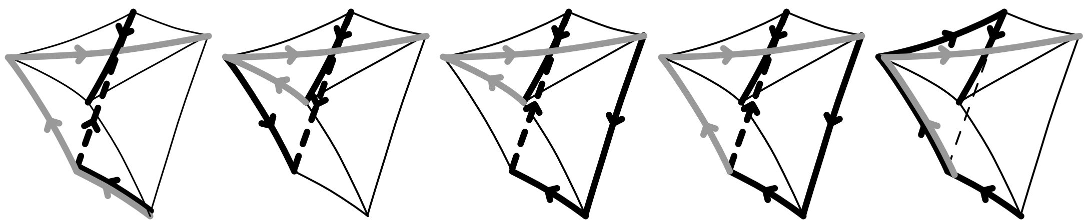

Figure 13. Examples of (a)–(e) of Proposition 3.4

Let $\varepsilon_{t,t} \cup \varepsilon_{t,h} \cup \varepsilon_{h,h} \cup \varepsilon_{h,t}$ denote the bracelet that bounds the quadrilateral spanned by $\alpha_{i_1} \cup \beta_{j_1}$; we label so that $\varepsilon_{t,h}$ is between a_{i_1}, b_{j_1+1} and $\varepsilon_{h,t}$ is between a_{i_1+1}, b_{j_1}, et cetera. As in the proof of Proposition 3.1 none of these arcs cross $\mathscr{A}_{i_1}, \mathscr{B}_{j_1}$. Similarly $l(\varepsilon_{t,t}) < l(\alpha_{i_1})$ or $l(\varepsilon_{h,h}) < l(\beta_{j_1})$ and all other such combinations. We have proved:

Lemma 3.3. *Either* $l(\varepsilon_{t,t}) < l(\alpha_{i_1})$ *or* $l(\varepsilon_{h,h}) < l(\beta_{j_1})$, *et cetera.*

Proposition 3.4. *Up to relabelling* $\alpha_* \leftrightarrow \beta_*$, *we have a contradiction*:
(a) *If* $\mathscr{A}'_{i_1} = \alpha_1, \ldots, \alpha_{i_1-1}, \varepsilon_{t,h}$, $\mathscr{B}'_{j_1} = \beta_1, \ldots, \beta_{j_1-1}, \varepsilon_{h,t}$ *are both chains.*

(b) *If either* $i_1 = 1$ *or* $\mathscr{A}'_{i_1} = a_1, \ldots, \alpha_{i_1-1}, \beta_1$ *is a chain and either* $\mathscr{B}'_{j_1} = \beta_1, \ldots, \beta_{j_1-1}, \varepsilon_{t,t}$ *or* $\mathscr{B}''_{j_1} = \beta_1, \ldots, \beta_{j_1-1}, \varepsilon_{h,t}$ *is a chain.*

(c) *If* $i_1 = 2$ *and* $\mathscr{A}'_2 = \alpha_1, \varepsilon_{h,h}$, $\mathscr{B}'_{j_1} = \beta_1, \ldots, \beta_{j_1-1}, \varepsilon_{t,t}$ *are both chains.*

(d) *If* $i_1 = 2$, $j_1 = 4$ *and* $\mathscr{A}'_2 = \alpha_1, \beta_2$, $\mathscr{B}'_2 = \beta_1, \alpha_2$, $\mathscr{B}''_2 = \beta_1, \varepsilon_{h,t}$ *and* $\mathscr{B}'_4 = \beta_1, \beta_2, \beta_3, \varepsilon_{t,h}$ *are all chains.*

(e) *If for some* $j < j_1$: $\mathscr{A}'_{i_1} = \alpha_1, \ldots, \alpha_{i_1-1}, \beta_j$, $\mathscr{B}'_j = \beta_1, \ldots, \beta_{j-1}, \alpha_{i_1}$, $\mathscr{B}''_j = \beta_1, \ldots, \beta_{j-1}, \varepsilon_{t,h}$ *and* $\mathscr{B}'_{j_1} = \beta_1, \ldots, \beta_{j_1-1}, \varepsilon_{h,t}$ *are all chains.*

Proof. For each part we have a contradiction by Lemma 3.3. For (a) we have $l(\alpha_{i_1}) \le l(\varepsilon_{t,h})$, $l(\beta_{j_1}) \le l(\varepsilon_{h,t})$. For (b) suppose $\mathscr{B}'_{j_1} = \beta_1, \ldots, \beta_{j_1-1}, \varepsilon_{t,t}$ is a chain. We then have $l(\alpha_{i_1}) = l(\beta_1) \le l(\varepsilon_{h,h})$, $l(\beta_{j_1}) \le l(\varepsilon_{t,t})$. For (c) we have $l(\alpha_2) \le l(\varepsilon_{h,h})$, $l(\beta_{j_1}) \le l(\varepsilon_{t,t})$. For (d) we have $l(\alpha_2) = l(\beta_2) \le l(\varepsilon_{h,t})$, $l(\beta_4) \le l(\varepsilon_{t,h})$. For (e) we have $l(\alpha_{i_1}) = l(\beta_j) \le l(\varepsilon_{t,h})$, $l(\beta_{j_1}) \le l(\varepsilon_{h,t})$. □

Up to relabelling we may suppose that $i_1 \le j_1$. Since $\alpha_{i_1}, \beta_{j_1}$ have distinct endpoints $a_{i_1}, a_{i_1+1}, b_{j_1}, b_{j_1+1}$ are all distinct.

First $i_1 = 1$. If $\{a_1, a_2\} \not\subset \{b_1, \ldots, b_{j_1-1}\}$ then either $\mathscr{B}'_{j_1} = \beta_1, \ldots, \beta_{j_1-1}, \varepsilon_{t,t}$ or $\mathscr{B}'_{j_1} = \beta_1, \ldots, \beta_{j_1-1}, \varepsilon_{h,t}$ is a chain and we can apply Proposition 3.4(b). So $\{a_1, a_2\} \subset \{b_1, \ldots, b_{j_1-1}\}$ and we have type (I) or (II). So $i_1 > 1$. Suppose that $b_{j_1+1} \notin \{a_1, \ldots, a_{i_1-1}\}$, $a_{i_1+1} \notin \{b_1, \ldots, b_{j_1-1}\}$. We then have that $\mathscr{A}'_{i_1} = \alpha_1, \ldots, \alpha_{i_1-1}, \varepsilon_{t,h}$, $\mathscr{B}'_{j_1} = \beta_1, \ldots, \beta_{j_1-1}, \varepsilon_{h,t}$ are both chains and we can apply Proposition 3.4(a). So we may suppose that either $b_{j_1+1} \in \{a_1, \ldots, a_{i_1-1}\}$ or $a_{i_1+1} \in \{b_1, \ldots, b_{j_1-1}\}$.

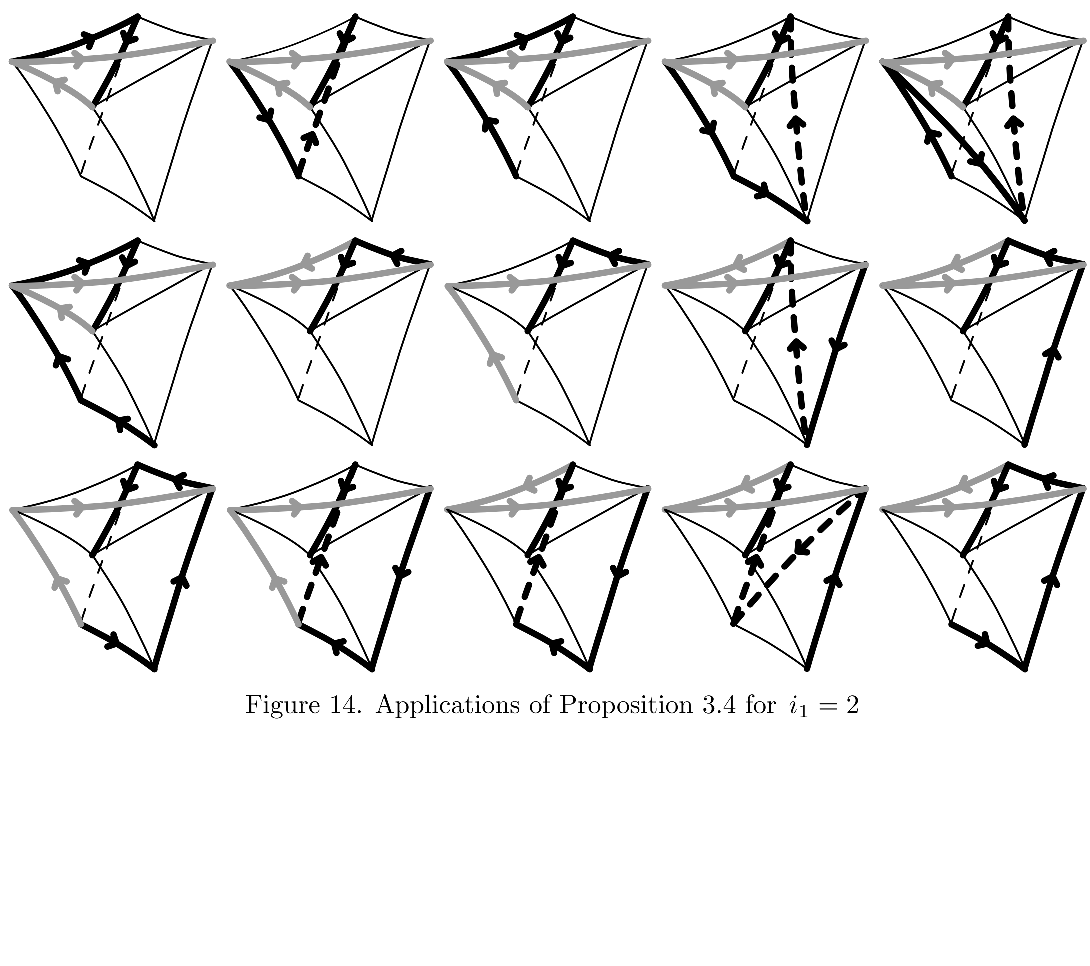

Figure 14. Applications of Proposition 3.4 for $i_1 = 2$

Next $i_1 = 2$. First we suppose $b_{j_1+1} = a_1$. If $a_2 \notin \{b_1, \ldots, b_{j_1-1}\}$ then $\mathscr{B}'_{j_1} = \beta_1, \ldots, \beta_{j_1-1}, \varepsilon_{t,t}$ is a chain. We note that $\varepsilon_{h,h}$ is between $b_{j_1+1} = a_1, a_3$ and so $\mathscr{A}'_2 = \alpha_1, \varepsilon_{h,h}$ is also a chain and we can apply Proposition 3.4(c). So $a_2 \in \{b_1, \ldots, b_{j_1-1}\}$.

For $j_1 = 2$: Proposition 3.4(b), see the first wire-frame, Figure 14. Suppose $j_1 = 3$. If $a_3 \in \{b_1, b_2\}$ type (I). If $a_3 \notin \{b_1, b_2\}$: Proposition 3.4(b), see the second and third wire-frames. For $j_1 = 4$ if $a_3 \in \{b_1, b_2, b_3\}$: type (I) or type (II). So $a_3 = b_6$. If $a_2 \in \{b_1, b_2\}$: Proposition 3.4(b), see the fourth and fifth wire-frames. If $a_2 = b_3$ then $\mathscr{A}'_2 = \alpha_1, \beta_3$, $\mathscr{B}'_3 = \beta_1, \beta_2, \alpha_2$, $\mathscr{B}''_3 = \beta_1, \beta_2, \varepsilon_{t,h}$ and $\mathscr{B}'_4 = \beta_1, \beta_2, \beta_3, \varepsilon_{h,t}$ are all chains, Proposition 3.4(e), see the sixth wire-frame.

Therefore $a_1 \neq b_{j_1+1}, a_3 \in \{b_1, \ldots, b_{j_1-1}\}$.

For $j_1 = 2$, $a_1 \neq b_3$, $a_3 = b_1$. If $a_1 = b_2$, Proposition 3.4(b), see the seventh wire-frame. So $a_1 \in \{b_4, b_5, b_6\}$, Proposition 3.4(c); see the eighth wire-frame.

For $j_1 = 3$, $a_1 \neq b_4$, $a_3 \in \{b_1, b_2\}$. If $a_2 \in \{b_1, b_2\}$, type (I). So $a_2 \in \{b_5, b_6\}$. If $a_1 \in \{b_1, b_2\}$, type (II). If $a_1 = b_3$, Proposition 3.4(b); see the ninth and tenth wire-frames. So $a_3 \in \{b_1, b_2\}$, $\{a_2, a_1\} = \{b_5, b_6\}$ and we can apply Theorem 3.7, see Figure 21.

For $j_1 = 4$, $a_1 \neq b_5$, $a_3 \in \{b_1, b_2, b_3\}$. If $a_2 \in \{b_1, b_2, b_3\}$, type (I) or (II). So $a_2 = b_6$. If $a_1 \in \{b_1, b_2, b_3\}$, type (I) or (II) unless $\{a_1, a_3\} = \{b_1, b_3\}$. If $(a_1, a_3) = (b_1, b_3)$, Proposition 3.4(b); see the eleventh wire-frame. If $(a_1, a_3) = (b_3, b_1)$, Proposition 3.4(d); see the 12th wire-frame. So $a_1 = b_4$, Proposition 3.4(b); see the 13th, 14th and 15th wire-frames.

Now $i_1 = 3$. First $j_1 = 3$. Up to relabelling we may suppose $b_4 \in \{a_1, a_2\}$. If $b_3 \in \{a_1, a_2\}$, type (I). So $b_3 \in \{a_5, a_6\}$. Likewise, if $b_2 \in \{a_1, a_2\}$, type (II).

Suppose $b_2 = a_3$. If $b_1 = a_4$, type (I). So $b_1 \neq a_4$, $\mathscr{A}'_3 = \alpha_1, \alpha_2, \beta_2$, $\mathscr{B}'_2 = \beta_1, \alpha_3$, $\mathscr{B}''_2 = \beta_1, \varepsilon_{t,h}$ and $\mathscr{B}'_3 = \beta_1, \beta_2, \varepsilon_{h,t}$ are all chains, Proposition 3.4(e), see the first to fourth wire-frames in Figure 15.

Suppose $b_2 = a_4$. If $b_1 \in \{a_2, a_3\}$, type (I) or (II). So $b_1 = a_1$ or $b_1 \in \{a_5, a_6\}$.

For $b_1 = a_1$ we have that $b_4 = a_2$. The arc set $\mathscr{A}_3 \cup \mathscr{B}_3$ divides $\mathscr{O}$ into four components having $6, 4, 3$ and 3 geodesic boundary pieces respectively. Label these by $\mathscr{O}_6$, $\mathscr{O}_4$, $\mathscr{O}_3$ and $\mathscr{O}'_3$ so that α_2 lies on the boundary of $\mathscr{O}_3$. Let c denote the cone point lying off $\mathscr{A}_3 \cup \mathscr{B}_3$. We note that both $\mathscr{A}'_2 = \alpha_1, \beta_1$, $\mathscr{B}'_2 = \beta_1, \alpha_1$ are chains and so α_2, β_2 are both shortest arcs.

Suppose $c \in \mathscr{O}_4$, see the fifth wire-frame. Let α'_1 (respectively β'_1) denote the arc between c and a_2 (respectively b_2) in $\mathscr{O}_4$. Both $\mathscr{A}'_3 = \alpha_1, \alpha_2, \varepsilon_{t,t}$, $\mathscr{B}'_3 = \beta_1, \beta_2, \varepsilon_{t,t}$ are chains and so $\max\{l(\alpha_3), l(\beta_3)\} \leq l(\varepsilon_{t,t})$. By Proposition 3.5(v), $l(\alpha'_1) < l(\alpha_1)$ or $l(\beta'_1) < l(\beta_1)$.

Now suppose $c \in \mathscr{O}_6$; see the sixth wire-frame. Let α'_3 (respectively β'_3) denote the arc between c and a_3 (respectively b_3) in $\mathscr{O}_6$. Again both $\mathscr{A}'_3 = \alpha_1, \alpha_2, \varepsilon_{t,t}$, $\mathscr{B}'_3 = \beta_1, \beta_2, \varepsilon_{t,t}$ are chains and so $\max\{l(\alpha_3), l(\beta_3)\} \leq l(\varepsilon_{t,t})$. Since α_2, β_2 are both shortest arcs $l(\alpha_2) = l(\beta_2)$. We can now apply Proposition 3.5(vii): $l(\alpha'_3) < l(\alpha_3)$ or $l(\beta'_3) < l(\beta_3)$. As $\mathscr{A}''_3 = \alpha_1, \alpha_2, \alpha'_3$, $\mathscr{B}''_3 = \beta_1, \beta_2, \beta'_3$ are both chains we have a contradiction.

For $c \in \mathscr{O}_3$ (equivalently $c \in \mathscr{O}'_3$) we can apply Theorems 2.12 and 3.6; see Figure 20.

For $b_1 \in \{a_5, a_6\}$ we first suppose $b_4 = a_1$. The arc set $\Upsilon = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \beta_2 \cup \beta_3$ divides $\mathscr{O}$ into three components, having $7, 4$ and 3 geodesic boundary pieces respectively. Label these components $\mathscr{O}_7$, $\mathscr{O}_4$ and $\mathscr{O}_3$ respectively. Either $b_1 \in \mathscr{O}_7$, see the seventh wire-frame, or $b_1 \in \mathscr{O}_3$, see the eighth wire-frame. For $b_1 \in \mathscr{O}_7$ (respectively $b_1 \in \mathscr{O}_3$) label the arc between b_1, b_3 in $\mathscr{O}_7$ (respectively $\mathscr{O}_3$) by β'_1. Since $\mathscr{A}'_3 = \alpha_1, \alpha_2, \varepsilon_{t,t}$, $\mathscr{B}'_3 = \beta_1, \beta_2, \varepsilon_{t,t}$ are both chains $\max\{l(\alpha_3), l(\beta_3)\} \leq l(\varepsilon_{t,t})$. By Proposition 3.5(iii), $l(\alpha_2) < l(\alpha_1)$ or $l(\beta'_1) < l(\beta_1)$ contradicting α_1, β_1 both being shortest arcs.

For this last example the same argument was adapted to the two different topological configurations. This also holds for all the remaining examples. Accordingly in Figure 15 we have pictured only one topological configuration to illustrate the argument used.

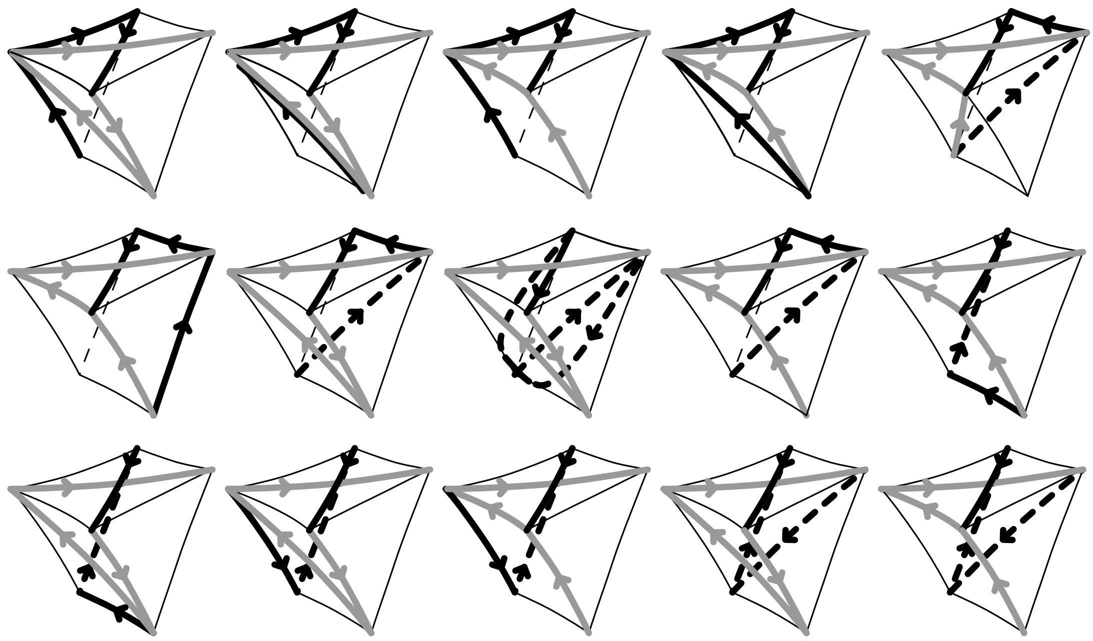

Figure 15. Applications of Proposition 3.5 for $i_1 = 3$, $j_1 = 3$

If $b_4 = a_2$ we can again apply Proposition 3.5(iii); see the ninth wire-frame.

So $b_2 \in \{a_5, a_6\}$.

If $b_1 = a_1$ then $b_4 = a_2$; see the tenth wire-frame. The arc set $\mathscr{A}_3 \cup \mathscr{B}_3$ divides $\mathscr{O}$ into three components having $6, 6$ and 3 geodesic boundary pieces respectively. Label these components $\mathscr{O}_6$, $\mathscr{O}'_6$ and $\mathscr{O}_3$ respectively so that α_2 lies on the boundary of $\mathscr{O}_6$. In $\mathscr{O}_6$, label by α'_1 the arc between b_1, a_3; by α''_1 the arc between b_2, a_2; and by α'''_1 the arc between b_2, a_3. Again $\mathscr{A}'_3 = \alpha_1, \alpha_2, \varepsilon_{t,t}$, $\mathscr{B}'_3 = \beta_1, \beta_2, \varepsilon_{t,t}$ are both chains $\max\{l(\alpha_3), l(\beta_3)\} \leq l(\varepsilon_{t,t})$. Since β_1, the arc disjoint from $\alpha_3 \cup \beta_3$, is a shortest arc, $l(\beta_1) \le l(\varepsilon_{h,t})$ and we can apply Proposition 3.5(i): $\max\{l(\alpha_1'), l(\alpha_1''), l(\alpha_1''')\} < l(\alpha_1)$ contradicting α_1 being a shortest arc.

If $b_1 = a_2$ then $b_4 = a_1$ and we can again apply Proposition 3.5(i); see the eleventh wire-frame. For $b_1 = a_3$ we can apply Proposition 3.4(b); see the 12th and 13th wire-frame. So $b_1 = a_4$ and we can apply Proposition 3.5(iii); see the 14th and 15th wire-frames.

Now $j_1 = 4$. First we suppose $b_5 \in \{a_1, a_2\}$. If $b_4 \in \{a_1, a_2\}$, type (I). So $b_4 \in \{a_5, a_6\}$. If $b_3 \in \{a_1, a_2\}$, type (II).

Suppose $b_3 = a_3$. If $\{b_1, b_2\} \ni a_4$, type (I) or (II). If $\{b_1, b_2\} \not\ni a_4$ we can apply Proposition 3.4(e); see the first and second wire-frames, Figure 16, where we have pictured the possibilities for $b_5 = a_1$.

Suppose $b_3 = a_4$. If $\{b_1, b_2\} \ni a_3$ or $b_2 = a_2$, type (I) or type (II). So $b_2 = a_1$ or $b_2 \in \{a_5, a_6\}$.

If $b_2 = a_1$ then $b_5 = a_2$, $b_1 \in \{a_5, a_6\}$ and we can apply Proposition 3.5(i); see the third wire-frame.

For $b_2 \in \{a_5, a_6\}$ either $b_1 = a_1$, $b_5 = a_2$ or $b_1 = a_2$, $b_5 = a_1$ and we can apply Proposition 3.5(i); see the fourth and fifth wire-frames.

So $b_3 \in \{a_5, a_6\}$.

Suppose $b_2 = a_1$ and so $b_5 = a_2$. Either $b_1 = a_3$ or $b_1 = a_4$.

For $b_1 = a_3$, $\{b_1, b_2\} = \{a_1, a_3\}$ and we can apply Proposition 2.10.

For $b_1 = a_4$ we note that $\mathscr{B}_2' = \beta_1, \alpha_1$ is a chain and so β_2 is a shortest arc. We can apply Proposition 3.5(i); see the sixth wire-frame.

Suppose $b_2 = a_2$ and so $b_5 = a_1$. Either $b_1 = a_3$ or $b_1 = a_4$.

For $b_1 = a_3$, see the seventh wire-frame. The arc set $\mathscr{A}_3 \cup \mathscr{B}_4$ divides $\mathscr{O}$ into three components having $6, 5$ and 4 geodesic boundary pieces respectively. Label these components $\mathscr{O}_6$, $\mathscr{O}_5$ and $\mathscr{O}_4$ respectively. In $\mathscr{O}_6$, label by α_1' (respectively α_1'') the arc between b_2, a_4 (respectively b_3, a_1). We have that $\mathscr{A}_3' = \alpha_1, \alpha_2, \varepsilon_{t,t}$, $\mathscr{B}_4' = \beta_1, \beta_2, \beta_3, \varepsilon_{h,t}$ are both chains $l(\alpha_3) \le l(\varepsilon_{t,t}), l(\beta_3) \le l(\varepsilon_{h,t})$. Now $\mathscr{B}_2' = \beta_1, \alpha_1$ is a chain and so β_2—the arc disjoint from $\alpha_3 \cup \beta_4$—is a shortest arc, $l(\beta_2) \le \min\{l(\varepsilon_{t,t}), l(\varepsilon_{t,h})\}$ and so we can apply Proposition 3.5(ii): $\max\{l(\alpha_1'), l(\alpha_1'')\} < l(\alpha_1)$ contradicting α_1 being a shortest arc.

For $b_1 = a_4$, type (II).

Suppose $b_2 = a_3$. If $b_1 = a_1$, we can apply Proposition 2.10. If $b_1 = a_2$ then $b_5 = a_1$ and we can apply Proposition 3.4(e); see the eighth wire-frame. If $b_1 = a_4$, type (I).

So $b_2 = a_4$.

If $b_1 = a_1$ then $b_5 = a_2$ and we note that $\mathscr{B}_2' = \beta_1, \alpha_1$ is a chain and so β_2 is a shortest arc. We can apply Proposition 3.5(iii) to contradict α_1, β_2 both being shortest arcs; see the ninth wire-frame.

So $b_1 \in \{a_2, a_3\}$, type (I) or (II). Therefore: $\{a_1, a_2\} \not\ni b_5$ and $a_4 \in \{b_1, b_2, b_3\}$.

If $a_3 \in \{b_1, b_2, b_3\}$, type (I) or (II). So $a_3 = b_6$.

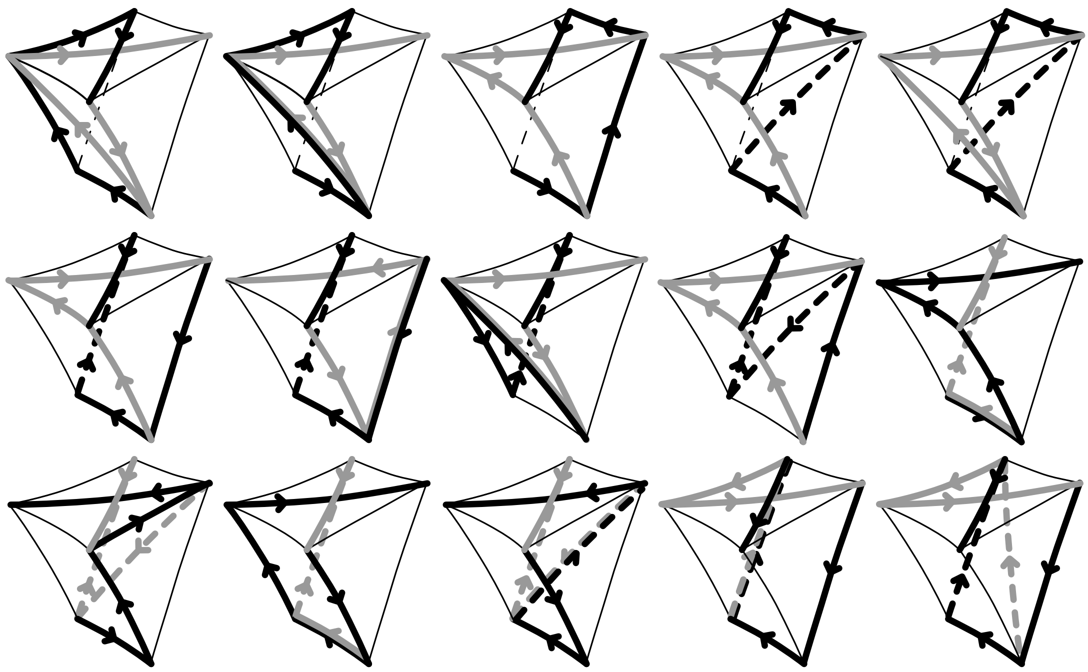

Figure 16. Applications of Propositions 3.4 and 3.5 for $i_1 = 3$, $j_1 = 4$

Suppose $a_2 = b_1$. If $a_4 = b_2$, type (II). If $a_4 = b_3$ then $a_1 = b_2$ or $a_1 = b_4$. We can apply Proposition 3.5(i) or (ii) respectively; see the tenth and eleventh wire-frames.

If $a_2 = b_2$, type (II).

Suppose $a_2 = b_3$. If $a_4 = b_2$, type (I). If $a_4 = b_1$ then $a_1 = b_2$ or $a_1 = b_4$. For $a_1 = b_2$ we can apply Proposition 3.5(i); see the 12th wire-frame.

For $a_1 = b_4$, see the 13th wire-frame. The arc set $\mathscr{A}_3 \cup \mathscr{B}_4$ divides $\mathscr{O}$ into three components having $6, 5$ and 4 geodesic boundary pieces respectively. Label these components $\mathscr{O}_6$, $\mathscr{O}_5$ and $\mathscr{O}_4$ respectively. In $\mathscr{O}_6$, label by β_1' (respectively β_1'') the arc between b_2, b_5 (respectively b_3, b_1). We have that $\mathscr{A}_3' = \alpha_1, \alpha_2, \varepsilon_{t,h}$, $\mathscr{B}_4' = \beta_1, \beta_2, \beta_3, \varepsilon_{t,t}$ are both chains and so $l(\alpha_3) \le l(\varepsilon_{t,h})$, $l(\beta_4) \le l(\varepsilon_{t,t})$. We know that $\mathscr{B}_2' = \beta_1, \varepsilon_{h,h}$ is a chain and so $l(\beta_2) \le l(\varepsilon_{h,h})$. If $l(\varepsilon_{h,h}) \le l(\varepsilon_{t,t})$ then $l(\beta_2) \le \min\{l(\varepsilon_{h,h}), l(\varepsilon_{t,t})\}$ and we can apply Proposition 3.5(ii): $\max\{l(\beta_1'), l(\beta_1'')\} < l(\beta_1)$ contradicting β_1 being a shortest arc. So suppose $l(\varepsilon_{t,t}) \le l(\varepsilon_{h,h})$. We can now apply Proposition 3.5(iv): $l(\beta_1') < l(\beta_1)$ or $l(\alpha_2) < l(\alpha_1)$ contradicting β_1, α_1 both being shortest arcs.

So $a_2 = b_4$ and we can apply Proposition 3.4(e), see the 14th and 15th wire-frame, where we pictured the possibilities for $a_4 = b_1$.

Finally we consider $i_1 = 4$, $j_1 = 4$. Up to relabelling, we may suppose that $b_5 \in \{a_1, a_2, a_3\}$. If $b_4 \in \{a_1, a_2, a_3\}$, type (I) or (II). So $b_4 = a_6$.

Suppose $b_3 \in \{a_1, a_2, a_3\}$. Either $\{b_5, b_3\} \ni a_2$, type (II) or $\{b_5, b_3\} = \{a_1, a_3\}$.

First: $(b_5, b_3) = (a_1, a_3)$. If $b_2 = a_2$ either $b_1 = a_4$ or $b_1 = a_5$ and we can apply Proposition 3.5(ii) or (i) respectively; see the first and second wire-frames, Figure 17. If $b_2 \in \{a_4, a_5\}$ either $b_1 = a_2$ and we can apply Proposition 2.10 or $b_1 \in \{a_4, a_5\}$, type (I).

Next: $(b_5, b_3) = (a_3, a_1)$. If $b_2 = a_2$ either $b_1 = a_4$ or $b_1 = a_5$ and we can apply Proposition 3.5(ii) or (i) respectively, see the third and fourth wire-frames. If $b_2 \in \{a_4, a_5\}$ either $b_1 = a_2$ and we can apply Proposition 2.10, or $b_1 \in \{a_4, a_5\}$, type (I).

Suppose $b_3 = a_4$. For $\{b_1, b_2\} \not\ni a_5$ we can apply Proposition 3.4(e); see the fifth and sixth wire-frames, where we have pictured the possibilities for $b_5 = a_1$. For $\{b_1, b_2\} \ni a_5$, type (I) or (II). So $b_3 = a_5$.

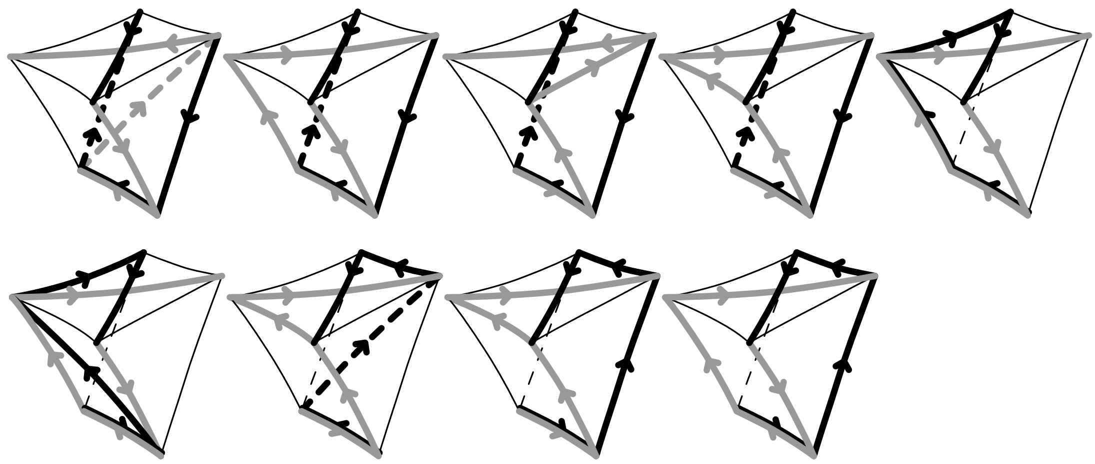

Figure 17. Applications of Propositions 3.4 and 3.5 for $i_1 = 4$, $j_1 = 4$

Suppose $b_2 = a_1$. If $b_1 = a_2$ we can apply Proposition 3.5(i); see the sixth wire-frame. If $b_1 = a_3$ we can apply Proposition 2.10. If $b_1 = a_4$, type (II).

Suppose $b_2 = a_2$. If $b_1 = a_1$ or $b_1 = a_3$ and we can apply Proposition 3.5(i); see the eighth and ninth wire-frames. If $b_1 = a_4$, type (II).

So $b_2 \in \{a_3, a_4\}$, type (I) or (II).

Let Z be an arc set that contains no crossing arcs and has the combinatorial pattern of the edge set of a triangular prism. Label cone points so that there is a Z arc between c_n, c_{n+1} and between c_n, c_{n+2} and between c_{n+1}, c_{n+3} for $n = 0, 2, 4$. Subscript addition is modulo 6. Label the Z arc between c_n, c_{n+1} by $\zeta_{n,n+1}$, et cetera. Label by $\zeta_{n,n+3}$ (respectively $\zeta_{n+1,n+2}$) the arc between c_n, c_{n+3} (respectively c_{n+1}, c_{n+2}) that does not cross any arc in Z for $n = 0, 2, 4$. In Figure 18 we have pictured an orbifold as triangular prism wire-frame with some of these arcs drawn in thick black.

Proposition 3.5. *We have the following*:

(i) *If* $\max\{l(\zeta_{2,5}), l(\zeta_{3,4})\} \leq l(\zeta_{3,5})$ *and* $l(\zeta_{0,1}) \leq l(\zeta_{4,5})$ *then* $\min\{l(\zeta_{0,3}), l(\zeta_{1,3}), l(\zeta_{1,2})\} < l(\zeta_{0,2})$.

(ii) *If* $l(\zeta_{2,5}) \leq l(\zeta_{3,5}), l(\zeta_{3,4}) \leq l(\zeta_{4,5})$ *and* $l(\zeta_{0,1}) \leq \min\{l(\zeta_{2,3}), l(\zeta_{4,5})\}$ *then* $\min\{l(\zeta_{0,3}), l(\zeta_{1,2})\} < l(\zeta_{0,2})$.

(iii) *If* $\max\{l(\zeta_{2,5}), l(\zeta_{3,4})\} \leq l(\zeta_{3,5})$ *then either* $l(\zeta_{0,3}) < l(\zeta_{0,2})$ *or* $l(\zeta_{5,1}) < l(\zeta_{4,1})$.

(iv) *If* $l(\zeta_{2,5}) \leq l(\zeta_{3,5})$ *and* $l(\zeta_{3,4}) \leq l(\zeta_{4,5}) \leq l(\zeta_{2,3})$ *then either* $l(\zeta_{0,3}) < l(\zeta_{0,2})$ *or* $l(\zeta_{5,1}) < l(\zeta_{4,1})$.

(v) *If* $\max\{l(\zeta_{2,5}), l(\zeta_{3,4})\} \leq l(\zeta_{3,5})$ *then either* $l(\zeta_{0,2}) < l(\zeta_{1,2})$ *or* $l(\zeta_{4,0}) < l(\zeta_{4,1})$.

(vi) *If* $l(\zeta_{2,5}) \leq l(\zeta_{3,5})$ *and* $l(\zeta_{3,4}) \leq l(\zeta_{4,5})$ *then either* $l(\zeta_{0,2}) < l(\zeta_{1,2})$ *or* $l(\zeta_{4,0}) < l(\zeta_{4,1})$.

(vii) *If* $\max\{l(\zeta_{2,5}), l(\zeta_{3,4})\} \leq l(\zeta_{3,5})$, $l(\zeta_{2,3}) = l(\zeta_{4,5})$ *and* $\zeta_{0,2}, \zeta_{4,0}$ *are shortest arcs, then* $l(\zeta_{1,3}) < l(\zeta_{3,4})$ *or* $l(\zeta_{5,1}) < l(\zeta_{2,5})$.

Proof. Choose the distinguished disjoint triple of arcs $\Gamma = \zeta_{0,1} \cup \zeta_{2,3} \cup \zeta_{4,5}$. We label the common perpendiculars to the pair of pants $\mathscr{O} \setminus \Gamma$ by $p_{0,2}, p_{2,4}, p_{4,0}$.

Cutting $\mathscr{O}$ open along $\Gamma, \zeta_{1,3}, \zeta_{5,1}$ we obtain a simply connected domain Ω. Choose a lift of Ω in the universal cover of the pair of pants $\mathscr{O} \setminus \Gamma$. Without confusion, we shall use the same labels for geodesics having non-trivial intersection with Ω as on $\mathscr{O}$. We label the orbits of cone points on the boundary of Ω in cyclic order: c_0, c_1, c_3, c_2, c'_3, c'_1, c'_5, c_4, c''_5, c''_1, so that $*_i \in \tilde{c_i}$. The points c_0, c_2, c_4 lie at the midpoints of edges, the other points lie at vertices. Label the component of $\widetilde{\zeta_{0,1}}$ containing c'_1 by $\zeta'_{0,1}$; see Figure 18.

Label the common perpendiculars to $\zeta_{0,1}, \zeta_{2,3}, \zeta_{4,5}$ by $p_{0,2}, p_{2,4}, p_{4,0}$. Let $\mathscr{H}$ denote the right hexagon bounded by $p_{2,4} \cup \zeta_{2,3} \cup p_{0,2} \cup \zeta_{0,1} \cup p_{0,4} \cup \zeta_{4,5}$. We shall refer to the *inside* and *outside* of $p_{0,2}$ so that $\mathscr{H}$ lies inside $p_{0,2}$. Similarly for $p_{2,4}, p_{4,0}$.

(i) We suppose that $\max\{l(\zeta_{2,5}), l(\zeta_{3,4})\} \leq l(\zeta_{3,5}), l(\zeta_{0,1}), l(\zeta_{0,1}) \leq l(\zeta_{4,5})$ and $l(\zeta_{0,3}) \geq l(\zeta_{0,2})$ and show that $l(\zeta_{1,2}) \leq l(\zeta_{0,2})$. Unless all the inequalities are equalities this inequality will be strict. If all the inequalities are equalities then we show that $l(\zeta_{1,3}) < l(\zeta_{0,2})$.

Consider $\mathscr{Q}'_{2,3}$ bounded by $\zeta_{4,5} \cup p_{2,4} \cup \zeta_{2,3} \cup \perp c_2c'_3$ and $\mathscr{Q}'_{5,4}$ bounded by $\zeta_{2,3} \cup p_{2,4} \cup \zeta_{4,5} \cup \perp c'_5c_4$, a pair of right quadrilaterals. Recall that $\perp XY$ denotes the bisector of a disjoint pair of points $X, Y \in \mathbf{H}^2$. Any right quadrilateral $\mathscr{Q}$ bounded by $A \cup B \cup C \cup D$, will be such that the A edge and D edge meet in an acute angle ($\mathscr{Q}$ is 'finite' or 'a trirectangle') or the A edge and D edge do not meet ($\mathscr{Q}$ is 'infinite').

Claim 1. Both $\mathscr{Q}'_{2,3}$, $\mathscr{Q}'_{5,4}$ are strictly outside $p_{2,4}$, c'_5 is on the $\zeta_{4,5}$ edge of $\mathscr{Q}'_{2,3}$ and c'_3 is on the $\zeta_{2,3}$ edge of $\mathscr{Q}'_{5,4}$.

If either c'_3 or c'_5 is inside $p_{2,4}$ then either $\angle c_2c'_3c'_5 \geq \pi/2$ or $\angle c'_3c'_5c_4 \geq \pi/2$ and so either $l(\zeta_{3,5}) < l(\zeta_{2,5})$ or $l(\zeta_{3,5}) < l(\zeta_{3,4})$. So both c'_3, c'_5 are strictly outside $p_{2,4}$. As above, we use $\angle xyz$ to denote the angle at the y vertex of the triangle spanned by xyz.

Likewise if either $\perp c_2c'_3$ or $\perp c'_5c_4$ is inside $p_{2,4}$ then either $l(\zeta_{3,5}) < l(\zeta_{2,5})$ or $l(\zeta_{3,5}) < l(\zeta_{3,4})$, since c'_3, c'_5 are strictly outside $p_{2,4}$. So $\perp c_2c'_3$, $\perp c'_5c_4$ and hence $\mathscr{Q}'_{2,3}$, $\mathscr{Q}'_{5,4}$ are both strictly outside $p_{2,4}$.

If $\mathscr{Q}'_{2,3}$ is finite and c'_5 is strictly beyond its acute vertex then $l(\zeta_{3,5}) < l(\zeta_{2,5})$. Similarly if $\mathscr{Q}'_{5,4}$ is finite and c'_3 is strictly beyond its acute vertex then $l(\zeta_{3,5}) < l(\zeta_{3,4})$. This completes the claim.

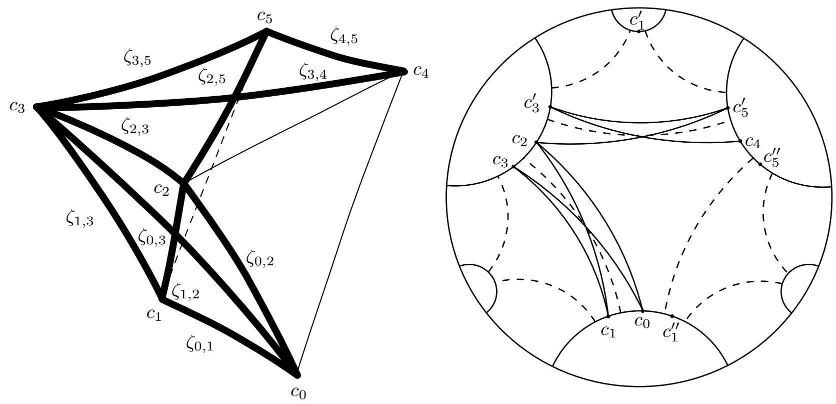

Figure 18. Arcs and lift for (i) and (ii)

Consider a third quadrilateral $\mathscr{Q}_{3,2}$ bounded by $\zeta_{0,1} \cup p_{0,2} \cup \zeta_{2,3} \cup \perp c_3 c_2$. We compare edge-lengths of $\mathscr{Q}_{3,2}$, $\mathscr{Q}'_{2,3}$. Since $d(c_3, c_2) = d(c_2, c'_3) = l(\zeta_{2,3}) = d(p_{0,2}, p_{2,4})$, the $\zeta_{2,3}$ edges of $\mathscr{Q}_{3,2}\mathscr{Q}'_{2,3}$ are the same length. Also $\mathscr{Q}_{3,2}$ is strictly inside $p_{0,2}$ since $\mathscr{Q}'_{2,3}$ is strictly outside $p_{2,4}$. Now $l(\zeta_{0,1}) \leq l(\zeta_{4,5})$ and so, by the geometry of right hexagons $l(p_{0,2}) \geq l(p_{2,4})$. That is, the $p_{0,2}$ edge of $\mathscr{Q}_{3,2}$ is longer than the $p_{2,4}$ edge of $\mathscr{Q}'_{2,3}$ and so, by the geometry of right quadrilaterals, the $\zeta_{0,1}$ edge of $\mathscr{Q}_{3,2}$ is longer than the $\zeta_{4,5}$ edge of $\mathscr{Q}'_{2,3}$.

If c_0 is outside $p_{0,2}$ or on the interior of the $\zeta_{0,1}$ edge of $\mathscr{Q}_{3,2}$, $l(\zeta_{0,3}) < l(\zeta_{0,2})$, contradicting our supposition. So $\mathscr{Q}_{3,2}$ is finite and c_0 beyond its $\zeta_{0,1}$ edge which is longer than the $\zeta_{4,5}$ edge of $\mathscr{Q}'_{2,3}$ which contains c'_5. So $d(c_0, p_{0,2}) \geq d(c'_5, p_{2,4})$. Since $\perp c'_5 c_4$ is strictly outside $p_{2,4}$ and $d(c_0, c_1) = l(\zeta_{0,1}) \leq l(\zeta_{4,5}) = d(c'_5 c_4)$, $\perp c_0 c_1$ is strictly inside $p_{0,2}$ and $d(\perp c_0 c_1, p_{0,2}) \geq d(\perp c'_5 c_4, p_{2,4})$.

Consider a fourth quadrilateral $\mathscr{Q}_{0,1}$ bounded by $\zeta_{2,3} \cup p_{0,2} \cup \zeta_{0,1} \cup \perp c_0 c_1$. We compare edge lengths of $\mathscr{Q}_{0,1}$, $\mathscr{Q}'_{5,4}$. We have shown that $\mathscr{Q}_{0,1}$ is strictly inside $p_{0,2}$ and that the $\zeta_{0,1}$ edge of $\mathscr{Q}_{0,1}$ is longer than the $\zeta_{4,5}$ edge of $\mathscr{Q}'_{5,4}$. Since $l(p_{0,2}) \geq l(p_{2,4})$, the $\zeta_{2,3}$ edge of $\mathscr{Q}_{0,1}$ is longer than the $\zeta_{2,3}$ edge of $\mathscr{Q}'_{5,4}$. As c'_3 is on the $\zeta_{2,3}$ edge of $\mathscr{Q}'_{5,4}$ and $d(c'_3, p_{2,4}) = d(c_2, p_{0,2})$, c_2 is on the $\zeta_{2,3}$ edge of $\mathscr{Q}_{0,1}$. So $l(\zeta_{1,2}) \leq l(\zeta_{0,2})$ as required.

Finally, $l(\zeta_{3,4}) = l(\zeta_{2,5}) = l(\zeta_{3,5})$, $l(\zeta_{0,1}) = l(\zeta_{4,5})$, $l(\zeta_{0,3}) = l(\zeta_{0,2})$ implies $l(\zeta_{1,3}) < l(\zeta_{0,2})$. So $\mathscr{Q}_{0,1}$ is finite and c_2 is at its acute vertex. Now c_3 is either outside $p_{0,2}$ or on the interior of the $\zeta_{2,3}$ edge of $\mathscr{Q}_{0,1}$: $l(\zeta_{1,3}) < l(\zeta_{0,3}) = l(\zeta_{0,2})$.

(ii) We suppose $l(\zeta_{2,5}) \leq l(\zeta_{3,5})$, $l(\zeta_{3,4}) \leq l(\zeta_{4,5})$, $l(\zeta_{0,1}) \leq \min\{l(\zeta_{2,3}), l(\zeta_{4,5})\}$ and $l(\zeta_{0,3}) \geq l(\zeta_{0,2})$ and show that $l(\zeta_{1,2}) < l(\zeta_{0,2})$.

Claim 2. $\mathscr{Q}'_{2,3}$, c'_5 are both strictly outside $p_{2,4}$, c'_5 is on the $\zeta_{4,5}$ edge of $\mathscr{Q}'_{2,3}$ and $d(c'_5, p_{2,4}) > d(c'_3, p_{2,4})$.

Suppose that $\mathscr{Q}'_{2,3}$ is strictly inside $p_{2,4}$.

If $\mathscr{Q}'_{2,3}$ is finite and c'_5 is beyond the acute vertex of $\mathscr{Q}'_{2,3}$ then $\angle c'_3c'_5c_4 > \pi/2$, $l(\zeta_{4,5}) < l(\zeta_{3,4})$. If c'_5 is on the interior of the $\zeta_{4,5}$ edge of $\mathscr{Q}'_{2,3}$ or outside $p_{2,4}$, $l(\zeta_{3,5}) < l(\zeta_{2,5})$. Similar arguments hold for $\mathscr{Q}'_{2,3}$ trivial or infinite.

Suppose $\mathscr{Q}'_{2,3}$ (and hence c'_3) is strictly outside $p_{2,4}$.

If c'_5 is inside $p_{2,4}$ we again have $\angle c'_3c'_5c_4 > \pi/2$, $l(\zeta_{4,5}) < l(\zeta_{3,4})$. If c'_5 is outside $p_{2,4}$ such that $d(c'_5, p_{2,4}) \le d(c'_3, p_{2,4})$ then $\angle c'_3c'_5c_4 \ge \angle c_2c'_3c'_5$. By inspection $\angle c_2c'_3c'_5 > \angle c_4c'_3c'_5$ so $\angle c'_3c'_5c_4 > \angle c_4c'_3c'_5$, $l(\zeta_{4,5}) < l(\zeta_{3,4})$. If $\mathscr{Q}'_{2,3}$ is finite and c'_5 is strictly beyond its acute vertex, $l(\zeta_{3,5}) < l(\zeta_{2,5})$. The claim follows.

Recall that $\mathscr{Q}_{3,2}$ is bounded by $\zeta_{0,1} \cup p_{0,2} \cup \zeta_{2,3} \cup \perp c_3c_2$. As $\mathscr{Q}'_{2,3}$ is strictly outside $p_{2,4}$ and the $\zeta_{2,3}$ edges of $\mathscr{Q}_{3,2}$, $\mathscr{Q}'_{2,3}$ are the same length: $\mathscr{Q}_{3,2}$ is strictly inside $p_{0,2}$. Again $l(\zeta_{0,1}) \le l(\zeta_{4,5})$ and so $l(p_{0,2}) \ge l(p_{2,4})$. So the $\zeta_{0,1}$ edge of $\mathscr{Q}_{3,2}$ is longer than the $\zeta_{4,5}$ edge of $\mathscr{Q}'_{2,3}$.

Recall that $\mathscr{Q}_{0,1}$ is bounded by $\zeta_{2,3} \cup p_{0,2} \cup \zeta_{0,1} \cup \perp c_0c_1$. We show that $\mathscr{Q}_{0,1}$ is strictly inside $p_{0,2}$. As with part (i), so that $l(\zeta_{0,3}) \ge l(\zeta_{0,2})$: $\mathscr{Q}_{3,2}$ is finite and c_0 is beyond its acute vertex. So $d(c_0, p_{0,2})$ is longer than the $\zeta_{0,1}$ edge of $\mathscr{Q}_{3,2}$ which is longer than the $\zeta_{4,5}$ edge of $\mathscr{Q}'_{2,3}$ which is longer than $d(c'_5, p_{2,4}) > d(c'_3, p_{2,4}) = d(c_2, p_{0,2})$. That is $d(c_0, p_{0,2}) > d(c_2, p_{0,2})$. Since $\perp c_3c_2$ is strictly inside $p_{0,2}$ and $d(c_0, c_1) = l(\zeta_{0,1}) \le l(\zeta_{2,3}) = d(c_3, c_2)$, $\perp c_0c_1$ is strictly inside $p_{0,2}$ and $d(\perp c_0c_1, p_{0,2}) > d(\perp c_3c_2, p_{0,2})$.

So $\mathscr{Q}_{3,2}$, $\mathscr{Q}_{0,1}$ are both inside $p_{0,2}$, the $\zeta_{0,1}$ edge of $\mathscr{Q}_{0,1}$ is strictly longer than the $\zeta_{2,3}$ edge of $\mathscr{Q}_{3,2}$. As $\mathscr{Q}_{3,2}$, $\mathscr{Q}_{0,1}$ have $p_{0,2}$ in common, the $\zeta_{2,3}$ edge of $\mathscr{Q}_{0,1}$ is strictly longer than the $\zeta_{0,1}$ edge of $\mathscr{Q}_{3,2}$ which is longer than the $\zeta_{4,5}$ edge of $\mathscr{Q}'_{2,3}$ which is strictly longer than $d(c'_5, p_{2,4}) > d(c'_3, p_{2,4}) = d(c_2, p_{0,2})$. So c_2 is on the interior of the $\zeta_{2,3}$ edge of $\mathscr{Q}_{0,1}$, $l(\zeta_{1,2}) < l(\zeta_{0,2})$.

(iii) We suppose that $\max\{l(\zeta_{2,5}), l(\zeta_{3,4})\} \le l(\zeta_{3,5})$ and show that either $l(\zeta_{0,3}) < l(\zeta_{0,2})$ or $l(\zeta_{5,1}) < l(\zeta_{4,1})$. From Claim 1, $\mathscr{Q}'_{2,3}$, $\mathscr{Q}'_{5,4}$ are both strictly outside $p_{2,4}$ and so $\mathscr{Q}_{3,2}$, $\mathscr{Q}''_{4,5}$ are strictly inside $p_{0,2}$, $p_{4,0}$ respectively. Here $\mathscr{Q}''_{4,5}$ denotes the right quadrilateral bounded by $\zeta_{0,1} \cup p_{4,0} \cup \zeta_{4,5} \perp c_4c''_5$.

Consider c_0, c''_1 on $\zeta_{0,1}$. Either c_0 is outside $p_{0,2}$ or c''_1 is outside $p_{4,0}$. So c_0 is strictly closer to c_3 than to c_2 or c''_1 is strictly closer to c''_5 than to c_4. That is either $l(\zeta_{0,3}) < l(\zeta_{0,2})$ *or* $l(\zeta_{5,1}) < l(\zeta_{4,1})$.

(iv) We suppose that $l(\zeta_{2,5}) \le l(\zeta_{3,5})$ and $l(\zeta_{3,4}) \le l(\zeta_{4,5}) \le l(\zeta_{2,3})$ and show that either $l(\zeta_{0,3}) < l(\zeta_{0,2})$ or $l(\zeta_{5,1}) < l(\zeta_{4,1})$.

From Claim 2, $\mathscr{Q}'_{2,3}$, c'_5 are both strictly outside $p_{2,4}$, $d(c'_5, p_{2,4}) > d(c'_3, p_{2,4})$. Since $d(c'_5, c_4) = l(\zeta_{4,5}) \le l(\zeta_{2,3}) = d(c_2, c'_3)$, $\mathscr{Q}'_{5,4}$ is strictly outside $p_{2,4}$. As with part (iii), we can conclude that either $l(\zeta_{0,3}) < l(\zeta_{0,2})$ or $l(\zeta_{5,1}) < l(\zeta_{4,1})$.

(v) We suppose that $\max\{l(\zeta_{2,5}), l(\zeta_{3,4})\} \le l(\zeta_{3,5})$ and show that either $l(\zeta_{0,2}) < l(\zeta_{1,2})$ or $l(\zeta_{4,0}) < l(\zeta_{4,1})$. From Claim l, c'_3, c'_5 are both strictly outside $p_{2,4}$ and so c_2 and c_4 are strictly inside $p_{0,2}$ and $p_{4,0}$ respectively.

Consider $\perp\ c_0c_1$, $\perp\ c_1''c_0$. Either $\perp\ c_0c_1$ is outside $p_{0,2}$ or $\perp\ c_1''c_0$ is outside $p_{4,0}$. So either c_2 is strictly closer to c_0 than to c_1 or c_4 is strictly closer to c_0 than to c_1''. That is either $l(\zeta_{0,2}) < l(\zeta_{1,2})$ *or* $l(\zeta_{4,0}) < l(\zeta_{4,1})$.

(vi) We suppose that $l(\zeta_{2,5}) \leq l(\zeta_{3,5})$, $l(\zeta_{3,4}) \leq l(\zeta_{4,5})$ and show that $l(\zeta_{0,2}) < l(\zeta_{1,2})$ or $l(\zeta_{4,0}) < l(\zeta_{4,1})$. From Claim 2, c_3' and c_5' are both strictly outside $p_{2,4}$. As with part (v) we can conclude that either $l(\zeta_{0,2}) < l(\zeta_{1,2})$ *or* $l(\zeta_{4,0}) < l(\zeta_{4,1})$.

(vii) We suppose that $\max\{l(\zeta_{2,5}), l(\zeta_{3,4})\} \leq l(\zeta_{3,5}), l(\zeta_{2,3}) = l(\zeta_{4,5})$ and that $\zeta_{0,2}$, $\zeta_{4,0}$ are both shortest arcs and show that $l(\zeta_{1,3}) < l(\zeta_{3,4})$ or $l(\zeta_{5,1}) < l(\zeta_{2,5})$.

By hypothesis $l(\zeta_{2,3}) = l(\zeta_{4,5})$ and so $\mathscr{O} \setminus \Gamma$ has rotational symmetry R exchanging boundary components $\zeta_{2,3}$, $\zeta_{4,5}$ and fixing $\zeta_{0,1}$, $p_{2,4}$ setwise. Gluing along $\zeta_{0,1}$ to recover $\mathscr{O} \setminus \zeta_{2,3} \cup \zeta_{4,5}$, this symmetry is respected, exchanging cone points c_0, c_1. In the universal cover, the rotational symmetry of $\mathscr{O} \setminus \zeta_{2,3} \cup \zeta_{4,5}$ lifts to a rotational symmetry R such that $R(\zeta_{2,3}) = \zeta_{4,5}$, $R(\zeta_{0,1}) = \zeta_{0,1}'$, $R(c_0) = c_1'$ and $R(p_{2,4}) = p_{2,4}$.

By Claim 1, $\perp\ c_2c_3'$ and $\perp\ c_5'c_4$ are both strictly outside $p_{2,4}$.

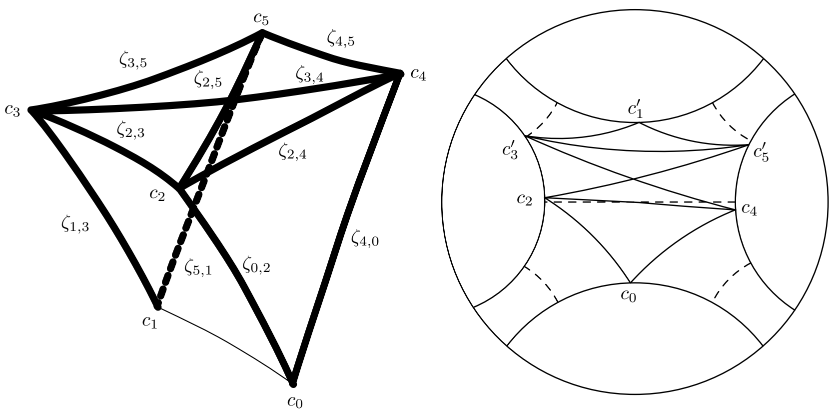

Figure 19. Arcs and lift for (vii)

Either (a) c_2, c_4 are both strictly inside $p_{2,4}$ *or* (b) one of c_2, c_4 is outside $p_{2,4}$. We shall give the argument for (b); a similar argument holds for (a). We suppose c_2 is outside $p_{2,4}$ and show that $l(\zeta_{1,3}) < l(\zeta_{3,4})$. (If c_4 is outside $p_{2,4}$, the same argument shows that $l(\zeta_{5,1}) < l(\zeta_{2,5})$.)

Claim 3. c_4 is inside $p_{2,4}$ and $d(c_2, p_{2,4}) \leq d(c_4, p_{2,4}) < l(\zeta_{4,5})/2$.

Consider the bisector $\perp\ c_0c_1'$. By rotational symmetry $\perp\ c_0c_1'$ passes through the midpoint of $p_{2,4}$. As $\zeta_{0,2}$ is a shortest arc, $c_2 \in \zeta_{2,3}$ is to the c_0 side of $\perp\ c_0c_1'$. So either $\zeta_{2,3}$ is strictly to the c_0 side of $\perp\ c_0c_1'$ or $\perp\ c_0c_1'$ intersects $\zeta_{2,3}$ outside $p_{2,4}$, beyond c_2.

By rotational symmetry, the former corresponds to $\zeta_{4,5} \ni c_4$ lying strictly to the c_1' side of $\perp\ c_0c_1'$. This contradicts $\zeta_{4,0}$ being a shortest arc. So $\perp\ c_0c_1'$ intersects $\zeta_{2,3}$ (respectively $\zeta_{4,5}$) outside (inside) $p_{2,4}$, beyond c_2 (between $p_{2,4}$,

c_4). As $\perp c_5'c_4$ is strictly outside $p_{2,4}$, $d(c_4,p_{2,4}) < d(c_5',p_{2,4})$ and so $d(c_4,p_{2,4}) < l(\zeta_{4,5})/2$. This completes the claim.

By rotational symmetry $R(c_4)$ is outside $p_{2,4}$ and $d(c_2,p_{2,4}) \le d\big(R(c_4),p_{2,4}\big) < l(\zeta_{2,3})/2$.

Claim 4. $d\big(R(c_4),c_1'\big) \le d\big(R(c_4),c_4\big)$.

We show that $d(c_4,c_0) \le d\big(c_4,R(c_4)\big)$, which by rotational symmetry is equivalent. By hypothesis $\zeta_{4,0}$ is a shortest arc and so $d(c_4,c_0) = l(\zeta_{4,0}) \le l(\zeta_{2,4}) = d(c_4,c_2)$. By Claim 3 and by inspection $d(c_4,c_2) \le d\big(c_4,R(c_4)\big)$.

Now c_3' is outside $p_{2,4}$ such that $d(c_3',p_{2,4}) = l(\zeta_{2,3}) + d(c_2,p_{2,4}) > l(\zeta_{2,3})/2 > d\big(R(c_4),p_{2,4}\big)$. So by Claim 4 and by inspection, $d(c_3',c_1') = l(\zeta_{1,3}) < l(\zeta_{3,4}) = d(c_3',c_4)$. □

We extend the arc set $K \cup \Lambda$ by $H = \cup \eta_l$ where η_l is between c_1, c_3 crossing only $\kappa_{0,l} \subset K$.

Theorem 3.6. *Either* $l(\kappa_{0,1}) < l(\eta_4)$ *or* $l(\kappa_{0,5}) < l(\kappa_{3,5})$ *or* $l(\lambda_2) < l(\lambda_0)$.

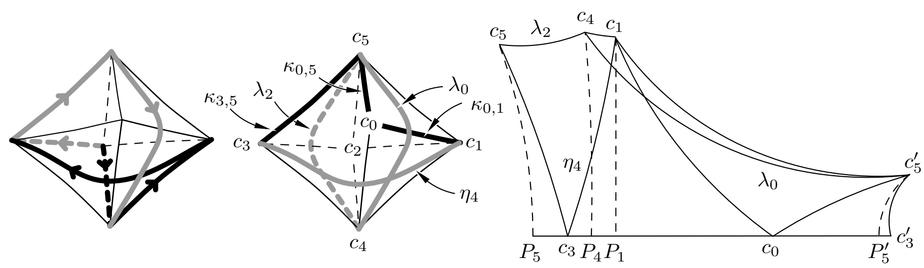

Figure 20. Application, arcs and lift of Ω in Theorem 3.6

Consider $\mathscr{A}_{i_1}, \mathscr{B}_{j_1}$ with $i_1 = 3$, $j_1 = 3$ and $b_4 = a_2$, $b_3 \in \{a_5, a_6\}$, $b_2 = a_4$, $b_1 = a_1$ such that $c \in \mathscr{O}_3$; see Figure 20. Since $\mathscr{A}_2' = \alpha_1, \beta_1$ is a chain, α_2 is a shortest arc. We set $\alpha_1 = \kappa_{2,3}$, $\alpha_2 = \kappa_{3,5}$, $\alpha_3 = \lambda_0$, $\beta_1 = \kappa_{2,4}$, $\beta_2 = \kappa_{1,4}$, $\beta_3 = \eta_4$. This arc set extends uniquely to the arc set $K \cup \Lambda \cup H$. Both $\mathscr{A}_3' = \alpha_1, \alpha_2, \lambda_1$, $\mathscr{A}_3'' = \alpha_1, \alpha_2, \lambda_3$ are chains, so $l(\lambda_0) \le \min\{l(\lambda_1), l(\lambda_3)\}$. Since α_1 is a shortest arc, Theorem 2.12 implies $\min\{l(\lambda_1), l(\lambda_3)\} \le l(\lambda_2)$. So $l(\lambda_0) \le l(\lambda_2)$. By Theorem 3.6 either $l(\kappa_{0,1}) < l(\eta_4)$ or $l(\kappa_{0,5}) < l(\kappa_{3,5})$ or $l(\lambda_2) < l(\lambda_0)$. As $\mathscr{B}_3' = \beta_1, \beta_2, \kappa_{0,1}$ is a chain and $\alpha_2 = \kappa_{3,5}$ is a shortest arc, we have a contradiction.

Proof. We suppose $l(\eta_4) \le l(\kappa_{0,1})$, $l(\kappa_{3,5}) \le l(\kappa_{0,5})$ and show that $l(\lambda_2) < l(\lambda_0)$. Cut $\mathscr{O}$ open along $\kappa_{1,5} \cup \kappa_{1,4} \cup \lambda_2$ and consider $\mathscr{O}_2'$, the component *not* containing c_2. Cut $\mathscr{O}_2'$ open along $\kappa_{3,0}$ so as to obtain an annulus. Making a further cut along $\kappa_{3,5}$ we obtain a simply connected domain Ω. Choose a lift of Ω in the universal cover of the annulus $\mathscr{O}_2' \setminus \kappa_{3,0}$.

Label the geodesics around the boundary of Ω by $\kappa_{3,0}$, $\kappa_{3,5}$, λ_2, $\kappa_{1,4}$, $\kappa_{1,5}$, $\kappa_{3,5}'$, in cyclic order. Give the other geodesics having non-trivial intersection with Ω the same labels as on $\mathscr{O}$. In the same cyclic order, label orbits of cone points around the boundary of Ω by c_0, c_3, c_5, c_4, c_1, c_5', c_3'. Let P_5, P_4, P_1, P_5' denote the perpendiculars to $\kappa_{3,0}$ from c_5, c_4, c_1, c_5' respectively.

Now: $l(\eta_4) \leq l(\kappa_{0,1})$ implies that P_1 is closer to c_3 than to c_0 and $l(\kappa_{3,5}) \leq l(\kappa_{0,5})$ implies that P_5' is closer to c_3' than to c_0. It follows that $d(P_1, P_5') \geq l(\kappa_{3,0})$.

The perpendiculars lie in the order P_5, P_4, P_1, P_5' and so $d(P_4, P_5') > d(P_1, P_5')$ and hence $d(P_5, P_4) < l(\kappa_{3,0})$. So the $\kappa_{3,0}$ edge of the birectangle $\mathscr{Q}_{2,3}$ is strictly shorter than the $\kappa_{3,0}$ edge of the birectangle $\mathscr{Q}_{0,3}$. Therefore $l(\lambda_2) < l(\lambda_0)$. ◻

Theorem 3.7. *If $l(\lambda_0) \leq 1(\lambda_2)$, $l(\kappa_{0,1}) \leq \min\{l(\kappa_{0,l}), l(\kappa_{3,l}), l(\eta_l)\}$ and $l(\kappa_{1,2}) = l(\kappa_{3,0}) \leq \min\{l(\kappa_{1,l}), l(\kappa_{2,l}), l(\kappa_{3,l})\}$, for $l = 4, 5$, then $\mathscr{O}$ is the octahedral orbifold.*

Consider $\mathscr{A}_{i_1}, \mathscr{B}_{j_1}$ with $i_1 = 2, j_1 = 3$ and $a_3 \in \{b_1, b_2\}$, $\{a_2, a_1\} = \{b_5, b_6\}$; see Figure 21. Suppose $a_3 = b_1$. Set $\alpha_1 = \kappa_{3,0}$, $\alpha_2 = \kappa_{0,1}$, $\beta_1 = \kappa_{1,2}$, $\beta_2 = \kappa_{2,5}$, $\beta_3 = \lambda_0$. This arc set extends uniquely to the arc set $K \cup \Lambda \cup H$. Each one of $\mathscr{B}_3' = \beta_1, \beta_2, \lambda_2$, $\mathscr{A}_2' = \alpha_1, \kappa_{0,l}$, $\mathscr{A}_2'' = \alpha_1, \kappa_{3,l}$, $\mathscr{A}_2''' = \alpha_1, \eta_l$ is a chain so $l(\lambda_0) \leq l(\lambda_2)$, $l(\kappa_{0,1}) \leq \{l(\kappa_{0,l}), l(\kappa_{3,l}), l(\eta_l)\}$ for $l = 4, 5$. Also $\kappa_{3,0} = \alpha_1$, $\kappa_{1,2} = \beta_1$ are both shortest arcs, so we can apply Theorem 3.7: $\mathscr{O}$ is the octahedral orbifold. This gives a contradiction since minimal chains on $\mathscr{O}ct$ lie in its set of shortest arcs, which contains no crossing arcs. Likewise for $a_3 = b_2$.

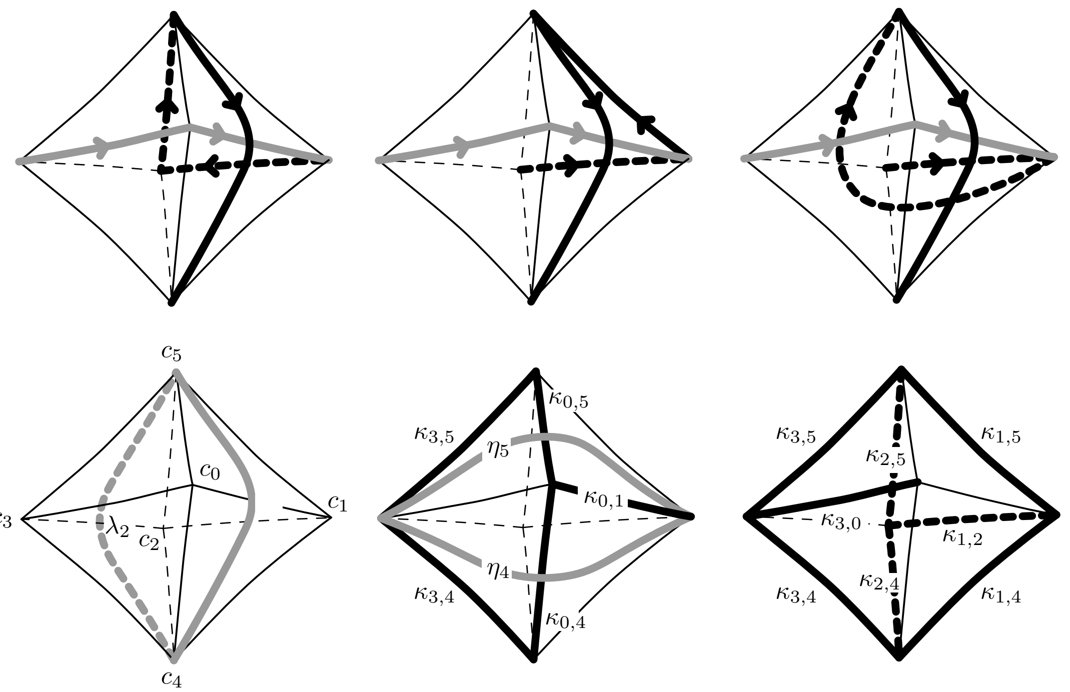

Figure 21. Applications and arc set for Theorem 3.7

Proof. Let $\Gamma = \kappa_{1,2} \cup \lambda_2 \cup \kappa_{3,0}$ a distinguished disjoint triple of arcs. Let $p_{3,1}$ denote the common perpendicular between $\kappa_{3,0}$, $\kappa_{1,2}$. The pair of pants $\mathscr{O} \setminus \Gamma$ has a rotational symmetry R exchanging $\kappa_{1,2}$, $\kappa_{3,0}$ and fixing λ_0, λ_2, $p_{3,1}$ setwise. Gluing along λ_2 so as to recover $\mathscr{O} \setminus \kappa_{1,2} \cup \kappa_{3,0}$ this symmetry is respected. So R exchanges the birectangles $\mathscr{Q}_{0,1} \leftrightarrow \mathscr{Q}_{0,3}$, $\mathscr{Q}_{2,1} \leftrightarrow \mathscr{Q}_{2,3}$. That is, $\mathscr{Q}_{0,1}$, $\mathscr{Q}_{0,3}$ (respectively $\mathscr{Q}_{2,1}$, $\mathscr{Q}_{2,3}$) are isometric.

As we have observed many times above, since $l(\lambda_0) \leq l(\lambda_2)$ the $\kappa_{3,0}$ edge of $\mathscr{Q}_{0,3}$ is shorter than the $\kappa_{3,0}$ edge of $\mathscr{Q}_{2,3}$. Here we observe that $l(\lambda_0) \leq l(\lambda_2)$ implies that $\angle c_l \mathscr{Q}_{0,1} \geq \angle c_l \mathscr{Q}_{2,1}$. Summing angles we have that $\angle c_4 \mathscr{Q}_{0,3} + \angle c_5 \mathscr{Q}_{0,3} \geq \angle c_4 \mathscr{Q}_{2,3} + \angle c_5 \mathscr{Q}_{2,3}$.

The set $\kappa_{1,2} \cup P_{4,1} \cup P_{4,3} \cup \kappa_{3,0} \cup P_{5,3} \cup P_{5,1}$ divides $\mathscr{O}$ into a pair of hexagons $\mathscr{H}_k \supset \lambda_k$ for $k = 0, 2$. We have shown that both hexagons have rotational symmetry and that the $\kappa_{1,2}$, $\kappa_{3,0}$ edges of $\mathscr{H}_0$ are shorter than the $\kappa_{1,2}$, $\kappa_{3,0}$ edges of $\mathscr{H}_2$. It follows that the $\kappa_{1,2}$, $\kappa_{3,0}$ edges of $\mathscr{H}_0$ are shorter than $l(\kappa_{1,2}) = l(\kappa_{3,0})$. Also $\angle c_4 \mathscr{H}_0 = \angle c_4 \mathscr{Q}_{0,1} + \angle c_4 \mathscr{Q}_{0,3} \geq \angle c_4 \mathscr{Q}_{2,1} + \angle c_4 \mathscr{Q}_{2,3} = \angle c_4 \mathscr{H}_2$. Since $\angle c_4 \mathscr{H}_0 + \angle c_4 \mathscr{H}_2 = \pi$ it follows that $\angle c_4 \mathscr{H}_0 = \angle c_5 \mathscr{H}_0 \geq \pi/2$.

Note. The hypotheses are symmetric up to an exchange of labels: $\kappa_{k,4} \leftrightarrow \kappa_{k,5}$ for $k = 0, 1, 2, 3$; $c_4 \leftrightarrow c_5$; and $\eta_4 \leftrightarrow \eta_5$. So we may suppose that $l(P_{4,1}) = l(P_{5,3}) \geq l(P_{4,3}) = l(P_{5,1})$.

Cut $\mathscr{O}$ open along $\kappa_{1,2} \cup \kappa_{1,4} \cup \kappa_{3,4} \cup \kappa_{3,0} \cup \kappa_{0,5} \cup \kappa_{2,5}$ to obtain a pair of simply connected domains Ω_0, Ω_2 containing λ_0, λ_2 respectively. Take a lift of Ω_0 to the universal cover of the pair of pants $\mathscr{O} \setminus \Gamma$. Without confusion use the same labels for geodesics having non-trivial intersection with Ω_0, for orbits of cone points lying on the boundary of Ω_0, for the common perpendicular between $\kappa_{1,2}, \kappa_{3,0}$, for the perpendiculars from c_4, c_5 to $\kappa_{1,2}, \kappa_{3,0}$ and for the rotational symmetric of hexagon $\mathscr{H}_0$ bounded by $\kappa_{1,2} \cup P_{4,1} \cup P_{4,3} \cup \kappa_{3,0} \cup P_{5,3} \cup P_{5,1}$. We may suppose that $\mathscr{H}_0$ has a bottom, a top, a left and a right, so that $\kappa_{1,2}$ is the *bottom*, $\kappa_{3,0}$ is the *top*, c_4 to the *left* and c_5 is to the *right*. Label the unlabelled vertices of $\mathscr{H}_0$: h_1, h_2, h_0, h_3 in anticlockwise order, beginning from bottom left.

The main part of the proof has two parts. The first part is to establish

Claim 1. c_1 is strictly between $P_{4,1}$, $P_{5,1}$.

For c_1 to the left of $P_{4,1}$ we show that $l(\kappa_{0,4}) < l(\kappa_{0,1})$. For c_1 right of $P_{5,1}$ a similar argument shows that $l(\kappa_{0,5}) < l(\kappa_{0,1})$.

We use a different pants decomposition of $\mathscr{O}$. Let $\Gamma' = \kappa_{3,0} \cup \kappa_{1,4} \cup \kappa_{2,5}$. Label the common perpendiculars associated to this pants decomposition by $p_{0,4}$, $p_{4,5}$ and $p_{5,0}$. Cut open along Γ', $\kappa_{1,2}$, $\kappa_{3,4}$ and $\kappa_{0,5}$ so as to obtain two simply connected domains: Ω_0, Ω_2, as above. Again take a lift of Ω_0 and use the same labels as before, with the additional labelling for the common perpendiculars to Γ'. So $\kappa_{1,4} \cup p_{4,5} \cup \kappa_{2,5} \cup p_{5,0} \cup \kappa_{3,0} \cup p_{0,4}$ bounds a right hexagon. Let p denote the common perpendicular to $\kappa_{1,4}, p_{5,0}$.

As c_1 is to the left of $P_{4,1}$, $\angle c_1 c_4 h_3 \geq \angle h_1 c_4 h_3 = \angle c_4 \mathscr{H}_0 \geq \pi/2$. So c_4 is to the $p_{4,5}$ side of $p_{0,4}$ or equivalently, $\perp c_1 c_4$ is closer to $p_{4,5}$ than to $p_{0,4}$. Now p is closer to $p_{0,4}$ than to $p_{4,5}$, as $l(\kappa_{3,0}) \leq l(\kappa_{2,5})$ and using the geometry of right hexagons. It follows that $\perp c_1c_4$ lies to the $p_{4,5}$ side of p and so $\perp c_1c_4$ does not intersect $\kappa_{3,0}$ and hence $l(\kappa_{0,4}) < l(\kappa_{0,1})$. This completes the claim.

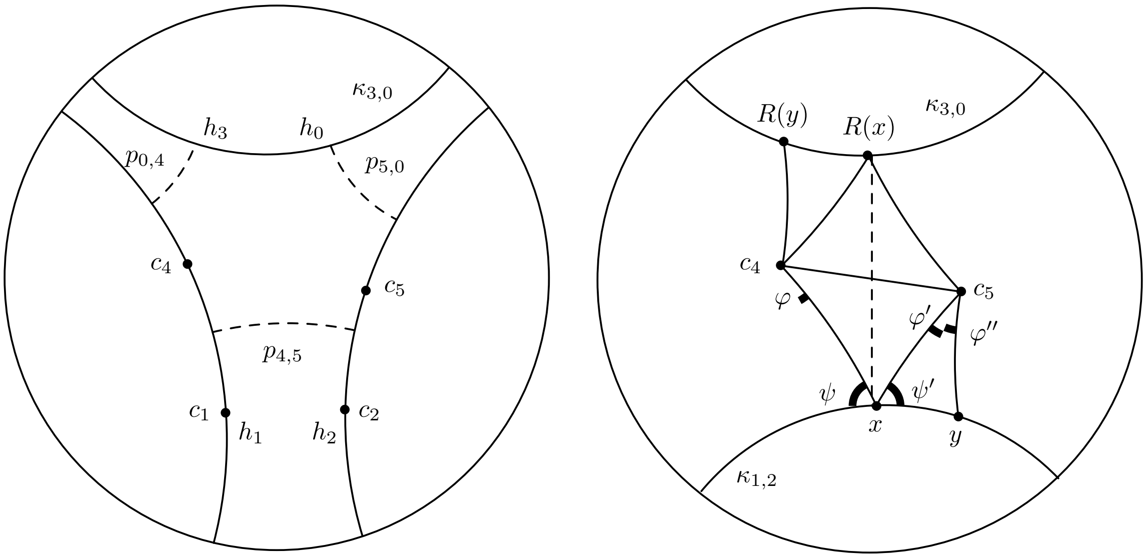

Figure 22. Claims 1 and 2.

In the second part, we show that $l(\kappa_{0,4}) \leq l(\kappa_{0,1})$ or $l(\kappa_{0,5}) \leq l(\kappa_{0,1})$ with equality *if and only if* a certain set of conditions are satisfied. At the end we show that this set of conditions implies that $\mathscr{O}$ is the octahedral orbifold.

The argument we use depends upon the positions of c_0, c_1: each part (i)–(viii) corresponds to a different configuration. However each part uses the following construction together with the angle comparison we establish in Claim 2.

For each part we define x to be a point on $\kappa_{1,2}$ strictly between $P_{4,1}, P_{5,1}$. We define y to be the point on $\kappa_{1,2}$ to the right of x such that $d(x,y) = l(\kappa_{1,2})$. Label angles so that $\varphi = \angle h_1c_4x$, $\psi = \angle c_4xh_1$, $\psi' = \angle h_2xc_5$, $\varphi' = \angle xc_5h_2$, $\varphi'' = \angle h_2c_5y$. Let $m_{1,2}$ (respectively $m_{3,0}$) denote the midpoint of h_1, h_2 (respectively h_3, h_0).

Claim 2. For (i), (ii), (iv), (v): $\angle R(x)c_4x \geq \pi/2 - \psi$ and for (iii), (vi), (vii), (viii): $\angle R(x)c_5x \geq \pi/2 - \psi'$.

By rotational symmetry we have $\angle R(x)c_4x = \angle R(x)c_5x$. Below we show that $\angle R(x)c_5x \geq \pi/2 - \psi'$ for (i)–(viii). For (i), (ii), (iv), (v) the point x is to the left of $m_{1,2}$. So $d(x, P_{4,1}) \leq d(x, P_{5,1})$. Since $l(P_{4,1}) \geq l(P_{5,1})$, it follows that $\psi \geq \psi'$. So the claim follows from the argument below.

Firstly: $\varphi \leq \varphi''$. Since $d(x,y) = l(\kappa_{1,2}) \geq d(P_{4,1}P_{5,1})$ it follows that $d(P_{4,1}, x) \leq d(P_{5,1}, y)$. Also $l(P_{4,1}) \geq l(P_{5,1})$.

Next: $\varphi' + \varphi'' \leq \psi'$. For (i), (ii), (iii), (v)(a), (viii) (respectively (v)(b), (vi)) we have that y is to the right of c_2 which is to the right of $P_{5,1}$ ($R(y)$ is to the left of c_3 which is to the left of $P_{4,3}$). It follows that $d(y, c_5) \geq d(c_2, c_5) = l(\kappa_{2,5}) \geq$

$l(\kappa_{1,2}) = d(y,x)$ $\big(d\big(R(y),c_4\big) \geq d(c_3,c_4) = l(\kappa_{3,4}) \geq l(\kappa_{3,0}) = d\big(R(y),R(x)\big)\big)$. This is equivalent to $\varphi' + \varphi'' \leq \psi'$.

For (iv), (vii) we have that x is to the left of c_1 which is to the left of $P_{5,1}$. It follows that $d(x,c_5) \geq d(c_1,c_5) = l(\kappa_{1,5}) \geq l(\kappa_{1,2}) = d(x,y)$. Also x is to the right of $m_{1,2}$ and so $d(x,P_{5,1}) \leq d(m_{1,2},P_{5,1}) = d(P_{4,1},P_{5,1})/2 \leq l(\kappa_{1,2})/2$. So $d(y,P_{5,1}) \geq d(x,P_{5,1})$, $d(y,c_5) \geq d(x,c_5) \geq d(x,y)$. Again, this is equivalent to $\varphi' + \varphi'' \leq \psi'$.

So $\angle R(x)c_4x = \angle c_4\mathscr{H}_0 - (\varphi+\varphi') \geq \pi/2-(\varphi+\varphi') \geq \pi/2-(\varphi''+\varphi') \geq \pi/2-\psi'$ and we are done.

Note. It is an elementary exercise to show that $m_{1,2}$ is to the right of $p_{3,1}$ and that $P_{0,1}$ (the perpendicular from c_0 to $\kappa_{1,2}$) is strictly between $P_{4,1}, P_{5,1}$.

First: c_1 to the left of $p_{3,1}$.

(i): $P_{0,1}$ to the left of c_1.

Let $x = c_1$. By inspection, $\angle c_0c_4x \geq \angle R(x)c_4x$ and by Claim 2, $\angle R(x)c_4x \geq \pi/2 - \psi$. Also by inspection, $\pi/2 - \psi \geq \angle c_0xc_4$. So $\angle c_0c_4c_1 \geq \angle c_0c_1c_4$ or equivalently $d(c_0,c_4) \leq d(c_0,c_1) : l(\kappa_{0,4}) \leq l(\kappa_{0,1})$.

(ii): $P_{0,1}$ between c_1, $p_{3,1}$.

Let $x = P_{0,1} \cap \kappa_{1,2}$. By inspection, $\angle c_0c_4x \geq \angle R(x)c_4x$ and by Claim 2, $\angle R(x)c_4x \geq \pi/2-\psi = \angle c_0xc_4$. So $d(c_0,c_4) \leq d(c_0,x)$. Now x is the closest point on $\kappa_{1,2}$ to c_0 so $d(c_0,x) \leq d(c_0,c_1) : l(\kappa_{0,4}) \leq l(\kappa_{0,1})$.

(iii): $P_{0,1}$ to the right of $p_{3,1}$.

Again let $x = P_{0,1}\cap\kappa_{1,2}$. By inspection, $\angle c_0c_5x \geq \angle R(x)c_5x$ and by Claim 2, $\angle R(x)c_5x \geq \pi/2 - \psi' = \angle c_0xc_5$. So $d(c_0,c_5) \leq d(c_0,x) \leq d(c_0,c_1) : l(\kappa_{0,5}) \leq l(\kappa_{0,1})$.

Next: c_1 to the right of $p_{3,1}$.

(iv): c_0 to the left of $p_{3,1}$, $\min\{d(c_0,p_{3,1}), d(c_1,p_{3,1})\} \geq d(m_{1,2},p_{3,1})$.

Let $x = m_{1,2}$. By inspection, $\angle c_0c_4x \geq \angle R(x)c_4x$ and by Claim 2, $\angle R(x)c_4x \geq \pi/2-\psi$. Again by inspection, $\pi/2-\psi \geq \angle c_0c_4x$. So $d(c_0,c_4) \leq d(c_0,x)$. Now $d(c_0,x) = d(c_0,m_{1,2}) \leq d(c_0,c_1) : l(\kappa_{0,4}) \leq l(\kappa_{0,1})$.

(v): c_0 to the left $p_{3,1}$, $\min\{d(c_0,p_{3,1}), d(c_1,p_{3,1})\} \leq d(m_{1,2},p_{3,1})$.

Either: (a) $d(c_1,p_{3,1}) \leq d(c_0,p_{3,1})$ or (b) $d(c_0,p_{3,1}) \leq d(c_1,p_{3,1})$.

For (a) (respectively (b)) let $x = c_1\big(R(c_0)\big)$. By inspection, $\angle c_0c_4x \geq \angle R(x)c_4x$ and by Claim 2, $\angle R(x)c_4x \geq \pi/2-\psi$. Again by inspection, $\pi/2-\psi \geq \angle c_0c_4x$. So $d(c_0,c_4) \leq d(c_0,c_1) : l(\kappa_{0,4}) \leq l(\kappa_{0,1})$.

(vi): $P_{0,1}$ between $p_{3,1}, c_1$ and to the left of $m_{1,2}$.

Let $x = P_{0,1}\cap\kappa_{1,2}$. If c_3 is to the right of $P_{4,3}$ we show that either $l(\kappa_{3,4}) < l(\kappa_{0,1})$ or $l(\eta_4) < l(\kappa_{0,1})$.

Suppose c_3 is between $P_{4,3}$, $m_{3,0}$. So $d(c_4,c_3) \leq d(c_4,m_{3,0})$. As $m_{1,2}$ is to the right of x, $d(c_5,m_{1,2}) \leq d(c_5,x)$. Since c_3 is to the right of $P_{4,3}$, c_0 is to the right of $P_{5,3}$. We can apply the same argument as in Claim 1 since $l(\kappa_{1,2}) \leq l(\kappa_{3,4})$: $\kappa_{1,2}$ lies strictly to the c_5 side of $\perp c_0c_5$. In particular $d(c_5,x) <$

$d(c_0, x)$. Again $d(c_0, x) \leq d(c_0, c_1)$. That is $d(c_4, c_3) \leq d(c_4, m_{3,0}) = d(c_5, m_{1,2}) \leq d(c_5, x) < d(c_0, x) \leq d(c_0, c_1)$: $l(\kappa_{3,4}) < l(\kappa_{0,1})$.

Next: c_3 to the right of $m_{3,0}$. So $\perp c_3c_0$ is to the right of $P_{5,3}$ and $\kappa_{1,2}$ lies strictly to the c_3 side of $\perp c_3c_0$, $d(c_3, c_1) < d(c_0, c_1)$: $l(\eta_4) < l(\kappa_{0,1})$.

So: c_3 is stricly to the left of $P_{4,3}$. By inspection, $\angle c_0c_5x \geq \angle R(x)c_5x$ and by Claim 2, $\angle R(x)c_5x \geq \pi/2 - \psi' = \angle c_0xc_5$. So $d(c_0, c_5) \leq d(c_0, x) \leq d(c_0, c_1)$: $l(\kappa_{0,5}) \leq l(\kappa_{0,1})$.

(vii): $P_{0,1}$ between $p_{3,1}$, c_1 and to the right of $m_{1,2}$.

Let $x = P_{0,1} \cap \kappa_{1,2}$. By inspection, $\angle c_0c_5x \geq \angle R(x)c_5x$ and by Claim 2, $\angle R(x)c_5x \geq \pi/2 - \psi' = \angle c_0xc_5$. Again it follows that $l(\kappa_{0,5}) \leq l(\kappa_{0,1})$.

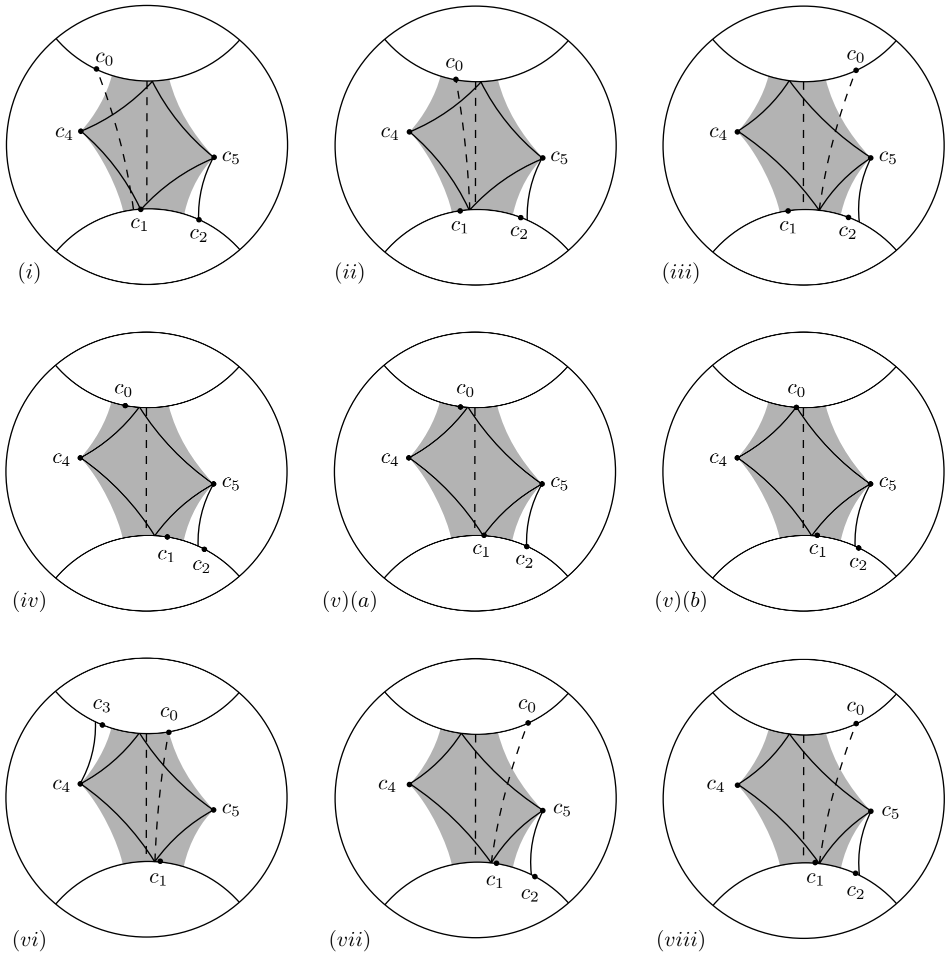

Figure 23. Configurations of c_0 and c_1

(viii): $P_{0,1}$ to the right of c_1: use the same argment as for (iii).

For each one of (i)–(viii): if $l(P_{4,1}) > l(P_{5,1})$ then $\varphi < \varphi''$ and so $l(\kappa_{0,4}) < l(\kappa_{0,1})$ or $l(\kappa_{0,4}) < l(\kappa_{0,1})$. Likewise if $l(\lambda_0) < l(\lambda_2)$ then $d(P_{4,1}, P_{5,1}) < l(\kappa_{1,2})$. So we may assume that $l(P_{4,1}) = l(P_{5,1})$ and $l(\lambda_0) = l(\lambda_2)$, $d(P_{4,1}, P_{5,1}) = l(\kappa_{1,2})$.

Moreover $l(P_{4,1}) = l(P_{5,1})$ implies $m_{1,2} = p_{3,1} \cap \kappa_{1,2}$, $m_{3,0} = p_{3,1} \cap \kappa_{3,0}$ and $l(\lambda_0) = l(\lambda_2)$ implies $\angle c_4 \mathscr{H}_0 = \pi/2$.

Consider (i). If $l(\kappa_{2,5}) > l(\kappa_{1,2})$ then $\varphi' + \varphi'' < \Psi'$ and so $l(\kappa_{0,4}) < l(\kappa_{0,1})$. So we may assume $l(\kappa_{2,5}) = l(\kappa_{1,2})$. Also if $x = c_1$ is strictly to the left of $p_{3,1}$ then $\angle c_0 c_4 x > \angle R(x) c_4 x$ and so $l(\kappa_{0,4}) < l(\kappa_{0,1})$. So we may assume $c_1 = p_{3,1} \cap \kappa_{1,2}$. Likewise $\angle c_0 c_4 x > \angle R(x) c_4 x$ if c_0 is strictly to the left of $p_{3,1}$. So we may assume $c_0 = p_{3,1} \cap \kappa_{3,0}$.

Checking through the proof of Claim 2 we now have $l(\kappa_{0,4}) = l(\kappa_{0,1})$. Indeed since $l(P_{4,1}) = l(P_{5,1})$ the hexagon $\mathscr{H}_0$ has reflective symmetries in $p_{3,1}$ and λ_0. As both $c_1 = p_{3,1} \cap \kappa_{1,2}$, $c_0 = p_{3,1} \cap \kappa_{3,0}$ so $l(\kappa_{0,4}) = l(\kappa_{0,5}) = l(\kappa_{1,4}) = l(\kappa_{1,5})$. Similarly for arcs on $\mathscr{H}_2$, which is isometric to $\mathscr{H}_0$ by an *orientation preserving* isometry. It follows that each arc $\kappa_{*,*}$ is the same length. As the arc set K has the combinatorial edge pattern of an octahedron, it is now not hard to show that $\mathscr{O}$ must be the octahedral orbifold.

We can argue similarly for each of the other parts (ii)–(viii). □

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