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GEOMETRIC INTERSECTION NUMBERS ON A FIVE-PUNCTURED SPHERE

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Abstract. Let $\mathscr{G}$ be the set of all simple closed geodesics on a five-punctured sphere Σ_5. In this article, we associate to each $\gamma \in \mathscr{G}$ four integers which are read off topologically from γ itself. These integers have three remarkable applications. First, the geometric intersection number of any two geodesics in $\mathscr{G}$ can be written explicitly in terms of the corresponding integers. Secondly, there is a homeomorphism of the completion of $\mathscr{G}$ onto a 3-sphere lying in $\mathbf{R}^4$ whose restriction to $\mathscr{G}$ is written explicitly in terms of these integers. Finally, these integers are related to trace polynomials of the corresponding transformations in a representation of $\pi_1(\Sigma_5)$ into $\mathrm{PSL}\,(2,\mathbf{C})$.

Introduction

According to Thurston, the set of all complete simple geodesics on a Riemann surface can be made into a topological space homeomorphic to a sphere whose dimension depends on the topology of the surface. By Thurston's result, the space $\overline{\mathscr{G}}_n$ of complete simple geodesics on an n-punctured sphere Σ_n with $n \geq 4$ is homeomorphic to a sphere of dimension $2n-7$.

In [4], the author introduced to each simple closed geodesic γ on Σ_4 a pair of integers $I_X(\gamma) \geq 0$ and $N(\gamma)$ whose absolute values are geometric intersection numbers of γ with a fixed pair of simple curves on Σ_4. With these integers, the author proved that the geometric intersection number of any two simple closed geodesics γ and δ on Σ_4 is

$$2|I_X(\gamma)N(\delta) - I_X(\delta)N(\gamma)|.$$

The geometric intersection formula above was used to prove the injectivity of a homeomorphism Ψ of $\overline{\mathscr{G}}_4$ onto the circle $\mathbf{R} \cup \{\infty\}$ with $\Psi(\gamma) = N(\gamma)/I_X(\gamma)$ for all simple closed geodesics γ. Moreover, if G is a Maskit four-punctured sphere group, and if $g \in G$ represents a simple closed geodesic γ on Σ_4, then the first two high-order terms of the trace polynomial of g are written explicitly in terms of $I_X(\gamma)$ and $N(\gamma)$.

The aim of this article is to generalize the results in [4] to the case of a five-punctured sphere.

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Similar trace formulas for once and twice punctured tori are proved using different methods in [6] and [7] respectively. However, the methods adopted in [4], [6], [7] and in this article are all based on the cutting sequence technique developed by Birman and Series [2].

In [7], the trace formulas are obtained by factoring a representation of the first fundamental group of a twice punctured torus $\mathscr{S}$ in $\mathrm{SL}(2,\mathbf{C})$ as a representation of the fundamental groupoid $\pi_{1,2}(\mathscr{S},p_1,p_2)$ on $\mathscr{S}$ with two basepoints p_1 and p_2, where one basepoint is chosen on each of the two cyclindrical subsurfaces obtained by cutting along a pair of disjoint curves, one passing through each of the punctures. The fundamental groupoid $\pi_{1,2}(\mathscr{S},p_1,p_2)$ is the groupoid of homotopy classes of paths in $\mathscr{S}$ with endpoints in the set $\{p_1,p_2\}$.

In addition to trace formulas, in [7] Keen, Parker and Series also provide a set of projective coordinates for the set of all simple closed geodesics on $\mathscr{S}$, called the $\pi_{1,2}$-coordinates. For every simple loop γ on $\mathscr{S}$, they consider the restriction of the integral weighted π_1-train track associated with γ to each cylinder, and call the restricted train track the integral weighted $\pi_{1,2}$-train track associated with γ by relating it to $\pi_{1,2}(\mathscr{S},p_1,p_2)$. The $\pi_{1,2}$-coordinates are integer functions of the integral weighted $\pi_{1,2}$-train tracks.

In this article, we shall give a set of projective coordinates to the set $\mathscr{G}$ of all simple closed geodesics on a five punctured sphere Σ_5 equipped with a hyperbolic metric. By using the coordinates, we provide a 3-sphere structure for the set $\overline{\mathscr{G}}$ of all complete simple geodesics on Σ_5.

To enumerate the set $\mathscr{G}$, we start with a Fuchsian representation G of the first fundamental group of Σ_5 acting on the upper half plane $\mathscr{U}$. The Fuchsian group G is generated by two parabolic transformations X and Y, and two hyperbolic transformations S and T.

In Section 2, we introduce four integer functions I_X, I_Y, N_S and N_T on $\mathscr{G}$. The integer functions I_X and I_Y are analogues of the integer function I_X defined in [4], and N_S and N_T are analogues of the integer function N defined in [4]. The values of I_X and I_Y are non-negative. The sign of N_S and that of N_T are determined by the symmetry of $\mathscr{D}$, where $\mathscr{D}$ is a fundamental domain for G acting on $\mathscr{U}$ with $\Gamma=\{S,S^{-1},T,T^{-1},X,X^{-1},Y,Y^{-1}\}$ the set of side pairings.

For every $\gamma\in\mathscr{G}$, the integers $I_X(\gamma)$, $I_Y(\gamma)$, $N_S(\gamma)$ and $N_T(\gamma)$ are read off from the lift of γ to $\mathscr{D}$. The lift of γ to $\mathscr{D}$ also determines words in elements of Γ representing γ, which are called Γ-words. We shall write Γ-words representing geodesics in $\mathscr{G}$ in a specific way, and call them cyclic semi-reduced Γ-words. In Section 2, we shall also relate these cyclic semi-reduced Γ-words to the integer functions I_X, I_Y, N_S and N_T.

By use of the integer functions I_X, I_Y, N_S and N_T, we prove a geometric intersection formula in Theorem 3.1. The geometric intersection formula says that if γ and δ are two geodesics in $\mathscr{G}$, then the geometric intersection number of γ

with δ is

$$2|I_X(\gamma)N_T(\delta) - I_X(\delta)N_T(\gamma)| + 2|I_Y(\gamma)N_S(\delta) - I_Y(\delta)N_S(\gamma)| + |I_{XY}(\gamma,\delta)| - I_{XY}(\gamma,\delta),$$

where $I_{XY}(\gamma,\delta) = \{I_X(\gamma) - I_Y(\gamma)\} \cdot \{I_X(\delta) - I_Y(\delta)\}$.

As a consequence of the geometric intersection formula, we obtain the geometric intersection numbers of six fixed geodesics in $\mathscr{G}$ with an arbitrary geodesic $\gamma \in \mathscr{G}$. These geometric intersection numbers will be called the elementary intersection numbers of γ.

The elementary intersection numbers are used to construct a homeomorphism Ψ of $\overline{\mathscr{G}}$ onto a 3-sphere Δ lying in $\mathbf{R}^6$ (Theorem 4.3). We start with a function of $\mathscr{G}$ into Δ which maps each $\gamma \in \mathscr{G}$ to the point whose coordinates are the elementary intersection numbers of γ. Then, by a continuity argument, we extend the function to obtain a continuous map Ψ from $\overline{\mathscr{G}}$ onto Δ. The injectivity of Ψ is proved by the geometric intersection formula.

By post composing Ψ by a map from $\mathbf{R}^6$ into $\mathbf{R}^4$, we obtain an embedding Φ of $\overline{\mathscr{G}}$ into $\mathbf{R}^4$ with

$$\Phi(\gamma) = \left(\frac{I_X(\gamma)}{\sigma(\gamma)}, \frac{N_T(\gamma)}{\sigma(\gamma)}, \frac{I_Y(\gamma)}{\sigma(\gamma)}, \frac{N_S(\gamma)}{\sigma(\gamma)}\right)$$

for every $\gamma \in \mathscr{G}$, where $\sigma(\gamma) = I_X(\gamma) + |N_T(\gamma)| + I_Y(\gamma) + |N_S(\gamma)|$ (Theorem 4.4).

In the final section, we first find for each $\gamma \in \mathscr{G}$ a cyclic semi-reduced Γ-word $W(\gamma)$ to represent it, and write the word explicitly; see Theorem 5.1, Corollary 5.2 and Theorem 5.3. Then, we consider the Maskit embedding of the Teichmüller space of Σ_5, which is a holomorphic family of Kleinian groups $G(\mu,\nu)$ parametrized by a subset $\mathscr{M}_5$ of $\mathbf{C}^2$. For every $(\mu,\nu) \in \mathscr{M}_5$, the group $G(\mu,\nu)$ uniformizes a five-punctured sphere and three thrice punctured spheres.

For every $\gamma \in \mathscr{G}$, let $W(\gamma;\mu,\nu) \in G(\mu,\nu)$ be the image of $W(\gamma)$ under the canonical isomorphism of G onto $G(\mu,\nu)$. The trace $\operatorname{tr} W(\gamma;\mu,\nu)$ of $W(\gamma;\mu,\nu)$ is a polynomial in μ and ν. For $\gamma \in \mathscr{G}$ with $m = I_X(\gamma) > 0$ or $n = I_Y(\gamma) > 0$, we prove in Theorem 5.5 that

$$\operatorname{tr} W(\gamma;\mu,\nu) = \pm\{\mu^{2m}\nu^{2n} + 4N_T(\gamma)\mu^{2m-1}\nu^{2n} + 2N_S(\gamma)\mu^{2m}\nu^{2n-1} + \cdots\}$$

whenever $m \geq n$, and

$$\operatorname{tr} W(\gamma;\mu,\nu) = \pm 4^{n-m}\{\mu^{2m}\nu^{2n} + 4N_T(\gamma)\mu^{2m-1}\nu^{2n} + 2N_S(\gamma)\mu^{2m}\nu^{2n-1} + \cdots\}$$

whenever $m \leq n$.

Together with the theory of pleating coordinates developed by Keen and Series [6], the trace formulas given above will be used to describe the shape of $\mathscr{M}_5$. The work will appear elsewhere.

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1. Preliminaries

1.1. The space of complete simple geodesics. Let Σ_5 be a 5-punctured sphere equipped with a hyperbolic metric. A loop on Σ_5 with no self intersections will be called a *simple loop*. An *essential simple loop* on Σ_5 is a simple loop which is neither homotopically trivial nor homotopically equivalent to a puncture of Σ_5. A finite union of pairwise disjoint essential simple loops on Σ_5 will be called a *multiple simple loop*.

Let $\mathscr{G}$ be the set of all free homotopy classes of non-oriented essential simple loops on Σ_5. Every element of $\mathscr{G}$ contains a unique geodesic γ on Σ_5. By abuse of notation, we shall also use γ for the free homotopy class containing γ.

Let $\mathscr{GL}$ be the set of all free homotopy classes of non-oriented multiple simple loops on Σ_5. It is clear that $\mathscr{G}$ is a subset of $\mathscr{GL}$.

Let α be a multiple simple loop on Σ_5. All connected components of α fall into at most two distinct free homotopy classes. There are integers $p \geq 0$ and $q \geq 0$ with $p+q > 0$ such that α has exactly p connected components freely homotopic to a $\gamma \in \mathscr{G}$, and has exactly q connected components freely homotopic to a $\gamma' \in \mathscr{G}$, where $\gamma \neq \gamma'$. We shall write $[\alpha] = p\gamma \oplus q\gamma'$, where $[\alpha]$ is the free homotopy class represented by α. Similarly, the free homotopy class represented by a curve β on Σ_5 will be denoted by $[\beta]$.

Let $[\mathscr{G}, \mathbf{R}_+]$ be the set of all functions from $\mathscr{G}$ into the set $\mathbf{R}_+$ of all non-negative real numbers. We provide $\mathscr{G}$ with the discrete topology, and provide $[\mathscr{G}, \mathbf{R}_+]$ with the compact-open topology. It is well known that $[\mathscr{G}, \mathbf{R}_+]$ is homeomorphic to the product space $\prod_{\gamma \in \mathscr{G}} \mathbf{R}_+^\gamma$, where each $\mathbf{R}_+^\gamma$ is a copy of $\mathbf{R}_+$.

Two elements f and g of $[\mathscr{G}, \mathbf{R}_+] - \{0\}$ are called *projectively equivalent* if there is a positive number t such that $f = tg$. Let $\mathrm{P}[\mathscr{G}, \mathbf{R}_+]$ be the set of all projective equivalence classes in $[\mathscr{G}, \mathbf{R}_+] - \{0\}$ provided with the quotient topology. Let π be the quotient map of $[\mathscr{G}, \mathbf{R}_+] - \{0\}$ onto $\mathrm{P}[\mathscr{G}, \mathbf{R}_+]$.

For any two curves α_1 and α_2 on Σ_5, let $\#(\alpha_1 \cap \alpha_2)$ denote the cardinality of the intersection $\alpha_1 \cap \alpha_2$. The *geometric intersection number* $i([\alpha_1], [\alpha_2])$ of $[\alpha_1]$ with $[\alpha_2]$ is defined by

$$i([\alpha_1], [\alpha_2]) = \min\{\#(\alpha_1' \cap \alpha_2') : [\alpha_j'] = [\alpha_j] \text{ for } j = 1, 2\}.$$

It follows immediately from the definition that if $[\alpha] = p\gamma \oplus q\gamma'$, then for any curve β on Σ_5

$$i([\alpha], [\beta]) = pi(\gamma, [\beta]) + qi\gamma', [\beta]),$$

where p and q are non-negative integers with $p+q > 0$, and where γ and γ' are disjoint geodesics in $\mathscr{G}$.

Each $\alpha \in \mathscr{GL}$ induces a function $\mathrm{I}_\alpha \colon \mathscr{G} \longrightarrow \mathbf{R}_+$ given by

$$\mathrm{I}_\alpha(\gamma) = i(\alpha, \gamma) \quad \text{for all } \gamma \in \mathscr{G}.$$

Let $\mathscr{I}\colon \mathscr{GL} \longrightarrow [\mathscr{G}, \mathbf{R}_+]$ be defined by

$$\mathscr{I}(\alpha) = \mathrm{I}_\alpha \quad \text{for all } \alpha \in \mathscr{GL}.$$

It is a well-known fact that the composition $\pi\mathscr{I}$ is injective; see [5]. This allows us to identify $\mathscr{GL}$ with $\pi\mathscr{I}(\mathscr{GL})$.

Let $\overline{\pi\mathscr{I}(\mathscr{GL})}$ and $\overline{\pi\mathscr{I}(\mathscr{G})}$ denote the closures of $\pi\mathscr{I}(\mathscr{GL})$ and $\pi\mathscr{I}(\mathscr{G})$ in $\mathrm{P}[\mathscr{G}, \mathbf{R}_+]$, respectively. Poénaru proved that $\overline{\pi\mathscr{I}(\mathscr{GL})} = \overline{\pi\mathscr{I}(\mathscr{G})}$, (Theorem 4 of [5] Exposé 4).

Note that an element $\mathscr{L}$ of $\mathrm{P}[\mathscr{G}, \mathbf{R}_+]$ is in $\overline{\pi\mathscr{I}(\mathscr{G})}$ if and only if for any l in $[\mathscr{G}, \mathbf{R}_+] - \{0\}$ with $\pi(l) = \mathscr{L}$ there is a sequence $\{t_k\}_{k=1}^{\infty}$ of positive numbers, and there is a sequence $\{\gamma_k\}_{k=1}^{\infty}$ of geodesics in $\mathscr{G}$ such that the sequence $\{t_k \, \mathrm{I}_{\gamma_k}\}_{k=1}^{\infty}$ converges to l. A sequence $\{l_k\}_{k=1}^{\infty}$ in $[\mathscr{G}, \mathbf{R}_+]$ is called *convergent* to $l \in [\mathscr{G}, \mathbf{R}_+]$ if for every $\gamma \in \mathscr{G}$ the sequence $\{l_k(\gamma)\}_{k=1}^{\infty}$ converges in $\mathbf{R}$ to $l(\gamma)$.

According to Thurston, $\overline{\pi\mathscr{I}(\mathscr{G})}$ is homeomorphic to a 3-sphere. In Section 4, we shall construct a homeomorphism of $\overline{\pi\mathscr{I}(\mathscr{G})}$ onto a 3-sphere lying in $\mathbf{R}^4$ (see Theorem 4.4).

1.2. Cyclic reduced words. To enumerate free homotopy classes in $\mathscr{GL}$, we consider the action of the fundamental group $\pi_1(\Sigma_5)$ on the upper half plane $\mathscr{U} = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$.

Let G be the subgroup of $\mathrm{PSL}(2, \mathbf{R})$ generated by the transformations:

$$X = \begin{pmatrix} 1 & 6 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 5 & 12 \\ 2 & 5 \end{pmatrix}.$$

For $j = 1, 2, 3$, let

$$C_j' = \{z \in \mathbf{C} : |2z + 2j - 1| = 1\} \quad \text{and} \quad C_j = \{z \in \mathbf{C} : |2z - (2j-1)| = 1\},$$

and let

$$C_4' = \{z \in \mathbf{C} : \operatorname{Re} z = -3\} \quad \text{and} \quad C_4 = \{z \in \mathbf{C} : \operatorname{Re} z = 3\}.$$

It is clear that $\mathscr{U}/G = \Sigma_5$, and that the domain $\mathscr{D} \subset \mathscr{U}$ bounded by C_j and C_j', $1 \le j \le 4$, is a fundamental domain for G acting on $\mathscr{U}$. We shall schematically draw $\mathscr{D}$ as a rectangular region shown in Figure 1, where the points on the boundary of $\mathscr{D}$ marked by "$\times$" correspond to punctures of Σ_5.

It is well known that every free homotopy class in $\mathscr{G}$ corresponds to a unique conjugacy class in G. We shall find a representative for each conjugacy class in G by using Birman and Series' cutting sequence technique [2].

Let Γ denote the set of all side pairings of $\mathscr{D}$, i.e.,

$$\Gamma = \{X, X^{-1}, Y, Y^{-1}, S, S^{-1}, T, T^{-1}\}.$$

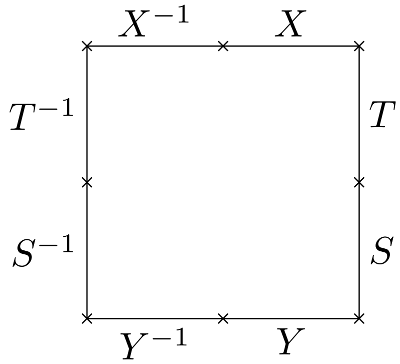

Figure 1. The fundamental domain $\mathscr{D}$.

For every $E \in \Gamma$, we label the common side s of $\mathscr{D}$ and $E(\mathscr{D})$ by E^{-1} on the side inside $\mathscr{D}$, and by E on the side inside $E(\mathscr{D})$; see Figure 1. The side s will be called the E*-side* of $\mathscr{D}$.

For every $g \in G$, the image $g(\mathscr{D})$ will be called a G*-translate* of $\mathscr{D}$. We transport the above side labelling to all G-translates of $\mathscr{D}$.

Let γ be an arbitrary closed curve on Σ_5. Let $\tilde{\gamma}$ be a lift of γ to $\mathscr{U}$ which projects to γ bijectively, and let $z_0 \in \mathscr{U}$ be an endpoint of $\tilde{\gamma}$. Without loss of generality, assume that there is a $g_0 \in G$ and there is a $\xi_0 \in \mathscr{D}$ such that $z_0 = g_0(\xi_0)$.

We orient $\tilde{\gamma}$ so that its initial point is z_0. The arc $\tilde{\gamma}$ cuts in order the G-translates $g_0(\mathscr{D})$, $g_1(\mathscr{D})$, $\ldots$, $g_k(\mathscr{D})$ of $\mathscr{D}$. Then the terminal point of $g_0^{-1}(\tilde{\gamma})$ is $g_0^{-1} \circ g_k(\xi_0)$, and γ is represented by $g = g_0^{-1} \circ g_k$.

For every integer j with $1 \le j \le k$, assume that the common side of $g_{j-1}(\mathscr{D})$ and $g_j(\mathscr{D})$ on the side inside $g_j(\mathscr{D})$ is labelled by $E_j \in \Gamma$. Then

$$E_j(D) = g_{j-1}^{-1}\big(g_j(D)\big),$$

or equivalently $E_j = g_{j-1}^{-1} \circ g_j$. Thus

$$g = g_0^{-1} \circ g_k = (g_0^{-1} \circ g_1) \circ (g_1^{-1} \circ g_2) \circ \cdots \circ (g_{k-1}^{-1} \circ g_k) = E_1 \circ E_2 \circ \cdots \circ E_k.$$

We call $E_1 \circ E_2 \circ \cdots \circ E_k$ a Γ*-word* representing γ.

From now on, we shall simply write the composition of a function f followed by the other function g as gf. Thus, we write

$$E_1 \circ E_2 \circ \cdots \circ E_k = \prod_{j=1}^{k} E_j.$$

A Γ-word $\prod_{j=1}^{k} E_j$ will be called *reduced* if $E_j \ne E_{j+1}^{-1}$ for $1 \le j \le k-1$. It is called *cyclically reduced* if in addition $E_1 \ne E_k^{-1}$.

Let γ be a simple loop on Σ_5. Using the above notation, for every integer j with $0 \le j \le k$, let l_j be the image of the intersection of $\tilde{\gamma}$ with $g_j(\overline{\mathscr{D}})$ mapped by g_j^{-1}, where $\overline{\mathscr{D}}$ is the relative closure of $\mathscr{D}$ in $\mathscr{U}$. The union $l_0 \cup l_k$ forms a simple arc in $\overline{\mathscr{D}}$ connecting the E_k^{-1}-side to the E_1-side. We shall simply write the simple arc as l_k. If $k > 1$ and if $1 \le j \le k-1$, then l_j is a simple arc in $\overline{\mathscr{D}}$ connecting the E_j^{-1}-side to the E_{j+1}-side. Each of these simple arcs $l_1, \ldots, l_k$ will be called a *strand* of γ.

Let α be a multiple simple loop on Σ_5. A strand of a connected component of α will be also called a *strand* of α.

A loop on Σ_5 will be called *reduced* if it is represented by a reduced Γ-word. A multiple simple loop α on Σ_5 will be called *reduced* if every connected component of α is reduced. It is easy to see that a simple loop or a multiple simple loop on Σ_5 is reduced if and only if every strand of the loop connects two different sides of $\mathscr{D}$.

If $\gamma \in \mathscr{G}$ is a geodesic, then every strand of γ is a hyperbolic geodesic arc, and thus every strand of γ must connect two different sides of $\mathscr{D}$ since $\mathscr{D}$ is a geodesic polygon. This proves that every simple closed geodesic on Σ_5 is a reduced loop. Thus every free homotopy class of multiple simple loops on Σ_5 contains a reduced one.

If $\gamma \in \mathscr{G}$ is a geodesic represented by a reduced Γ-word W, then γ is also represented by an arbitrary cyclic permutation of W. If $\gamma' \in \mathscr{G}$ is a geodesic which has the same underlying set as γ but with opposite orientation, then γ' is represented by W^{-1}. Because we are only interested in non-oriented simple loops, we shall identify all reduced Γ-words which are cyclic permutations of W or cyclic permutations of W^{-1}, and call any one of them a *cyclic reduced* Γ-*word* representing γ and its free homotopy class. Every cyclic reduced Γ-word is cyclically reduced.

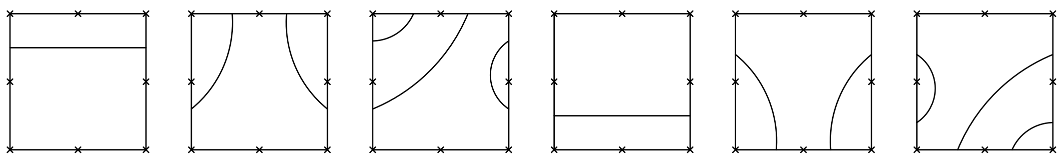

Figure 2. From the left to the right: γ_{11}, γ_{12}, γ_{13}, γ_{21}, γ_{22}, γ_{23}.

As examples, let $\gamma_{jk} \in \mathscr{G}$ be the geodesics given in Figure 2. Each γ_{jk} is represented by a cyclic reduced Γ-word W_{jk} as follows:

$$W_{11} = T, \quad W_{12} = X^{-1}S, \quad W_{13} = XT^{-1}S,$$
$$W_{21} = S, \quad W_{22} = Y^{-1}T, \quad W_{23} = S^{-1}YT.$$

For simplicity, we shall also write $\gamma_{11} = \gamma_T$ and $\gamma_{21} = \gamma_S$.

1.3. Subwords and admissible subarcs. The purpose of this subsection is to find some necessary conditions for cyclic reduced Γ-words representing geodesics in $\widehat{\mathscr{G}} = \mathscr{G} - \{\gamma_S, \gamma_T\}$ from the geometry of the corresponding geodesics.

Let $\gamma \in \widehat{\mathscr{G}}$ be a geodesic represented by a cyclic reduced Γ-word $W(\gamma)$ given by

$$W(\gamma) = \prod_{j=1}^{k} E_j.$$

Note that $k > 1$ since $\gamma \in \widehat{\mathscr{G}}$. For any two integers j, l with $1 \le j \le k$ and $1 \le l \le k$, the reduced Γ-word

$$W' = E_j \cdots E_{j+l-1} \tag{1}$$

will be called a *subword* of $W(\gamma)$, where $E_{j+i} = E_{j+i-k}$ whenever $1 \le i \le l$ and $i + j > k$.

Now, we shall relate W' to γ geometrically. For every i, let l_i be the strand of γ connecting the E_{i-1}^{-1}-side to the E_i-side, where $E_{i-1} = E_k$ if $i = 1$. Assume that $1 \le l < k$, i.e., $W' \ne W(\gamma)$. We think that W' "represents" a subarc γ' of γ. We choose γ' to be the projection of the union $\bigcup_{i=j}^{j+l-1} l_i$ to Σ_5. Each of the arcs l_j, $\dots$, l_{j+l-1} is called a *strand* of γ'.

The subarc γ' has two distinct endpoints. One of the two endpoints is the projection of the endpoint of l_j on the E_{j-1}^{-1}-side, and the other endpoint is the projection of the endpoint of l_{j+l-1} on the E_{j+l-1}-side.

The word given in equation (1) is not clear enough to indicate that γ' has an endpoint which is the projection of a point lying on the E_{j-1}^{-1}-side. Also, to be different from cyclic reduced words representing simple closed geodesics, we shall write the reduced Γ-word representing γ' as

$$\vec{E}_{j-1} W' = \vec{E}_{j-1} E_j \cdots E_{j+l-1}, \tag{2}$$

where $\vec{E}_{j-1}$ is to indicate that $\vec{E}_{j-1} W'$ is not cyclic, and one of the endpoints of γ' is the projection of a point on the E_{j-1}^{-1}-side.

A subarc of a geodesic $\gamma \in \mathscr{G}$ will be called *admissible* if either it is γ itself, or it is represented by a reduced Γ-word as given in equation (2).

Remark 1.1. Let $\gamma \in \widehat{\mathscr{G}}$ be a geodesic represented by a cyclic reduced Γ-word $W(\gamma)$. From now on, for $\varepsilon = \pm 1$, $E \in \Gamma$, $E_1, E_2 \in \Gamma - \{E^{\pm 1}\}$, and an integer $k > 1$, we shall write

$$E_1 \underbrace{E^{\varepsilon} \cdots E^{\varepsilon}}_{k\,\text{times}} E_2 = E_1 E^{k\varepsilon} E_2$$

if above word is a subword of $W(\gamma)$.

By the same reasoning as that in [4, Section 3], there are no admissible subarcs of γ represented by any one of the following words:

$$\begin{array}{llll} \vec{X}^{\varepsilon}X^{\varepsilon}, & \vec{Y}^{\varepsilon}Y^{\varepsilon}, & \vec{T}^{\delta}X^{\varepsilon}T^{\delta}, & \vec{S}^{\delta}Y^{\varepsilon}S^{\delta}, \\ \vec{X}^{\varepsilon}T^{k}X^{\delta}, & \vec{Y}^{\varepsilon}S^{k}Y^{\delta}, & \vec{T}^{\varepsilon}S^{\delta}T^{\varepsilon}, & \vec{S}^{\varepsilon}T^{\delta}S^{\varepsilon}, \end{array}$$

where ε, $\delta \in \{1,-1\}$, and $k \neq 0$ is an integer. Thus none of the following is a subword of $W(\gamma)$:

$$\begin{array}{llll} X^{\varepsilon}X^{\varepsilon}, & Y^{\varepsilon}Y^{\varepsilon}, & T^{\delta}X^{\varepsilon}T^{\delta}, & S^{\delta}Y^{\varepsilon}S^{\delta}, \\ X^{\varepsilon}T^{k}X^{\delta}, & Y^{\varepsilon}S^{k}Y^{\delta}, & T^{\varepsilon}S^{\delta}T^{\varepsilon}, & S^{\varepsilon}T^{\delta}S^{\varepsilon}. \end{array}$$

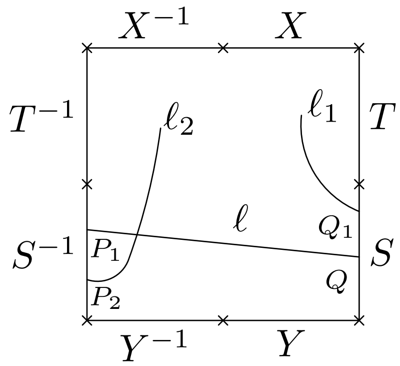

Figure 3.

Proposition 1.1. *Let $\gamma \in \mathscr{G}$ be a geodesic represented by a cyclic reduced Γ-word W, and let $k \neq 0$ be an integer.*

(i) *If $E_1, E_2 \in \{T^{\pm 1}, X^{\pm 1}\}$, and if $E_1 S^k E_2$ is a subword of W, then $|k| = 1$.*

(ii) *If $E_1, E_2 \in \{S^{\pm 1}, Y^{\pm 1}\}$, and if $E_1 T^k E_2$ is a subword of W, then $|k| = 1$.*

Proof. We shall prove the statement (i). The statement (ii) will follow by a similar argument.

Assume that $k > 0$. We choose once for all an orientation on the S^{-1}-side. Let ζ be the fixed point of the transformation $S^{-1}T$. If P and P' are two distinct points lying on the S^{-1}-side, and if P lies between P' and ζ, then we write $P \prec P'$. This gives an orientation to the S-side as well. For any two distinct points Q and Q' lying on the S-side, if $S^{-1}(Q) \prec S^{-1}(Q')$, then we write $Q \prec Q'$.

Let γ' be the admissible subarc of γ represented by $\vec{E}_1 S^k E_2$. Let l_1 be the strand of γ' joining the E_1^{-1}-side to the S-side with the endpoint Q_1 on the S-side. Let l_2 be the strand of γ' joining the S^{-1}-side to the E_2-side with the endpoint P_2 on the S^{-1}-side.

Suppose that $k > 1$. Then γ' has a strand l joining the S^{-1}-side to the S-side with the endpoint $P_1 = S^{-1}(Q_1)$ on the S^{-1}-side. Let Q be the endpoint of l on the S-side. Since γ is simple, we have $Q_1 \prec Q$ (see Figure 3). But, now, we have $P_1 \prec P_2$. This implies that l_2 intersects l which is a contradiction. Hence, $k = 1$.

By the same reasoning as above, one proves that $k = -1$ if $k < 0$.

1.4. π_1-train tracks. In Section 3, we shall need π_1-*train tracks* introduced by Birman and Series (see [1]). A π_1-train track τ on $\mathscr{D}$ is a collection of mutually disjoint simple arcs l_j in $\overline{\mathscr{D}}$ with endpoints lying on the sides of $\mathscr{D}$ such that

(i) except endpoints each l_j is contained in $\mathscr{D}$,

(ii) each l_j joins two distinct sides of $\mathscr{D}$, and

(iii) each pair of distinct sides of $\mathscr{D}$ are connected by at most one l_j.

A π_1-train track τ on $\mathscr{D}$ is called *integral weighted* if every arc in τ is assigned a non-negative integer.

Every reduced multiple simple loop α on Σ_5 can be associated with an integral weighted π_1-train track as described below.

We choose for each $E \in \Gamma$ a point $P(E)$ on the E-side of $\mathscr{D}$ so that $P(E^{-1})$ and $P(E)$ are identified by the transformation E.

For any two distinct $E_1, E_2 \in \Gamma$, let $n_\alpha(E_1, E_2)$ be the number of strands of α connecting the E_1-side to the E_2-side of $\mathscr{D}$. If $n_\alpha(E_1, E_2) > 0$, then we collapse all strands of α which connect the E_1-side to the E_2-side into a single arc from $P(E_1)$ to $P(E_2)$ weighted by the integer $n_\alpha(E_1, E_2)$. These weighted arcs form the required integral weighted π_1-train track $\tau(\alpha)$ on $\mathscr{D}$ (see [1, Theorem 1.3]).

It is clear that if α and β are freely homotopic reduced multiple simple loops on Σ_5, then $n_\alpha(E_1, E_2) = n_\beta(E_1, E_2)$ whenever $E_1, E_2 \in \Gamma$ are distinct, and thus $\tau(\alpha) = \tau(\beta)$. Since every free homotopy class of multiple simple loops on Σ_5 contains a reduced one, we may write

$$n_{[\alpha]}(E_1, E_2) = n_\alpha(E_1, E_2)$$

whenever α is a reduced multiple simple loop on Σ_5, and call $n_{[\alpha]}(E_1, E_2)$ the *number of strands* of $[\alpha]$ connecting the E_1-side to the E_2-side. Similarly, we write

$$\tau([\alpha]) = \tau(\alpha).$$

Let $[\alpha]$, $[\alpha_1]$ and $[\alpha_2]$ be any three elements of $\mathscr{GL}$. If, as subsets of $\overline{\mathscr{D}}$, $\tau([\alpha])$ is the union of $\tau([\alpha_1])$ and $\tau([\alpha_2])$, and if there are two fixed non-negative integers p and q with $p + q > 0$ satisfying

$$n_{[\alpha]}(E_1, E_2) = pn_{[\alpha_1]}(E_1, E_2) + qn_{[\alpha_2]}(E_1, E_2)$$

for any two distinct $E_1, E_2 \in \Gamma$, then we shall write

$$[\alpha] = p[\alpha_1] + q[\alpha_2].$$

From the definition, we see that $[\alpha] = p\gamma + q\gamma'$ if $[\alpha] = p\gamma \oplus q\gamma'$, where $p \geq 0$, $q \geq 0$ are integers with $p + q > 0$, and where $\gamma, \gamma' \in \mathscr{G}$ are disjoint geodesics.

2. Four integer functions

In Section 4, we shall construct a homeomorphism Φ of $\overline{\pi\mathscr{I}(\mathscr{G}\mathscr{L})}$ onto a 3-sphere lying in $\mathbf{R}^4$. For $\alpha \in \mathscr{G}\mathscr{L}$, the value $\Phi(\alpha)$ is written in terms of four integers $I_X(\alpha) \geq 0$, $I_Y(\alpha) \geq 0$, $N_S(\alpha)$ and $N_T(\alpha)$. The sign of $N_S(\alpha)$ and that of $N_T(\alpha)$ are determined by the geometry of α. The integers $I_X(\alpha)$, $I_Y(\alpha)$, $|N_S(\alpha)|$ and $|N_T(\alpha)|$ are numbers of strands of α.

The integer functions I_X and I_Y are analogues of the integer function I_X given in [4], and the integer functions N_S and N_T are analogues of the integer function N given in [4]. In this section, we shall define the integer functions I_X, I_Y, N_S and N_T, and discuss their properties.

2.1. Elementary intersection numbers. For the construction of the homeomorphism Φ, we shall start with a homeomorphism Ψ of $\overline{\pi\mathscr{I}(\mathscr{G}\mathscr{L})}$ onto a 3-sphere lying in $\mathbf{R}^6$ whose value at every $\alpha \in \mathscr{G}\mathscr{L}$ is written in terms of the geometric intersection numbers of α with the six geodesics γ_{jk} given in Figure 2. These six geometric intersection numbers $i(\alpha, \gamma_{jk})$ will be called the *elementary intersection numbers* of α.

To compute elementary intersection numbers, we consider the projections of the sides of $\mathscr{D}$ to Σ_5. For $E \in \{S, T, X, Y\}$, the E-side of $\mathscr{D}$ projects to Σ_5 a simple curve β_E connecting exactly two punctures. Write

$$I_E(\alpha) = i(\alpha, [\beta_E])$$

for all $\alpha \in \mathscr{G}\mathscr{L}$. Note that

$$I_E(\alpha) = \#\{\text{strands of } \alpha \text{ which meet the } E\text{-side (or the } E^{-1}\text{-side)}\}.$$

Thus, we have

$$\begin{aligned} i(\alpha, \gamma_{11}) &= 2I_X(\alpha), \quad & i(\alpha, \gamma_{21}) &= 2I_Y(\alpha), \\ i(\alpha, \gamma_{12}) &= 2I_T(\alpha), \quad & i(\alpha, \gamma_{22}) &= 2I_S(\alpha). \end{aligned} \tag{3}$$

We shall prove later that the elementary intersection numbers of α can be written in terms of $I_X(\alpha)$, $I_Y(\alpha)$, $N_S(\alpha)$ and $N_T(\alpha)$ (see Corollary 3.4). This allows us to construct the homeomorphism Ψ by use of the functions I_X, I_Y, N_S and N_T.

For later use, we extend the integer functions I_E to admissible subarcs of geodesics in $\mathscr{G}$ as follows. For $E \in \Gamma$, and for an arbitrary admissible subarc γ' of a geodesic $\gamma \in \mathscr{G}$, let

$$I_E(\gamma') = \#(\text{strands of } \gamma' \text{ which meet the } E\text{-side of } \mathscr{D}).$$

Note that $I_E(\gamma) = I_{E^{-1}}(\gamma)$ for $\gamma \in \mathscr{G}$ and for $E \in \Gamma$.

2.2. Cyclic semi-reduced Γ-words. Let $\gamma \in \widehat{\mathscr{G}} = \mathscr{G} - \{\gamma_T, \gamma_S\}$ be represented by a cyclic reduced Γ-word $W(\gamma)$. We have known that for $E \in \{S, T, X, Y\}$ the integer $I_E(\gamma)$ is the number of strands of γ which meet the E-side. We may also relate the number $I_E(\gamma)$ to $W(\gamma)$ as follows

$$I_E(\gamma) = \text{the total number of the letters } E \text{ and } E^{-1} \text{ appearing in } W(\gamma).$$

Therefore, to compute the elementary intersection numbers of $\gamma \in \widehat{\mathscr{G}}$ is equivalent to finding a cyclic reduced Γ-word representing γ.

In general, it is not easy to write cyclic reduced Γ-words representing geodesics in $\mathscr{G}$ explicitly. Therefore, we shall introduce *cyclic semi-reduced* Γ-words. Cyclic semi-reduced Γ-words also work for our purposes. To compute geometric intersection numbers, we only need a partial description of cyclic semi-reduced Γ-words, which will be given in Section 2.5. The complete description is given in Section 5.

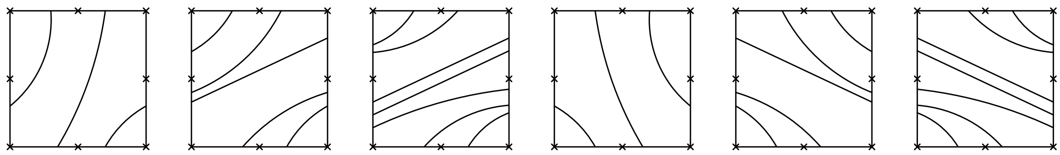

Figure 4. From the left to the right: α_1, α_2, α_3, α_4, α_5, α_6.

To motivate the definition of cyclic semi-reduced Γ-words, we consider the geodesics represented by the following cyclic reduced Γ-words:

$$\begin{aligned} &W_1 = XS^{-1}Y, && W_2 = TX^{-1}S^{-1}Y^{-1}S, && W_3 = TXT^{-1}S^{-2}YS, \\ &W_4 = X^{-1}SY^{-1}, && W_5 = T^{-1}XSYS^{-1}, && W_6 = T^{-1}X^{-1}TS^2Y^{-1}S^{-1}. \end{aligned}$$

Let α_j be the geodesic represented by W_j for $1 \le j \le 6$ (see Figure 4). By defining the zero power E^0 of the transformation E to be the identity transformation for $E = S$ or T, we may rewrite above words as

$$W_j = T^{r_j} X^{\omega_j} T^{t_j} S^{p_j} Y^{\varepsilon_j} S^{q_j}, \tag{4}$$

where $\chi_i = (r_i, \omega_i, t_i, p_i, \varepsilon_i, q_i)$ are given below:

$$\begin{aligned} &\chi_1 = (0, 1, 0, -1, 1, 0), && \chi_2 = (1, -1, 0, -1, -1, 1), \\ &\chi_3 = (1, 1, -1, -2, 1, 1), && \chi_4 = (0, -1, 0, 1, -1, 0), \\ &\chi_5 = (-1, 1, 0, 1, 1, -1), && \chi_6 = (-1, -1, 1, 2, -1, -1). \end{aligned}$$

From the word given in (4), we have

$$I_X(\alpha_j) = 1, \quad I_Y(\alpha_j) = 1, \quad I_S(\alpha_j) = |p_j| + |q_j| \quad \text{and} \quad I_T(\alpha_j) = |r_j| + |t_j|.$$

Now, we define the cyclic semi-reduced Γ-words representing geodesics in $\widehat{\mathscr{G}}$ as follows. Let $\gamma \in \widehat{\mathscr{G}}$ be a geodesic represented by a cyclic reduced Γ-word $W(\gamma)$. If $Y^\varepsilon E$ or EY^ε is a subword of $W(\gamma)$ with $\varepsilon = \pm 1$ and $E \in \{X^{\pm 1}, T^{\pm 1}\}$, we shall write

$$Y^\varepsilon E = Y^\varepsilon S^0 E \quad \text{and} \quad EY^\varepsilon = ES^0 Y^\varepsilon.$$

Similarly, if $E \in \{Y^{\pm 1}, S^{\pm 1}\}$, and if $X^\varepsilon E$ or EX^ε is a subword of $W(\gamma)$, then we write

$$X^\varepsilon E = X^\varepsilon T^0 E \quad \text{and} \quad EX^\varepsilon = ET^0 X^\varepsilon.$$

The resulting cyclic Γ-word will be called *semi-reduced*, still denoted by $W(\gamma)$.

2.3. Four automorphisms of $\mathscr{GL}$. Let α_j be the geodesics given in Section 2.2, and let W_j be the corresponding cyclic semi-reduced Γ-words. By considering the symmetry of the fundamental domain $\mathscr{D}$, we realize that for $1 \le j \le 3$ the words W_{j+3} are the images of W_j under the automorphism Θ_1 of G defined by

$$\Theta_1(E) = E^{-1} \quad \text{for } E \in \{S, T, X, Y\}.$$

There is another automorphism Θ_2 of G obtained from the symmetry of $\mathscr{D}$ defined by

$$\Theta_2(S) = T, \quad \Theta_2(T) = S, \quad \Theta_2(X) = Y, \quad \Theta_2(Y) = X.$$

For $j = 1$ or 2, the automorphism Θ_j induces an orientation reversing homeomorphism of Σ_5 onto itself which is also denoted by Θ_j. If $\gamma \in \mathscr{G}$ is a geodesic, let $\Theta_j(\gamma)$ denote the free homotopy class in $\mathscr{G}$ represented by the image of γ mapped by Θ_j. This defines an injective function, still denoted by Θ_j, of $\mathscr{G}$ onto itself such that if W is a cyclic reduced (or semi-reduced) Γ-word representing $\gamma \in \mathscr{G}$, then $\Theta_j(\gamma)$ is represented by $\Theta_j(W)$.

For instance, we have $\Theta_1(\alpha_j) = \alpha_{j+3}$ for $1 \le j \le 3$. For every integer j with $1 \le j \le 6$, the geodesic $\Theta_2(\alpha_j)$ is represented by the word

$$\Theta_2(W_j) = S^{r_j} Y^{\omega_j} S^{t_j} T^{p_j} X^{\varepsilon_j} T^{q_j},$$

where W_j is the cyclic semi-reduced Γ-word given in (4).

Now, we extend the functions Θ_1 and Θ_2 to $\mathscr{GL}$ by defining

$$\Theta_j(a\gamma \oplus b\gamma') = a\Theta_j(\gamma) \oplus b\Theta_j(\gamma')$$

for $j = 1, 2$, where $a \ge 0$ and $b \ge 0$ are integers with $a + b > 0$, and where γ and γ' are disjoint geodesics in $\mathscr{G}$.

With the two maps Θ_1 and Θ_2, we may simplify the argument on finding cyclic semi-reduced Γ-words by considering subsets of $\mathscr{G}$ which are related by Θ_1 and Θ_2. Let

$\mathscr{GL}_S^+ = \{\alpha \in \mathscr{GL} : \alpha$ has no strands joining the S^{-1}-side to the Y^ε-side, $\varepsilon = \pm 1\}$, and let

$$\mathscr{GL}_S^- = \Theta_1(\mathscr{GL}_S^+),\ \mathscr{GL}_T^+ = \Theta_2(\mathscr{GL}_S^-) \text{ and } \mathscr{GL}_T^- = \Theta_1(\mathscr{GL}_T^+) = \Theta_2(\mathscr{GL}_S^+).$$

For $E = S$ or T, let $\mathscr{G}_E^+ = \mathscr{GL}_E^+ \cap \mathscr{G}$ and $\mathscr{G}_E^- = \mathscr{GL}_E^- \cap \mathscr{G}$.

Note that for $E = S$ or T the sets $\mathscr{GL}_E^+$ and $\mathscr{GL}_E^-$ are not disjoint since

$$a\gamma_S \oplus b\gamma_T \in \mathscr{GL}_E^+ \cap \mathscr{GL}_E^-,$$

where $a \geq 0$ and $b \geq 0$ are integers with $a + b > 0$.

The following proposition is an immediate consequence of the definition.

Proposition 2.1. *If $\alpha \in \mathscr{GL}$, then $I_E(\Theta_1(\alpha)) = I_E(\alpha)$ for $E \in \{S, T, X, Y\}$ and*

$$I_X\big(\Theta_2(\alpha)\big) = I_Y(\alpha), \quad I_Y(\Theta_2(\alpha)) = I_X(\alpha),$$
$$I_S\big(\Theta_2(\alpha)\big) = I_T(\alpha), \quad I_T(\Theta_2(\alpha)) = I_S(\alpha).$$

Taking a further step to investigate the relations among the geodesics α_1, α_2 and α_3, we found that the geodesics α_1, α_2 and α_3 are related by the automorphisms $\mathscr{T}_1$ and $\mathscr{T}_2$ of G defined by

$$\mathscr{T}_1\colon \quad S \longrightarrow S, \quad T \longrightarrow T, \quad X \longrightarrow X, \qquad Y \longrightarrow Y^{-1}S,$$
$$\mathscr{T}_2\colon \quad S \longrightarrow S, \quad T \longrightarrow T, \quad X \longrightarrow X^{-1}T, \quad Y \longrightarrow Y.$$

From the definition, we obtain

$$\Theta_2\mathscr{T}_1\Theta_2 = \mathscr{T}_2 \quad \text{and} \quad \Theta_1\mathscr{T}_j\Theta_1 = \mathscr{T}_j^{-1} \quad \text{for } j = 1, 2.$$

For $j = 1$ or 2, the automorphism $\mathscr{T}_j$ induces an orientation preserving homeomorphism of Σ_5 onto itself, denoted by $\mathscr{T}_j$ as well. The homeomorphism $\mathscr{T}_1$ interchanges the two punctures on Σ_5 corresponding to the fixed point of Y and the fixed point of $Y^{-1}S$, and leaves the other punctures invariant. The homeomorphism $\mathscr{T}_2$ interchanges the two punctures on Σ_5 corresponding to the fixed point of X and the fixed point of $X^{-1}T$, and leaves the other punctures invariant.

Each $\mathscr{T}_j$ also induces an injective function of $\mathscr{G}$ onto itself so that if W is a cyclic reduced (or semi-reduced) Γ-word representing $\gamma \in \mathscr{G}$, then $\mathscr{T}_j(\gamma)$ is represented by $\mathscr{T}_j(W)$. Now, α_1, α_2 and α_3 are related by $\mathscr{T}_1$ and $\mathscr{T}_2$ as follows:

$$\mathscr{T}_1\mathscr{T}_2^{-1}(\alpha_1) = \alpha_2 \quad \text{and} \quad \mathscr{T}_1\mathscr{T}_2^{-1}(\alpha_2) = \alpha_3.$$

Like Θ_1 and Θ_2, the functions $\mathscr{T}_1$ and $\mathscr{T}_2$ extend to $\mathscr{GL}$ defined by

$$\mathscr{T}_j(a\gamma \oplus b\gamma') = a\mathscr{T}_j(\gamma) \oplus b\mathscr{T}_j(\gamma'), \quad j = 1, 2,$$

where $a \geq 0$ and $b \geq 0$ are integers with $a + b > 0$, and where γ and γ' are disjoint geodesics in $\mathscr{G}$.

Proposition 2.2. *Let $\alpha \in \mathscr{GL}$.*
(i) *If $I_Y(\alpha) = 0$, then $\mathscr{T}_1(\alpha) = \alpha$.*
(ii) *If $I_X(\alpha) = 0$, then $\mathscr{T}_2(\alpha) = \alpha$.*
(iii) *If k is an integer, and if $E = X$ or Y, then $I_E\big(\mathscr{T}_1^k(\alpha)\big) = I_E(\alpha) = I_E\big(\mathscr{T}_2^k(\alpha)\big)$.*

Proof. For the proof of (i) and (ii), it suffices to consider the case where $\alpha \in \mathscr{G}$. Let W be a cyclic semi-reduced Γ-word representing α. If $I_Y(\alpha) = 0$, then Y and Y^{-1} are not subwords of W, and $\mathscr{T}_1(W) = W$. This proves that α is invariant under $\mathscr{T}_1$. Similarly, α is invariant under $\mathscr{T}_2$ if $I_X(\alpha) = 0$.

Since γ_{11} and γ_{21} are invariant under $\mathscr{T}_1$ and $\mathscr{T}_2$, we have

$$i\big(\mathscr{T}_j^k(\alpha), \gamma_{m1}\big) = i\big(\alpha, \mathscr{T}_j^{-k}(\gamma_{m1})\big) = i(\alpha, \gamma_{m1})$$

for $j, m \in \{1, 2\}$. Now, the statement (iii) follows from equation (3).

2.4. Definition of the integer functions N_S and N_T. Let $\gamma \in \mathscr{G}$ be a geodesic. If $\gamma \in \mathscr{G}_S^+$, let

$$\begin{aligned} N_S(\gamma) = \#&(\text{strands of } \gamma \text{ joining the } S\text{-side and the } S^{-1}\text{-side}) \\ &+ \#(\text{strands of } \alpha \text{ joining the } S\text{-side and the } Y^{\varepsilon}\text{-side}) \end{aligned}$$

for $\varepsilon = \pm 1$. If $\gamma \in \mathscr{G}_T^+$, let

$$\begin{aligned} N_T(\gamma) = \#&(\text{strands of } \gamma \text{ joining the } T\text{-side and the } T^{-1}\text{-side}) \\ &+ \#(\text{strands of } \alpha \text{ joining the } T^{-1}\text{-side and the } X^{\varepsilon}\text{-side}) \end{aligned}$$

for $\varepsilon = \pm 1$. For $E = S$ or T, if $\gamma \in \mathscr{G}_E^-$, let $N_E(\gamma) = -N_E\big(\Theta_1(\gamma)\big)$.

From the definition, we have

Proposition 2.3. *If $\gamma \in \widehat{\mathscr{G}}$, then $N_S(\gamma) = -N_T\big(\Theta_2(\gamma)\big)$ and $N_T(\gamma) = -N_S\big(\Theta_2(\gamma)\big)$.*

For two integers $a \geq 0$ and $b \geq 0$ with $a + b > 0$, let

$$N_S(a\gamma_S \oplus b\gamma_T) = a \quad \text{and} \quad N_T(a\gamma_S \oplus b\gamma_T) = b.$$

Next, if $\gamma \in \widehat{\mathscr{G}}$ is a geodesic disjoint from γ_S, let

$$N_S(a\gamma_S \oplus b\gamma) = a \quad \text{and} \quad N_T(a\gamma_S \oplus b\gamma) = bN_T(\gamma).$$

If $\gamma \in \widehat{\mathscr{G}}$ is a geodesic disjoint from γ_T, let

$$N_S(a\gamma_T \oplus b\gamma) = bN_S(\gamma) \quad \text{and} \quad N_T(a\gamma_T \oplus b\gamma) = a.$$

Finally, if γ_1 and γ_2 are disjoint geodesics in $\widehat{\mathscr{G}}$, we define

$$N_E(a\gamma_1 \oplus b\gamma_2) = aN_E(\gamma_1) + bN_E(\gamma_2) \quad \text{for } E = S, T.$$

To interpret $N_S(\alpha)$ and $N_T(\alpha)$ geometrically for $\alpha \in \mathscr{GL}$, we need

Lemma 2.4. *If γ_1 and γ_2 are disjoint geodesics in $\widehat{\mathscr{G}}$, then*

$$N_S(\gamma_1)N_S(\gamma_2) \geq 0 \quad \text{and} \quad N_T(\gamma_1)N_T(\gamma_2) \geq 0.$$

Proof. We shall prove $N_T(\gamma_1)N_T(\gamma_2) \geq 0$. This implies, by Proposition 2.3, that $N_S(\gamma_1)N_S(\gamma_2) \geq 0$. First, note that if $\gamma \in \widehat{\mathscr{G}}$ with $N_T(\gamma) \neq 0$, then $I_X(\gamma) > 0$.

Suppose that $N_T(\gamma_1) > 0$ and $N_T(\gamma_2) < 0$. Then γ_1 has a strand l_1 joining the T^{-1}-side to the X^{ε}-side with $\varepsilon = \pm 1$, and has a strand l_1' joining the $X^{-\varepsilon}$-side to some E-side with $E \in \{T^{-1}, S^{\pm 1}, Y^{\pm 1}\}$ so that its endpoint on $X^{-\varepsilon}$-side is identified with that of l_1 on the X^{ε}-side by the transformation X^{ε}.

Similarly, γ_2 has a strand l_2 joining the T-side to the X^{δ}-side with $\delta = \pm 1$, and has a strand l_2' joining the $X^{-\delta}$-side to some E'-side with $E' \in \{T, S^{\pm 1}, Y^{\pm 1}\}$ so that its endpoint on the $X^{-\delta}$-side is identified with that of l_2 on the X^{δ}-side by the transformation X^{δ}.

Since $l_1 \cup l_1'$ must intersect $l_2 \cup l_2'$, then $i(\gamma_1, \gamma_2) > 0$. Contradiction!

Now, for $\alpha \in \mathscr{GL}$ we have

$$\begin{aligned}|N_S(\alpha)| &= \#(\text{strands of } \alpha \text{ joining the } S\text{-side and the } S^{-1}\text{-side}) \\ &\quad + \#(\text{strands of } \alpha \text{ joining the } S^{\delta}\text{-side and the } Y^{\varepsilon}\text{-side}); \\ |N_T(\alpha)| &= \#(\text{strands of } \alpha \text{ joining the } T\text{-side and the } T^{-1}\text{-side}) \\ &\quad + \#(\text{strands of } \alpha \text{ joining the } T^{\delta}\text{-side and the } X^{\varepsilon}\text{-side}),\end{aligned}$$

where $\delta, \varepsilon = \pm 1$.

Proposition 2.5. *Let $\alpha \in \mathscr{GL}$.*

(i) *If $I_X(\alpha) > 0$, then $N_T(\alpha) \geq 0$ whenever $\alpha \in \mathscr{GL}_T^+$, and $N_T(\alpha) \leq 0$ whenever $\alpha \in \mathscr{GL}_T^-$. Thus, $N_T\big(\Theta_1(\alpha)\big) = -N_T(\alpha)$.*

(ii) *If $I_Y(\alpha) > 0$, then $N_S(\alpha) \geq 0$ whenever $\alpha \in \mathscr{GL}_S^+$, and $N_S(\alpha) \leq 0$ whenever $\alpha \in \mathscr{GL}_S^-$. Thus, $N_S\big(\Theta_1(\alpha)\big) = -N_S(\alpha)$.*

(iii) *If $I_X(\alpha)I_Y(\alpha) > 0$, then*

$$N_S(\alpha) = -N_T\big(\Theta_2(\alpha)\big) \quad \text{and} \quad N_T(\alpha) = -N_S\big(\Theta_2(\alpha)\big).$$

Proof. The statement (ii) will follow from (i) by considering $\Theta_2(\alpha)$. The statement (iii) is a consequence of (i) and (ii). It remains to prove the statement (i).

Write $\alpha = a\gamma_1 \oplus b\gamma_2$, where $a \geq 0$ and $b \geq 0$ are integers with $a + b > 0$, and where γ_1 and γ_2 are disjoint geodesics in $\mathscr{G}$. If $ab = 0$, then the statement (i) holds trivially since $I_X(\alpha) > 0$.

Assume that $ab > 0$. Since $I_X(\alpha) > 0$, then $\gamma_1 \neq \gamma_T$ and $\gamma_2 \neq \gamma_T$. If $\gamma_1 = \gamma_S$, then $I_X(\gamma_2) > 0$, and $N_T(\alpha) = bN_T(\gamma_2)$. Now, the assertion follows from the definition of the function N_T on $\widehat{\mathscr{G}}$.

Similarly, the statement (i) is true if $\gamma_2 = \gamma_S$. If $\gamma_1 \neq \gamma_S$ and $\gamma_2 \neq \gamma_S$, the proof is completed by Lemma 2.4.

2.5. Relating N_S and N_T to cyclic semi-reduced Γ-words. Now, we shall explain how to determine $N_S(\gamma)$ and $N_T(\gamma)$ from a cyclic semi-reduced Γ-word W representing $\gamma \in \widehat{\mathscr{G}}$. Note that $I_X(\gamma) > 0$ or $I_Y(\gamma) > 0$.

If $I_Y(\gamma) = n > 0$, then there are exactly n triples of integers $(p_i, \varepsilon_i, q_i)$ with $\varepsilon_i = \pm 1$ such that $E_i S^{p_i} Y^{\varepsilon_i} S^{q_i} E_i'$ is a subword of W for every integer $i \in \{1, \ldots, n\}$, where $E_i, E_i' \in \{T^{\pm 1}, X^{\pm 1}, Y^{\pm 1}\}$. From Remark 1.1, we have $E_i, E_i' \in \{T^{\pm 1}, X^{\pm 1}\}$ for every i. Thus W must be of the form

$$W = \prod_{i=1}^{n} S^{p_i} Y^{\varepsilon_i} S^{q_i} W_i, \tag{5}$$

where each W_i is a semi-reduced Γ-word of the form

$$W_i = \prod_{i=1}^{m_i} E_{ij}$$

with $E_{i1}, E_{im_i} \in \{T^{\pm 1}, X^{\pm 1}\}$, and $E_{ij} \neq Y^{\pm 1}$ whenever $1 < j < m_i$.

If $I_X(\gamma) = n > 0$, then $I_Y(\Theta_2(\gamma)) = n$, and γ is represented by a cyclic semi-reduced Γ-word as given in equation (5). Thus γ is represented by a cyclic semi-reduced Γ-word W of the form

$$W = \prod_{i=1}^{n} T^{p_i} X^{\varepsilon_i} T^{q_i} W_i, \tag{6}$$

where $\varepsilon = \pm 1$, where p_i and q_i are integers, and where each W_i is a semi-reduced Γ-word of the form

$$W_i = \prod_{i=1}^{m_i} E_{ij}$$

with $E_{i1}, E_{im_i} \in \{S^{\pm 1}, Y^{\pm 1}\}$, and $E_{ij} \neq X^{\pm 1}$ whenever $1 < j < m_i$.

Before continuing our discussion, we shall find necessary conditions for the integers p_i and q_i given in (5) and (6).

Lemma 2.6. *Let $\varepsilon = \pm 1$, let p and q be integers, let $\gamma \in \widehat{\mathscr{G}}$, and let W be a cyclic semi-reduced Γ-word representing γ.*

(i) *If $W' = ES^p Y^\varepsilon S^q E'$ is a subword of W with $E, E' \in \{X^\pm, T^\pm\}$, then*

$$-1 \leq (p+q)\varepsilon \leq 0.$$

Moreover, $p \leq 0$ and $q \geq 0$ when $\gamma \in \mathscr{G}_S^+$, and $p \geq 0$ and $q \leq 0$ when $\gamma \in \mathscr{G}_S^-$.

(ii) *If $W' = ET^p X^\varepsilon T^q E'$ is a subword of W with $E, E' \in \{Y^\pm, S^\pm\}$, then*

$$-1 \leq (p+q)\varepsilon \leq 0.$$

Moreover, $p \geq 0$ and $q \leq 0$ when $\gamma \in \mathscr{G}_T^+$, and $p \leq 0$ and $q \geq 0$ when $\gamma \in \mathscr{G}_T^-$.

Proof. For the proof of (i), we may assume that $\varepsilon = 1$ and $\gamma \in \mathscr{G}_S^+$. By the definition of $\mathscr{G}_S^+$, we have $p \le 0$ and $q \ge 0$.

We rewrite W' as $W' = ES^{-p}Y^{\varepsilon}S^{q}E' = ES^{-p}YS^{q}E'$, where $p \ge 0$ and $q \ge 0$. If $q > p$, then $\mathscr{T}_1^{-2p}(W') = EYS^{q-p}E'$ is a subword of $\mathscr{T}_1^{-2p}(W)$, and $\mathscr{T}_1^{-2p}(\gamma)$ is not simple. Contradiction!

If $p > q+1$, then $\mathscr{T}_1^{-2q}(W') = ES^{-p+q}YE'$. This implies that $\mathscr{T}_1^{-2q}(\gamma)$ has a strand joining the S-side to the S^{-1}-side, and has a strand joining the Y^{-1}-side to the E'-side with $E' \in \{T^{\pm}, X^{\pm}\}$. This is impossible. Therefore, $q \le p \le q+1$.

By considering $\mathscr{T}_2$, the statement (ii) will follow by a similar argument.

Proposition 2.7. *Let $\gamma \in \widehat{\mathscr{G}}$ be a geodesic, and let W be a cyclic semi-reduced Γ-word representing γ.*

(i) *If W is of the form given in equation* (5)*, then $N_S(\gamma) = \sum_{i=1}^{n}(q_i - p_i)$.*

(ii) *If W is of the form given in equation* (6)*, then $N_T(\gamma) = \sum_{i=1}^{n}(p_i - q_i)$.*

Proof. From Proposition 2.3, the statement (ii) follows from the statement (i). On the other hand, since $N_S\big(\Theta_1(\gamma)\big) = -N_S(\gamma)$, we may assume that $\gamma \in \mathscr{G}_S^+$. Thus $p_i \le 0$ and $q_i \ge 0$ for all i by Lemma 2.6.

For every i, let γ_i be the admissible subarc of γ represented by $\vec{E}_i W_i E_i'$, where

$$E_i = \begin{cases} S & \text{if } q_i > 0, \\ Y^{\varepsilon_i} & \text{if } q_i = 0, \end{cases} \quad \text{and} \quad E_i' = \begin{cases} S^{-1} & \text{if } p_i < 0, \\ Y^{\varepsilon_{i+1}} & \text{if } p_i = 0. \end{cases}$$

From the definition of W_i, we know that each γ_i neither has strands connecting the S-side to the Y-side, nor has strands connecting the S-side to the Y^{-1}-side. From Proposition 1.1, each γ_i has no strands joining the S-side and the S^{-1}-side. Thus $N_S(\gamma)$ is completely determined by the subwords $S^{p_i}Y^{\varepsilon_i}S^{q_i}$, $1 \le i \le n$.

Using notation given in equation (5), for every i let γ_i' be the admissible subarc represented by $\vec{E}_{(i-1)m_{i-1}}S^{p_i}Y^{\varepsilon_i}S^{q_i}$, and let

$$\begin{aligned} N_i^{(1)} &= \#(\text{strands of } \gamma_i' \text{ connecting the } S\text{-side and the } S^{-1}\text{-side}), \\ N_i^{(2)} &= \#(\text{strands of } \gamma_i' \text{ connecting the } S\text{-side and the } Y\text{-side}) \\ &\quad + \#(\text{strands of } \gamma_i' \text{ connecting the } S\text{-side and the } Y^{-1}\text{-side}). \end{aligned}$$

Since $-1 \le (p_i + q_i)\varepsilon_i \le 0$ for every i, then

$$(N_i^{(1)}, N_i^{(2)}) = \begin{cases} (q_i - p_i - 2, 2) & \text{if } q_i - p_i > 2, \\ (0, q_i - p_i) & \text{if } q_i - p_i \le 2. \end{cases}$$

Thus

$$N_S(\gamma) = \sum_{i=1}^{n}(N_i^{(1)} + N_i^{(2)}) = \sum_{i=1}^{n}(q_i - p_i).$$

At the end of this section, we shall investigate how the integers $N_S(\mathscr{T}_j^k(\gamma))$ and $N_T\big(\mathscr{T}_j^k(\gamma)\big)$ relate to the integers $N_S(\gamma)$ and $N_T(\gamma)$ for $j = 1$ or 2, where $k \ne 0$ is an integer.

Proposition 2.8. *Let $\gamma \in \mathscr{G}$, and let k be an arbitrary integer. Then*
(i) $N_S\big(\mathscr{T}_1^k(\gamma)\big) = N_S(\gamma) + kI_Y(\gamma)$ *and* $N_S\big(\mathscr{T}_2^k(\gamma)\big) = N_S(\gamma)$;
(ii) $N_T\big(\mathscr{T}_1^k(\gamma)\big) = N_T(\gamma)$ *and* $N_T\big(\mathscr{T}_2^k(\gamma)\big) = N_T(\gamma) - kI_X(\gamma)$.

Proof. The proposition holds trivially for $\gamma = \gamma_T$ and for $\gamma = \gamma_S$. In the following, we assume that $\gamma \in \widehat{\mathscr{G}}$.

Since $\Theta_2\mathscr{T}_1\Theta_2 = \mathscr{T}_2$, then the equations in (ii) follow from that given in (i) by Proposition 2.1 and Proposition 2.3.

Now, we shall only prove the equations given in (i) for $k = \pm 1$. Then the proof of the proposition is completed by applying mathematical induction to $|k|$.

If $I_Y(\gamma) = 0$, then $N_S(\gamma) = 0$. From Proposition 2.2, we have $I_Y\big(\mathscr{T}_j^k(\gamma)\big) = 0$ for $j = 1, 2$. Thus $N_S\big(\mathscr{T}_j^k(\gamma)\big) = 0$, and the equations in (i) hold.

Let $I_Y(\gamma) = n > 0$. Assume that $\gamma \in \mathscr{G}_S^+$. Then γ is represented by a cyclic semi-reduced Γ-word W of the form

$$W = \prod_{i=1}^{n} S^{-p_i} Y^{\varepsilon_i} S^{q_i} W_i,$$

where $\varepsilon = \pm 1$, $p_i \geq 0$, $q_i \geq 0$ are integers, and where each W_i is a semi-reduced Γ-word as given in equation (5). Since

$$\mathscr{T}_1(W) = \prod_{i=1}^{n} S^{-p_i'} Y^{-\varepsilon_i} S^{q_i'} W_i \quad \text{and} \quad \mathscr{T}_1^{-1}(W) = \prod_{i=1}^{n} S^{-p_i''} Y^{-\varepsilon_i} S^{q_i''} W_i,$$

with $p_i' + q_i' = p_i + q_i + 1$ and $p_i'' + q_i'' = p_i + q_i - 1$, from Proposition 2.7 we have

$$N_S(\mathscr{T}_1(\gamma)) = \sum_{i=1}^{n} (p_i' + q_i') = n + \sum_{i=1}^{n} (p_i + q_i) = N_S(\gamma) + I_Y(\gamma) \quad \text{and}$$
$$N_S(\mathscr{T}_1^{-1}(\gamma)) = \sum_{i=1}^{n} (p_i'' + q_i'') = -n + \sum_{i=1}^{n} (p_i + q_i) = N_S(\gamma) - I_Y(\gamma).$$

Let $W_i' = \mathscr{T}_2(W_i)$ and $W_i'' = \mathscr{T}_2^{-1}(W_i)$ for every i. By the definition of W_i and that of $\mathscr{T}_2$, we easily see that W_i' and W_i'' have the same form as W_i has. Since

$$\mathscr{T}_2(W) = \prod_{i=1}^{n} S^{-p_i} Y^{\varepsilon_i} S^{q_i} W_i' \quad \text{and} \quad \mathscr{T}_2^{-1}(W) = \prod_{i=1}^{n} S^{-p_i} Y^{\varepsilon_i} S^{q_i} W_i'',$$

then

$$N_S\big(\mathscr{T}_2(\gamma)\big) = N_S(\mathscr{T}_2^{-1}(\gamma)) = \sum_{i=1}^{n} (p_i + q_i) = N_S(\gamma).$$

If $\gamma \in \mathscr{G}_S^-$, then $\Theta_1(\gamma) \in \mathscr{G}_S^+$, and

$$\begin{aligned}
N_S\big(\mathscr{T}_1(\gamma)\big) &= -N_S\big(\Theta_1\mathscr{T}_1(\gamma)\big) = -N_S\big(\mathscr{T}_1^{-1}\Theta_1(\gamma)\big)\\
&= -\{N_S\big(\Theta_1(\gamma)\big) - I_Y\big(\Theta_1(\gamma)\big)\} = N_S(\gamma) + I_Y(\gamma);\\
N_S\big(\mathscr{T}_1^{-1}(\gamma)\big) &= -N_S\big(\Theta_1\mathscr{T}_1^{-1}(\gamma)\big) = -N_S\big(\mathscr{T}_1\Theta_1(\gamma)\big)\\
&= -\big\{N_S\big(\Theta_1(\gamma)\big) + I_Y\big(\Theta_1(\gamma)\big)\big\} = N_S(\gamma) - I_Y(\gamma);\\
N_S\big(\mathscr{T}_2^k(\gamma)\big) &= -N_S\big(\Theta_1\mathscr{T}_2^k(\gamma)\big) = -N_S\big(\mathscr{T}_2^{-k}\Theta_1(\gamma)\big)\\
&= -N_S\big(\Theta_1(\gamma)\big) = N_S(\gamma) \quad \text{for } k = \pm 1.
\end{aligned}$$

3. Geometric intersection numbers

In this section, we shall prove the geometric intersection formula (see Theorem 3.1). The geometric intersection formula will be used to prove the injectivity of a homeomorphism Ψ of $\overline{\pi\mathscr{I}(\mathscr{G}\mathscr{L})}$ onto a 3-sphere. The homeomorphism Ψ will be constructed with elementary intersection numbers. From the geometric intersection formula, we obtain the elementary intersection numbers of geodesics in $\mathscr{G}$. Then we will get elementary intersection numbers of $\alpha \in \mathscr{G}\mathscr{L}$.

3.1. The geometric intersection formula. The main work of this subsection is to prove the following theorem:

Theorem 3.1 (Geometric intersection formula). *If γ_1 and γ_2 are two simple closed geodesics on Σ_5, then*

$$\begin{aligned}
i(\gamma_1,\gamma_2) = 2|I_X(\gamma_1)N_T(\gamma_2) - I_X(\gamma_2)N_T(\gamma_1)| + 2|I_Y(\gamma_1)N_S(\gamma_2) - I_Y(\gamma_2)N_S(\gamma_1)|\\
+ |I_{XY}(\gamma_1,\gamma_2)| - I_{XY}(\gamma_1,\gamma_2),
\end{aligned}$$

where $I_{XY}(\gamma_1,\gamma_2) = \{I_X(\gamma_1) - I_Y(\gamma_1)\}\cdot\{I_X(\gamma_2) - I_Y(\gamma_2)\}$.

As a consequence of the geometric intersection formula, we obtain the elementary intersection numbers of geodesics in $\mathscr{G}$ as follows.

Corollary 3.2. *If $\gamma \in \mathscr{G}$, then*

$$\begin{aligned}
&i(\gamma,\gamma_{12}) = 2|N_T(\gamma)| + |I_Y(\gamma) - I_X(\gamma)| + I_Y(\gamma) - I_X(\gamma),\\
&i(\gamma,\gamma_{13}) = 2|N_T(\gamma) - I_X(\gamma)| + |I_Y(\gamma) - I_X(\gamma)| + I_Y(\gamma) - I_X(\gamma),\\
&i(\gamma,\gamma_{22}) = 2|N_S(\gamma)| + |I_X(\gamma) - I_Y(\gamma)| + I_X(\gamma) - I_Y(\gamma), \quad \textit{and}\\
&i(\gamma,\gamma_{23}) = 2|N_S(\gamma) - I_Y(\gamma)| + |I_X(\gamma) - I_Y(\gamma)| + I_X(\gamma) - I_Y(\gamma).
\end{aligned}$$

Proof of the geometric intersection formula. It is easy to see that the geometric intersection formula is valid if γ_1 or γ_2 is in $\{\gamma_T, \gamma_S\}$. It remains to prove the formula for $\gamma_1, \gamma_2 \in \widehat{\mathscr{G}}$.

For every integer k, write $F_k = \mathscr{T}_2^{-k}\mathscr{T}_1^{k}$. From Proposition 2.8, we obtain

$$I_{XY}(\gamma_1,\gamma_2) = I_{XY}\big(F_k(\gamma_1), F_k(\gamma_2)\big),$$
$$I_X(\gamma_1)N_T(\gamma_2) - I_X(\gamma_2)N_T(\gamma_1) = I_X\big(F_k(\gamma_1)\big)N_T\big(F_k(\gamma_2)\big) - I_X\big(F_k(\gamma_2)\big)N_T\big(F_k(\gamma_1)\big),$$
$$I_Y(\gamma_1)N_S(\gamma_2) - I_Y(\gamma_2)N_S(\gamma_1) = I_Y\big(F_k(\gamma_1)\big)N_S\big(F_k(\gamma_2)\big) - I_Y\big(F_k(\gamma_2)\big)N_S\big(F_k(\gamma_1)\big)$$

for all integers k. From Proposition 2.2 and Proposition 2.8, there is an integer $k > 0$ such that

$$N_T\big(F_k(\gamma_j)\big) \geq 2I_X(\gamma_j) = 2I_X\big(F_k(\gamma_j)\big) \text{ and } N_S\big(F_k(\gamma_j)\big) \geq 2I_Y(\gamma_j) = 2I_Y\big(F_k(\gamma_j)\big)$$

for $j = 1, 2$; thus we may assume that

$$N_T(\gamma_j) \geq 2I_X(\gamma_j) \quad \text{and} \quad N_S(\gamma_j) \geq 2I_Y(\gamma_j).$$

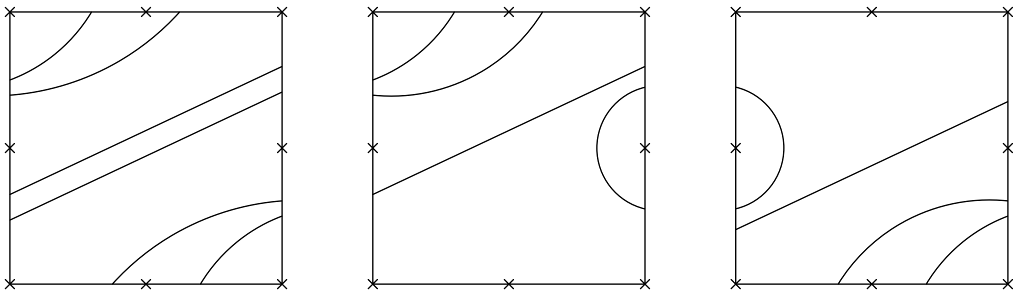

Figure 5. From the left to the right: τ_1, τ_2, τ_3.

If $\beta \in \widehat{\mathscr{G}}$ is a geodesic with $N_T(\beta) \geq 2I_X(\beta)$ and $N_S(\beta) \geq 2I_Y(\beta)$, then β lies in $\mathscr{G}_S^+ \cap \mathscr{G}_T^+$, and β can be written as

$$\beta = p\gamma_S + q\gamma_T + r\tau_1 + s\tau_2 \quad \text{or} \quad \beta = p\gamma_S + q\gamma_T + r\tau_1 + s\tau_3,$$

where p, q, r and s are non-negative integers with $p+q+r+s > 0$, and where τ_1, τ_2 and τ_3 are geodesics represented by the following cyclic reduced Γ-words (see Figure 5):

$$W(\tau_1) = S^{-1}Y^{-1}STXT^{-1}, \quad W(\tau_2) = S^{-1}TXT^{-1} \quad \text{and} \quad W(\tau_3) = S^{-1}Y^{-1}ST.$$

Let $\mathscr{GL}_1$ be the set of all elements of $\mathscr{GL}$ of the form $p\gamma_S + q\gamma_T + r\tau_1 + s\tau_2$, and let $\mathscr{GL}_2$ be the set of all elements of $\mathscr{GL}$ of the form $p\gamma_S + q\gamma_T + r\tau_1 + s\tau_3$, where p, q, r and s are non-negative integers with $p+q+r+s > 0$.

Let $\mathscr{D}$ be the fundamental domain for G given in Section 1.2. Let $\mathscr{R}$ denote the reflection in the imaginary axis. Let l^* be the semi-circle contained in $\mathscr{D}$

joining the fixed point of $S^{-1}T$ to the fixed point of TS^{-1}. Note that l^* is invariant under $\mathscr{R}$. Let

P^* be the point of intersection of l^* with the imaginary axis,
$\mathscr{D}^+$ be the connected component of $\mathscr{D} - l^*$ lying above l^*,
$\mathscr{D}^-$ be the connected component of $\mathscr{D} - l^*$ lying below l^*,
Σ_5^+ and Σ_5^- be the projections of $\mathscr{D}^+$ and $\mathscr{D}^-$ to Σ_5, respectively,
$\mathscr{S}_4^+$ be the four-punctured sphere obtained from $\mathscr{D}^+ - \{P^*\}$ by identifying the boundary points of $\mathscr{D}^+ - \{P^*\}$ via X, T and $\mathscr{R}$,
$\mathscr{S}_4^-$ be the four-punctured sphere obtained from $\mathscr{D}^- - \{P^*\}$ by identifying the boundary points of $\mathscr{D}^- - \{P^*\}$ via Y, S and $\mathscr{R}$, and
γ^* be the projection of l^* to Σ_5, which is the common boundary of Σ_5^+ and Σ_5^-. The free homotopy class containing γ^* is also denoted by γ^*.

The fixed point ζ of $S^{-1}T$ projects to a puncture ζ^+ on $\mathscr{S}_4^+$, and projects to a puncture ζ^- on $\mathscr{S}_4^-$. Let $[\zeta^+]$ denote the free homotopy class of simple loops on $\mathscr{S}_4^+$ enclosing ζ^+, and let $[\zeta^-]$ denote the free homotopy class of simple loops on $\mathscr{S}_4^-$ enclosing ζ^-. It is obvious that $i([\zeta^+], \alpha) = 0$ for all free homotopy classes α of multiple simple loops on $\mathscr{S}_4^+$, and that $i([\zeta^-], \beta) = 0$ for all free homotopy classes β of multiple simple loops on $\mathscr{S}_4^-$.

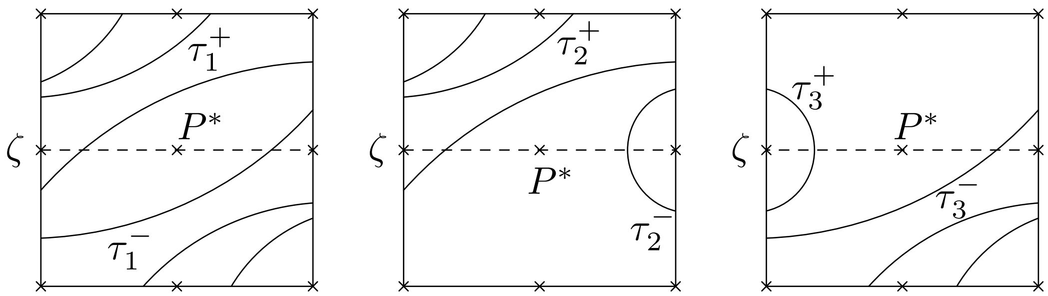

Figure 6. τ_j^+ and τ_j^- for $j = 1, 2, 3$.

For any reduced simple loop α in the free homotopy class $\gamma \in \mathscr{G}$, let

$$\alpha^+ = \alpha \cap \Sigma_5^+ \quad \text{and} \quad \alpha^- = \alpha \cap \Sigma_5^-.$$

We shall call a connected component of the lift of α^+ to $\mathscr{D}$ a strand of α^+, and call a connected component of the lift of α^- to $\mathscr{D}$ a strand of α^-. Let

$$\gamma^+ = \{\alpha^+ : \alpha \text{ is a reduced simple loop in the free homotopy class } \gamma\} \quad \text{and}$$
$$\gamma^- = \{\alpha^- : \alpha \text{ is a reduced simple loop in the free homotopy class } \gamma\}.$$

See Figure 6 for examples of γ^+ and γ^-. When there is no risk of confusion, we shall also use γ^+ and γ^- to represent any curve in them. Since the geodesic γ_T is disjoint from Σ_5^-, we shall also write $\gamma_T^+ = \gamma_T$. Similarly, write $\gamma_S^- = \gamma_S$.

If $\gamma = a\gamma_T + b\gamma_S + c\tau_1 + d\tau_2$ is an arbitrary geodesic in $\widehat{\mathscr{G}} \cap \mathscr{GL}_1$, then γ^- has $2d$ strands whose union is homotopic to d copies of τ_2^-. We shall call such strands τ_2^--*type* strands of γ^-.

If $\gamma = a\gamma_T + b\gamma_S + c\tau_1 + d\tau_3$ is an arbitrary geodesic in $\widehat{\mathscr{G}} \cap \mathscr{GL}_2$, then γ^+ has $2d$ strands whose union is homotopic to d copies of τ_3^+. We shall call such strands τ_3^+-*type* strands of γ^+.

Let $\gamma \in \widehat{\mathscr{G}} \cap (\mathscr{GL}_1 \cup \mathscr{GL}_2)$ be a geodesic, and write

$$\gamma = a\gamma_T + b\gamma_S + c\tau_1 + d\tau_2 \quad \text{or} \quad \gamma = a\gamma_T + b\gamma_S + c\tau_1 + d\tau_3.$$

Then $i(\gamma, \gamma^*) = 2(c+d)$ since

$$i(a\gamma_T + b\gamma_S + c\tau_1 + d\tau_2, \gamma^*) = 2(c+d) = i(a\gamma_T + b\gamma_S + c\tau_1 + d\tau_3, \gamma^*).$$

Set $k = c + d$. Every simple closed curve α in the homotopy class γ is homotopic to a simple loop $\hat{\alpha}$ with the following properties:

(i) The lift of $\hat{\alpha}$ to $\mathscr{D}$ intersects $l^* - \{P^*\}$ at $P_1, \dots, P_k$, $P_1', \dots, P_k'$ with $P_j' = \mathscr{R}(P_j)$.

(ii) The endpoints of strands of $\hat{\alpha}$ coincide with that of α. Then $\hat{\alpha}^+$ projects to $\mathscr{S}_4^+$ a multiple simple loop $\tilde{\alpha}^+$, and $\hat{\alpha}^-$ projects to $\mathscr{S}_4^-$ a multiple simple loop $\tilde{\alpha}^-$. Let $\tilde{\gamma}^+$ denote the free homotopy class of multiple simple loops on $\mathscr{S}_4^+$ represented by $\tilde{\alpha}^+$, and let $\tilde{\gamma}^-$ denote the free homotopy class of multiple simple loops on $\mathscr{S}_4^-$ represented by $\tilde{\alpha}^-$.

If $\gamma = a\gamma_T + b\gamma_S + c\tau_1 + d\tau_2$ with $c + d > 0$, then

$$\tilde{\gamma}^+ = a\gamma_T + (c+d)\tilde{\tau}_1^+ \quad \text{and} \quad \tilde{\gamma}^- = \{b\gamma_S + c\tilde{\tau}_1^-\} \oplus d[\zeta^-].$$

If $\gamma = a\gamma_T + b\gamma_S + c\tau_1 + d\tau_3$ with $c + d > 0$, then

$$\tilde{\gamma}^+ = \{a\gamma_T + c\tilde{\tau}_1^+\} \oplus d[\zeta^+] \quad \text{and} \quad \tilde{\gamma}^- = b\gamma_S + (c+d)\tilde{\tau}_1^-.$$

Now we are in the position to compute $i(\gamma_1, \gamma_2)$ for $\gamma_1, \gamma_2 \in \widehat{\mathscr{G}} \cap (\mathscr{GL}_1 \cup \mathscr{GL}_2)$. Without loss of generality, we may assume that all points of intersection of γ_1 and γ_2 are not on γ^*.

Case 1. Assume that $\gamma_1, \gamma_2 \in \widehat{\mathscr{G}} \cap \mathscr{GL}_1$. Clearly, $I_{XY}(\gamma_1, \gamma_2) \geq 0$ and $|I_{XY}(\gamma_1, \gamma_2)| - I_{XY}(\gamma_1, \gamma_2) = 0$. By applying suitable homotopy maps to γ_1 and γ_2, we may assume that τ_2^--type strands of γ_1^- are disjoint from γ_2, and that τ_2^--type strands of γ_2^- are disjoint from γ_1. Then by Theorem 2.6 of [4] we obtain

$$\begin{aligned} i(\gamma_1, \gamma_2) &= i(\gamma_1^+, \gamma_2^+) + i(\gamma_1^-, \gamma_2^-) = i(\tilde{\gamma}_1^+, \tilde{\gamma}_2^+) + i\tilde{\gamma}_1^-, \tilde{\gamma}_2^-) \\ &= 2|I_X(\gamma_1)N_T(\gamma_2) - I_X(\gamma_2)N_T(\gamma_1)| + 2|I_Y(\gamma_1)N_S(\gamma_2) - I_Y(\gamma_2)N_S(\gamma_1)| \\ &= 2|I_X(\gamma_1)N_T(\gamma_2) - I_X(\gamma_2)N_T(\gamma_1)| + 2|I_Y(\gamma_1)N_S(\gamma_2) - I_Y(\gamma_2)N_S(\gamma_1)| \\ &\quad + |I_{XY}(\gamma_1, \gamma_2)| - I_{XY}(\gamma_1, \gamma_2). \end{aligned}$$

Case 2. If $\gamma_1, \gamma_2 \in \widehat{\mathscr{G}} \cap \mathscr{GL}_2$, then $\Theta_1\Theta_2(\gamma_1)$ and $\Theta_1\Theta_2(\gamma_2)$ are both in $\widehat{\mathscr{G}} \cap \mathscr{GL}_1$, and the geometric intersection formula is valid for this case by Proposition 2.1.

Case 3. Assume that $\gamma_1 \in \widehat{\mathscr{G}} \cap \mathscr{G}\mathscr{L}_1$ and $\gamma_2 \in \widehat{\mathscr{G}} \cap \mathscr{G}\mathscr{L}_2$. Write

$$\gamma_1 = a\gamma_T + b\gamma_S + c\tau_1 + d\tau_2 \quad \text{and} \quad \gamma_2 = a'\gamma_T + b'\gamma_S + c'\tau_1 + d'\tau_3,$$

where $dd' > 0$. Clearly, $I_{XY}(\gamma_1, \gamma_2) < 0$ and

$$|I_{XY}(\gamma_1, \gamma_2)| - I_{XY}(\gamma_1, \gamma_2) = 2dd'.$$

Write the union of τ_2^--type strands of γ_1^- as $d\tau_2^-$, and write the union of τ_3^+-type strands of γ_2^+ as $d'\tau_3^+$.

To compute $i(d\tau_2^-, \gamma_2^-) + i(\gamma_1^+, d'\tau_3^+)$, we need the orientation on the S-side and that on the S^{-1}-side (see the proof of Proposition 1.1). Also, we need an orientation to the T-side and an orientation to the T^{-1}-side.

Recall that ζ is the fixed point of the transformation $S^{-1}T$. If P and P' are two distinct points on the T^{-1}-side, and if P lies between ζ and P', then we write $P \prec P'$. For any two distinct points Q and Q' on the T-side, if $T^{-1}(Q) \prec T^{-1}(Q')$, then we write $Q \prec Q'$.

Let $m = a' + 2c' + d'$ and $n = b' + 2c' + 2d'$. Let

$P_1 \prec \cdots \prec P_m$ be the endpoints of strands of γ_2 on the T-side,
$Q_1 \prec \cdots \prec Q_n$ be the endpoints of the strands of γ_2 on the S-side,
$L_j^{(2)}$ be the strand of γ_2 with P_j an endpoint, $1 \le j \le d'$,
$l_j^{(2)}$ be the strand of γ_2 with Q_j an endpoint, $1 \le j \le d'$,
$A_1 \prec \cdots \prec A_d$ be the first d points on the S-side where the lift of γ_1 meets,
A'_j be the point on the S^{-1}-side identified with A_j by S^{-1}, $1 \le j \le d$,
$L_j^{(1)}$ be the strand of γ_1 with A'_j an endpoint, $1 \le j \le d$, and
$l_j^{(1)}$ be the strand of γ_1 with A_j an endpoint, $1 \le j \le d$.

Note that $L_j^{(1)}$ connects the S^{-1}-side to the T-side, and each $l_j^{(1)}$ connects the S-side to the T-side. Let B_j be the endpoint of $l_j^{(1)}$ on the T-side. It is clear that $B_1 \prec \cdots \prec B_d$.

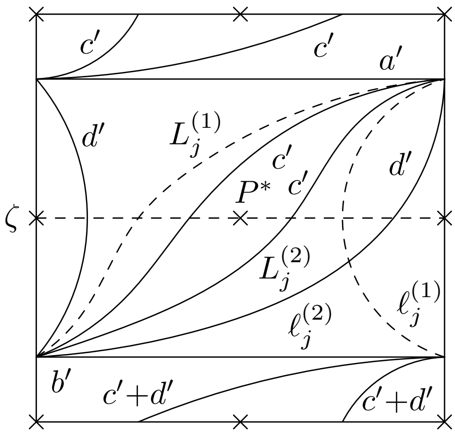

Figure 7.

Without loss of generality, we assume that $i(\gamma_1^+, d'\tau_3^+) = 0$, and that the union L of all $L_j^{(1)}$ is disjoint from γ_2 (see Figure 7). Then

$$P_{d'} \prec B_1 \prec \cdots \prec B_d \prec P_{d'+1} \quad \text{and} \quad Q_{d'} \prec A_1 \prec \cdots \prec A_d \prec Q_{d'+1}.$$

This implies that each $l_j^{(1)}$ intersects all $L_i^{(2)}$ and all $l_i^{(2)}$ transversally. Then

$$i(d\tau_2^-, \gamma_2^-) = 2dd'.$$

By Theorem 2.6 of [4] again, we complete the proof of Theorem 3.1 as follows:

$$\begin{aligned} i(\gamma_1, \gamma_2) &= i(\gamma_1^+, \gamma_2^+) + i(\gamma_1^-, \gamma_2^-) \\ &= i(\tilde\gamma_1^+, \tilde\gamma_2^+) + i(\tilde\gamma_1^-, \tilde\gamma_2^-) + i(d\tau_2^-, \gamma_2^-) + i(\gamma_1^+, d'\tau_3^+) \\ &= i\big(a\gamma_T + (c+d)\tilde\tau_1^+, a'\gamma_T + c'\tilde\tau_1^+\big) \\ &\quad + i\big(b\gamma_S + c\tilde\tau_1^-, b'\gamma_S + (c'+d')\tilde\tau_1^-\big) + 2dd' \\ &= 2|I_X(\gamma_1)N_T(\gamma_2) - I_X(\gamma_2)N_T(\gamma_1)| + 2|I_Y(\gamma_1)N_S(\gamma_2) - I_Y(\gamma_2)N_S(\gamma_1)| \\ &\quad + |I_{XY}(\gamma_1, \gamma_2)| - I_{XY}(\gamma_1, \gamma_2). \end{aligned}$$

3.2. Elementary intersection numbers of multiple simple loops. In the rest of this section, we shall prove the following proposition.

Proposition 3.3. *If $\alpha \in \mathscr{GL}$, and if k is an integer, then*

$$\begin{aligned} &i\big(\mathscr{T}_j^k(\alpha), \gamma_{11}\big) = i(\alpha, \gamma_{11}), \quad i\big(\mathscr{T}_j^k(\alpha), \gamma_{21}\big) = i(\alpha, \gamma_{21}) \quad \textit{for } j = 1, 2, \\ &i\big(\mathscr{T}_1^k(\alpha), \gamma_{1j}\big) = i(\alpha, \gamma_{1j}), \quad i\big(\mathscr{T}_2^k(\alpha), \gamma_{2j}\big) = i(\alpha, \gamma_{2j}) \quad \textit{for } j = 2,\ 3, \\ &i\big(\mathscr{T}_2^k(\alpha), \gamma_{12}\big) = 2|N_T(\alpha) - kI_X(\alpha)| + |I_Y(\alpha) - I_X(\alpha)| + I_Y(\alpha) - I_X(\alpha), \\ &i\big(\mathscr{T}_2^k(\alpha), \gamma_{13}\big) = 2|N_T(\alpha) - (k+1)I_X(\alpha)| + |I_Y(\alpha) - I_X(\alpha)| + I_Y(\alpha) - I_X(\alpha), \\ &i\big(\mathscr{T}_1^k(\alpha), \gamma_{22}\big) = 2|N_S(\alpha) + kI_Y(\alpha)| + |I_X(\alpha) - I_Y(\alpha)| + I_X(\alpha) - I_Y(\alpha), \quad \textit{and} \\ &i\big(\mathscr{T}_1^k(\alpha), \gamma_{23}\big) = 2|N_S(\alpha) + (k-1)I_Y(\alpha)| + |I_X(\alpha) - I_Y(\alpha)| + I_X(\alpha) - I_Y(\alpha). \end{aligned}$$

By letting $k = 0$ in the last four equations of the above proposition, we have

Corollary 3.4 (Elementary intersection numbers). *If $\alpha \in \mathscr{GL}$, then*

$$\begin{aligned} &i(\alpha, \gamma_{12}) = 2|N_T(\alpha)| + |I_Y(\alpha) - I_X(\alpha)| + I_Y(\alpha) - I_X(\alpha), \\ &i(\alpha, \gamma_{13}) = 2|N_T(\alpha) - I_X(\alpha)| + |I_Y(\alpha) - I_X(\alpha)| + I_Y(\alpha) - I_X(\alpha), \\ &i(\alpha, \gamma_{22}) = 2|N_S(\alpha)| + |I_X(\alpha) - I_Y(\alpha)| + I_X(\alpha) - I_Y(\alpha), \quad \textit{and} \\ &i(\alpha, \gamma_{23}) = 2|N_S(\alpha) - I_Y(\alpha)| + |I_X(\alpha) - I_Y(\alpha)| + I_X(\alpha) - I_Y(\alpha). \end{aligned}$$

Lemma 3.5. *Let γ and $\gamma' \in \widehat{\mathscr{G}}$ be disjoint geodesics, and let $\alpha = a\gamma \oplus b\gamma'$, where $a \geq 0$ and $b \geq 0$ are integers with $a+b>0$. Then for all integers k*

$$N_T\big(\mathscr{T}_1^k(\alpha)\big) = N_T(\alpha), \quad N_T\big(\mathscr{T}_2^k(\alpha)\big) = N_T(\alpha) - kI_X(\alpha),$$
$$N_S\big(\mathscr{T}_2^k(\alpha)\big) = N_S(\alpha), \quad N_S\big(\mathscr{T}_1^k(\alpha)\big) = N_S(\alpha) + kI_Y(\alpha).$$

Proof. Since $N_E(\alpha) = aN_E(\gamma) + bN_E(\gamma')$ for $E = S$ or T, from Proposition 2.8 we obtain

$$\begin{aligned} N_T\big(\mathscr{T}_1^k(\alpha)\big) &= aN_T\big(\mathscr{T}_1^k(\gamma)\big) + bN_T\big(\mathscr{T}_1^k(\gamma')\big) \\ &= aN_T(\gamma) + bN_T(\gamma') = N_T(\alpha) \quad \text{and} \\ N_T\big(\mathscr{T}_2^k(\alpha)\big) &= aN_T\big(\mathscr{T}_2^k(\gamma)\big) + bN_T\big(\mathscr{T}_2^k(\gamma')\big) \\ &= a\{N_T(\gamma) - kI_X(\gamma)\} + b\{N_T(\gamma') - kI_X(\gamma')\} = N_T(\alpha) - kI_X(\alpha). \end{aligned}$$

Similarly, $N_S\big(\mathscr{T}_2^k(\alpha)\big) = N_S(\alpha)$ and $N_S\big(\mathscr{T}_1^k(\alpha)\big) = N_S(\alpha) + kI_Y(\alpha)$.

Lemma 3.5. *If γ and γ' are two disjoint geodesics in $\widehat{\mathscr{G}}$, then*

$$\begin{aligned} \big(N_T(\gamma) - I_X(\gamma)\big)\big(N_T(\gamma') - I_X(\gamma')\big) &\geq 0, \\ \big(N_T(\gamma) + I_X(\gamma)\big)\big(N_T(\gamma') + I_X(\gamma')\big) &\geq 0, \\ \big(N_S(\gamma) - I_Y(\gamma)\big)\big(N_S(\gamma') - I_Y(\gamma')\big) &\geq 0, \\ \big(N_S(\gamma) + I_Y(\gamma)\big)\big(N_S(\gamma') + I_Y(\gamma')\big) &\geq 0. \end{aligned}$$

Proof. We shall prove that $\big(N_T(\gamma) - I_X(\gamma)\big)\big(N_T(\gamma') - I_X(\gamma')\big) \geq 0$. The other three inequalities will follow by a similar argument.

From Lemma 2.4, we have $N_T(\gamma)N_T(\gamma') \geq 0$, then

$$\big(N_T(\gamma) - I_X(\gamma)\big)\big(N_T(\gamma') - I_X(\gamma')\big) \geq 0 \quad \text{when } N_T(\gamma) \leq 0.$$

Now, consider the case where $N_T(\gamma) \geq 0$, and suppose that

$$\big(N_T(\gamma) - I_X(\gamma)\big)\big(N_T(\gamma') - I_X(\gamma')\big) < 0.$$

Without loss of generality, we assume that

$$N_T(\gamma) > I_X(\gamma) \quad \text{and} \quad 0 \leq N_T(\gamma') < I_X(\gamma').$$

There is a strand l_1 of γ joining the X-side to the T^{-1}-side, and there is a strand l_2 of γ joining the X^{-1}-side to the T^{-1}-side.

Let $m = I_X(\gamma') > 0$. There exist m strands $L_1, \dots, L_m$ of γ' with endpoints on the X^{-1}-side.

If every L_j connects the X^{-1}-side to the T^{-1}-side, then $N_T(\gamma') \geq m = I_X(\gamma')$. This is a contradiction to the assumption. Therefore, there is an integer j such that L_j connects the X^{-1}-side to the E-side with $E \neq T^{-1}$. This implies $L_j \cap (l_1 \cup l_2) \neq \emptyset$. This is impossible since γ and γ' are disjoint.

Lemma 3.7. *Let $\gamma, \gamma' \in \mathscr{G}$ be two disjoint geodesics, and let $\alpha = a\gamma \oplus b\gamma'$, where $a \geq 0$ and $b \geq 0$ are integers with $a+b>0$. Then*

$$\{I_X(\gamma) - I_Y(\gamma)\} \cdot \{I_X(\gamma') - I_Y(\gamma')\} \geq 0,$$

and thus

$$\begin{aligned} |I_X(\alpha) - I_Y(\alpha)| + I_X(\alpha) - I_Y(\alpha) &= a\{|I_X(\gamma) - I_Y(\gamma)| + I_X(\gamma) - I_Y(\gamma)\} \\ &\quad + b\{|I_X(\gamma') - I_Y(\gamma')| + I_X(\gamma') - I_Y(\gamma')\}; \\ |I_Y(\alpha) - I_X(\alpha)| + I_Y(\alpha) - I_X(\alpha) &= a\{|I_Y(\gamma) - I_X(\gamma)| + I_Y(\gamma) - I_X(\gamma)\} \\ &\quad + b\{|I_Y(\gamma') - I_X(\gamma')| + I_Y(\gamma') - I_X(\gamma')\}. \end{aligned}$$

Proof. If $\gamma \in \{\gamma_T, \gamma_S\}$ or $\gamma' \in \{\gamma_T, \gamma_S\}$, then

$$\{I_X(\gamma) - I_Y(\gamma)\} \cdot \{I_X(\gamma') - I_Y(\gamma')\} = 0.$$

In the following, we assume that $\gamma, \gamma' \in \widehat{\mathscr{G}}$.

Now, choose an integer $k>0$ such that

$$\begin{aligned} N_T\big(\mathscr{T}_2^{-k}\mathscr{T}_1^{k}(\gamma)\big) &\geq 2I_X(\gamma) = 2I_X\big(\mathscr{T}_2^{-k}\mathscr{T}_1^{k}(\gamma)\big), \\ N_S\big(\mathscr{T}_2^{-k}\mathscr{T}_1^{k}(\gamma)\big) &\geq 2I_Y(\gamma) = 2I_Y\big(\mathscr{T}_2^{-k}\mathscr{T}_1^{k}(\gamma)\big), \\ N_T\big(\mathscr{T}_2^{-k}\mathscr{T}_1^{k}(\gamma')\big) &\geq 2I_X(\gamma') = 2I_X\big(\mathscr{T}_2^{-k}\mathscr{T}_1^{k}(\gamma')\big), \\ N_S\big(\mathscr{T}_2^{-k}\mathscr{T}_1^{k}(\gamma')\big) &\geq 2I_Y(\gamma') = 2I_Y\big(\mathscr{T}_2^{-k}\mathscr{T}_1^{k}(\gamma')\big). \end{aligned}$$

Since for $E = X$ or Y

$$\begin{aligned} I_E\big(\mathscr{T}_2^{-k}\mathscr{T}_1^{k}(\alpha)\big) &= aI_E\big(\mathscr{T}_2^{-k}\mathscr{T}_1^{k}(\gamma)\big) + bI_E\big(\mathscr{T}_2^{-k}\mathscr{T}_1^{k}(\gamma')\big) \\ &= aI_E(\gamma) + bI_E(\gamma') = I_E(\alpha), \end{aligned}$$

we may assume that

$$N_T(\gamma) \geq 2I_X(\gamma), \quad N_S(\gamma) \geq 2I_Y(\gamma), \quad N_T(\gamma') \geq 2I_X(\gamma'), \quad N_S(\gamma') \geq 2I_Y(\gamma').$$

Let $\mathscr{GL}_1$ and $\mathscr{GL}_2$ be the subsets of $\mathscr{GL}$ given in the proof of Theorem 3.1. If γ and γ' both are in $\mathscr{GL}_1$, write

$$\gamma = p\gamma_S + q\gamma_T + r\tau_1 + s\tau_2 \quad \text{and} \quad \gamma' = p'\gamma_S + q'\gamma_T + r'\tau_1 + s'\tau_2.$$

Then

$$\{I_X(\gamma) - I_Y(\gamma)\} \cdot \{I_X(\gamma') - I_Y(\gamma')\} = ss' \geq 0.$$

Similarly,

$$\{I_X(\gamma) - I_Y(\gamma)\} \cdot \{I_X(\gamma') - I_Y(\gamma')\} \geq 0$$

if γ and γ' both are in $\mathscr{GL}_2$.

Finally, assume that $\gamma \in \mathscr{GL}_1$ and $\gamma' \in \mathscr{GL}_2$, and write

$$\gamma = p\gamma_S + q\gamma_T + r\tau_1 + s\tau_2 \quad \text{and} \quad \gamma' = p'\gamma_S + q'\gamma_T + r'\tau_1 + s'\tau_3.$$

If $ss' > 0$, then $i(\gamma, \gamma') > 0$. This is impossible. Thus $ss' = 0$. This implies that both γ and γ' are either in $\mathscr{GL}_1$ or in $\mathscr{GL}_2$, and completes the proof.

Proof of Proposition 3.3. It follows from equation (3) and Proposition 2.2, we have

$$i\big(\mathscr{T}_j^k(\alpha),\gamma_{11}\big)=i(\alpha,\gamma_{11}),\quad i\big(\mathscr{T}_j^k(\alpha),\gamma_{21}\big)=i(\alpha,\gamma_{21})\quad \text{for } j=1,2.$$

Since γ_{1j} is invariant under $\mathscr{T}_1$, and since γ_{2j} is invariant under $\mathscr{T}_2$ for $j=2,3$, then

$$\begin{aligned} i\big(\mathscr{T}_1^k(\alpha),\gamma_{1j}\big)&=i\big(\alpha,\mathscr{T}_1^{-k}(\gamma_{1j})\big)=i(\alpha,\gamma_{1j}),\quad \text{and}\\ i\big(\mathscr{T}_2^k(\alpha),\gamma_{2j}\big)&=i\big(\alpha,\mathscr{T}_2^{-k}(\gamma_{2j})\big)=i(\alpha,\gamma_{2j}).\end{aligned}$$

It remains to prove the last four equations given in the proposition. In the following, a and b are assumed to be non-negative integers with $a+b>0$.

If $\alpha=a\gamma_S\oplus b\gamma_T$, then α is invariant under $\mathscr{T}_j$ for $j=1,2$, and $I_E(\alpha)=0$ for $E=X$, Y. Thus the equations hold trivially.

Let $\gamma\in\widehat{\mathscr{G}}$ be a geodesic disjoint from γ_S. If $\alpha=a\gamma\oplus b\gamma_S$, then

$$I_Y(\gamma)=0=I_Y(\alpha),\quad N_S(\gamma)=0\quad\text{and}\quad N_S(\alpha)=b.$$

Since $I_Y(\gamma)=0$, then γ is invariant under $\mathscr{T}_1$, and so is α. From Corollary 3.2 and Lemma 3.7, we have

$$|I_X(\alpha)-I_Y(\alpha)|+I_X(\alpha)-I_Y(\alpha)=2aI_X(\gamma)$$

and

$$\begin{aligned} i\big(\mathscr{T}_1^k(\alpha),\gamma_{22}\big)&=ai(\gamma,\gamma_{22})+bi(\gamma_S,\gamma_{22})=2aI_X(\gamma)+2b\\ &=2|N_S(\alpha)+kI_Y(\alpha)|+|I_X(\alpha)-I_Y(\alpha)|+I_X(\alpha)-I_Y(\alpha);\\ i\big(\mathscr{T}_1^k(\alpha),\gamma_{23}\big)&=ai(\gamma,\gamma_{23})+bi(\gamma_S,\gamma_{23})=2aI_X(\gamma)+2b\\ &=2|N_S(\alpha)+(k-1)I_Y(\alpha)|+|I_X(\alpha)-I_Y(\alpha)|+I_X(\alpha)-I_Y(\alpha).\end{aligned}$$

Since γ_S is invariant under $\mathscr{T}_2$, and $i(\gamma_S,\gamma_{1j})=0$ for $j=1,2$, then

$$i\big(\mathscr{T}_2^k(\alpha),\gamma_{1j}\big)=ai\big(\mathscr{T}_2^k(\gamma),\gamma_{1j}\big).$$

Since $I_X(\alpha)=aI_X(\gamma)$, and since $N_T(\alpha)=aN_T(\gamma)$, from Corollary 3.2 and Lemma 3.5 we have

$$\begin{aligned} i\big(\mathscr{T}_2^k(\alpha),\gamma_{12}\big)&=2|N_T(\alpha)-kI_X(\alpha)|+|I_Y(\alpha)-I_X(\alpha)|+I_Y(\alpha)-I_X(\alpha);\\ i\big(\mathscr{T}_2^k(\alpha),\gamma_{13}\big)&=2|N_T(\alpha)-(k+1)I_X(\alpha)|+|I_Y(\alpha)-I_X(\alpha)|+I_Y(\alpha)-I_X(\alpha).\end{aligned}$$

By a similar argument as above, one proves that the last four equations hold for $\alpha=a\gamma\oplus b\gamma_T$, where $\gamma\in\widehat{\mathscr{G}}$ is a geodesic disjoint from γ_T.

Finally, we consider the free homotopy classes $\alpha = a\gamma \oplus b\gamma'$, where γ and γ' are disjoint geodesics in $\widehat{\mathscr{G}}$. If $ab = 0$, the equations hold trivially by Corollary 3.2.

Assume that $a > 0$ and $b > 0$. Then $I_X(\alpha)I_Y(\alpha) > 0$. Otherwise, say $I_Y(\alpha) = 0$, we have $I_Y(\gamma) = I_Y(\gamma') = 0$. This is impossible since any two distinct simple closed geodesics on a four-punctured sphere must meet (see [4, Theorem 2.5] and [4, Theorem 2.6]).

Note that the last three equations given in the proposition follow from the equation

$$i\big(\mathscr{T}_2^k(\alpha), \gamma_{12}\big) = 2|N_T(\alpha) - kI_X(\alpha)| + |I_Y(\alpha) - I_X(\alpha)| + I_Y(\alpha) - I_X(\alpha).$$

Since

$$i\big(\mathscr{T}_2^k(\alpha), \gamma_{13}\big) = i\big(\mathscr{T}_2^k(\alpha), \mathscr{T}_2^{-1}(\gamma_{12})\big) = i\big(\mathscr{T}_2^{k+1}(\alpha), \gamma_{12}\big),$$

then

$$i\big(\mathscr{T}_2^k(\alpha), \gamma_{13}\big) = 2|N_T(\alpha) - (k+1)I_X(\alpha)| + |I_Y(\alpha) - I_X(\alpha)| + I_Y(\alpha) - I_X(\alpha).$$

Because $\mathscr{T}_1^k = \Theta_2\mathscr{T}_2^k\Theta_2$, from Propositions 2.1 and 2.4 we obtain

$$\begin{aligned} i\big(\mathscr{T}_1^k(\alpha), \gamma_{22}\big) &= i\big(\Theta_2\mathscr{T}_2^k\Theta_2(\alpha), \gamma_{22}\big) = i\big(\mathscr{T}_2^k\Theta_2(\alpha), \Theta_2(\gamma_{22})\big) = i\big(\mathscr{T}_2^k\Theta_2(\alpha), \gamma_{12}\big) \\ &= 2|N_T\big(\Theta_2(\alpha)\big) - kI_X\big(\Theta_2(\alpha)\big)| + |I_Y\big(\Theta_2(\alpha)\big) - I_X\big(\Theta_2(\alpha)\big)| \\ &\quad + I_Y\big(\Theta_2(\alpha)\big) - I_X\big(\Theta_2(\alpha)\big) \\ &= 2|-N_S(\alpha) - kI_Y(\alpha)| + |I_X(\alpha) - I_Y(\alpha)| + I_X(\alpha) - I_Y(\alpha) \\ &= 2|N_S(\alpha) + kI_Y(\alpha)| + |I_X(\alpha) - I_Y\alpha)| + I_X(\alpha) - I_Y(\alpha) \end{aligned}$$

and

$$\begin{aligned} i\big(\mathscr{T}_1^k(\alpha), \gamma_{23}\big) &= i\big(\mathscr{T}_1^k(\alpha), \mathscr{T}_1(\gamma_{22})\big) = i\big(\mathscr{T}_1^{k-1}(\alpha), \gamma_{22}\big) \\ &= 2|N_S(\alpha) + (k-1)I_Y(\alpha)| + |I_X(\alpha) - I_Y(\alpha)| + I_X(\alpha) - I_Y(\alpha). \end{aligned}$$

Now, we shall prove the equation

$$i(\mathscr{T}_2^k(\alpha), \gamma_{12}) = 2|N_T(\alpha) - kI_X(\alpha)| + |I_Y(\alpha) - I_X(\alpha)| + I_Y(\alpha) - I_X(\alpha).$$

From Proposition 2.8, Lemma 3.5 and Lemma 3.7, we obtain

$$\begin{aligned} i\big(\mathscr{T}_2(\alpha), \gamma_{12}\big) &= ai\big(\mathscr{T}_2(\gamma), \gamma_{12}\big) + bi\big(\mathscr{T}_2(\gamma'), \gamma_{12}\big) \\ &= 2a|N_T\big(\mathscr{T}_2(\gamma)\big)| + 2b|N_T\big(\mathscr{T}_2(\gamma')\big)| \\ &\quad + a\big\{|I_Y\big(\mathscr{T}_2(\gamma)\big) - I_X\big(\mathscr{T}_2(\gamma)\big)| + I_Y\big(\mathscr{T}_2(\gamma)\big) - I_X\big(\mathscr{T}_2(\gamma)\big)\big\} \\ &\quad + b\big\{|I_Y\big(\mathscr{T}_2(\gamma')\big) - I_X\big(\mathscr{T}_2(\gamma')\big)| + I_Y\big(\mathscr{T}_2(\gamma')\big) - I_X\big(\mathscr{T}_2(\gamma')\big)\big\} \\ &= 2a|N_T(\gamma) - I_X(\gamma)| + 2b|N_T(\gamma') - I_X(\gamma')| \\ &\quad + a\{|I_Y(\gamma) - I_X(\gamma)| + I_Y(\gamma) - I_X(\gamma)\} \\ &\quad + b\{|I_Y(\gamma') - I_X(\gamma')| + I_Y(\gamma') - I_X(\gamma')\} \\ &= 2|a\{N_T(\gamma) - I_X(\gamma)\} + b\{N_T(\gamma') - I_X(\gamma')\}| \\ &\quad + |I_Y(\alpha) - I_X(\alpha)| + I_Y(\alpha) - I_X(\alpha) \\ &= 2|N_T(\alpha) - I_X(\alpha)| + |I_Y(\alpha) - I_X(\alpha)| + I_Y(\alpha) - I_X(\alpha). \end{aligned}$$

If $k>1$, by Lemma 3.5 we have

$$\begin{aligned} i\big(\mathscr{T}_2^k(\alpha),\gamma_{12}\big) &= 2|N_T\big(\mathscr{T}_2^{k-1}(\alpha)\big) - I_X\big(\mathscr{T}_2^{k-1}(\alpha)\big)| \\ &\quad + |I_Y\big(\mathscr{T}_2^{k-1}(\alpha)\big) - I_X\big(\mathscr{T}_2^{k-1}(\alpha)\big)| \\ &\quad + I_Y\big(\mathscr{T}_2^{k-1}(\alpha)\big) - I_X\big(\mathscr{T}_2^{k-1}(\alpha)\big) \\ &= 2|N_T(\alpha) - kI_X(\alpha)| + |I_Y(\alpha) - I_X(\alpha)| + I_Y(\alpha) - I_X(\alpha). \end{aligned}$$

By the same reasoning as above, one shows

$$i\big(\mathscr{T}_2^{-1}(\alpha),\gamma_{12}\big) = 2|N_T(\alpha) + I_X(\alpha)| + |I_Y(\alpha) - I_X(\alpha)| + I_Y(\alpha) - I_X(\alpha).$$

Thus for $k>1$

$$\begin{aligned} i\big(\mathscr{T}_2^{-k}(\alpha),\gamma_{12}\big) &= 2|N_T\big(\mathscr{T}_2^{-k+1}(\alpha)\big) - I_X\big(\mathscr{T}_2^{-k+1}(\alpha)\big)| \\ &\quad + |I_Y\big(\mathscr{T}_2^{-k+1}(\alpha)\big) + I_X\big(\mathscr{T}_2^{-k+1}(\alpha)\big)| \\ &\quad + I_Y\big(\mathscr{T}_2^{-k+1}(\alpha)\big) - I_X\big(\mathscr{T}_2^{-k+1}(\alpha)\big) \\ &= 2|N_T(\alpha) + kI_X(\alpha)| + |I_Y(\alpha) - I_X(\alpha)| + I_Y(\alpha) - I_X(\alpha). \end{aligned}$$

4. A homeomorphism of $\overline{\pi\mathscr{I}(\mathscr{G})}$ onto a 3-sphere

Now, we are ready to construct a homeomorphism of $\overline{\pi\mathscr{I}(\mathscr{G})}$ onto a 3-sphere. Let $\Pi = \{(r_1, r_2, \dots, r_6) \in \mathbf{R}_+^6 : r_1 + r_2 + \cdots + r_6 = 1\}$, and let $\mathscr{C} = \Pi_1 \cup \Pi_2 \cup \Pi_3$, where

$$\begin{aligned} \Pi_1 &= \{(r_1, r_2, r_3) \in \mathbf{R}_+^3 : r_2 + r_3 = r_1\}, \\ \Pi_2 &= \{(r_1, r_2, r_3) \in \mathbf{R}_+^3 : r_1 + r_3 = r_2\}, \\ \Pi_3 &= \{(r_1, r_2, r_3) \in \mathbf{R}_+^3 : r_1 + r_2 = r_3\}. \end{aligned}$$

Following Poénaru ([5], Exposé 4), we shall first construct a function Ψ of $\mathscr{I}(\mathscr{GL})$ into $(\mathscr{C}\times\mathscr{C})\cap\Pi$ so that its extension to $\pi^{-1}\pi\mathscr{I}(\mathscr{GL})$ satisfies

$$\Psi(t\,\mathrm{I}_\alpha) = \Psi(\mathrm{I}_\alpha) \quad \text{for } \alpha\in\mathscr{GL} \text{ and for } t>0.$$

Thus Ψ induces a function on $\pi\mathscr{I}(\mathscr{GL})$, also denoted by Ψ.

By using a continuity argument, we extend Ψ to $\overline{\pi\mathscr{I}(\mathscr{G})}$, and prove that Ψ is a homeomorphism of $\overline{\pi\mathscr{I}(\mathscr{G})}$ onto a 3-sphere lying in $\mathbf{R}^6$ (Theorem 4.3). Finally, by postcomposing Ψ by a function from $\mathbf{R}^6$ into $\mathbf{R}^4$, we will get a homeomorphism of $\overline{\pi\mathscr{I}(\mathscr{G})}$ into a 3-sphere lying in $\mathbf{R}^4$ (Theorem 4.4).

4.1. The definition of Ψ on $\mathscr{GL}$. For integers $i\in\{1,2\}$ and $j\in\{1,2,3\}$, and for $\alpha\in\mathscr{GL}$, let

$$x_{ij}(\alpha) = \frac{i(\alpha,\gamma_{ij})}{\lambda(\alpha)}, \quad \text{where} \quad \lambda(\alpha) = \sum_{i=1}^{2}\sum_{j=1}^{3} i(\alpha,\gamma_{ij}),$$

and let $\psi_1\colon \mathscr{GL} \longrightarrow \mathbf{R}_+^6$ be defined by

$$\psi_1(\alpha) = \big(x_{11}(\alpha), x_{12}(\alpha), x_{13}(\alpha), x_{21}(\alpha), x_{22}(\alpha), x_{23}(\alpha)\big).$$

Note that the image of ψ_1 lies in Π since $\sum_{i=1}^{2}\sum_{j=1}^{3} x_{ij}(\alpha) = 1$ for all $\alpha \in \mathscr{GL}$.

To construct a function of $\mathscr{GL}$ into $(\mathscr{C} \times \mathscr{C}) \cap \Pi$, we form the sum

$$\rho(\alpha) = 2\{I_X(\alpha)+I_Y(\alpha)+|N_T(\alpha)|+|N_T(\alpha)-I_X(\alpha)|+|N_S(\alpha)|+|N_S(\alpha)-I_Y(\alpha)|\}.$$

From Corollary 3.4, we have $0 < \rho(\alpha) \le \lambda(\alpha)$ for all $\alpha \in \mathscr{GL}$, and

$$\frac{\rho(\alpha)}{\lambda(\alpha)} = 1 - \frac{4|I_X(\alpha) - I_Y(\alpha)|}{\lambda(\alpha)} = 1 - 2|x_{11}(\alpha) - X_{21}(\alpha)|.$$

Thus $|x_{11}(\alpha) - x_{21}(\alpha)| < \frac{1}{2}$ for all $\alpha \in \mathscr{GL}$, and the image of ψ_1 is contained in the set $\mathscr{E} = \{(r_1, r_2, r_3, r_4, r_5, r_6) \in \Pi : |r_1 - r_4| < \frac{1}{2}\}$.

Let

$$\mathscr{E}^+ = \{(r_1, r_2, r_3, r_4, r_5, r_6) \in \Pi : 0 \le r_1 - r_4 < \tfrac{1}{2}\} \quad \text{and}$$
$$\mathscr{E}^- = \{(r_1, r_2, r_3, r_4, r_5, r_6) \in \Pi : 0 \le r_4 - r_1 < \tfrac{1}{2}\}.$$

Let $\psi_2\colon \mathscr{E} \longrightarrow \mathbf{R}^6$ be defined by $\psi_2(r_1, r_2, r_3, r_4, r_5, r_6) = (t_1, t_2, t_3, t_4, t_5, t_6)$, where

$$t_j = \begin{cases} \dfrac{r_j}{1-2(r_1-r_4)} & \text{for } j = 1,2,3,4 \text{ and } (r_1,r_2,r_3,r_4,r_5,r_6) \in \mathscr{E}^+, \\ \dfrac{r_j - r_1 + r_4}{1-2(r_1-r_4)} & \text{for } j = 5,6 \text{ and } (r_1,r_2,r_3,r_4,r_5,r_6) \in \mathscr{E}^+, \\ \dfrac{r_j}{1-2(r_4-r_1)} & \text{for } j = 1,4,5,6 \text{ and } (r_1,r_2,r_3,r_4,r_5,r_6) \in \mathscr{E}^-, \\ \dfrac{r_j + r_1 - r_4}{1-2(r_4-r_1)} & \text{for } j = 2,3 \text{ and } (r_1,r_2,r_3,r_4,r_5,r_6) \in \mathscr{E}^-. \end{cases}$$

It is clear that ψ_2 is continuous on $\mathscr{E}$ with

$$\psi_2(\mathscr{E}^+) \subset \Pi^+ = \{(t_1,t_2,t_3,t_4,t_5,t_6) \in \Pi : t_1 \ge t_4\} \quad \text{and}$$
$$\psi_2(\mathscr{E}^-) \subset \Pi^- = \{(t_1,t_2,t_3,t_4,t_5,t_6) \in \Pi : t_1 \le t_4\}.$$

A direct computation proves that ψ_2 is an injective function onto Π with the inverse $\psi_2^{-1}(t_1,t_2,t_3,t_4,t_5,t_6) = (r_1,r_2,r_3,r_4,r_5,r_6) \in \mathscr{E}$, where

$$r_j = \begin{cases} \dfrac{t_j}{1+2(t_1-t_4)} & \text{for } j = 1,2,3,4, \text{ and } (t_1,t_2,t_3,t_4,t_5,t_6) \in \Pi^+, \\ \dfrac{t_j + t_1 - t_4}{1+2(t_1-t_4)} & \text{for } j = 5,6, \text{ and } (t_1,t_2,t_3,t_4,t_5,t_6) \in \Pi^+, \\ \dfrac{t_j}{1+2(t_4-t_1)} & \text{for } j = 1,4,5,6, \text{ and } (t_1,t_2,t_3,t_4,t_5,t_6) \in \Pi^-, \\ \dfrac{t_j - t_1 + t_4}{1+2(t_4-t_1)} & \text{for } j = 2,3, \text{ and } (t_1,t_2,t_3,t_4,t_5,t_6) \in \Pi^-. \end{cases}$$

This proves that ψ_2 is a homeomorphism of $\mathscr{E}$ onto Π.

Let Ψ be the composition of ψ_1 followed by ψ_2. We shall prove that Ψ maps $\mathscr{GL}$ into $\Delta = (\mathscr{C} \times \mathscr{C}) \cap \Pi$. For $\alpha \in \mathscr{GL}$, write

$$\begin{aligned}&\big(\xi_{11}(\alpha), \xi_{12}(\alpha), \xi_{13}(\alpha), \xi_{21}(\alpha), \xi_{22}(\alpha), \xi_{23}(\alpha)\big)\\ &\qquad = \psi_2\big(x_{11}(\alpha), x_{12}(\alpha), x_{13}(\alpha), x_{21}(\alpha), x_{22}(\alpha), x_{23}(\alpha)\big).\end{aligned}$$

From the definition of $\rho(\alpha)$, we have

$$\begin{aligned}\xi_{11}(\alpha) &= \frac{2I_X(\alpha)}{\rho(\alpha)}, & \xi_{12}(\alpha) &= \frac{2|N_T(\alpha)|}{\rho(\alpha)}, & \xi_{13}(\alpha) &= \frac{2|N_T(\alpha) - I_X(\alpha)|}{\rho(\alpha)},\\ \xi_{21}(\alpha) &= \frac{2I_Y(\alpha)}{\rho(\alpha)}, & \xi_{22}(\alpha) &= \frac{2|N_S(\alpha)|}{\rho(\alpha)}, & \xi_{23}(\alpha) &= \frac{2|N_S(\alpha) - I_Y(\alpha)|}{\rho(\alpha)}.\end{aligned}$$

For simplicity, write $N_T = N_T(\alpha)$, $N_S = N_S(\alpha)$, $I_X = I_X(\alpha)$, $I_Y = I_Y(\alpha)$, and $\xi_{ij} = \xi_{ij}(\alpha)$ for all $\alpha \in \mathscr{GL}$. Then

$$\begin{aligned} N_T \le 0 &\implies \xi_{11} + \xi_{12} = \xi_{13}, & N_S \le 0 &\implies \xi_{21} + \xi_{22} = \xi_{23},\\ 0 \le N_T \le I_X &\implies \xi_{11} - \xi_{12} = \xi_{13}, & 0 \le N_S \le I_Y &\implies \xi_{21} - \xi_{22} = \xi_{23},\\ N_T \ge I_X &\implies -\xi_{11} + \xi_{12} = \xi_{13}, & N_S \ge I_Y &\implies -\xi_{21} + \xi_{22} = \xi_{23}.\end{aligned}$$

Therefore, $\Psi(\mathscr{GL}) \subset \Delta$.

4.2. A homeomorphism of Δ onto a 3-sphere. In this subsection, we shall prove that $\Delta = (\mathscr{C} \times \mathscr{C}) \cap \Pi$ is homeomorphic to a 3-sphere.

Let A be the invertible linear transformation of $\mathbf{R}^3$ onto itself carrying the vectors $(1,0,1)$, $(1,1,0)$ and $(0,1,1)$ to the vectors $(1,0,1)$, $\big(-\frac12, \frac12\sqrt3, 1\big)$ and $\big(-\frac12, -\frac12\sqrt3, 1\big)$ in this order. The matrix representation of A is

$$A = \begin{pmatrix} \frac12 & -1 & \frac12\\ \frac12\sqrt3 & 0 & -\frac12\sqrt3\\ \frac12 & \frac12 & \frac12\end{pmatrix} \quad \text{with the inverse} \quad A^{-1} = \begin{pmatrix} \frac13 & \sqrt3^{-1} & \frac23\\ \frac{-2}{3} & 0 & \frac23\\ \frac13 & -\sqrt3^{-1} & \frac23\end{pmatrix}.$$

Let $\mathscr{C}' = A(\mathscr{C})$. Note that if $(x_1, x_2, x_3) = A(r_1, r_2, r_3) \in \mathscr{C}'$, then $x_3 \ge 0$. Let

$$\begin{aligned} L_1 &= \{(t, 0, t) \in \mathbf{R}^3 : t \ge 0\},\\ L_2 &= \big\{\big(-\tfrac12 t, \tfrac12\sqrt3\, t, t\big) \in \mathbf{R}^3 : t \ge 0\big\} \quad \text{and}\\ L_3 &= \big\{\big(-\tfrac12 t, -\tfrac12\sqrt3\, t, t\big) \in \mathbf{R}^3 : t \ge 0\big\}.\end{aligned}$$

By a direct computation, one proves easily that $\Pi_1' = A(\Pi_1)$ lies on the plane $x_1 + \sqrt3\, x_2 = x_3$ bounded by L_1 and L_2, $\Pi_2' = A(\Pi_2)$ lies on the plane $2x_1 + x_3 = 0$

bounded by L_2 and L_3, and $\Pi'_3 = A(\Pi_3)$ lies on the plane $\sqrt{3}\,x_2 + x_3 = x_1$ bounded by L_1 and L_3. By the definition, $\mathscr{C}' = \Pi'_1 \cup \Pi'_2 \cup \Pi'_3$. Let J be the linear transformation of $\mathbf{R}^6$ onto itself represented by the following matrix

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Then J is a homeomorphism of $\mathbf{R}^6$ onto itself with

$$\Pi' = J(\Pi) = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbf{R}^6 : x_3 + x_6 = \tfrac{1}{2}\},$$

and $J(\Delta) = (\mathscr{C}' \times \mathscr{C}') \cap \Pi' = \Delta'$.

It is clear that the orthogonal projection η: $\mathbf{R}^3 \longrightarrow \mathbf{R}^2$ defined by

$$\eta(x_1, x_2, x_3) = (x_1, x_2)$$

restricted to $\mathscr{C}'$ is a homeomorphism onto $\mathbf{R}^2$. Then the projection ϕ: $\mathbf{R}^6 \longrightarrow \mathbf{R}^4$ defined by

$$\phi(x_1, x_2, x_3, x_4, x_5, x_6) = \big(\eta(x_1, x_2, x_3), \eta(x_4, x_5, x_6)\big)$$

restricted to $\mathscr{C}' \times \mathscr{C}'$ is a homeomorphism onto $\mathbf{R}^2 \times \mathbf{R}^2 \cong \mathbf{R}^4$. Let

$$\mathbf{B} = (\mathscr{C}' \times \mathscr{C}') \cap \{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbf{R}^6 : x_3 + x_6 \le \tfrac{1}{2}\}.$$

Now, we shall prove that $\phi(\mathbf{B})$ is bounded and convex, and has non-empty interior. This implies that $\phi(\mathbf{B})$ is homeomorphic to the closed unit ball

$$\{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 \le 1\}.$$

By the definition of $\mathbf{B}$, as a subspace of $\mathscr{C}' \times \mathscr{C}'$, the boundary of $\mathbf{B}$ is Δ', then $\phi(\Delta')$ is homeomorphic to a 3-sphere, and so is Δ.

Let R be the rotation in $\mathbf{R}^3$ with the matrix representation

$$\begin{pmatrix} \cos\frac{2}{3}\pi & -\sin\frac{2}{3}\pi & 0 \\ \sin\frac{2}{3}\pi & \cos\frac{2}{3}\pi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\Pi'_j \times \Pi'_k = R^{j-1}(\Pi'_1) \times R^{k-1}(\Pi'_1) = (R^{j-1} \times R^{k-1})(\Pi'_1 \times \Pi'_1)$$

for $j, k \in \{1, 2, 3\}$, where $R^{j-1} \times R^{k-1}$ is the linear transformation of $\mathbf{R}^6$ onto itself represented by the following matrix

$$\begin{pmatrix} R^{j-1} & 0 \\ 0 & R^{k-1} \end{pmatrix}.$$

It easy to see that

$$(R^{j-1} \times R^{k-1})(0,0,r,0,0,s) = (0,0,r,0,0,s)$$

for any two real numbers r and s. Since the normal vector $(0,0,1,0,0,1)$ of Π' is invariant under $R^{j-1} \times R^{k-1}$, and since the point $(0,0,\frac{1}{4},0,0,\frac{1}{4})$ of Π' is fixed by $R^{j-1} \times R^{k-1}$, then Π' is invariant under $R^{j-1} \times R^{k-1}$, and thus

$$\phi(\mathbf{B}) = \bigcup_{j=1}^{3} \bigcup_{k=1}^{3} \phi\big((R^{j-1} \times R^{k-1})(V)\big),$$

where

$$\begin{aligned} V &= \{(x_1,x_2,x_3,x_4,x_5,x_6) \in \Pi_1' \times \Pi_1' : x_3 + x_6 \le \tfrac{1}{2}\} \\ &= \{(x_1,x_2,x_3,x_4,x_5,x_6) \in \Pi_1' \times \Pi_1' : x_1 + \sqrt{3}\,x_2 + x_4 + \sqrt{3}\,x_5 \le \tfrac{1}{2}\}. \end{aligned}$$

Clearly, V is bounded. This proves that $\phi(\mathbf{B})$ is bounded since $R^{j-1} \times R^{k-1}$ is a Euclidean isometry.

To prove the convexity of $\phi(\mathbf{B})$, we consider any two distinct points Q and Q' of $\mathbf{B}$ with coordinates $(x_1,x_2,x_3,x_4,x_5,x_6)$ and $(x_1',x_2',x_3',x_4',x_5',x_6')$ respectively. Let

$$P_1 = (x_1,x_2,x_3), \quad P_2 = (x_4,x_5,x_6), \quad P_1' = (x_1',x_2',x_3') \quad \text{and} \quad P_2' = (x_4',x_5',x_6'),$$

and let $\overline{P_jP_j'}$ denote the line segment connecting P_j to P_j' for $j = 1,2$. The vertical plane in $\mathbf{R}^3$ containing $\overline{P_jP_j'}$ intersects $\mathscr{C}'$ in a polygonal curve σ_j with parametric equation $f_j(t)$, $0 \le t \le 1$, so that $f_j(0) = P_j$ and $f_j(1) = P_j'$. Note that $\eta(\sigma_j) = \eta(\overline{P_jP_j'})$. The curve

$$L = \{(f_1(t), f_2(t)) \in \mathbf{R}^3 \times \mathbf{R}^3 : 0 \le t \le 1\}$$

lies on $\mathscr{C}' \times \mathscr{C}'$ connecting Q to Q', and $\phi(L)$ is a line segment in $\phi(\mathbf{B})$ with $\phi(Q)$ and $\phi(Q')$ as its endpoints. Therefore, $\phi(\mathbf{B})$ is convex.

Note that $(\Pi_1' \times \Pi_1') \cap \Pi'$ is contained in the hyperplane in $\mathbf{R}^6$ of equation

$$x_1 + \sqrt{3}\,x_2 + x_4 + \sqrt{3}\,x_5 = \tfrac{1}{2},$$

then the distance from the origin to $(\Pi_1' \times \Pi_1') \cap \Pi'$ is at least $1/4\sqrt{2}$. This implies that $\phi(\mathbf{B})$ contains the closed ball centered at the origin with radius $1/4\sqrt{2}$, and $\phi(\mathbf{B})$ has non-empty interior. The proof is complete.

4.3. The extension of Ψ to $\overline{\pi\mathscr{I}(\mathscr{G})}$. Now, we are going to extend the map Ψ to $\overline{\pi\mathscr{I}(\mathscr{G})} = \overline{\pi\mathscr{I}(\mathscr{G}\mathscr{L})}$.

For every $\alpha \in \mathscr{G}\mathscr{L}$, we define $x_{ij}(\mathrm{I}_\alpha) = x_{ij}(\alpha)$. Since each x_{ij} is homogeneous, then x_{ij} extends naturally to $\pi^{-1}\pi\mathscr{I}(\mathscr{G}\mathscr{L})$ defined by $x_{ij}(t\,\mathrm{I}_\alpha) = x_{ij}(\mathrm{I}_\alpha)$ for all $t > 0$ and for all $\alpha \in \mathscr{G}\mathscr{L}$. Thus each x_{ij} induces a well-defined map, also denoted by x_{ij}, on $\pi\mathscr{I}(\mathscr{G}\mathscr{L})$ defined by $x_{ij}\big(\pi(\mathrm{I}_\alpha)\big) = x_{ij}(\mathrm{I}_\alpha)$.

For an arbitrary $\mathscr{L} \in \pi^{-1}\overline{\pi\mathscr{I}(\mathscr{G})}$, there is a sequence $\{t_n\}_{n=1}^\infty$ of positive numbers, and there is a sequence $\{\gamma_n\}_{n=1}^\infty$ in $\mathscr{G}$ such that $\{t_n\,\mathrm{I}_{\gamma_n}\}_{n=1}^\infty$ converges to $\mathscr{L}$. Thus

$$t_n i(\gamma_n, \gamma_{ij}) = t_n\,\mathrm{I}_{\gamma_n}(\gamma_{ij}) \to \mathscr{L}(\gamma_{ij})$$

as $n \to \infty$ for $i = 1, 2$ and for $j = 1, 2, 3$. This implies

$$\lim_{n\to\infty} x_{ij}(t_n\,\mathrm{I}_{\gamma_n}) = \frac{\mathscr{L}(\gamma_{ij})}{\sum_{k=1}^2 \sum_{l=1}^3 \mathscr{L}(\gamma_{kl})}$$

for $i = 1, 2$ and for $j = 1, 2, 3$. Let $\lambda\colon \pi^{-1}\overline{\pi\mathscr{I}(\mathscr{G})} \longrightarrow \mathbf{R}_+$ be defined by

$$\lambda(\mathscr{L}) = \sum_{k=1}^{2}\sum_{l=1}^{3} \mathscr{L}(\gamma_{kl}) \quad \text{for all } \mathscr{L} \in \pi^{-1}\overline{\pi\mathscr{I}(\mathscr{G})},$$

and let $x_{ij}\colon \pi^{-1}\overline{\pi\mathscr{I}(\mathscr{G})} \longrightarrow \mathbf{R}_+$ be defined by

$$x_{ij}(\mathscr{L}) = \frac{\mathscr{L}(\gamma_{ij})}{\sum_{k=1}^2 \sum_{l=1}^3 \mathscr{L}(\gamma_{kl})} \quad \text{for all } \mathscr{L} \in \pi^{-1}\overline{\pi\mathscr{I}(\mathscr{G})}.$$

It is easy to see that each x_{ij} is continuous on $\pi^{-1}\overline{\pi\mathscr{I}(\mathscr{G})}$ with $x_{ij}(t\mathscr{L}) = x_{ij}(\mathscr{L})$ for all $t > 0$ and for all $\mathscr{L} \in \pi^{-1}\overline{\pi\mathscr{I}(\mathscr{G})}$.

Since the restriction of π to $\pi^{-1}\overline{\pi\mathscr{I}\mathscr{G})}$ is a quotient map onto $\overline{\pi\mathscr{I}(\mathscr{G})}$, then each x_{ij} extends to $\overline{\pi\mathscr{I}(\mathscr{G})}$ a continuos map given by $x_{ij}\big(\pi(\mathscr{L})\big) = x_{ij}(\mathscr{L})$ for $\mathscr{L}$ in $\pi^{-1}\overline{\pi\mathscr{I}(\mathscr{G})}$. This gives a continuous map of $\overline{\pi\mathscr{I}(\mathscr{G})}$ into $\mathbf{R}_+^6$ whose restriction to $\mathscr{G}\mathscr{L}$ is ψ_1. We also use ψ_1 for this continuous map on $\overline{\pi\mathscr{I}(\mathscr{G})}$, and let $\Psi = \psi_2\psi_1$ as before.

Proposition 4.1. *The function Ψ maps $\overline{\pi\mathscr{I}(\mathscr{G})}$ continuously onto Δ.*

Clearly, Ψ is a continuous map of $\overline{\pi\mathscr{I}(\mathscr{G})}$ into Π. Since $\Psi(\mathscr{G}) \subset \Delta$, and since Δ is closed in $\mathbf{R}^6$, then $\Psi\big(\overline{\pi\mathscr{I}(\mathscr{G})}\big) \subset \Delta$.

To complete the proof of Proposition 4.1, we have to show that $\Psi\big(\pi\mathscr{I}(\mathscr{G}\mathscr{L})\big)$ is dense in Δ since Ψ is continuous and $\overline{\pi\mathscr{I}(\mathscr{G})} = \overline{\pi\mathscr{I}(\mathscr{G}\mathscr{L})}$ is compact.

A point $(r_1, r_2, r_3, r_4, r_5, r_6)$ of $\mathbf{Q}^6$ will be called a *rational point*, where $\mathbf{Q}$ is the set of all rational numbers.

Lemma 4.2. *Every rational point of* $\Pi \cap (\Pi_2 \times \Pi_2)$ *lies in* $\Psi\big(\pi\mathscr{I}(\mathscr{G}\mathscr{L})\big)$.

Proof. Let $(v_1/u, v_2/u, v_3/u, v_4/u, v_5/u, v_6/u)$ be any rational point of $(\Pi_2 \times \Pi_2) \cap \Pi$, where $u > 0$ and all $v_j \geq 0$ are even integers. Note that

$$2(v_2 + v_5) = u, \quad v_1 + v_3 = v_2 \quad \text{and} \quad v_4 + v_6 = v_5.$$

We want to show that there are non-negative integers a, b, c and d with $a+b+c+d > 0$ such that

$$\left(\frac{v_1}{u}, \frac{v_2}{u}, \frac{v_3}{u}, \frac{v_4}{u}, \frac{v_5}{u}, \frac{v_6}{u}\right) = \begin{cases} \Psi\big(\mathscr{T}_1^{-1}\mathscr{T}_2(a\gamma_T + b\gamma_S + c\tau_1 + d\tau_2)\big) & \text{if } v_1 \geq v_4, \\ \Psi\big(\mathscr{T}_1^{-1}\mathscr{T}_2(a\gamma_T + b\gamma_S + c\tau_1 + d\tau_3)\big) & \text{if } v_1 \leq v_4, \end{cases}$$

where τ_1, τ_2 and τ_3 are the geodesics given in the proof of Theorem 3.1.

Let $\alpha = \mathscr{T}_1^{-1}\mathscr{T}_2(a\gamma_T + b\gamma_S + c\tau_1 + d\tau_2)$. From Proposition 3.3 and Corollary 3.4, we have

$$\begin{aligned} I_X(\alpha) &= c + d, \\ N_T(\alpha) &= a + c + d, \\ I_Y(\alpha) &= c, \\ N_S(\alpha) &= b + c, \quad \text{and} \\ \rho(\alpha) &= 2(2a + 2b + 4c + 2d). \end{aligned}$$

If $v_1 \geq v_4$, by solving the following equations for a, b, c and d

$$\begin{aligned} 2(c+d) &= 2I_X(\alpha) = v_1, \\ 2(a+c+d) &= 2N_T(\alpha) = v_2, \\ 2c &= 2I_Y(\alpha) = v_4, \\ 2(b+c) &= 2N_S(\alpha) = v_5, \end{aligned}$$

we have

$$a = \tfrac{1}{2}(v_2 - v_1), \quad b = \tfrac{1}{2}(v_5 - v_4), \quad c = \tfrac{1}{2}v_4 \quad \text{and} \quad d = \tfrac{1}{2}(v_1 - v_4).$$

A direct computation gives $\rho(\alpha) = 2(v_2 + v_5) = u$,

$$2|N_T(\alpha) - I_X(\alpha)| = v_2 - v_1 = v_3 \quad \text{and} \quad 2|N_S(\alpha) - I_Y(\alpha)| = v_5 - v_4 = v_6.$$

This proves

$$\Psi(\alpha) = \left(\frac{v_1}{u}, \frac{v_2}{u}, \frac{v_3}{u}, \frac{v_4}{u}, \frac{v_5}{u}, \frac{v_6}{u}\right).$$

Next, assume that $v_1 \leq v_4$. Let α be given as above such that

$$\Psi(\alpha) = \left(\frac{v_4}{u}, \frac{v_5}{u}, \frac{v_6}{u}, \frac{v_1}{u}, \frac{v_2}{u}, \frac{v_3}{u}\right).$$

Since $\mathscr{T}_2\Theta_2 = \Theta_2\mathscr{T}_1$, $\Theta_1\mathscr{T}_1^{-1} = \mathscr{T}_1\Theta_1$ and $\Theta_1\mathscr{T}_2^{-1} = \mathscr{T}_2\Theta_1$, then

$$\begin{aligned}\mathscr{T}_1^{-1}\mathscr{T}_2(a\gamma_T + b\gamma_S + c\tau_1 + d\tau_3) &= \mathscr{T}_1^{-1}\mathscr{T}_2\Theta_1\Theta_2(a\gamma_T + b\gamma_S + c\tau_1 + d\tau_2)\\ &= \Theta_1\Theta_2\mathscr{T}_2\mathscr{T}_1^{-1}(a\gamma_T + b\gamma_S + c\tau_1 + d\tau_2)\\ &= \Theta_1\Theta_2\mathscr{T}_1^{-1}\mathscr{T}_2(a\gamma_T + b\gamma_S + c\tau_1 + d\tau_3)\\ &= \Theta_1\Theta_2(\alpha).\end{aligned}$$

Let $\beta = \Theta_1\Theta_2(\alpha)$. It follows immediately from Proposition 2.1 that

$$I_X(\beta) = I_Y(\alpha), \quad I_Y(\beta) = I_X\alpha), \quad N_T(\beta) = N_S(\alpha) \quad \text{and} \quad N_S(\beta) = N_T(\alpha)$$

and

$$\Psi(\beta) = \big(\xi_{21}(\alpha), \xi_{22}(\alpha), \xi_{23}(\alpha), \xi_{11}(\alpha), \xi_{12}(\alpha), \xi_{13}(\alpha)\big) = \left(\frac{v_1}{u}, \frac{v_2}{u}, \frac{v_3}{u}, \frac{v_4}{u}, \frac{v_5}{u}, \frac{v_6}{u}\right).$$

Proof of Proposition 4.1. We shall prove that $\Psi\big(\pi\mathscr{I}(\mathscr{GL})\big)$ is dense in Δ by showing that every rational point of Δ is in $\Psi\big(\pi\mathscr{I}(\mathscr{GL})\big)$, and this completes the proof.

Let $\zeta = (v_1/u, v_2/u, v_3/u, v_4/u, v_5/u, v_6/u)$ be an arbitrary rational point of Δ, where $u > 0$ and all $v_j \geq 0$ are even integers. There are non-negative integers m and n such that

$$mv_1 \leq v_2 < (m+1)v_1 \quad \text{and} \quad nv_4 \leq v_5 < (n+1)v_4.$$

Let

$$\zeta_1 = \left(\frac{v_1}{u}, \frac{v_2}{u}, \frac{v_3}{u}\right) \quad \text{and} \quad \zeta_2 = \left(\frac{v_4}{u}, \frac{v_5}{u}, \frac{v_6}{u}\right).$$

Set $v_j' = v_j$ for $j = 1,\ 4$, set

$$v_2' = \begin{cases} v_2 + (m+1)v_1 & \text{if } \zeta_1 \in \Pi_1 \cup \Pi_2,\\ -v_2 + (m+1)v_1 & \text{if } \zeta_1 \in \Pi_3,\end{cases}$$

$$v_3' = \begin{cases} v_2 + mv_1 & \text{if } \zeta_1 \in \Pi_1 \cup \Pi_2,\\ -v_2 + mv_1 & \text{if } \zeta_1 \in \Pi_3,\end{cases}$$

$$v_5' = \begin{cases} v_5 + (n+1)v_4 & \text{if } \zeta_2 \in \Pi_1 \cup \Pi_2,\\ -v_5 + (n+1)v_4 & \text{if } \zeta_2 \in \Pi_3,\end{cases}$$

$$v_6' = \begin{cases} v_5 + nv_4 & \text{if } \zeta_2 \in \Pi_1 \cup \Pi_2,\\ -v_5 + nv_4 & \text{if } \zeta_2 \in \Pi_3,\end{cases}$$

and set $w = \sum_{j=1}^{6} v_j'$. Then $w > 0$ and all $v_j' \geq 0$ are even integers,

$$|v_2' - (m+1)v_1| = v_2, \quad |v_5' - (n+1)v_4| = v_5,$$

and

$$|v_2' - (m+2)v_1| = \begin{cases} |v_2 - v_1| = v_3 & \text{if } \zeta_1 \in \Pi_1 \cup \Pi_2, \\ |-v_2 - v_1| = v_3 & \text{if } \zeta_1 \in \Pi_3, \end{cases}$$

$$|v_5' - (n+2)v_4| = \begin{cases} |v_5 - v_4| = v_6 & \text{if } \zeta_2 \in \Pi_1 \cup \Pi_2, \\ |-v_5 - v_4| = v_6 & \text{if } \zeta_2 \in \Pi_3. \end{cases}$$

As $v_2' = v_1' + v_3'$ and $v_5' = v_4' + v_6'$, the point $(v_1'/w, v_2'/w, v_3'/w, v_4'/w, v_5'/w, v_6'/w)$ is a rational point in $\Pi \cap (\Pi_2 \times \Pi_2)$.

From the proof Lemma 4.2 we know that there is an $\alpha \in \mathscr{GL}$ with $N_T(\alpha) \geq I_X(\alpha)$ and $N_S(\alpha) \geq I_Y(\alpha)$ such that

$$\begin{aligned} 2I_X(\alpha) &= v_1', \\ 2N_T(\alpha) &= v_2', \\ 2\{N_T(\alpha) - I_X(\alpha)\} &= v_3', \\ 2I_Y(\alpha) &= v_4', \\ 2N_S(\alpha) &= v_5', \\ 2\{N_S(\alpha) - I_Y(\alpha)\} &= v_6'. \end{aligned}$$

Let $\alpha' = \mathscr{T}_2^{m+1}\mathscr{T}_1^{-n-1}(\alpha)$. From Lemma 3.5,

$$\begin{aligned} 2I_X(\alpha') &= 2I_X(\alpha) = v_1, \\ 2I_Y(\alpha') &= 2I_Y(\alpha) = v_4, \\ 2|N_T(\alpha')| &= |2\{N_T(\alpha) - (m+1)I_X(\alpha)\}| = |v_2' - (m+1)v_1| = v_2, \\ 2|N_T(\alpha') - I_X(\alpha')| &= |2\{N_T(\alpha) - (m+2)I_X(\alpha)\}| = |v_2' - (m+2)v_1| = v_3, \\ 2|N_S(\alpha')| &= |2\{N_S(\alpha) - (n+1)I_Y(\alpha)\}| = |v_5' - (n+1)v_4| = v_5, \\ 2|N_S(\alpha') - I_Y(\alpha')| &= |2\{N_S(\alpha) - (n+2)I_Y(\alpha)\}| = |v_5' - (n+2)v_4| = v_6. \end{aligned}$$

Thus $\Psi(\alpha') = \zeta$.

4.4. The injectivity of Ψ. So far, we have proved that Ψ maps $\overline{\pi\mathscr{I}(\mathscr{G})}$ onto the 3-sphere Δ. Next, we shall prove that Ψ is injective on $\overline{\pi\mathscr{I}(\mathscr{G})}$. This proves the following theorem.

Theorem 4.3. *The map Ψ is a homeomorphism of $\overline{\pi\mathscr{I}(\mathscr{G})}$ onto Δ, and then $\overline{\pi\mathscr{I}(\mathscr{G})}$ is homeomorphic to a 3-sphere.*

Since ψ_2 is a homeomorphism of $\mathscr{E}$ onto Π, it remains to show that ψ_1 is injective on $\overline{\pi\mathscr{I}(\mathscr{G})}$.

Let $\mathscr{L}_1, \mathscr{L}_2 \in \pi^{-1}\overline{\pi\mathscr{I}(\mathscr{G})}$ with $\psi_1(\pi(\mathscr{L}_1)) = \psi_1(\pi(\mathscr{L}_2))$. There exist sequences $\{t_n\}$ and $\{s_n\}$ of positive numbers, and there exist sequences $\{\alpha_n\}$ and $\{\beta_n\}$ of elements in $\mathscr{G}$ such that

$$\lim_{n\to\infty} t_n \,\mathrm{I}_{\alpha_n} = \mathscr{L}_1 \quad \text{and} \quad \lim_{n\to\infty} s_n \,\mathrm{I}_{\beta_n} = \mathscr{L}_2.$$

Set $p = \lambda(\mathscr{L}_1)/\lambda(\mathscr{L}_2)$. By assumption, for $i = 1, 2$ and for $j = 1, 2, 3$, we have

$$\mathscr{L}_1(\gamma_{ij}) = p\mathscr{L}_2(\gamma_{ij}), \quad \text{or, equivalently,} \quad \lim_{n\to\infty} t_n \, \mathrm{I}_{\alpha_n}(\gamma_{ij}) = \lim_{n\to\infty} ps_n \, \mathrm{I}_{\beta_n}(\gamma_{ij}).$$

We shall complete the proof by showing that

$$\lim_{n\to\infty} t_n \, \mathrm{I}_{\alpha_n}(\gamma) = \lim_{n\to\infty} ps_n \, \mathrm{I}_{\beta_n}(\gamma) \quad \text{for all } \gamma \in \mathscr{G}.$$

Since

$$\lim_{n\to\infty} t_n I_X(\alpha_n) = \lim_{n\to\infty} t_n \, \mathrm{I}_{\alpha_n}(\gamma_{11}) = \lim_{n\to\infty} ps_n \, \mathrm{I}_{\beta_n}(\gamma_{11}) = \lim_{n\to\infty} ps_n I_X(\beta_n), \quad \text{and}$$
$$\lim_{n\to\infty} t_n I_Y(\alpha_n) = \lim_{n\to\infty} t_n \, \mathrm{I}_{\alpha_n}(\gamma_{21}) = \lim_{n\to\infty} ps_n \, \mathrm{I}_{\beta_n}(\gamma_{21}) = \lim_{n\to\infty} ps_n I_Y(\beta_n),$$

then, by using the geometric intersection formula, we only have to show that

$$\lim_{n\to\infty} t_n |I_X(\alpha_n)N_T(\gamma) - I_X(\gamma)N_T(\alpha_n)| = \lim_{n\to\infty} ps_n |I_X(\beta_n)N_T(\gamma) - I_X(\gamma)N_T(\beta_n)|$$

and

$$\lim_{n\to\infty} t_n |I_Y(\alpha_n)N_S(\gamma) - I_Y(\gamma)N_S(\alpha_n)| = \lim_{n\to\infty} ps_n |I_Y(\beta_n)N_S(\gamma) - I_Y(\gamma)N_S(\beta_n)|.$$

To simplify notation, set $A_n = t_n I_X(\alpha_n)$, $B_n = ps_n I_X(\beta_n)$, $C_n = t_n N_T(\alpha_n)$, $D_n = ps_n N_T(\beta_n)$, $I = I_X(\gamma)$ and $N = N_T(\gamma)$. Thus

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n \quad \text{and} \quad \lim_{n\to\infty} |C_n| = \lim_{n\to\infty} |D_n|.$$

It is clear that

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} D_n \quad \text{if} \quad \lim_{n\to\infty} |C_n| = \lim_{n\to\infty} |D_n| = 0.$$

If

$$\lim_{n\to\infty} |C_n| = \lim_{n\to\infty} |D_n| \neq 0,$$

by the continuity of Ψ we may choose α_n and β_n so that $C_n D_n > 0$, and then we also have

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} D_n.$$

The inequality

$$\big||A_n N - C_n I| - |B_n N - D_n I|\big| \leq |A_n - B_n| \cdot |N| + |C_n - D_n| \cdot I$$

proves that

$$\lim_{n\to\infty} \{|A_n N - C_n I| - |B_n N - D_n I|\} = 0,$$

or equivalently,

$$\lim_{n\to\infty} t_n |I_X(\alpha_n)N_T(\gamma) - I_X(\gamma)N_T(\alpha_n)| = \lim_{n\to\infty} ps_n |I_X(\beta_n)N_T(\gamma) - I_X(\gamma)N_T(\beta_n)|.$$

By the same reasoning, one shows that

$$\lim_{n\to\infty} t_n |I_Y(\alpha_n)N_S(\gamma) - I_Y(\gamma)N_S(\alpha_n)| = \lim_{n\to\infty} ps_n |I_Y(\beta_n)N_S(\gamma) - I_Y(\gamma)N_S(\beta_n)|.$$

The proof is complete.

4.5. An embedding of h $\overline{\pi\mathscr{I}(\mathscr{G})}$ into $\mathbf{R}^4$. Let $\mathscr{C} = \Pi_1 \cup \Pi_2 \cup \Pi_3$ be the set given at the beginning of this section, and let φ: $\mathscr{C} \longrightarrow \mathbf{R}^2$ be defined by

$$\varphi(r_1, r_2, r_3) = \begin{cases} (r_1, r_2) & \text{if } (r_1, r_2, r_3) \in \Pi_1 \cup \Pi_2, \\ (r_1, -r_2) & \text{if } (r_1, r_2, r_3) \in \Pi_3. \end{cases}$$

It is easy to see that $(r_1, r_2, r_3) \in (\Pi_1 \cup \Pi_2) \cap \Pi_3$ if and only if $r_2 = 0$. This implies that φ is continuous on $\mathscr{C}$. Moreover, φ is injective as proved below.

Let (r_1, r_2, r_3) and (t_1, t_2, t_3) be two points of $\mathscr{C}$, $\varphi(r_1, r_2, r_3) = \varphi(t_1, t_2, t_3)$. By the definition, we have $r_1 = t_1$. Also, we see easily that $r_2 = 0$ if and only if $t_2 = 0$. If $r_2 = 0$, then $(r_1, r_2, r_3), (t_1, t_2, t_3) \in \Pi_3$, and thus $(r_1, r_2, r_3) = (t_1, t_2, t_3)$. Assume that $r_2 t_2 \neq 0$, i.e. $r_2 > 0$ and $t_2 > 0$. Then either

$$\begin{aligned} (r_1, r_2) &= \varphi(r_1, r_2, r_3) = \varphi(t_1, t_2, t_3) = (t_1, t_2), \quad \text{or} \\ (r_1, -r_2) &= \varphi(r_1, r_2, r_3) = \varphi(t_1, t_2, t_3) = (t_1, -t_2), \end{aligned}$$

and thus $(r_1, r_2, r_3) = (t_1, t_2, t_3)$. Therefore, φ is injective.

By the definition of Π_1, Π_2 and Π_3, we obtain the inverse of φ immediately given by $\varphi^{-1}(t_1, t_2) = (t_1, |t_2|, |t_1 - t_2|)$ for all $(t_1, t_2) \in \varphi(\mathscr{C})$.

Since $r_1 + r_2 + r_4 + r_5 > 0$ whenever $(r_1, r_2, r_3, r_4, r_5, r_6) \in \Delta$, then the function ψ_3: $\Delta \longrightarrow \mathbf{R}^4$ defined by

$$\psi_3(r_1, r_2, r_3, r_4, r_5, r_6) = \left(\frac{\varphi(r_1, r_2, r_3)}{r_1 + r_2 + r_4 + r_5}, \frac{\varphi(r_4, r_5, r_6)}{r_1 + r_2 + r_4 + r_5} \right)$$

is continuous on Δ. We shall prove that ψ_3 is injective.

Let $(r_1, r_2, r_3, r_4, r_5, r_6)$ and $(t_1, t_2, t_3, t_4, t_5, t_6)$ be any two points of Δ with

$$\psi_3(r_1, r_2, r_3, r_4, r_5, r_6) = \psi_3(t_1, t_2, t_3, t_4, t_5, t_6).$$

Write

$$\begin{aligned} \varphi(r_1, r_2, r_3) &= (r_1', r_2'), \\ \varphi(r_4, r_5, r_6) &= (r_4', r_5'), \\ \varphi(t_1, t_2, t_3) &= (t_1', t_2') \quad \text{and} \\ \varphi(t_4, t_5, t_6) &= (t_4', t_5'). \end{aligned}$$

Then $r_j = r_j'$ and $t_j = t_j'$ for $j = 1, 4$; $r_j = |r_j'|$ and $t_j = |t_j'|$ for $j = 2, 5$;

$$r_3 = |r_1' - r_2'|, \quad r_6 = |r_4' - r_5'|, \quad t_3 = |t_1' - t_2'|, \quad t_6 = |t_4' - t_5'|.$$

Let

$$p = \frac{r_1 + r_2 + r_4 + r_5}{t_1 + t_2 + t_4 + t_5} = \frac{r_1' + |r_2'| + r_4' + |r_5'|}{t_1' + |t_2'| + t_4' + |t_5'|}.$$

By assumption, $r_j' = p t_j'$ for $j = 1, 2, 4, 5$. Since $\sum_{j=1}^{6} r_j = \sum_{j=1}^{6} t_j = 1$, then

$$\begin{aligned} 1 &= r_1' + |r_2'| + |r_1' - r_2'| + r_4' + |r_5'| + |r_4' - r_5'| \\ &= p\{t_1' + |t_2'| + |t_1' - t_2'| + t_4' + |t_5'| + |t_4' - t_5'|\} = p. \end{aligned}$$

Therefore, $(r_1, r_2, r_3, r_4, r_5, r_6) = (t_1, t_2, t_3, t_4, t_5, t_6)$.

From Theorem 4.3 together with the above discussion, we have shown the following theorem.

Theorem 4.4. *The composition* Φ *of* Ψ *followed by* ψ_3 *is a homeomorphism of* $\overline{\pi\mathscr{I}(\mathscr{G})}$ *onto a* 3*-sphere lying in* $\mathbf{R}^4$. *Moreover,*

$$\Phi(\alpha) = \left(\frac{I_X(\alpha)}{\sigma(\alpha)}, \frac{N_T(\alpha)}{\sigma(\alpha)}, \frac{I_Y(\alpha)}{\sigma(\alpha)}, \frac{N_S(\alpha)}{\sigma(\alpha)}\right) \quad \textit{for all } \alpha \in \mathscr{G}\mathscr{L},$$

where $\sigma(\alpha) = I_X(\alpha) + |N_T(\alpha)| + I_Y(\alpha) + |N_S(\alpha)|$.

5. Words for geodesics in $\widehat{\mathscr{G}}$ and their traces

In this section, we consider the Maskit embedding $\mathscr{M}_5$ of the Teichmüller space of Σ_5, which is a family of regular B-groups $G(\mu,\nu)$ parametrized by complex numbers μ and ν. Each $G(\mu,\nu)$ representing a five-punctured sphere and three thrice-punctured spheres. The regular set $\Omega(\mu,\nu)$ of $G(\mu,\nu)$ has a unique simply connected component $\Omega_0(\mu,\nu)$ invariant under $G(\mu,\nu)$ such that $\Omega_0(\mu,\nu)/G(\mu,\nu)$ is a five-punctured sphere. Every geodesic $\gamma \in \widehat{\mathscr{G}}$ corresponds to a cyclic semi-reduced Γ-word $W(\gamma;\mu,\nu)$ in $G(\mu,\nu)$. The trace $\operatorname{tr} W(\gamma;\mu,\nu)$ is a polynomial in μ and ν. The main work of this section is to compute the high order terms of the trace polynomials $\operatorname{tr} W(\gamma;\mu,\nu)$. This section is a part of the author's Ph.D. thesis [3].

5.1. Cyclic semi-reduced Γ-words for geodesics in $\widehat{\mathscr{G}}$. In this subsection, we shall give a complete description of cyclic semi-reduced Γ-words representing geodesics in $\widehat{\mathscr{G}}$. Furthermore, we shall write them in exactly two canonical forms. This reduces the difficulty of computing the high-order terms of the trace polynomials $\operatorname{tr} W(\gamma;\mu,\nu)$.

From Proposition 2.7 and [4, Theorem 3.2], we have

Theorem 5.1. *Let* $\gamma \in \widehat{\mathscr{G}}$. *If* $I_Y(\gamma) = 0$, *then* γ *is represented by a cyclic semi-reduced* Γ*-word of the form*

$$\prod_{i=1}^{m} T^{r_i} X^{\omega_i} T^{t_i} S^{\delta_i},$$

where $\delta_i, \omega_i \in \{1,-1\}$, $m = I_X(\gamma) = I_S(\gamma)$, *and* r_i *and* t_i *are integers satisfying the following conditions*:

(i) $-1 \le (r_i + t_i)\omega_i \le 0$ *and* $-1 \le (r_{i+1} + t_i)\delta_i \le 0$, *where* $r_{m+1} = r_1$.
(ii) $|r_i|, |t_i| \in \{r, r+1\}$, *where* $r = \min\{|r_i|, |t_i| : i = 1, \ldots, m\}$.
(iii) $r_i \ge 0$, $t_i \le 0$ *whenever* $\gamma \in \mathscr{G}_T^+$, *and* $r_i \le 0$, $t_i \ge 0$ *whenever* $\gamma \in \mathscr{G}_T^-$.
(iv) $\sum_{i=1}^{m} (r_i - t_i) = N_T(\gamma)$.

By considering the function Θ_2, we have

Corollary 5.2. *Let $\gamma \in \widehat{\mathscr{G}}$. If $I_X(\gamma) = 0$, then γ is represented by a cyclic semi-reduced Γ-word of the form*

$$\prod_{i=1}^{n} S^{p_i} Y^{\varepsilon_i} S^{q_i} T^{\delta_i},$$

where $\delta_i, \varepsilon_i \in \{1, -1\}$, $n = I_Y(\gamma) = I_T(\gamma)$, and p_i and q_i are integers satisfying the following conditions:

(i) $-1 \le (p_i + q_i)\varepsilon_i \le 0$ *and* $-1 \le (p_{i+1} + q_i)\delta_i \le 0$, *where* $p_{n+1} = p_1$.

(ii) $|p_i|, |q_i| \in \{p, p+1\}$, *where* $p = \min\{|p_i|, |q_i| : i = 1, \dots, n\}$.

(iii) $p_i \le 0$, $q_i \ge 0$ *whenever* $\gamma \in \mathscr{G}_S^+$, *and* $p_i \ge 0$, $q_i \le 0$ *whenever* $\gamma \in \mathscr{G}_S^-$.

(iv) $\sum_{i=1}^{n} (q_i - p_i) = N_S(\gamma)$.

In the following, we assume that $\gamma \in \widehat{\mathscr{G}}$ with $I_X(\gamma) I_Y(\gamma) > 0$. From Proposition 2.1, we may assume that $\gamma \in \mathscr{G}_S^+$ with $I_X(\gamma) \ge I_Y(\gamma)$. Let $I_Y(\gamma) = n$. Then γ is represented by a cyclic semi-reduced Γ-word W of the form

$$W = \prod_{i=1}^{n} S^{-p_i} Y^{\varepsilon_i} S^{q_i} W_i,$$

where $\varepsilon_i = \pm 1$, where $p_i \ge 0$ and $q_i \ge 0$ are integers, and where each W_i is a semi-reduced Γ-word as given in equation (5). Since

$$\mathscr{T}_1^2(W) = \prod_{i=1}^{n} S^{-p_i - 1} Y^{\varepsilon_i} S^{q_i + 1} W_i,$$

by considering the geodesic $\mathscr{T}_1^2(\gamma)$ we may assume that $p_i > 0$ and $q_i > 0$ for all i.

Now, we shall determine the subwords W_i. Note that each W_i is always followed by S^{-1} since $p_{i+1} > 0$ for each i, where $p_{n+1} = p_1$. Consider the admissible subarc γ_i represented by the reduced word $\widetilde{W_i} = \vec{S} W_i S^{-1}$. Note that

$$I_X(\gamma_i) = I_{X^{-1}}(\gamma_i) > 0, \quad I_Y(\gamma_i) = I_{Y^{-1}}(\gamma_i) = 0 \quad \text{and} \quad I_{S^{-1}}(\gamma_i) = 2 + I_S(\gamma_i),$$

for every i, and that

$$I_X(\gamma) = \sum_{i=1}^{n} I_X(\gamma_i).$$

To simplify notation, for every fixed i we write $a = m_i$ and write

$$\widetilde{W_i} = \vec{S} E_1 \cdots E_a S^{-1}.$$

Let l be the strand of γ_i joining the S^{-1}-side to the E_1-side, and let l' be the strand of γ_i joining the E_a^{-1}-side to the S^{-1}-side. Let P_0 and P_0' be the endpoints of l and l' on the S^{-1}-side respectively, and let Q_0 be the point on the S-side such that $Q_0 = S(P_0)$.

Claim. If P is the endpoint of a strand of γ_i on the S^{-1}-side, and if $P \neq P_0$ and $P \neq P_0'$, then $P \prec P_0$ and $P \prec P_0'$.

Proof of the claim. Note that such a point P exists only when $I_{S^{-1}}(\gamma_i) > 2$. Let $Q = S(P)$. Then Q is an endpoint of a strand L of γ_i connecting the S-side to the E-side for some $E \in \{X^{\pm 1}, T^{\pm 1}\}$.

If $P_0 \prec P$, then $Q_0 \prec Q$. By the definition of W_i and that of Q_0, the point Q_0 is an endpoint of a strand L_0 of γ connecting the S-side and the E'-side with $E' \in \{S^{-1}, Y^{\pm 1}\}$. This implies that L_0 intersects L. This is impossible since γ is simple. Hence, $P \prec P_0$. Similarly, $P \prec P_0'$. The proof of the claim is complete.

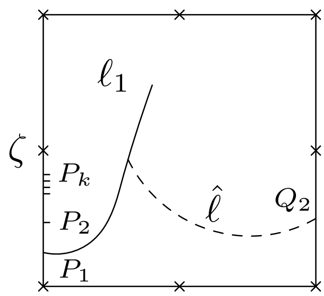

Figure 8.

Let $P_k \prec \cdots \prec P_1$ be all the points where the lift of γ_i to $\mathscr{D}$ meets the S^{-1}-side, where $k = I_{S^{-1}}(\gamma_i) \geq 2$. From the above claim, we have $\{P_1, P_2\} = \{P_0, P_0'\}$.

Let l_1 be the strand of γ_i with P_1 an endpoint, and let A_1 be the other endpoint of l_1. Note that A_1 lies on the E-side for some $E \in \{X^{\pm 1}, T^{\pm 1}\}$. Let $Q_2 = S(P_2)$. Since $I_Y(\gamma_i) = I_{Y^{-1}}(\gamma_i) = 0$, there is a simple arc $\hat{l} \subset \mathscr{D}$ joining Q_2 to A_1 which is disjoint from all strands of γ_i except possibly l_1 (see Figure 8).

Let $\hat{\gamma}_i$ be the curve on Σ_5 obtained from γ_i by replacing l_1 by $\hat{l}$. Clearly, $\hat{\gamma}_i$ is a simple loop in $\widehat{\mathscr{G}}$ with $I_Y(\hat{\gamma}_i) = 0$ and $I_X(\hat{\gamma}_i) = I_X(\gamma_i)$.

By Theorem 5.1, the free homotopy class $[\hat{\gamma}_i]$ is represented by a cyclic semi-reduced Γ-word $\widehat{W}_i$ of the form

$$\widehat{W}_i = \prod_{j=1}^{m_i'} T^{r_{ij}} X^{\omega_{ij}} T^{t_{ij}} S^{\delta_{ij}},$$

where $m_i' = I_X([\hat{\gamma}_i]) = I_X(\gamma_i)$, and r_{ij}, t_{ij}, ω_{ij} and δ_{ij} are integers satisfying the conditions given in Theorem 5.1.

Let $\hat{\gamma}_i$ be oriented so that the initial point of the projection of $\hat{l}$ to Σ_5 is the projection of A_1, and the terminal point is the projection of Q_2. We write $\widehat{W}_i$ so that $\widehat{W}_i$ represents the oriented closed curve $\hat{\gamma}_i$. Then $\delta_{im_i'} = 1$, and

$$\widetilde{W}_i = \vec{S} \left(\prod_{j=1}^{m_i'} T^{r_{ij}} X^{\omega_{ij}} T^{t_{ij}} S^{\delta_{ij}'} \right) S^{-1},$$

where $\delta'_{im'_i} = 0$ and $\delta'_{ij} \in \{1, -1\}$ for $1 \le j < m'_i$, and thus

$$W = \prod_{i=1}^{n} S^{-p_i} Y^{\varepsilon_i} S^{q_i} \left(\prod_{j=1}^{m'_i} T^{r_{ij}} X^{\omega_{ij}} T^{t_{ij}} S^{\delta'_{ij}} \right). \tag{7}$$

Theorem 5.3. *Let* $\gamma \in \widehat{\mathscr{G}}$ *with* $m = I_X(\gamma) > 0$ *and* $n = I_Y(\gamma) > 0$.

(A) *If* $m \ge n$, *then* γ *is represented by a cyclic semi-reduced* Γ*-word* $W(\gamma)$ *of the form*

$$W(\gamma) = \prod_{i=1}^{n} S^{p_i} Y^{\varepsilon_i} S^{q_i} \left(\prod_{j=1}^{m_i} T^{r_{ij}} X^{\omega_{ij}} T^{t_{ij}} S^{\delta_{ij}} \right),$$

where $\varepsilon_i, \omega_{ij} \in \{1, -1\}$, $m_i > 0$, *and* p_i, q_i, r_{ij}, t_{ij} *and* δ_{ij} *are integers satisfying the following conditions*:

(i) $\sum_{i=1}^{n} m_i = m$.

(ii) *For* $1 \le i \le n$, $\delta_{im_i} = 0$, *and if* $m_i > 1$, *then* $|\delta_{ij}| = 1$ *for* $1 \le j < m_i$.

(iii) *For* $1 \le i \le n$,

$$-1 \le (p_i + q_i)\varepsilon_i \le 0 \quad \textit{and} \quad |p_i|, |q_i| \in \{p, p+1\},$$

where $p = \min\{|p_i|, |q_i| : 1 \le i \le n\}$. *Moreover,* $p_i \le 0$, $q_i \ge 0$ *for all* i *when* $\gamma \in \mathscr{G}_S^+$, *and* $p_i \ge 0$, $q_i \le 0$ *for all* i *when* $\gamma \in \mathscr{G}_S^-$.

(iv) *For* $1 \le i \le n$ *and* $1 \le j \le m_i$,

$$-1 \le (r_{ij} + t_{ij})\omega_{ij} \le 0 \quad \textit{and} \quad |r_{ij}|, |t_{ij}| \in \{r, r+1\},$$

where $r = \min\{|r_{ij}|, |t_{ij}| : 1 \le i \le n, 1 \le j \le m_i\}$. *Moreover,* $r_{ij} \le 0$, $t_{ij} \ge 0$ *when* $\gamma \in \mathscr{G}_T^-$, *and* $r_{ij} \ge 0$, $t_{ij} \le 0$ *when* $\gamma \in \mathscr{G}_T^+$.

(v) $N_S(\gamma) = \sum_{i=1}^{n} (q_i - p_i)$ *and* $N_T(\gamma) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} (r_{ij} - t_{ij})$.

(B) *If* $n \ge m$, *then* γ *is represented by a cyclic semi-reduced* Γ*-word* $W(\gamma)$ *of the form*

$$W(\gamma) = \prod_{i=1}^{m} T^{r_i} X^{\omega_i} T^{t_i} \left(\prod_{j=1}^{n_i} S^{p_{ij}} Y^{\varepsilon_{ij}} S^{q_{ij}} T^{\delta_{ij}} \right),$$

where ε_{ij}, $\omega_i \in \{1, -1\}$, $n_i > 0$, *and* r_i, t_i, p_{ij}, q_{ij} *and* δ_{ij} *are integers satisfying the following conditions*:

(i) $\sum_{i=1}^{m} n_i = n$.

(ii) *For* $1 \le i \le m$, $\delta_{in_i} = 0$, *and if* $n_i > 1$, *then* $\delta_{ij} = \pm 1$ *for* $1 \le j < n_i$.

(iii) *For* $1 \le i \le m$,

$$-1 \le (r_i + t_i)\omega_i \le 0 \quad \textit{and} \quad |r_i|, |t_i| \in \{r, r+1\},$$

where $r = \min\{|r_i|, |t_i| : 1 \le i \le m\}$. *Moreover,* $r_i \le 0$, $t_i \ge 0$ *for all* i *when* $\gamma \in \mathscr{G}_T^-$, *and* $r_i \ge 0$, $t_i \le 0$ *for all* i *when* $\gamma \in \mathscr{G}_T^+$.

(iv) *For* $1 \le i \le m$ *and* $1 \le j \le n_i$,

$$-1 \le (p_{ij} + q_{ij})\varepsilon_{ij} \le 0 \quad \textit{and} \quad |p_{ij}|, |q_{ij}| \in \{p, p+1\},$$

where $p = \min\{|p_{ij}|, |q_{ij}| : 1 \le i \le m,\ 1 \le j \le n_i\}$. *Moreover,* $p_{ij} \le 0$, $q_{ij} \ge 0$ *when* $\gamma \in \mathscr{G}_S^+$, *and* $p_{ij} \ge 0$, $q_{ij} \le 0$ *when* $\gamma \in \mathscr{G}_S^-$.

(v) $N_T(\gamma) = \sum_{i=1}^m (r_i - t_i)$ *and* $N_S(\gamma) = \sum_{i=1}^m \sum_{j=1}^{n_i} (q_{ij} - p_{ij})$.

Remark 5.1. If $I_X(\gamma) = I_Y(\gamma) = n$, then

$$W(\gamma) = \prod_{i=1}^n S^{p_i} Y^{\varepsilon_i} S^{q_i} T^{r_i} X^{\omega_i} T^{t_i}.$$

Proof of Theorem 5.3. From Propositions 2.1 and 2.3, the assertion (B) will follow from (A) by considering the geodesic $\Theta_2(\gamma)$. Thus, we shall assume that $m \ge n$. On the other hand, since $I_E\big(\Theta_1(\gamma)\big) = I_E(\gamma)$ for $E \in \{X, Y\}$, we may assume that $\gamma \in \mathscr{G}_S^+$.

Let W be a cyclic semi-reduced Γ-word representing γ. Then $\mathscr{T}_1^2(W)$ is of the form as given in equation (7):

$$\mathscr{T}_1^2(W) = \prod_{i=1}^n S^{-p_i} Y^{\varepsilon_i} S^{q_i} \left(\prod_{j=1}^{m_i} T^{r_{ij}} X^{\omega_{ij}} T^{t_{ij}} S^{\delta_{ij}} \right)$$

with $p_i > 0$ and $q_i > 0$ for all i, and thus

$$W = \prod_{i=1}^n S^{-p_i'} Y^{\varepsilon_i} S^{q_i'} \left(\prod_{j=1}^{m_i} T^{r_{ij}} X^{\omega_{ij}} T^{t_{ij}} S^{\delta_{ij}} \right),$$

where $p_i' = p_i - 1 \ge 0$ and $q_i' = q_i - 1 \ge 0$ for $i = 1, \ldots, n$.

It follows from Proposition 2.7 that

$$N_S(\gamma) = \sum_{i=1}^n (q_i' - p_i') \quad \text{and} \quad N_T(\gamma) = \sum_{i=1}^n \sum_{j=1}^{m_i} (r_{ij} - t_{ij}).$$

This proves condition (v).

It remains to prove that if γ is represented by the word W given in (A), then

(iii)′ $|p_i|, |q_i| \in \{p, p+1\}$ for $1 \le i \le n$, and
(iv)′ $|r_{ij}|, |t_{ij}| \in \{r, r+1\}$ for $1 \le i \le n$ and $1 \le j \le m_i$,

where

$$p = \min\{|p_i|, |q_i| : 1 \le i \le n\} \quad \text{and} \quad r = \min\{|r_{ij}|, |t_{ij}| : 1 \le i \le n,\ 1 \le j \le m_i\}.$$

Note that the other conditions follow from Lemma 2.6.

We shall prove condition (iii)$'$. Condition (iv)$'$ will follow by a similar argument. By applying a cyclic permutation to the word W, we may assume that $p = \min\{|p_1|, |q_1|\}$. By considering W^{-1}, we may assume that $\varepsilon_1 = 1$.

Without loss of generality, we assume that $\gamma \in \mathscr{G}_S^+$, and write

$$W = \prod_{i=1}^{n} S^{-p_i} Y^{\varepsilon_i} S^{q_i} \left(\prod_{j=1}^{m_i} T^{r_{ij}} X^{\omega_{ij}} T^{t_{ij}} S^{\delta_{ij}} \right), \quad p_i, q_i \geq 0 \text{ for all } i.$$

Since $q_1 - p_1 = (q_1 - p_1)\varepsilon_1 \leq 0$, then $p = q_1$.

There is nothing to prove if $n = 1$. Assume that $n > 1$. Suppose that there is an $i_0 > 1$ such that $\max\{p_{i_0}, q_{i_0}\} > p + 1$.

$$\mathscr{T}_1^{-2p}(W) = \prod_{i=1}^{n} S^{-p_i'} Y^{\varepsilon_i} S^{q_i'} \left(\prod_{j=1}^{m_i} T^{r_{ij}} X^{\omega_{ij}} T^{t_{ij}} S^{\delta_{ij}} \right),$$

where $p_i' = p_i - p$ and $q_i' = q_i - p$ for all i.

Let $\gamma' = \mathscr{T}_1^{-2p}(\gamma)$. Since $q_1' = q_1 - p = 0$, then γ' has a strand joining the Y^{-1}-side to the E-side for some $E \in \{X^{\pm}, T^{\pm}\}$. On the other hand, $\max\{p_{i_0}', q_{i_0}'\} > 1$, then γ' has a strand joining the S-side to the S^{-1}-side. This is impossible! The proof is complete.

5.2. Trace polynomials. In what follows, let G be the subgroup of $\mathrm{PSL}(2, \mathbf{C})$ generated by the following four parabolic transformations:

$$S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix},$$

$$X = \begin{pmatrix} 1+4i & 16 \\ 1 & 1-4i \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1+4i & 4 \\ 4 & 1-4i \end{pmatrix}.$$

By using Maskit's first combination theorem ([8, Theorem VII.C.2]), one can prove that G is a regular B-group representing a five-punctured sphere and three thrice punctured spheres. The regular set of G has a simply connected component Ω_0 invariant under G such that $\Omega_0/G = \Sigma_5$. Such a Kleinian group G will be called a *Maskit five-punctured group.*

There is a connected and simply connected fundamental domain $\mathscr{D}$ for G acting on Ω_0 (see Figure 9) with $\Gamma = \{S^{\pm 1}, T^{\pm 1}, X^{\pm 1}, Y^{\pm 1}\}$ the set of side pairings. The domain $\mathscr{D}$ may be schematically drawed as in Figure 1 with sides labelled as before. Thus every geodesic in $\mathscr{G}$ is represented by a cyclic semi-reduced Γ-word given in Theorem 5.1, Corollary 5.2 or Theorem 5.3.

Now, we consider the quasiconformal conjugates of G. Let f be a quasiconformal automorphism of $\widehat{\mathbf{C}}$ such that fGf^{-1} is a Kleinian group. If f is normalized

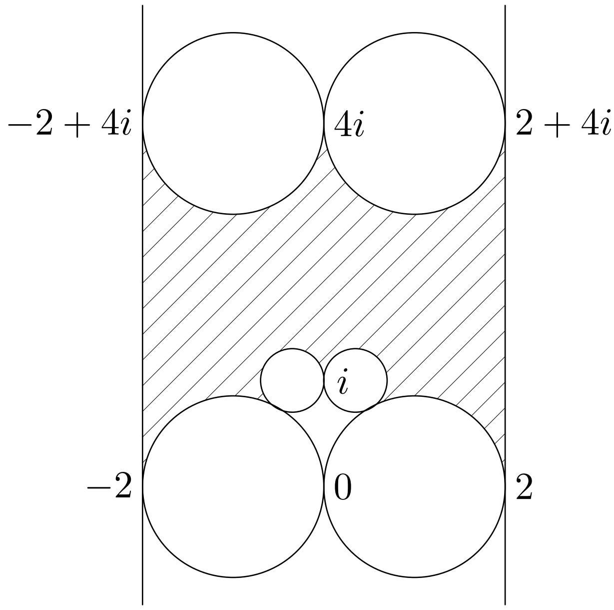

Figure 9. The fundamental domain $\mathscr{D}$.

to fix $0, 1$ and ∞, then fGf^{-1} is the subgroup of $\mathrm{PSL}(2, \mathbf{C})$ generated by S, T, X_μ and Y_ν, where

$$X_\mu = \begin{pmatrix} 1+\mu & -\mu^2 \\ 1 & 1-\mu \end{pmatrix} \quad \text{and} \quad Y_\nu = \begin{pmatrix} 1+2\nu & 4 \\ -\nu^2 & 1-2\nu \end{pmatrix}$$

with complex numbers μ and ν satisfying $|\mu| \geq 1$, $|\nu| \geq \frac{1}{2}$ and $|\mu\nu + 2| \geq 1$.

For any two non-zero complex numbers μ and ν, let $G(\mu, \nu)$ be the subgroup of $\mathrm{PSL}(2, \mathbf{C})$ generated by S, T, X_μ and Y_ν. We refer to the set $\mathscr{M}_5$ of all $(\mu, \nu) \in \mathbf{C}^2$ with $\operatorname{Im}\mu > 0$ and $\operatorname{Im}\nu > 0$ such that $G(\mu, \nu)$ is a Maskit five-punctured group as the Maskit embedding of the Teichmüller space of Σ_5.

For every $(\mu, \nu) \in \mathscr{M}_5$, let $\rho_{(\mu,\nu)}\colon G \longrightarrow G(\mu, \nu)$ be the isomorphism defined by

$$\rho_{(\mu,\nu)}(S) = S, \quad \rho_{(\mu,\nu)}(T) = T, \quad \rho_{(\mu,\nu)}(X) = X_\mu \quad \text{and} \quad \rho_{(\mu,\nu)}(Y) = Y_\nu.$$

For every $\gamma \in \widehat{\mathscr{G}}$, let $W(\gamma) \in G$ be a cyclic semi-reduced Γ-word representing γ, and let $W(\gamma; \mu, \nu) = \rho_{(\mu,\nu)}\big(W(\gamma)\big)$. Write the trace polynomial $\operatorname{tr} W(\gamma; \mu, \nu)$ as

$$F(\gamma; \mu, \nu) = \operatorname{tr} W(\gamma; \mu, \nu) = a_1\mu^r\nu^s + a_2\mu^{r-1}\nu^s + a_3\mu^r\nu^{s-1} + O(r+s-2),$$

where $a_1 \neq 0$, a_2 and a_3 are integers, and where $O(r+s-2)$ is a polynomial in μ and ν of degree $\leq r+s-2$. We call $a_1\mu^r\nu^s + a_2\mu^{r-1}\nu^s + a_3\mu^r\nu^{s-1}$ the *high order terms* of $F(\gamma; \mu, \nu)$.

If $I_Y(\gamma)=0$ and $I_X(\gamma)=m>0$, then from [4, Theorem 3.4] we have

$$F(\gamma;\mu,\nu)=\pm\{\mu^{2m}+4N_T(\gamma)\mu^{2m-1}\}+O(\mu^{2m-2}), \tag{8}$$

where $O(\mu^{2m-2})$ is a polynomial in μ of degree $\le 2m-2$.

If $I_X(\gamma)=0$ and $I_Y(\gamma)=n>0$, then from Lemma 5.4(ii) given below we have

$$F(\gamma;\mu,\nu)=\pm 4^n\{\nu^{2n}+2N_S(\gamma)\nu^{2n-1}\}+O(\nu^{2n-2}), \tag{9}$$

where $O(\nu^{2n-2})$ is a polynomial in ν of degree $\le 2n-2$.

Lemma 5.4. *If $\gamma\in\widehat{\mathscr{G}}$ with $I_X(\gamma)=m$ and $I_Y(\gamma)=n$, then*
(i) $F\big(\Theta_1(\gamma);\mu,\nu\big)=F(\gamma;-\mu,-\nu)$,
(ii) $F\big(\Theta_2(\gamma);\mu,\nu\big)=F(\gamma;-2\nu,-\frac12\mu)$,
(iii) $F\big(\mathscr{T}_1(\gamma);\mu,\nu\big)=(-1)^nF(\gamma;\mu,\nu+1)$,
(iv) $F\big(\mathscr{T}_1^{-1}(\gamma);\mu,\nu\big)=(-1)^nF(\gamma;\mu,\nu-1)$,
(v) $F\big(\mathscr{T}_2(\gamma);\mu,\nu\big)=(-1)^mF(\gamma;\mu-2,\nu)$, *and*
(vi) $F\big(\mathscr{T}_2^{-1}(\gamma);\mu,\nu\big)=(-1)^mF(\gamma;\mu+2,\nu)$.

Proof. Let

$$C_1=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}\quad\text{and}\quad C_2=\begin{pmatrix} 0 & -2i\\ 1/2i & 0\end{pmatrix},$$

and let $\chi_j(A)=C_jAC_j^{-1}$ for all $A\in \mathrm{PSL}(2,\mathbf{C})$. Set $\rho_j=\chi_j\Theta_j$. A direct computation gives

$$\begin{gathered}\rho_j(S)=S,\qquad \rho_j(T)=T,\qquad \rho_1(X_\mu)=X_{-\mu},\\ \rho_1(Y_\nu)=Y_{-\nu},\quad \rho_2(X_\mu)=X_{-2\nu},\quad \rho_2(Y_\nu)=Y_{-\mu/2}.\end{gathered}$$

By a similar argument as that in the proof of Lemma 3.3 of [4], the assertions (i) and (ii) will follow.

Since the transformations S, T and X_μ are invariant under $\mathscr{T}_1$, and since

$$\mathscr{T}_1(Y_\nu)=Y_\nu^{-1}S=-Y_{\nu+1}\quad\text{and}\quad \mathscr{T}_1^{-1}(Y_\nu)=SY_\nu^{-1}=-Y_{\nu-1},$$

then (iii) and (iv) are valid. From (ii) and (iii), we have

$$\begin{aligned}F\big(\mathscr{T}_2(\gamma);\mu,\nu\big)&=F\big(\Theta_2\mathscr{T}_1\Theta_2(\gamma);\nu,\mu\big)=F\big(\mathscr{T}_1\Theta_2(\gamma);-2\nu,-\tfrac12\mu\big)\\ &=(-1)^{I_Y\left(\Theta_2(\gamma)\right)}F\big(\Theta_2(\gamma);-2\nu,-\tfrac12\mu+1\big)=(-1)^mF(\gamma;\mu-2,\nu).\end{aligned}$$

This proves (v). Similarly, the equation given in (vi) will follow from (ii) and (iv).

In the rest of this section, we shall compute the high-order terms of $F(\gamma;\mu,\nu)$ for $\gamma \in \widehat{\mathscr{G}}$ with $I_X(\gamma)I_Y(\gamma) > 0$.

Let $I_X(\gamma) = m$ and $I_Y(\gamma) = n$. Assume that $m \geq n$, and that $\gamma \in \mathscr{G}_T^-$. Then γ is represented by a cyclic semi-reduced Γ-word given below:

$$W = \prod_{i=1}^{n} S^{p_i} Y^{\varepsilon_i} S^{q_i} \Big(\prod_{j=1}^{m_i} T^{-r_{ij}} X^{\omega_{ij}} T^{t_{ij}} S^{\delta_{ij}} \Big),$$

where $r_{ij}, t_{ij} \geq 0$. Note that

$$N_T(\gamma) = -\sum_{i=1}^{n}\sum_{j=1}^{m_i}(r_{ij}+t_{ij}) \quad \text{and} \quad N_S(\gamma) = \sum_{i=1}^{n}(q_i - p_i).$$

For integers $r \geq 0$, $t \geq 0$, p and q, and for $\omega, \delta, \varepsilon \in \{1,-1\}$, we have:

$$T^{-r}X^{\omega}T^{t} = \begin{pmatrix} \omega\mu + 1 - 4r\omega & -\omega\mu^2 + 4(r+t)\omega\mu + \text{const.} \\ \omega & -\omega\mu + 1 + 4t\omega \end{pmatrix},$$

$$S^{p}Y^{\varepsilon}S^{q} = \begin{pmatrix} 2\varepsilon\nu + 1 + 4\varepsilon q & 4\varepsilon \\ -\varepsilon\nu^2 + 2\varepsilon(p-q)\nu + \text{const.} & -2\varepsilon\nu + 1 + 4\varepsilon p \end{pmatrix},$$

$$T^{-r}X^{\omega}T^{t}S^{\delta} =$$

$$\begin{pmatrix} -\omega\delta\mu^2 + (1 + 4(r+t)\delta)\omega\mu + \text{const.} & -\omega\mu^2 + 4(r+t)\omega\mu + \text{const.} \\ -\omega\delta\mu + \text{const.} & -\omega\mu + 1 + 4t\omega \end{pmatrix}.$$

For $i = 1,\ldots,n$, let $\xi_i = \omega_{i1}$ when $m_i = 1$, let

$$\xi_i = \Big(\prod_{j=1}^{m_i}\omega_{ij}\Big)\Big(\prod_{j=1}^{m_i-1}\delta_{ij}\Big) \quad \text{when } m_i > 1,\ \lambda_i = 4\sum_{j=1}^{m_i}(r_{ij}+t_{ij}),$$

and let

$$W_i = \prod_{j=1}^{m_i} T^{-r_{ij}} X^{\omega_{ij}} T^{t_{ij}} S^{\delta_{ij}} = \begin{pmatrix} a_i(\mu) & b_i(\mu) \\ c_i(\mu) & d_i(\mu) \end{pmatrix}.$$

If $m_i = 1$, then

$$\begin{aligned} a_i(\mu) &= \xi_i(\mu + \text{const.}) = \xi_i(\mu^{2m_i-1} + \cdots), \\ b_i(\mu) &= -\xi_i(\mu^2 - \lambda_i\mu + \text{const.}) = -\xi_i(\mu^{2m_i} - \lambda_i\mu^{2m_i-1} + \cdots), \\ c_i(\mu) &= \xi_i = \xi_i(\mu^{2m_i-2} + \cdots), \quad \text{and} \\ d_i(\mu) &= -\xi_i(\mu + \text{const.}) = -\xi_i(\mu^{2m_i-1} + \cdots). \end{aligned}$$

By induction, one can show that for $m_i \geq 1$

$$\begin{aligned} a_i(\mu) &= (-1)^{m_i}\xi_i(-\mu^{2m_i-1}+\cdots), \\ b_i(\mu) &= (-1)^{m_i}\xi_i(\mu^{2m_i}-\lambda_i\mu^{2m_i-1}+\cdots), \\ c_i(\mu) &= (-1)^{m_i}\xi_i(-\mu^{2m_i-2}+\cdots), \quad \text{and} \\ d_i(\mu) &= (-1)^{m_i}\xi_i(\mu^{2m_i-1}+\cdots). \end{aligned}$$

For every $i = 1, \ldots, n$, let

$$S^{p_i}Y^{\varepsilon_i}S^{q_i}W_i = \begin{pmatrix} \tilde{a}_i(\mu,\nu) & \tilde{b}_i(\mu,\nu) \\ \tilde{c}_i(\mu,\nu) & \tilde{d}_i(\mu,\nu) \end{pmatrix},$$

and for every n let

$$\prod_{i=1}^{n} S^{p_i}Y^{\varepsilon_i}S^{q_i}W_i = \begin{pmatrix} A_n(\mu,\nu) & B_n(\mu,\nu) \\ C_n(\mu,\nu) & D_n(\mu,\nu) \end{pmatrix}.$$

A direct computation gives:

$$\deg \tilde{a}_i = 2m_i, \quad \deg \tilde{b}_i = 2m_i + 1 = \deg \tilde{c}_i, \quad \deg \tilde{d}_i = 2m_i + 2$$

and

$$\tilde{d}_i(\mu,\nu) = (-1)^{m_i-1}\xi_i\varepsilon_i\big(\nu^2\mu^{2m_i} - \lambda_i\nu^2\mu^{2m_i-1} + 2(q_i-p_i)\nu\mu^{2m_i} + \cdots\big).$$

By applying induction to n, we have

$$\begin{aligned} \deg A_n(\mu,\nu) &= 2(n-1) + 2\sum_{i=1}^{n} m_i, \\ \deg B_n(\mu,\nu) &= 2n - 1 + 2\sum_{i=1}^{n} m_i = \deg C_n(\mu,\nu), \\ \deg D_n(\mu,\nu) &= 2n + 2\sum_{i=1}^{n} m_i, \end{aligned}$$

and the high-order terms of $D_n(\mu,\nu)$ are determined by

$$\prod_{i=1}^{n}\tilde{d}_i(\mu,\nu) = \prod_{i=1}^{n}(-1)^{m_i-1}\xi_i\varepsilon_i(\nu^2\mu^{2m_i} - \lambda_i\nu^2\mu^{2m_i-1} + 2(q_i-p_i)\nu\mu^{2m_i} + \cdots).$$

Since $F(\gamma;\mu,\nu) = A_n(\mu,\nu) + D_n(\mu,\nu)$ and $\deg A_n(\mu,\nu) < \deg D_n(\mu,\nu) - 1$, then the high-order terms of $F(\gamma;\mu,\nu)$ are determined by $D_n(\mu,\nu)$.

For any two polynomials

$$f(\mu,\nu) = a_1\mu^r\nu^s + a_2\mu^{r-1}\nu^s + a_3\mu^r\nu^{s-1} + \cdots \quad \text{and}$$
$$g(\mu,\nu) = b_1\mu^{r'}\nu^{s'} + b_2\mu^{r'-1}\nu^{s'} + b_3\mu^{r'}\nu^{s'-1} + \cdots,$$

the high-order terms of the polynomial $f(\mu,\nu)g(\mu,\nu)$ is

$$a_1b_1\mu^{r+r'}\nu^{s+s'} + (a_1b_2 + a_2b_1)\mu^{r+r'-1}\nu^{s+s'} + (a_1b_3 + a_3b_1)\mu^{r+r'}\nu^{s+s'-1}.$$

Thus, we have

$$\begin{aligned} F(\gamma;\mu,\nu) &= \pm\Big\{\nu^{2n}\mu^{2m} - \Big(\sum_{i=1}^{n}\lambda_i\Big)\nu^{2n}\mu^{2m-1} + 2\Big(\sum_{i=1}^{n}(q_i - p_i)\Big)\mu^{2m}\nu^{2n-1} + \cdots\Big\} \\ &= \pm\{\mu^{2m}\nu^{2n} + 4N_T(\gamma)\mu^{2m-1}\nu^{2n} + 2N_S(\gamma)\mu^{2m}\nu^{2n-1} + \cdots\}. \end{aligned}$$

From Proposition 2.1 and Lemma 5.4, the above equations are also valid for $\gamma \in \mathscr{G}_T^+$ with $I_X(\gamma) \geq I_Y(\gamma)$.

If $n = I_Y(\gamma) \geq I_X(\gamma) = m$, then, by Proposition 2.1 and Lemma 5.4 again, we have

$$\begin{aligned} F(\gamma;\mu,\nu) &= F\big(\Theta_2(\gamma); -2\nu, -\tfrac{1}{2}\mu\big) \\ &= \pm 4^{n-m}\{\mu^{2m}\nu^{2n} + 4N_T(\gamma)\mu^{2m-1}\nu^{2n} + 2N_S(\gamma)\mu^{2m}\nu^{2n-1} + \cdots\}. \end{aligned}$$

Summing up above discussion together with equations (8) and (9), we have proved the following theorem.

Theorem 5.5 (trace formula). *Let $\gamma \in \widehat{\mathscr{G}}$ with $I_X(\gamma) = m$ and $I_Y(\gamma) = n$. If $m \geq n$, then*

$$F(\gamma;\mu,\nu) = \pm\{\mu^{2m}\nu^{2n} + 4N_T(\gamma)\mu^{2m-1}\nu^{2n} + 2N_S(\gamma)\mu^{2m}\nu^{2n-1} + \cdots\}.$$

If $m \leq n$, then

$$F(\gamma;\mu,\nu) = \pm 4^{n-m}\{\mu^{2m}\nu^{2n} + 4N_T(\gamma)\mu^{2m-1}\nu^{2n} + 2N_S(\gamma)\mu^{2m}\nu^{2n-1} + \cdots\}.$$

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